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OPTIMALITY CONDITIONS FOR THE MINIMIZATION OF A QUADRATIC WITH TWO QUADRATIC CONSTRAINTS*

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Abstract. The trust region method has been proven to be very successful in both unconstrained and constrained optimization. It requires the global minimum of a general quadratic function subject to ellipsoid constraints. In this paper, we generalize the trust region subproblem by allowing two general quadratic constraints. Conditions and properties of its solution are discussed.

Key words. trust region method, global minimizer, constrained optimization, subproblem, quadratic constraint

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1. Introduction.

Many trust region algorithms for constrained optimization require solving subproblems of the following form:

$$\min\{q(x) : \| Dx \|_2 \leq \delta, \| A^T x + c \|_2 \leq \xi, x \in \Re^n\}, \tag{1.1}$$

where $q : \Re^n \to R$ is a quadratic model of the objective function in a neighborhood of the current iterate, D is a positive definite scaling matrix, $c \in \Re^m$ is a vector whose elements are the values of the constraints, $A^T \in \Re^{m\times n}$ is the Jacobian matrix of the constraints computed at the current iterate, and the numbers δ and ξ are determined by the trust region method (for example, see [1] and [11]). For unconstrained optimization problems, the trust region subproblem is to minimize a quadratic function in an ellipsoid, namely

$$\min\{q(x) : \| Dx \|_2 \leq \delta,\ x \in \Re^n\}. \tag{1.2}$$

Many results for problem (1.2) have been obtained, including Gay [4], Moré and Sorensen [10], Martínez [7], and Sorensen [12]. Most authors study the global minimizer of (1.2), but Martínez [7] also studies local minimizers of (1.2). One motivation for studying nonglobal local minimizers is that a global minimizer of (1.1) at which the constraint $||A^T x + c|| \leq \xi$ is inactive must be a local minimizer of (1.2) (see [7]).

Problem (1.1) has also been studied by many researchers; for example, see Celis, Dennis, and Tapia [1], Crouzeix, Martínez, Legaz, and Seeger [2], Heinkenschloss [5], Yuan [15], [16], Zhang [17], and the references therein. It is interesting to note that unlike the case of one constraint, for the two constraint case it is possible that the Hessian of the Lagrangian has negative eigenvalues, even when only one constraint is active at the global minimizer. For details, see Yuan [15].

Several extensions of problem (1.1) are of interest. The simplest type is to consider the problem

$$\min\{q(x) : c_1(x) \leq 0, c_2(x) \leq 0, x \in \Re^n\}, \tag{1.3}$$

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where $q(x)$, $c_1(x)$, and $c_2(x)$ are quadratic functions. Several special cases of (1.3) have been discussed in the literature. For example, Heinkenschloss [5] considered the case that $q(x)$, $c_1(x)$, $c_2(x)$ are all convex quadratics; Martínez and Santos [8] considered (1.3) with a general quadratic $q(x)$ and c_1, c_2 strictly convex quadratics. More can also be found in [15], [16], [17]. In this paper we consider the case where $q(x)$, $c_1(x)$, and $c_2(x)$ all are general quadratic functions. Our paper is motivated by a recent work of Moré [9], in which he studied the problem of minimizing a quadratic function subject to one general quadratic constraint which has the form

$$\min\{q(x) : c(x) \leq 0, x \in \Re^n\}, \tag{1.4}$$

where $q(x), c(x)$ are general quadratic functions. Stern and Wolkowicz [13] also studied the above problem with a two-sided (upper and lower bound) quadratic constraint; they also discussed the characterizations of optimality and gave some conditions for the existence of solutions.

Our paper can be viewed as a generalization of Yuan [15] from convex constraints to general constraints. Our results are also related to Martínez [7], as his analysis on nonglobal local minimizers of problem (1.2) are applicable to problem (1.1) when the constraint $||A^T x + c|| \leq \xi$ is inactive at the solution. However, our results are more general because we study general quadratic functions $c_1(x)$ and $c_2(x)$ while Martínez [7] and Yuan [15] require convex constraints.

Throughout this paper, we assume that the object function $q(x)$ and the constrained functions $c_1(x)$ and $c_2(x)$ are all quadratic:

$$q(x) = \gamma + w^T x + \frac{1}{2} x^T Q x, \tag{1.5}$$

$$c_1(x) = \gamma_1 + w_1^T x + \frac{1}{2} x^T C_1 x, \tag{1.6}$$

$$c_2(x) = \gamma_2 + w_2^T x + \frac{1}{2} x^T C_2 x, \tag{1.7}$$

where $\gamma, \gamma_1, \gamma_2 \in \Re$, $w, w_1, w_2 \in \Re^n$, and Q, C_1, C_2 are symmetric matrices in $\Re^{n \times n}$. We also use the following notations:

$$E_1 = \{x : x \in \Re^n,\ c_1(x) \leq 0\}, \tag{1.8}$$

$$E_2 = \{x : x \in \Re^n,\ c_2(x) \leq 0\}, \tag{1.9}$$

$$E = E1 \cap E_2. \tag{1.10}$$

Some of our results depend on the following conditions:

$$\inf_{x \in E_1} \{c_2(x)\} < 0 < \sup_{x \in E_1} \{c_2(x)\}, \tag{1.11}$$

$$\inf_{x \in E_2} \{c_1(x)\} < 0 < \sup_{x \in E_2} \{c_1(x)\}, \tag{1.12}$$

which can be viewed as a generalization of a condition given by Moré for one constraint problem (see (2.3) below). The above conditions are not restrictive for problem (1.3). In fact, if the left part of (1.11) is not true, it follows from Theorem 3.2 of [9] (given as Theorem 2.3 below) that there exists $\lambda \in \Re^+$ such that $c_2(x) + \lambda c_1(x)$ is equal to a convex quadratic, which means that $C_2 + \lambda C_1$ is positive semidefinite. Then problem

(1.3) reduces to minimizing $q(x)$ subject to $c_1(x) = 0$ in the subspace $N_{C_2+\lambda C_1}$. If the right inequality of (1.11) fails, (1.3) reduces to the one constraint problem studied by Moré [9]. Therefore, it is no loss of generality in assuming (1.11)–(1.12).

The paper is organized as follows. In the next section we state some known results which we will use repeatedly in the paper. In section 3, we give a condition that ensures the existence of a global minimizer and derive some optimality conditions for problem (1.3) when both constraints are active and the gradients are zeros at the solution. In order to further our analysis, we also explore some relations between optimality and certain definiteness of matrix pencils. In section 4, we consider optimality for problem (1.3) when $q(x)$, $c_1(x)$, and $c_2(x)$ are all general quadratics. Necessary conditions for local minimizers and global minimizers are given. It is shown that the Hessian of the Lagrangian at the solution has at most one negative eigenvalue if the Jacobian of the constraints is not zero and that for some special cases it has no negative eigenvalue. These results are not trivial, as directly applying standard second order necessary conditions can only show that the Hessian of the Lagrangian has at most two negative eigenvalues. A few remarks are also made in last section.

2. Some important results.

In this section we state some known results which will be used in our analysis.

