A Scaled Central Path for Linear Programming

Ya-xiang Yuan*

State Key Laboratory of Scientific and Engineering Computing,

Institute of Computational Mathematics and Scientific/Engineering Computing,

Chinese Academy of Sciences, P. O. Box 2719, Beijing 100080, China

Email-addresses: yyx@lsec.cc.ac.cn

dedicated to the 80-th birthday of late Professor Kang FENG

Abstract

Interior point methods are very efficient methods for solving large scale linear programming problems. The central path plays a very important role in interior point methods. In this paper we propose a new central path, which scales the variables. Thus it has the advantage of forcing the path has roughly the same distance from each active constraint boundary nearly the solution.

1 Introduction

Interior point methods are one of the most intensive studied topics in optimization. Thousands of publications have been appeared on interior point methods. A very good recent review is given by [4]. Interior point methods have very good theoretical properties including the nice polynomial complexity property. And more important is that numerous applications have shown that interior point methods are very efficient for solving large sparse linear programming problems. Interior point methods have been proved to be indispensable to semi-definite programming, another class of important optimization problems. Interior point methods have also applied to nonlinear programming and nonlinear complementary problems. For example, please see ([1, 2, 3]).

Path following algorithms are a class of very important interior point methods for linear programming. Consider the following standard linear programming problem

$$\min c^T x \tag{1.1}$$

subject to

$$Ax = b, \qquad x \geq 0, \tag{1.2}$$

*This work is partially supported by Chinese NNSF grants 19731010 and the Knowledge Innovation Program of Chinese Academy of Sciences

where $c \in \Re^n$, $b \in \Re^m$ and $A \in \Re^{m \times n}$. The dual problem for the above linear program can be written as

$$\max \; b^T y \tag{1.3}$$

subject to

$$A^T y + s = c, \qquad s \geq 0, \tag{1.4}$$

where $y \in \Re^m$ are the dual variables and $s \in \Re^n$ are the slack variables. If both the prime problem (1.1)-(1.2) and the dual problem (1.3)-(1.4) have feasible solutions, then both problems have optimal solutions. And, in this case, for any solution x^* of the primal problem and any solution (y^*, s^*) of the dual problem, we have that

$$c^T x^* = b^T y^*, \tag{1.5}$$

(see, [5]). For any point (x, y, x) that satisfies (1.2) and (1.4), it follows that

$$c^T x - b^T y = (A^T y + s)^T x - (Ax)^T y = s^T x \geq 0. \tag{1.6}$$

Thus, a solution is obtained as long as the complementarity gap $s^T x$ is zero. Because both x and s are nonnegative, the condition $s^T x = 0$ is equivalent to $x_i s_i = 0$ for all $i = 1, ..., n$. Let $X = diag[x_1, x_2, ..., x_n]$, relation $s^T x = 0$ can be expressed as $Xs = 0$. Thus, we can write the optimal conditions in the following form

$$\begin{aligned} Ax &= b & (1.7)\\ A^T y + s &= c & (1.8)\\ Xs &= 0 & (1.9)\\ (x, s) &\geq 0. & (1.10) \end{aligned}$$

Define the set

$$\mathcal{F} = \{(x, y, s) \; Ax = b, A^T y + s = c, x \geq 0, s \geq 0\}, \tag{1.11}$$

which is the direct product of the primal feasible set and the dual feasible set. Interior point methods generate iterate point in the interior of the region $\mathcal{F}$, that is

$$int(\mathcal{F}) = \{(x, y, s) \; Ax = b, A^T y + s = c, x > 0, s > 0\}. \tag{1.12}$$

The central path defined by the following system

$$\begin{aligned} Ax &= b & (1.13)\\ A^T y + s &= c & (1.14)\\ Xs &= \mu e & (1.15)\\ (x, s) &> 0, & (1.16) \end{aligned}$$

where e is a vector whose elements are all 1, and $\mu > 0$ is a parameter. It is easy to see that system (1.13)-(1.16) is a perturbation of the optimal condition (1.7)-(1.10). Let

$(x(\mu), y(\mu), s(\mu)$ be on the central path, it can be shown that $x(\mu)$ is a solution of the penalized problem

$$\min c^T x - \mu \sum_{i=1}^{n} \log(x_i) \tag{1.17}$$

subject to

$$Ax = b. \tag{1.18}$$

Many interior point methods use the central path. Some algorithms explicitly use the central path as they force the interate points follow the central path. Even for some algorithm that do not use the central path directly in the algorithm statements, the central path is used for convergence analyses (see, [5]).

