A Box-Shaped Cyclically Reduced Operator

Chen Greif
University of British Columbia

Joint work with Rob Hocking, UBC

Chinese Academy of Sciences
Beijing
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The Convection-Diffusion Equation

$$\mathcal{L}u = f\,,$$

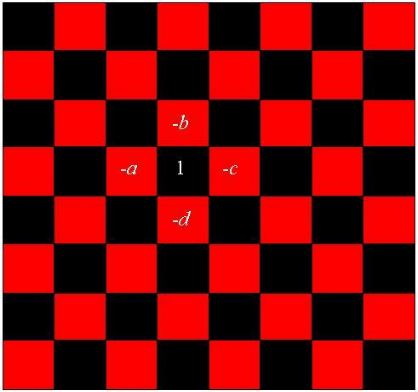
in $d = 1,\ 2$ or 3 dimensions with appropriate boundary conditions, where $\mathcal{L}$ is a d-dimensional partial differential operator.

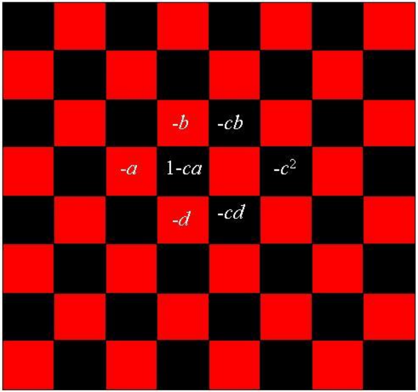
Model Problem: convection-diffusion equation

$$\nabla \cdot (p\,\nabla)\,u + q \cdot \nabla u = f,$$

where p and q are functions in d variables, possibly with variable coefficients. For simplicity, assume Dirichlet boundary conditions.

$q = (\sigma, \tau, \mu)$ for 3D constant coefficients.


-b
-a
1
-c
-d


-b
-cb
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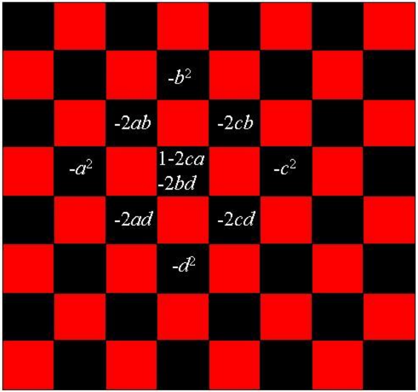
1-ca
-c2
-d
-cd


$-b^2$
$-2ab$
$-2cb$
$-a^2$
$1-2ca$
$-2bd$
$-c^2$
$-2ad$
$-2cd$
$-d^2$

2D: Elman and Golub (1990-1992);
3D: Greif and Varah (1998-2000)

Cyclically reduced operators work on a 'skewed' grid and have a big stencil:

$$\begin{aligned} R_2\, u_{i,j} = \; & (a^2 - 2be - 2cd)\, u_{i,j} - e^2\, u_{i,j+2} - 2de\, u_{i+1,j+1} - c^2\, u_{i-2,j} \\ & - d^2\, u_{i+2,j} - 2bc\, u_{i-1,j-1} - b^2\, u_{i,j-2} - 2ce\, u_{i-1,j+1} - 2bd\, u_{i+1,j-1}. \end{aligned}$$

$$\begin{aligned} R_3\, u_{i,j,k} = \; & (a^2 - 2be - 2cd - 2fg)\, u_{i,j,k} - f^2\, u_{i,j,k-2} - 2ef\, u_{i,j+1,k-1} \\ & - 2cf\, u_{i-1,j,k-1} - 2df\, u_{i+1,j,k-1} - 2bf\, u_{i,j-1,k-1} - e^2\, u_{i,j+2,k} \\ & - 2de\, u_{i+1,j+1,k} - c^2\, u_{i-2,j,k} - d^2\, u_{i+2,j,k} - 2bc\, u_{i-1,j-1,k} \\ & - b^2\, u_{i,j-2,k} - 2eg\, u_{i,j+1,k+1} - 2cg\, u_{i-1,j,k+1} - 2ce\, u_{i-1,j+1,k} \\ & - 2bd\, u_{i+1,j-1,k} - 2dg\, u_{i+1,j,k+1} - 2bg\, u_{i,j-1,k+1} - g^2\, u_{i,j,k+2}. \end{aligned}$$

(BUT: they work on a smaller number of dof's and have great spectral properties!)