THEOREM 2.1 (see Moré [9]). *If $A \in \Re^{n\times n}$ and $C \in \Re^{n\times n}$ are symmetric matrices, then $A + \lambda C$ is positive definite for some $\lambda \in \Re$ if and only if*

$$w \neq 0, \quad w^T C w = 0 \Longrightarrow \qquad w^T A w > 0. \tag{2.1}$$

THEOREM 2.2 (see Moré [9]). *Assume that $A \in \Re^{n\times n}$ and $C \in \Re^{n\times n}$ are symmetric matrices and that C is indefinite. Then*

$$w^T C w = 0 \Longrightarrow \qquad w^T A w \geq 0 \tag{2.2}$$

if and only if $A + \lambda C$ is positive semidefinite for some $\lambda \in \Re$.

THEOREM 2.3 (see Moré [9]). *Let $q(x)$ and $c(x)$ be quadratic functions defined on $\Re^n$. Assume that*

$$\inf_{x\in\Re^n} c(x) < 0 < \sup_{x\in\Re^n} c(x) \tag{2.3}$$

holds and that $\nabla^2 c \neq 0$. A vector x^* is a global minimizer of the problem

$$\min\{q(x) : c(x) = 0, x \in \Re^n\} \tag{2.4}$$

if and only if $c(x^*) = 0$ and there is a multiplier $\lambda^* \in \Re$ such that the Kuhn–Tucker condition

$$\nabla q(x^*) + \lambda^* \nabla c(x^*) = 0 \tag{2.5}$$

is satisfied with

$$\nabla^2 q(x^*) + \lambda^* \nabla^2 c(x^*) \tag{2.6}$$

positive semidefinite.

THEOREM 2.4 (see Yuan [15]). *Let $C, D \in \Re^{n\times n}$ be two symmetric matrices and let A and B be two closed sets in $\Re^n$ such that $A \cup B = \Re^n$. If we have*

$$x^T C x \geq 0, x \in A, \quad x^T D x \geq 0, x \in B, \tag{2.7}$$

then there exists a $t \in [0, 1]$ such that the matrix $tC + (1-t)D$ is positive semidefinite.

3. Optimality and matrices pencils. In this section, we first give a condition which implies that the global minimum of problem (1.3) can be attained. Then, we study a special case of (1.3) when both constraints are active and the Jacobian of the constraints are zero at the solution.

Denote

$$S_1 = \{x : x \in \Re^n, x^T C_1 x \le 0\}, \tag{3.1}$$

$$S_2 = \{x : x \in \Re^n, x^T C_2 x \le 0\}, \tag{3.2}$$

$$S = S_1 \cap S_2. \tag{3.3}$$

LEMMA 3.1. *Assume that the feasible set* (1.10) *is nonempty*; *if*

$$x \neq 0, x \in S \Longrightarrow x^T Q x > 0, \tag{3.4}$$

then (1.3) *has a global minimizer.*

Proof. If problem (1.3) does not have a global minimizer, then there exists $\{x_k, k = 1, 2, ...\}$ such that $\lim_{k\to\infty} ||x_k|| \to \infty$ and

$$q(x_k) \le q(x_1),\ c_1(x_k) \le 0;\ c_2(x_k) \le 0. \tag{3.5}$$

Let $d_k = \frac{x_k}{||x_k||}$. Without loss of generality (w.l.o.g.), we assume that $\lim_{k\to\infty} d_k = d_0$. It then follows from (1.5)–(1.7) and (3.5) that

$${d_0}^T Q d_0 \le 0,\ {d_0}^T C_1 d_0 \le 0,\ {d_0}^T C_2 d_0 \le 0,$$

which contradicts (3.4). Thus, the lemma is true. □

It should be noted that (3.4) is not a necessary condition for problem (1.3) to have a global minimizer. For example, let $x = (x_1, x_2)^T \in \Re^2$; we define $q(x) = x_1^2 - x_2^2$, $c_1(x) = x_2$, and $c_2(x) = \frac{1}{2}x_1 - x_2$. This problem has a global minimizer $(0,0)^T$. Obviously $S = \Re^2$, but for $\bar{x} = (0,1)^T \in S$, $\bar{x}^T Q \bar{x} < 0$ holds.

Lemma 3.1 indicates that there are connections between Q, C_1, C_2, and the global minimizer of (1.3). Moré [9] and Stern and Wolkowicz [13] have derived relations between matrix pencils and the optimization problem with one general quadratic constraint. In the rest of this section, we will discuss the relation between matrix pencils and a special case of problem (1.3) when both constraints are active and the Jacobian of the constraints are zero at the solution.

Assume that x^* is a local minimizer of problem (1.3) at which $c_1(x^*) = c_2(x^*) = 0$ and $\nabla c_1(x^*) = \nabla c_2(x^*) = 0$. It is easy to see that the null vector 0 is a local minimizer of the following problem:

$$\min\{q(x^* + x) : x \in S\}, \tag{3.6}$$

where S is defined by (3.3).

For any A which is an $n \times n$ symmetric matrix $A \in \Re^{n\times n}$, we define $N_A = \{x : x^T A x = 0\}$. Denote $F = N_{C_1} \cap N_{C_2}$. The following result is the first conclusion of the main theorem of Uhlig [14].

THEOREM 3.2. *Assume that* $A, B \in \Re^{n\times n}$ *and* $n \ge 3$; *then, there exist* $\alpha,\ \beta \in \Re$ *satisfying* $\alpha^2 + \beta^2 > 0$ *such that* $\alpha A + \beta B$ *is positive definite if and only if* $N_A \cap N_B = \{0\}$.

In what follows we state a result about the pair of matrices (C_1, C_2).

LEMMA 3.3. *$\alpha C_1 + \beta C_2$ is indefinite for any $\alpha,\ \beta \in \Re$ satisfying $\alpha^2 + \beta^2 > 0$ if and only if*

$$\inf_{x \in N_{C_1}} \{x^T C_2 x\} < 0 < \sup_{x \in N_{C_1}} \{x^T C_2 x\}, \tag{3.7}$$

$$\inf_{x \in N_{C_2}} \{x^T C_1 x\} < 0 < \sup_{x \in N_{C_2}} \{x^T C_1 x\}. \tag{3.8}$$

Proof. First suppose that (3.7)–(3.8) hold. For any $\alpha,\ \beta \in \Re$ satisfying $\alpha^2 + \beta^2 > 0$, w.l.o.g. assume $\alpha > 0$. It follows from (3.8) that

$$\inf_{x \in N_{C_2}} x^T(\alpha C_1 + \beta C_2)x \ < 0 < \sup_{x \in N_{C_2}} x^T(\alpha C_1 + \beta C_2)x, \tag{3.9}$$

which shows that $\alpha C_1 + \beta C_2$ is indefinite.

Now we assume that $\alpha C_1 + \beta C_2$ is indefinite for any $\alpha,\ \beta \in \Re$ satisfying $\alpha^2 + \beta^2 > 0$. If (3.7)–(3.8) is not true, there is no loss of generality in assuming that

$$\inf_{x \in N_{C_1}} x^T C_2 x = 0. \tag{3.10}$$

Our assumption that $\alpha C_1 + \beta C_2$ is indefinite for any $\alpha,\ \beta \in \Re^n$ satisfying $\alpha^2 + \beta^2 > 0$ implies that C_1 is indefinite; thus, it follows from (3.10) and Theorem 2.2 that there exists $\lambda \in \Re$ such that $C_2 + \lambda C_1$ is positive semidefinite, which is a contradiction. This completes our proof. ▯

For the special problem (3.6), conditions (1.11) and (1.12) are equivalent to

$$\inf_{x \in S_1} \{x^T C_2 x\} < 0 < \sup_{x \in S_1} \{x^T C_2 x\}, \tag{3.11}$$

$$\inf_{x \in S_2} \{x^T C_1 x\} < 0 < \sup_{x \in S_2} \{x^T C_1 x\}. \tag{3.12}$$

We see that our conditions (3.11)–(3.12) are strictly weaker than (3.7)–(3.8). If

$$C_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix}, \tag{3.13}$$

then (3.11)–(3.12) are satisfied, but (3.7)–(3.8) fail.