Because of the importance of the central path in the designs and analyses of interior point methods, we study the center path. From the views of complementary conditions, we propose a new central path. We believe that this path can also be used to construct new interior point methods.

A new central path is derived in the next section, and in Section 3 we proposed two ways to compute search directions based on the new central path and we also give a brief discussion on how our ideas can be further extended.

2 A Scaled Central Path

The condition (1.15) of the central path requires that each element of Xs are the same when they approach zero. The condition replaces the complementarity condition

$$x_i s_i = 0, \qquad (i = 1, ..., n) \tag{2.1}$$

by the relation

$$x_i s_i = \mu, \qquad (i = 1, ..., n). \tag{2.2}$$

This is reasonable because if one element of $x_i s_i$ approaches zero much faster than the other elements, the point (x, y, s) is much closer to the boundary of feasible set of (1.13), (1.10) and (1.3) than to the solution (x^*, y^*, s^*). Roughly speaking, the condition (2.2) force the point (x, y, s) not too close to any particular active boundary that has the form $x_i = 0$ or $s_i = 0$.

Suppose at the solution point (x^*, y^*, s^*) the strictly complementary condition hold, namely

$$x_i^* + s_i^* > 0, \qquad (i = 1, ..., n). \tag{2.3}$$

When both the primal problem and the dual problem are feasible, solutions satisfying (2.3) always exist (see, [5]). Assume that J^* is the subset of $\{1, 2, ..., n\}$ such that

$$x_i^* = 0, \qquad i \in J^* \tag{2.4}$$

and

$$s_i^* = 0, \qquad i \notin J^*. \tag{2.5}$$

Thus, when a point (x, y, s) is close to the solution (x^*, y^*, s^*), the distance from the iterate point (x, y, s) to each active boundary near the solution is $x_i = 0$ $(i \in J^*)$ and $s_i = 0$ $(i \notin J^*)$. If we have

$$x_i = \mu, \qquad i \in J^* \tag{2.6}$$

and

$$s_i^* = \mu, \qquad i \notin J^*, \tag{2.7}$$

then the distances from the point (x, y, s) to all active boundary are the same. But J^* is not known unless the problem is not solved. Strict complementarity condition (2.3) indicates that near the solution (2.6) and (2.7) are equivalent to

$$\min[x_i, s_i] = \mu, \qquad (i = 1, ..., n). \tag{2.8}$$

The path that satisfies (1.13), (1.14), (1.16) and (2.8) can be regarded as a "strictly central" path of the feasible region (1.12) near the solution (x^*, y^*, s^*) as it has the same distance to any active boundary. Unfortunately the function $\min[x_i, s_i]$ is a nonsmooth function. Therefore we use the approximation

$$\frac{x_i s_i}{x_i + s_i} \approx \min[x_i, s_i]. \tag{2.9}$$

This formula is a very good approximation if $\min[x_i, s_i] << \max[x_i, s_i]$, which is exactly the case near the solution provided that (2.3) is true.

Thus, we can define a new central path by the following system

$$\begin{aligned} Ax &= b && (2.10)\\ A^T y + s &= c && (2.11)\\ Xs &= \mu(x + s) && (2.12)\\ (x, s) &> 0. && (2.13) \end{aligned}$$

We call this central path the *scaled central path*. It is interesting to study the theoretical properties of the scaled central path, and numerical behaviour of the interior point methods based on the scaled central path. In the following, we demonstrate that the scaled central path can be used to obtain search directions for solving linear programming problems.