Questions to Investigate

1. Can we make the cyclically reduced operator more appealing in terms of stencil and ease of implementation?
2. Multigrid and cyclic reduction (ongoing work with Yavneh)

Reduction Using a 'Skewed' Operator (G., Hocking [2009-2010])

Mesh Reynolds numbers by

$$\gamma = \frac{\sigma h}{2}, \qquad \delta = \frac{\tau h}{2}, \qquad \eta = \frac{\mu h}{2}.$$

$$F^{+} u_{i,j} = a u_{i,j} + b u_{i,j-1} + c u_{i-1,j} + d u_{i+1,j} + e u_{i,j+1},$$

where

$$a = 4,\ b = -1 - \delta,\ c = -1 - \gamma,\ d = -1 + \gamma,\ e = -1 + \delta.$$

$$F^{\times} u_{i,j} = a u_{i,j} + b u_{i+1,j+1} + c u_{i-1,j+1} + d u_{i-1,j-1} + e u_{i+1,j-1},$$

after scaling by $2h^2$, where

$$a = 4,\ b = -1+\gamma+\delta,\ c = -1-\gamma+\delta,\ d = -1-\gamma-\delta,\ e = -1+\gamma-\delta.$$

Similar idea (though more subtle) for 3D.

Figure: Four color ordering applied to a 7×7 grid. Use F^{+} for yellow and blue, and $F^{\times}$ for red and green.

The Linear System

The $(2n+1)^2 \times (2n+1)^2$ linear system can be written in block form as

$$\begin{pmatrix} D_1 & B_1 & 0 & 0 \\ B_2 & D_2 & 0 & 0 \\ B_3 & B_4 & D_3 & 0 \\ B_5 & B_6 & 0 & D_4 \end{pmatrix} \begin{pmatrix} u^{red} \\ u^{green} \\ u^{blue} \\ u^{yellow} \end{pmatrix} = \begin{pmatrix} 2h^2 f^{red} \\ 2h^2 f^{green} \\ h^2 f^{blue} \\ h^2 f^{yellow} \end{pmatrix},$$

where the matrices D_1, D_2, D_3, and D_4 are diagonal. We apply a block elimination procedure to obtain a reduced (Schur complement) system of size $n^2 \times n^2$, involving only the green points:

$$(D_2 - B_2 D_1^{-1} B_1) u^{green} = 2h^2 (f^{green} - B_2 D_1^{-1} f^{red}).$$

Proposition. The difference equation for the operator for the green points is equivalent to standard finite difference discretization of the differential equation

$$-\left[\left(1+\frac{\sigma^2h^2}{4}\right)u_{xx}+\left(1+\frac{\tau^2h^2}{4}\right)u_{yy}\right]+\sigma u_x+\tau u_y=f+O(h^2).$$

Proposition. Suppose the new cyclic reduction process is applied to the 2D convection-diffusion problem, yielding a box shaped 9 point operator. Then the operator is identical to the standard cyclically reduced operator applied to a grid rotated 45 degrees clockwise with mesh spacing $\sqrt{2}h$.

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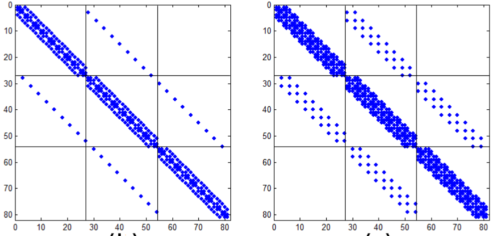
Block Orderings: 2D

25	26	27	52	53	54	79	80	81
22	23	24	49	50	51	76	77	78
19	20	21	46	47	48	73	74	75
16	17	18	43	44	45	70	71	72
13	14	15	40	41	42	67	68	69
10	11	12	37	38	39	64	65	66
7	8	9	34	35	36	61	62	63
4	5	6	31	32	33	58	59	60
1	2	3	28	29	30	55	56	57

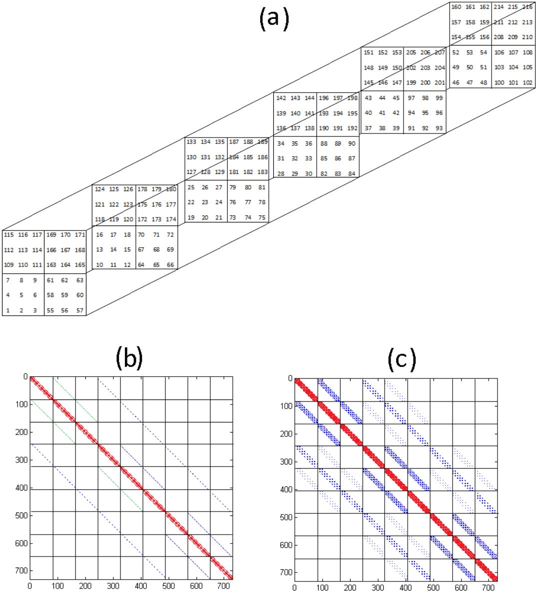
(a)



(b) (c)

Figure: (a) 3-line ordering applied to a 9×9 grid. The grid blocks are separated by lines. (b) The induced block structure of the 5-point operator matrix. (c) The induced block structure of the 9-point operator matrix.