One direct consequence of (3.11)–(3.12) is the following lemma.

LEMMA 3.4. *If* (3.11)–(3.12) *hold, then*

$$\operatorname{span}(S_1 \ \cap \ S_2) = \Re^n. \tag{3.14}$$

Proof. By (3.11)–(3.12), both C_1 and C_2 are indefinite. If $\max(x^T C_1 x, x^T C_2 x) \geq 0$ for every $x \in \Re^n$, it follows from Theorem 2.4 that there exists $\lambda \in (0, 1)$ such that $C_1 + \lambda(C_1 - C_2)$ is positive semidefinite, which implies

$$x^T C_2 x \geq 0 \quad \text{whenever } x^T C_1 x \leq 0 \tag{3.15}$$

and

$$x^T C_1 x \geq 0 \quad \text{whenever } x^T C_2 x \leq 0. \tag{3.16}$$

Inequalities (3.15)–(3.16) contradict (3.11)–(3.12). Thus, there exists $\bar{x} \in \Re^n$ such that

$$\bar{x}^T C_1 \bar{x} < 0, \ \ \bar{x}^T C_2 \bar{x} < 0. \tag{3.17}$$

Define

$$\delta(\bar{x},\ \varepsilon) = \{x : \|\ x - \bar{x}\ \| \leq \varepsilon\ \}. \tag{3.18}$$

It follows from (3.17) and the continuity of quadratic functions that for sufficiently small $\epsilon > 0$,

$$x \in S_1 \cap S_2 \quad \forall\ x \in\ \delta(\bar{x}, \epsilon). \tag{3.19}$$

The above relation implies (3.14). This proves our lemma. □

The above lemma implies the following result.

LEMMA 3.5. *If* (3.11)–(3.12) *hold and if* $y^* = 0$ *is a local minimizer of* (3.6), *then* $\nabla q(x^*) = 0$.

Proof. Because $y^* = 0$ is a local minimizer of (3.6), it follows that

$$x^T \nabla q(x^*) \geq 0 \quad \forall\ x \in S_1 \cap S_2. \tag{3.20}$$

Due to $S_1 \cap S_2 = -(S_1 \cap S_2)$, (3.20) implies that

$$x^T \nabla q(x^*) = 0 \quad \forall x \in S_1 \cap S_2. \tag{3.21}$$

It follows from (3.21) and (3.14) that $\nabla q(x^*) = 0$. □

Motivated by the results of Morè (see Theorems 2.1–2.3), one may guess that if $x^* = 0$ is a global minimizer of problem (3.6) and conditions (3.11)–(3.12) hold, then there may exist $\alpha, \beta \in \Re$ such that $Q + \alpha C_1 + \beta C_2$ is positive definite or semidefinite. However, our next example shows that even when conditions (3.7)–(3.8) are true and $x^* = 0$ is a global minimizer of problem (3.6), $Q + \alpha C_1 + \beta C_2$ may be indefinite for any $\alpha, \beta \in \Re$.

Example 1.

$$\min\{-(x^2 + y^2)/3 + y^2 - x^2 - 2xy :\ x^2 - y^2 \leq 0, 2xy \leq 0\}. \tag{3.22}$$

For this problem, we have

$$Q = -\frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.23}$$

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.24}$$

It is easy to show that $\alpha C_1 + \beta C_2$ is indefinite for any $\alpha,\ \beta \in \Re$ satisfying $\alpha^2 + \beta^2 > 0$. Thus, conditions (3.7)–(3.8) hold. One can also easily verify that $x^* = 0$ is a unique solution of problem (3.22). However, for any $\alpha, \beta \in \Re$, it holds that $Q + \alpha C_1 + \beta C_2 = -\frac{1}{3}I + (\alpha - 1)C_1 + (\beta - 1)C_2$, which implies that $Q + \alpha C_1 + \beta C_2$ cannot be positive definite or semidefinite.

To study the optimal conditions at a global minimizer of (3.6), we also need the following result due to Hestenes and Mcshane [6].

LEMMA 3.6. *Let* $C_1, C_2 \in \Re^{n\times n}$ *be symmetric matrices satisfying* (3.7)–(3.8). *Let* $m(\alpha, \beta)$ *be the least eigenvalue of the matrix* $Q + \alpha C_1 + \beta C_2$. *Then, there exists* $(\alpha_0, \beta_0) \in \Re^2$ *which maximizes the function* $m(\alpha, \beta)$.

Our next result is a small modification of Lemma B in [6]. For completeness, we rewrite it and give a detailed proof.

LEMMA 3.7. *Assume the matrices C_1 and C_2 satisfying* (3.7)–(3.8) *and* $(\alpha_0, \beta_0) \in \Re^2$ *maximize the function* $m(\alpha, \beta)$. *Set* $m_0 = m(\alpha_0, \beta_0)$, *with* X *as the subspace spanned by all the eigenvectors of the matrix* $Q + \alpha_0 C_1 + \beta_0 C_2$ *related to* m_0. *Then for any linear space L which contains* X, *there is no* $\alpha C_1 + \beta C_2$ *positive definite on* L.

Proof. Assume there exists $\bar{C} = \alpha C_1 + \beta C_2$ positive definite on L. Let K be the unit sphere $x^T x = 1$, and L_1 is the set of points in L on K. Choose $b > 0$ such that $x^T \bar{C} x > b$ on L_1, and let $\bar{N}$ be a neighborhood of L_1 related to K on which $x^T \bar{C} x > b$, m_1 is the minimum of $x^T(Q + \alpha_0 C_1 + \beta_0 C_2)x$ on the closed set $K - \bar{N}$; then, $m_1 > m_0$. It follows that for a sufficiently small positive constant t one will have

$$x^T(Q + \alpha_0 C_1 + \beta_0 C_2 + t\bar{C})x > m_0 \tag{3.25}$$

on $K - \bar{N}$. But,

$$x^T(Q + \alpha_0 C_1 + \beta_0 C_2 + t\bar{C})x > m_0 + tb \tag{3.26}$$

on $\bar{N}$. Thus, it holds that $m(\alpha_0 + t\alpha, \beta_0 + t\beta) > m(\alpha_0, \beta_0)$, which contradicts the choice of (α_0, β_0). This proves the lemma. ∎

Now we can give one of our main results in this section.

THEOREM 3.8. *If* (3.7)–(3.8) *hold and if* $y^* = 0$ *is a local minimizer of problem* (3.6), *then* $\nabla q(x^*) = 0$ *and there exist* $\alpha, \beta \in \Re$ *such that* $Q + \alpha C_1 + \beta C_2$ *has at most two negative eigenvalues.*

Proof. It follows from Lemma 3.5 that $\nabla q(x^*) = 0$. Because $\nabla q(x^*) = 0$ and the optimality of $y^* = 0$, we have that $x^T Q x \geq 0$ for all $x \in S$.