3 Search Directions

Now we give a brief discussion on how to use the scaled central path (2.10)-(2.13) to computer search directions. There are many ways to obtain the search directions. Assume the current iterate point (x, y, s) is in the set (1.12). Let the search direction be (d_x, d_y, d_s). Directly applying the Newton's method to the scaled central path we obtain:

$$\begin{aligned} Ad_x &= 0 && (3.1)\\ A^T d_y + d_s &= 0 && (3.2)\\ (X - \gamma\mu I)d_s + (S - \gamma\mu I)d_x &= \gamma\mu(x + s) - Xs, && (3.3) \end{aligned}$$

where $S = diag[s_1, s_2, ..., s_n]$. Here

$$\mu = x^T s/(\|x\|_1 + \|s\|_1), \tag{3.4}$$

and γ should be a positive number in $(0, 1)$. We can write (2.12) in the following form

$$X(X + S)^{-1}s = \mu e. \tag{3.5}$$

Replacing (2.12) by (3.5) and applying Newton's method again, we obtain

$$\begin{aligned} Ad_x &= 0 && (3.6)\\ A^T d_y + d_s &= 0 && (3.7)\\ S^2 d_x + X^2 d_s &= \gamma\mu(X + S)(x + s) - XS(x + s). && (3.8) \end{aligned}$$

Here

$$\mu = x^T(X + S)^{-1}s/n \tag{3.9}$$

and γ, as above, is a positive number in $(0, 1)$.

Similar to the central path (1.13)-(1.16), the scaled central path also has a *log* penalty property. Let $(x(\mu), y(\mu), s(\mu))$ be a point in the scaled central path (2.10)-(2.13), we can see that $x(\mu)$ is a minimum of

$$\min c^T x - \mu \sum_{i=1}^{n} [x_i + \mu \log(x_i - \mu)] \tag{3.10}$$

subject to

$$Ax = b. \tag{3.11}$$

The object function in the above problem can be viewed as the sum of a slightly perturbated original objective function $(c - \mu e)^T x$ and a shifted penalty function $\sum \log(x_i - \mu)$. However, it is not clear what are the advantages of using the penalty parameter to perturbate the objective function and to shift *log* penalty functions. Inspired from this observation, we can consider the following general penalty function formulation:

$$\min(c - \mu_1 e)^T x + \mu \sum_{i=1}^{n} \log(x_i - \mu_2) \tag{3.12}$$

subject to

$$Ax = b. \tag{3.13}$$

The scaled central path corresponds to the case when $\mu_1 = \mu_2 = \mu$. And the standard central path corresponds to the case when $\mu_1 = \mu_2 = 0$. It is obvious that new central paths can be obtained if we choose different μ_1 and μ_2.

We have proposed a new central path for linear programming, and we believe that this central path can be used to construct new interior point methods. Our analyses also indicate that our ideas can be extended to define different central path.

References

[1] F. Alizadeh, J.-P.A. Haeberly and M.L. Overton, " A new primal-dual interior point method for semidefinite programming", in: J.G. Lewis, ed., *Proceedings of the Fifth SIAM Conference on Applied Linear Algebra*, (SIAM, Philadelphia, 1994), pp. 113-117.

[2] T.F. Coleman and Y. Li, "A trust region and affine scaling interior point method for nonconvex minimization with linear inequality constraints", *Math. Program.* 88(2000) 1-31.

[3] J.Y. Han and Y.B. Zhao, "Two interior-point methods for nonlinear $P_*(\tau)$-complementarity problems,, *J. Optim. Theory Appl.*, (1999), 659–679.

[4] S.J. Wright, "Recent developments in interior-point methods", in: M.J.D. Powell and S. Scholtes, eds., *System Modelling and Optimization*, (Kluwer, Boston, 2000), pp. 311–333.

[5] Y. Ye, *Interior Point Algorithms: Theory and Analysis*, (John Wiley and Sons, Chichester, 1997).