Block Orderings: 3D



Techniques for Computing Bounds on Convergence: Permutations and Eigenvalue Computations

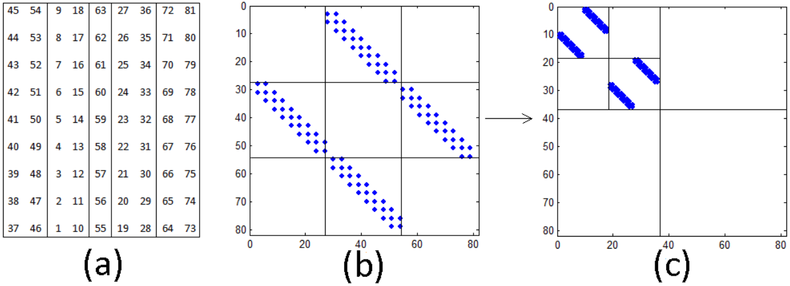


Figure: When unknowns are reordered as in (a), the matrix N_{kL} (b) takes on block diagonal form (c).

Block Jacobi Bound

$$\rho_{kL} \leq \frac{2bc(1+\cos(2\pi h))}{a^2 - 4bc(1+\cos\left(\frac{\pi}{k+1}\right))(1+\cos(2\pi h))}[1+\delta_{1k}(2\cos(2\pi h)-1)].$$

In the case $k = 1$, the bound is attained.

	7-pt $1P$	7-pt $2P$	19-pt $2P$	27-pt $1P$	27-pt $2P$
7-pt $1P$	1				
7-pt $2P$	2	1			
19-pt $2P$	2.67	1.34	1		
27-pt $1P$	5.35	2.67	2	1	
27-pt $2P$	9.14	4.57	3.42	1.71	1

Table: Relative asymptotic convergence rates for different combinations of discretization scheme and grid ordering. The number in row i and column j is the ratio of the asymptotic convergence rate of method i to method j, using Taylor expansions.

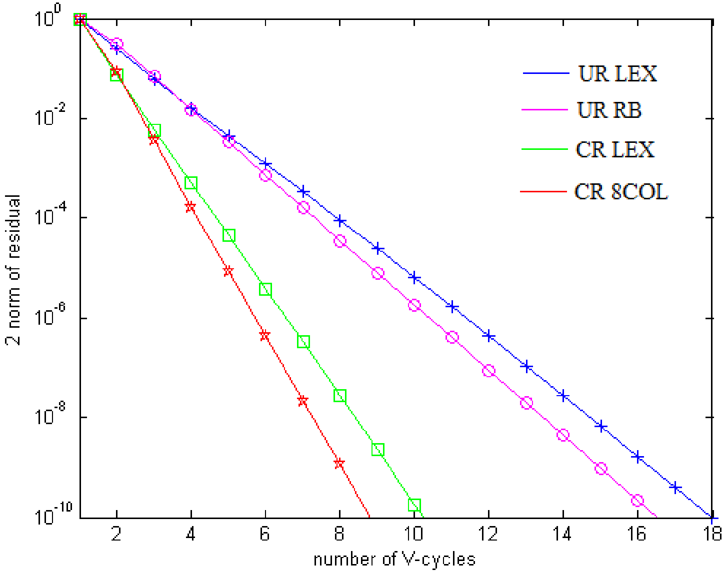
(a) 2D

1-line	n=129	n=257	n=513
$+$	40	65	115
$\diamond$	18	30	57
$\square$	14	23	42
2-line	n=129	n=257	n=513
$+$	28	45	85
$\diamond$	20	37	63
$\square$	15	25	43

(b) 3D

1-plane	n=17	n=33	n=65
7-point	7	12	23
19-point	5	9	14
27-point	3	6	11
2-plane	n=17	n=33	n=65
7-point	7	13	24
19-point	5	8	14
27-point	3	6	11

Table: Variable coefficient problem: number of iterations for GMRES preconditioned with ILU(0.01) to bring the norm of the relative residual down by a factor of 10^{-4}.



10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
2 norm of residual
2
4
6
8
10
12
14
16
18
number of V-cycles
UR LEX
UR RB
CR LEX
CR 8COL

Conclusions

- We have introduced new cyclically reduced operators for the discrete convection-diffusion equation in 2D and 3D.
- The operators are derived by applying multicolor orderings (2^d colors in d dimensions) and assigning different discretizations to different colors, so that an elimination of all but one color is possible.
- Simple stencils on a cartesian grid: a 9-point operator in the 2D case and a 27-point operator in the 3D case. The operators maintain the same good spectral properties that other cyclically reduced operators have, and at the same time, the structure of their stencil makes them easy to implement.
- We have analyzed the effect of block orderings in conjunction with our new operator, and have found improvements in convergence rates.