If the theorem is not true, assume that for any $\alpha, \beta \in \Re$, $Q + \alpha C_1 + \beta C_2$ has three or more negative eigenvalues. Let (α_0, β_0) maximize the function $m(\alpha, \beta)$, and let L be the subspace spanned by the eigenvectors of the matrix $Q + \alpha_0 C_1 + \beta_0 C_2$ corresponding to its negative eigenvalues. For example, $L = \operatorname{span}\{x_1, x_2, \ldots, x_l : (Q + \alpha_0 C_1 + \beta_0 C_2)x_i = a_i x_i, a_i < 0, \|x_i\|_2 = 1\}$ and $l = \dim(L) \geq 3$. It follows that in L, we have

$$Q + \alpha_0 C_1 + \beta_0 C_2 = \sum_{i=1}^{l} a_i x_i x_i^T. \tag{3.27}$$

If there exists $x_0 \in F \neq 0$ in L, w.l.o.g. we assume that $\|x_0\|_2 = 1$. Then, by the definition of F, we get

$$x_0^T(Q + \alpha_0 C_1 + \beta_0 C_2)x_0 \geq 0, \tag{3.28}$$

which contradicts the definition of L. It follows that $F \cap L = \{0\}$. However, since $l \geq 3$, it follows from Theorem 3.2 that there exist $\alpha, \beta \in \Re$ such that $\alpha C_1 + \beta C_2$ is positive definite on L, which contradicts Lemma 3.7. ∎

In fact, under the conditions of Theorem 3.8, let α_0 and β_0 as defined in Lemma 3.6 and $m_0 = m(\alpha, \beta)$ denote L_1 as the subspace spanned by the eigenvectors of $Q + \alpha_0 C_1 + \beta_0 C_2$ related to m_0. If $m_0 < 0$, then by Theorem 3.8 we have $\dim(L_1) < 3$. By Lemma 3.7, $\dim(L_1) \neq 1$; thus, it must hold that $\dim(L_1) = 2$. This can also be verified by our Example 1, where $Q + C_1 + C_2 = -\frac{1}{3}I$, $m(1, 1) = -\frac{1}{3}$. But for any $(\alpha, \beta) \in \Re^2$, $Q + \alpha C_1 + \beta C_2 = -\frac{1}{3}I + (\alpha - 1)C_1 + (\beta - 1)C_2$. If $(\alpha, \beta) \neq (1, 1)$, then $(\alpha - 1)C_1 + (\beta - 1)C_2$ is indefinite, which implies that the least eigenvalue of $Q + \alpha C_1 + \beta C_2$ is less than $-\frac{1}{3}$. Thus, for Example 1, it holds that $m_0 = m(1, 1) = -\frac{1}{3}$.

Now we only consider the case $m_0 < 0$ under the conditions of Theorem 3.8. Let L_1 be the subspace defined by $L_1 = \{x \in \Re^n : (Q + \alpha_0 C_1 + \beta_0 C_2)x = m_0 x\}$. It easy to see that in L_1, $Qx = (-\alpha_0 C_1 - \beta_0 C_2 + m_0 I)x$. Thus, $x^* = 0$ is a global minimizer of the following problem:

$$\min\{x^T(-\alpha_0 C_1 - \beta_0 C_2 + m_0 I)x : \; x^T C_1 x \le 0, x^T C_2 x \le 0, \; x \in L_1\}. \tag{3.29}$$

Since $m_0 < 0$, $x^T C_1 x$ and $x^T C_2 x$ vanish simultaneously only at the point 0. By Lemma 3.7, there is no $\alpha, \beta \in \Re$ such that $\alpha C_1 + \beta C_2$ is positive definite. Thus, in L_1 we have

$$x^T C_1 x \ne 0 \quad \forall x^T C_2 x = 0, x \ne 0 \tag{3.30}$$

and

$$x^T C_2 x \ne 0 \quad \forall x^T C_1 x = 0, x \ne 0. \tag{3.31}$$

If $x^T C_1 x > 0$ for all $x^T C_2 x = 0, x \ne 0 \in L_1$, then it follows from Theorem 2.1 that there exists $\lambda \in \Re$ such that $C_1 + \lambda C_2$ is positive definite, which is a contradiction. Therefore, there exists $\bar{x} \in L_1$ such that

$$\bar{x}^T C_2 \bar{x} = 0, \; \bar{x}^T C_1 \bar{x} \le 0. \tag{3.32}$$

The fact that $\bar{x}^T(-\alpha_0 C_1 + m_0 I)\bar{x} \ge 0$ implies that $\alpha_0 > 0$. Similarly, one can show that $\beta_0 > 0$.

If conditions (3.11)–(3.12) are true and (3.7)–(3.8) fail, then we have the following result.

THEOREM 3.9. *If* (3.11)–(3.12) *hold and* (3.7)–(3.8) *fail and if $y^* = 0$ is a local minimizer of problem* (3.6)*, then $\nabla q(x^*) = 0$ and there exist $\lambda_1, \lambda_2 \in \Re$ such that $Q + \lambda_1 C_1 + \lambda_2 C_2$ is positive semidefinite.*

Proof. It follows from Lemma 3.5 that $\nabla q(x^*) = 0$. Since conditions (3.7)–(3.8) are not satisfied, it follows from Lemma 3.3 that there exist $\alpha, \beta \in \Re$ such that $\alpha^2 + \beta^2 \ne 0$ and that $\alpha C_1 + \beta C_2$ is positive semidefinite. Without loss of generality, we assume that $\alpha \ne 0$. Define $\lambda = \beta/\alpha$. First we assume that $\alpha > 0$, which implies that $C_1 + \lambda C_2$ is positive semidefinite. This leads to the following two cases: if $\lambda > 0$ then $x^T C_1 x \le 0 \Longrightarrow x^T C_2 x \ge 0$, which contradicts (3.11)–(3.12); if $\lambda < 0$ then $x^T C_1 x \le 0 \Longrightarrow x^T C_2 x \le 0$, which contradicts (3.11).

Now we assume that $\alpha < 0$, which implies that $C_1 + \lambda C_2$ is negative semidefinite. If $\lambda > 0$ then $x^T C_1 x = 0 \Longrightarrow x^T C_2 x \le 0$; $y^* = 0$ is a local minimizer of problem $\min\{x^T Q x : \; x^T C_1 x = 0, \; x \in \Re^n\}$. Thus, our theorem follows from Theorem 2.2. If $\lambda \le 0$ then $x^T C_2 x \le 0 \Longrightarrow x^T C_1 x \le 0$, which contradicts (3.12). Therefore, the theorem is proved. ∎

If C_i is positive definite then we can choose the corresponding Lagrange multiplier λ_i large enough so that $Q + \lambda_i C_i$ is positive definite. But in the case that C_1 is positive semidefinite and $N_{C_1} \ne \emptyset$, then even (3.11) holds, and there may be no $\lambda_1, \lambda_2 \in \Re$ such that $Q + \lambda_1 C_1 + \lambda_2 C_2$ is positive semidefinite. This can be verified by the following example.

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

C_1 is semidefinite and (3.11) holds; $y^* = 0$ is a global minimizer of

$$\min\{x^T Q x : x \in\ S_1 \cap S_2\}, \tag{3.33}$$

but for any $\lambda_1,\ \lambda_2 \in \Re$, $Q + \lambda_1 C_1 + \lambda_2 C_2$ is not positive semidefinite. For the case where $C_1,\ C_2$ are indefinite, if (3.11)–(3.12) do not hold, Theorem 3.9 may also fail. For example,

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now we give a lemma which will be used in the next section.

LEMMA 3.10. *If $C \in \Re^{n\times n}$ is a symmetric indefinite matrix, then* $\operatorname{span}(N_C) = \Re^n$.

Proof. Without loss of generality, we assume that

$$C = \operatorname{diag}\left(\alpha_1, \ldots, \alpha_I; -\beta_1, \ldots, -\beta_J; 0, \ldots, 0\right), \tag{3.34}$$

where $\alpha_i (i = 1, \ldots, I)$ and $\beta_j (j = 1, \ldots, J)$ are positive numbers and $I \geq 1,\ J \geq 1$. It is easy to see that

$$\frac{\beta_j}{\alpha_1} e_1 + e_{I+j} \in N_C \quad (j = 1, \ldots, J), \tag{3.35}$$

$$e_i - \frac{\alpha_i}{\beta_1} e_{I+1} \in N_C \quad (i = 1, \ldots, I), \tag{3.36}$$

$$e_k \in N_C \quad (k = I + J + 1, \ldots, n), \tag{3.37}$$

and these vectors are linearly independent. Thus, $\operatorname{span}(N_C) = \Re^n$. □

The following result is a direct consequence of Theorem 2.1.

COROLLARY 3.11. *If $y^* = 0$ is an isolated minimizer of the problem*

$$\min\{x^T Q x + g^T x :\ x^T C x = 0\}, \tag{3.38}$$

then there exists $\lambda \in \Re$ such that $Q + \lambda C$ is positive definite.

Proof. For any nonzero $x \in \Re^n$ such that $x^T C x = 0$, we have $(-x)^T C(-x) = 0$; thus, our assumption implies that

$$x^T Q x = \frac{1}{2}(x^T Q x + g^T x) + \frac{1}{2}(-x^T Q(-x) + g^T(-x)) > 0. \tag{3.39}$$

Therefore, the corollary follows from Theorem 2.1. □

Similarly, we can show the following theorem.

THEOREM 3.12. *If $y^* = 0$ is an isolated minimizer of problem* (3.6) *and conditions* (3.7)–(3.8) *fail, then there exist $\lambda_1,\ \lambda_2 \in \Re$ such that $Q + \lambda_1 C_1 + \lambda_2 C_2$ is positive definite.*

Proof. For any feasible point x of (3.6), the point $-x$ is also a feasible point. Thus, $y^* = 0$ is also an isolated local minimizer of (3.33). Therefore, we have that $x^T Q x > 0$ for all nonzero x, which satisfies $x^T C_1 x = x^T C_2 x = 0$. Since conditions (3.7)–(3.8) are not satisfied, w.l.o.g. we assume that (3.7) is not true. First, we assume that

$$\sup_{x \in N_{C_1}} x^T C_2 x = 0. \tag{3.40}$$

Thus, $y^* = 0$ is also the unique global minimizer of

$$\min\{x^T Qx : \ x^T C_1 x = 0\}. \tag{3.41}$$

It follows from Theorem 2.1 that there exists $\lambda \in \Re$ such that $Q + \lambda C_1$ is positive definite.

To complete our proof, we assume that

$$\min_{x \in N_{C_1}} x^T C_2 x = 0. \tag{3.42}$$

We consider three different cases: C_1 is positive semidefinite, negative semidefinite, or indefinite.

If C_1 is positive semidefinite, then the feasible region $\{x^T C_1 x \leq 0\}$ is the subspace N_{C_1}. Thus, the null vector 0 is an isolated local minimizer of

$$\min\{x^T Qx : \ x^T C_2 x = 0, \quad x \in N_{C_1}\}, \tag{3.43}$$

which shows that there exists $\mu \in \Re$ such that $Q + \mu C_2$ is positive definite in N_{C_1}. Thus,

$$x \neq 0, \quad x^T C_1 x = 0 \quad \Longrightarrow x^T(Q + \mu C_2)x > 0. \tag{3.44}$$

Hence, there exists $\lambda \in \Re$ such that $Q + \mu C_2 + \lambda C_1$ is positive definite.

If C_1 is negative semidefinite, we have that

$$x \neq 0, \quad x^T C_2 x = 0 \ \Longrightarrow x^T Qx > 0. \tag{3.45}$$

Therefore, it follows from Theorem 2.1 that there exists $\mu \in \Re$ such that $Q + \mu C_2$ is positive definite.

Finally, we consider if C_1 is indefinite. It follows from (3.42) and Theorem 2.2 that there exists $\alpha \in \Re$ such that $C_2 + \alpha C_1$ is positive semidefinite. Because $y^* = 0$ is an isolated local minimizer of (3.6), we have for all $x \in N_{C_2 + \alpha C_1}$,

$$x \neq 0, \quad x^T C_1 x = 0 \quad \Longrightarrow \quad x^T Qx > 0. \tag{3.46}$$

Thus, it follows from Theorem 2.1 that there exists $\beta \in \Re$ such that $Q + \beta C_1$ is positive definite in the subspace $N_{C_2 + \alpha C_1}$. Using Theorem 2.1 again, we can show that there exists $\gamma \in \Re$ such that $Q + \beta C_1 + \gamma(C_2 + \alpha C_1)$ is positive definite. Hence, the theorem is true. □

4. Optimal conditions. In this section we mainly give necessary conditions for minimizers of problem (1.3). Necessary conditions for optimality are already given in the previous section, when both constraints are active and gradients of the constraints are zeros at the solution.

First, the following result is obvious.

THEOREM 4.1. *Assume that* $c_1(x^*) < 0$ *and* $c_2(x^*) < 0$. x^* *is a local minimizer of problem* (1.3) *if and only if* $\nabla q(x^*) = 0$ *and* Q *is positive semidefinite.*

Hence, in the following we assume that at least one of the constraints is active at a minimizer.

If only one constraint is active at the global minimizer x^*, w.l.o.g. we assume that

$$c_1(x^*) = 0, \ c_2(x^*) < 0. \tag{4.1}$$

THEOREM 4.2. *Assume that* (4.1) *holds and that* x^* *is a local minimizer of problem* (1.3). *If* $\nabla c_1(x^*) \neq 0$, *then there exists* $\lambda_1 \in \Re^+$ *such that*

$$\nabla q(x^*) + \lambda_1 \, \nabla c_1(x^*) = 0 \tag{4.2}$$

holds and $Q + \lambda_1 C_1$ *has at most one negative eigenvalue. If* $\nabla c_1(x^*) = 0$, *then* $\nabla q(x^*) = 0$ *and there exists* $\lambda_1 \in \Re^+$ *such that* $Q + \lambda_1 C_1$ *is positive semidefinite.*

Proof. If $\nabla c_1(x^*) \neq 0$, (4.2) follows from the Kuhn–Tucker theory. It follows from the second order necessary condition (see, for example, Fletcher [3]) that $Q + \lambda_1 C_1$ is positive semidefinite in the subspace

$$W = \{d : \nabla c_1(x^*)^T d = 0, \ d \in \Re^n \ \}. \tag{4.3}$$

Therefore, $Q + \lambda_1 C_1$ has at most at most one negative eigenvalue.

Now we assume that $\nabla c_1(x^*) = 0$, since $y^* = 0$ is a local minimizer of

$$\min_{d \in S_1} q(x^* + d). \tag{4.4}$$

Thus, it follows that

$$d^T \nabla q(x^*) = 0 \quad \forall d \in N_{C_1}.$$

The fact that $\nabla c_1(x^*) = 0$ and (1.12) imply that C_1 is indefinite. Lemma 3.10 shows that $\text{span}(N_{C_1}) = \Re^n$, which gives $d^T \nabla q(x^*) = 0$ for all $d \in \Re^n$. Therefore, $\nabla q(x^*) = 0$. This shows that (4.2) holds for all $\lambda_1 \in \Re$. $\nabla q(x^*) = 0$, Theorem 2.4, and the fact that $y^* = 0$ solves (4.4) imply that there exists $\lambda_1 \in \Re^+$ such that $Q + \lambda_1 C_1$ is positive semidefinite. ∎

Using the second order necessary conditions, it can be proven that the Hessian of the Lagrangian has at most two negative eigenvalues if both constraints $c_1(x) \leq 0$ and $c_2(x) \leq 0$ are active at the solution.

For convex problems, Yuan [15] shows that the Hessian of the Lagrangian has at most only one negative eigenvalue at a global minimizer. In the following, Yuan's results are extended to general cases. For the rest of this section we assume that x^* is a global minimizer of problem (1.3) and both constraints are active at x^*, which means that $c_1(x^*) = c_2(x^*) = 0$. First, we consider the case when $\nabla c_1(x^*)$ and $\nabla c_2(x^*)$ are linearly independent.

THEOREM 4.3. *If* x^* *is a global minimizer of problem* (1.3) *and if* $\nabla c_1(x^*)$ *and* $\nabla c_2(x^*)$ *are linearly independent, then there exist* $\lambda_1, \lambda_2 \in \Re^+$ *such that*

$$\nabla q(x^*) + \lambda_1 \nabla c_1(x^*) + \lambda_2 \nabla c_2(x^*) = 0 \tag{4.5}$$

and $Q + \lambda_1 C_1 + \lambda_2 C_2$ *has at least* $n - 1$ *nonnegative eigenvalues.*

Proof. Let $\lambda_1, \lambda_2 \in \Re^+$ be the corresponding Lagrange multipliers $H = Q + \lambda_1 C_1 + \lambda_2 C_2$. Then, by the second order necessary condition we know that

$$x^T H x \geq 0 \quad \forall x \in \Re^n, \ x \perp \nabla c_1(x^*), \ x \perp \nabla c_2(x^*).$$

If H has two negative eigenvalues, similar to Yuan [15], there exist $e_1, \ e_2 \in \Re^n$ such that for all nonzero $d \in \text{span}\{e_1, \ e_2\}$, $d^T H d < 0$. Because x^* is the global minimizer, $(0,0)^T$ is the unique solution of

$$\hat{c}_1(\alpha, \beta) = c_1(x^* + \alpha e_1 + \beta e_2) = 0, \quad \hat{c}_2(\alpha, \beta) = c_2(x^* + \alpha e_1 + \beta e_2) = 0,$$

where $(\alpha,\ \beta) \in \Re^2,\ x = x^* + \alpha e_1 + \beta e_2$. So the curves $\hat{c}_1(\alpha,\beta) = 0,\ \hat{c}_2(\alpha,\beta) = 0$ meet only at $(0,0)$. Define $\bar{F}$ as the set of all feasible points that are connected to $(0,0)$; thus, the boundary of $\bar{F}$ consists of two curves. One is $\hat{c}_1(\alpha,\beta) = 0$; the other is $\hat{c}_2(\alpha,\beta) = 0$. Let the asymptotic direction of these two curves be $\bar{d}_1,\ \bar{d}_2$; then we have

$$\bar{d}_1{}^T \nabla^2 \hat{c}_1 \bar{d}_1 = 0;\ \bar{d}_1{}^T \nabla^2 \hat{c}_2 \bar{d}_1 \le 0, \tag{4.6}$$

$$\bar{d}_2{}^T \nabla^2 \hat{c}_1 \bar{d}_2 \le 0;\ \bar{d}_2{}^T \nabla^2 \hat{c}_2 \bar{d}_2 = 0. \tag{4.7}$$

Due to the optimality of x^*, we know that

$$\bar{d}_1{}^T \nabla^2 \hat{q}(x^*) \bar{d}_1 \ge 0;\ \bar{d}_2{}^T \nabla^2 \hat{q}(x^*) \bar{d}_2 \ge 0, \tag{4.8}$$

where $\hat{q}(\alpha,\beta) = q(x^* + \alpha e_1 + \beta e_2)$. Since $d^T H d < 0$ for all nonzero $d \in \text{span}\{e_1,\ e_2\}$, it follows that

$$\bar{d}_1{}^T \nabla^2 \hat{c}_2 \bar{d}_1 < 0;\ \bar{d}_2{}^T \nabla^2 \hat{c}_1 \bar{d}_2 < 0. \tag{4.9}$$

By considering a sequence of interior points of $\bar{F}$, one can see that for any direction d between $\bar{d}_1,\ \bar{d}_2$,

$$d^T \nabla^2 \hat{c}_2 d < 0;\ d^T \nabla^2 \hat{c}_1 d < 0. \tag{4.10}$$

Otherwise, assume there exists $d \in \text{int}(K),\ d^T \nabla^2 \hat{c}_1 d = 0$; then, $(\alpha,\beta)\nabla^2 \hat{c}_1 (\alpha,\beta)^T$ has a local maximum at d. Hence, $\nabla^2 \hat{c}_1$ is negative semidefinitive, which shows that $\hat{c}_1(\alpha,\beta) = 0$ is a parabolic curve. Because the two curves have only one cross and the asymptotic direction of a parabolic curve is the same one, we know that $\bar{d}_1$ is parallel to $\bar{d}_2$, which contradicts (4.9). Hence, there exists a cone K whose boundary direction is $\bar{d}_1,\ \bar{d}_2$, and for any interior direction of K, (4.10) holds. Now, for large enough $t > 0,\ -td$ is a feasible point. Because the two curves meet only at $(0,0)$, $-td \notin \bar{F}$. Let the connected part of the feasible set which includes $-td$ be $\hat{F}$; then, $\bar{F} \cap \hat{F} = \phi$. Because $(0,0)$ is the unique cross of two curves, the boundary of $\hat{F}$ is defined by only one curve. Without loss of generality, assume that the boundary is defined by $\hat{c}_1(\alpha,\beta) = 0$. Let the asymptotic directions of $\hat{F}$ be $\hat{d}_1,\ \hat{d}_2$, and the corresponding cone is $\hat{K}$. Since (4.10) holds for all $\bar{d} \in K$, it holds that $-K \subset \hat{K}$, so $-\bar{d}_2 \in \hat{K}$. Furthermore, for all $\hat{d} \in \hat{K}$ we have

$$\hat{d}^T \nabla^2 \hat{c}_2 \hat{d} \le 0, \quad \hat{d}^T \nabla^2 \hat{c}_1 \hat{d} \le 0.$$

One can also show that there exists no $\hat{d} \in \hat{K}$ such that

$$\hat{d}^T \nabla^2 \hat{c}_2 \hat{d} = 0, \quad \hat{d}^T \nabla^2 \hat{c}_1 \hat{d} = 0$$

and that

$$\hat{d}_1{}^T \nabla^2 \hat{c}_1 \hat{d}_1 = 0,\ \hat{d}_2{}^T \nabla^2 \hat{c}_1 \hat{d}_2 = 0, \tag{4.11}$$

$$\hat{d}_1{}^T \nabla^2 \hat{c}_2 \hat{d}_1 < 0,\ \hat{d}_2{}^T \nabla^2 \hat{c}_2 \hat{d}_2 < 0. \tag{4.12}$$

Hence, $-\bar{d}_2$ is an interior direction of $\hat{K}$, which implies that $\hat{c}_2(\alpha,\beta) = 0$ is a parabolic curve. This contradicts (4.9). So, H has at most one negative eigenvalue. $\square$

The condition that the Hessian of the Lagrangian has at most one negative eigenvalue is not a sufficient condition for x^* being a local minimizer. For example, point $(1,1,0)^T$ is a Kuhn–Tucker point of the following 3-dimensional problem:

$$\min -4y + (x-1)^2 + y^2 - 10z^2 \tag{4.13}$$

$$\text{s.t. } x^2 + y^2 + z^2 \leq 2, \tag{4.14}$$

$$(x-2)^2 + y^2 + z^2 \leq 2. \tag{4.15}$$

It is easy to see that the Lagrange multipliers are $(1,1)$. The Hessian of the Lagrangian is

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{pmatrix},$$

which has $2(= n-1)$ positive eigenvalues. But one can easily show that the point $(1,1,0)^T$ is not a local minimizer because the second order necessary condition is not satisfied.

In the following we deal with the case when $\nabla c_1(x^*)$ and $\nabla c_2(x^*)$ are linearly dependent. Because we have already studied the case when $\nabla c_1(x^*) = \nabla c_2(x^*) = 0$ in the previous section, we can assume that either $\nabla c_1(x^*)$ or $\nabla c_2(x^*)$ is not zero. Without loss of generality, we assume that $\nabla c_1(x^*) \neq 0$ and $\nabla c_2(x^*) = \alpha \nabla c_1(x^*)$ for the rest of the section. First we discuss the case when $\alpha > 0$.

THEOREM 4.4. *If x^* is a global minimizer of problem* (1.3) *and if there exists $\alpha > 0$ such that $\nabla c_2(x^*) = \alpha \nabla c_1(x^*) \neq 0$, then there exist $\lambda_1, \lambda_2 \in \Re^+$ such that* (4.5) *holds and the matrix $Q + \lambda_1 C_1 + \lambda_2 C_2$ is positive semidefinite.*

Proof. Since $\nabla c_2(x^*) = \alpha \nabla c_1(x^*) \neq 0$ for some $\alpha > 0$, the optimality of x^* implies that $d^T \nabla q(x^*) \geq 0$ for all d such that $d^T \nabla c_1(x^*) < 0$. Therefore, there exists $\beta \leq 0$ such that $\nabla q(x^*) = \beta \nabla c_1(x^*)$. If $\beta < 0$, there is no loss of generality in assuming that $\nabla c_1(x^*) = \nabla c_2(x^*) = -\nabla q(x^*) \neq 0$. First, we show that

$$\max(x^T(Q + C_1)x, x^T(Q + C_2)x) \geq 0 \quad \forall x \in \Re^n. \tag{4.16}$$

If it fails, there exists $\hat{d} \in \Re^n$ such that

$$\hat{d}^T \nabla c_1(x^*) \neq 0, \quad \hat{d}^T(Q + C_1)\hat{d} < 0, \quad \hat{d}^T(Q + C_2)\hat{d} < 0. \tag{4.17}$$

The fact that x^* is a global minimizer of (1.3) and (4.17) imply that either $\hat{d}^T C_1 \hat{d}$ or $\hat{d}^T C_2 \hat{d}$ is not zero. Thus, we can choose $\lambda \neq 0 \in \Re$ so that

$$c_1(x^* + \lambda \hat{d}) = 0, \quad c_2(x^* + \lambda \hat{d}) \leq 0 \tag{4.18}$$

or

$$c_1(x^* + \lambda \hat{d}) \leq 0, \quad c_2(x^* + \lambda \hat{d}) = 0. \tag{4.19}$$

Without loss of generality, we assume that (4.18) is true; it then follows that

$$q(x^* + \lambda \hat{d}) - q(x^*) = \lambda^2 \hat{d}^T (Q + C_1)\hat{d} < 0, \tag{4.20}$$

which is a contradiction. Thus, (4.16) holds. Hence, our theorem follows from (4.16) and Theorem 2.4.

If $\beta = 0$, (4.5) holds for $\lambda_1 = \lambda_2 = 0$. The optimality of x^* implies that $d^T Q d \geq 0$ for all d such that $d^T \nabla c_1(x^*) < 0$. Since $\text{span}\{d : d^T \nabla c_1(x^*) < 0\} = \Re^n$, it follows that Q is positive semidefinite. □

In what follows we will consider the case when $\nabla c_2(x^*) = \alpha \nabla c_1(x^*)$ for some $\alpha \leq 0$.

THEOREM 4.5. *Assume that x^* is a global minimizer of problem* (1.3) *and that $c_1(x)$ and $c_2(x)$ satisfy* (1.11)–(1.12)*. If $\nabla c_1(x^*) \neq 0$ and $\nabla c_2(x^*) = \alpha \nabla c_1(x^*)$ for some $\alpha \leq 0$ and $\nabla q(x^*) = \gamma \nabla c_1(x^*)$, then there exist $\lambda_1, \lambda_2 \in \Re^+$ so that* (4.5) *holds and $Q + \lambda_1 C_1 + \lambda_2 C_2$ is positive semidefinite.*

Proof. First, we consider the case when $\nabla c_2(x^*) = \alpha \nabla c_1(x^*)$ for some $\alpha < 0$. Without loss of generality, we assume that $\nabla c_1(x^*) = -\nabla c_2(x^*) \neq 0$ and $\gamma \leq 0$. Now we show that

$$\max(x^T (Q - \gamma C_1) x,\ x^T (C_1 + C_2) x) \geq 0 \quad \forall x \in \Re^n. \tag{4.21}$$

Otherwise, we can choose $\hat{d} \in \Re^n$ such that

$$\hat{d}^T \nabla c_1(x^*) < 0, \quad \hat{d}^T (Q - \gamma C_1) \hat{d} < 0, \quad \hat{d}^T (C_1 + C_2) \hat{d} < 0. \tag{4.22}$$

If $\hat{d}^T C_1 \hat{d} = 0$, then $\hat{d}^T Q \hat{d} < 0, \hat{d}^T C_2 \hat{d} < 0$. We can let $\lambda \in \Re^+$ sufficiently large so that $\lambda \hat{d}$ is feasible and $q(x^* + \lambda \hat{d}) - q(x^*) < 0$, which is a contradiction. If $\hat{d}^T C_1 \hat{d} \neq 0$, we can choose $\lambda \in \Re$ so that

$$c_1(x^* + \lambda \hat{d}) = 0, \quad c_2(x^* + \lambda \hat{d}) < 0. \tag{4.23}$$

It follows that

$$q(x^* + \lambda \hat{d}) - \gamma c_1(x^* + \lambda \hat{d}) - q(x^*) = q(x^* + \lambda \hat{d}) - q(x^*) = \lambda^2 \hat{d}^T (Q - \gamma C_1) \hat{d} < 0, \tag{4.24}$$

which is a contradiction. Thus, (4.21) holds. Since conditions (1.11)–(1.12) imply that $C_1 + C_2$ cannot be positive semidefinite, our theorem follows from (4.21) and Theorem 2.4.

Now we turn to the case when $\nabla c_2(x^*) = 0$. The assumptions in our theorem imply that $\nabla q(x^*) = \gamma \nabla c_1(x^*)$ for some $\gamma \leq 0$. By a similar process, we can show that

$$\max(x^T (Q - \gamma C_1) x, x^T C_2 x) \geq 0 \quad \forall x \in \Re^n, \tag{4.25}$$

which means that our theorem still holds when $\nabla c_2(x^*) = 0$. □

In the above two theorems, we have discussed optimal properties of the Hessian of a generalized Lagrangian functions when $\nabla c_1(x^*)$ and $\nabla c_2(x^*)$ are linearly dependent and $\nabla q(x^*) \in \text{span}\{\nabla c_1(x^*)\}$. But if $\nabla q(x^*) \notin \text{span}\{\nabla c_1(x^*)\}$, then the Kuhn–Tucker theory and (4.5) fail. In this case, we need to assume that x^* is a unique solution to continue our analysis.

THEOREM 4.6. *Assume that x^* is a unique global minimizer of problem* (1.3) *and that $c_1(x)$ and $c_2(x)$ satisfy* (1.11)–(1.12)*. If $\nabla c_1(x^*) \neq 0$ and $\nabla c_2(x^*) = -\alpha \nabla c_1(x^*)$ for some $\alpha \geq 0$ and if $\nabla q(x^*)$ and $\nabla c_1(x^*)$ are linearly independent, then there exist $\lambda_1, \lambda_2 \in \Re$ such that $Q + \lambda_1 C_1 + \lambda_2 C_2$ has at least $n - 1$ positive eigenvalues.*

Proof. Let W be defined by (4.3). It follows from the definition of x^* that $y^* = 0$ is the unique solution of the following problem:

$$\min\{x^T Q x + x^T \nabla q(x^*) :\ x^T C_1 x \leq 0,\ x^T C_2 x \leq 0,\ x \in W\}. \tag{4.26}$$

We now show that

$$\max(x^T C_1 x, x^T C_2 x) \geq 0 \quad \forall x \in W. \tag{4.27}$$

Otherwise, there exists $x \in W$ such that

$$x^T C_1 x < 0, \; x^T C_2 x < 0. \tag{4.28}$$

Without loss of generality, we assume that $x^T \nabla q(x^*) \leq 0$. Therefore, we can choose sufficiently small $\epsilon > 0$ such that

$$\bar{x}^T \nabla q(x^*) < 0, \; \bar{x}^T C_1 \bar{x} \leq 0, \; \bar{x}^T C_2 \bar{x} \leq 0, \; \bar{x} = x - \epsilon \nabla q(x^*), \tag{4.29}$$

which contradicts the basic assumptions of the theorem. Thus, (4.27) is true. It follows from Theorem 3.12 that there exist $\lambda_1, \; \lambda_2 \in \Re$ such that $Q + \lambda_1 C_1 + \lambda_2 C_2$ is positive definite in W. This proves our theorem. □

5. Discussion. We have shown that the Hessian of the Lagrangian at the solution of problem (1.3) has at most only one negative eigenvalue if the Jacobian of the constraints is not zero. For some special cases, it is shown that the Hessian is positive semidefinite or definite. We have also derived some relations between matrix pencils and optimality. The necessary conditions given in the paper are stronger than the standard second order necessary condition, which says the Hessian is positive semidefinite in the null space of the constraint gradients. It is pointed out that the necessary conditions obtained are not sufficient conditions for optimality. It is interesting to investigate whether there are sufficient conditions that are weaker than the standard second order sufficient condition, which requires the Hessian of the Lagrangian to be positive definite at the null space of the constraint gradients. We believe that our theoretical results will help us to understand problem (1.3) better; they also will be useful for development of numerical algorithms for trust region subproblems.

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REFERENCES

[1] M. R. Celis, J. E. Dennis, Jr., and R. A. Tapia, *A trust region strategy for nonlinear equality constrained optimization*, in Numerical Optimization, P. T. Boggs, R. H. Byrd, and R. B. Schnabel, eds., SIAM, Philadelphia, PA, 1985, pp. 71–82.

[2] J. P. Crouzeix, J. E. Martínez-Legaz, and A. Seeger, *An Alternative Theorem for Quadratic Forms and Extensions*, Preprint 98, Department of Math, University of Barcelona, Spain, 1991.

[3] R. Fletcher, *Practical Methods of Optimization*, 2nd ed., John Wiley and Sons, Chichester, 1987.

[4] D. M. Gay, *Computing optimal locally constrained steps*, SIAM J. Sci. Statist. Comput., 2 (1981), pp. 186–197.

[5] M. Heinkenschloss, *On the Solution of a Two Ball Trust Region Subproblem*, Tech. rep. 92-16, Universität Trier, Trier, Germany, 1992.

[6] M. R. Hestenes and E. J. Mcshane, *A theorem on quadratic forms and its application in the calculus of variations*, Trans. Amer. Math. Soc., 40 (1940), pp. 501–512.

[7] J. M. Martínez, *Local minimizers of quadratic functions on Euclidean balls and spheres*, SIAM J. Optim., 4 (1994), pp. 159–176.

[8] J. M. Martínez and S.A. Santos, *A Trust Region Method for Minimization on Arbitrary Domains*, Tech. rep., State University of Campinas, Campinas, Brazil, 1991.

[9] J. J. Moré, *Generalization of the trust region problem*, Optim. Methods Software, 2 (1993), pp. 189–209.

[10] J. J. Moré and D. Sorensen, *Computing a trust region step*, SIAM J. Sci. Statist. Comput., 4 (1983), pp. 553–572.

[11] M. J. D. Powell and Y. Yuan, *A trust region algorithm for equality constrained optimization*, Math. Programming, 49 (1991), pp. 189–211.

[12] D. C. Sorensen, *Newton's method with a model trust region modification*, SIAM J. Numer. Anal., 19 (1982), pp. 409–426.

[13] R. Stern and H. Wolkowicz, *Indefinite Trust Region Subproblems and Nonsymmetric Eigenvalue Perturbations*, Tech. rep. CORR 92-38, Department of Combinatorics and Optimization, University of Waterloo, Canada, 1993.

[14] F. Uhlig, *A recurring theorem about pairs of quadratic forms and extensions: A survey*, Linear Algebra Appl., 25 (1979), pp. 219–237.

[15] Y. Yuan, *On a subproblem of trust region algorithms for constrained optimization*, Math. Programming, 47 (1990), pp. 53–63.

[16] Y. Yuan, *A dual algorithm for minimizing a quadratic function with two quadratic constraints*, J. Comput. Math., 9 (1991), pp. 348–359.

[17] Y. Zhang, *Computing a Celis–Dennis–Tapia trust-region step for equality constrained optimization*, Math. Programming, 55 (1992), pp. 109–124.