VEMs: A New Weapon in Scientific Computing

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Feng Kang Distinguished Lecture

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Outline

The VEM results presented here are form joint works with

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Cour

As Henri Poincaré once remarked, "*solution of a mathematical problem*" is a phrase of indefinite meaning. Pure mathematicians sometimes are satisfied with showing that "the non-existence of a solution implies a logical contradiction", while engineers might consider a "numerical result" as the only reasonable goal. Such one sided views seem to reflect human limitations rather than objective values. In itself mathematics is an indivisible organism uniting theoretical contemplation and active application. (*R. Courant*)

MSO

Scientific Computing from a Mathematical point of view

- The practical interest of Scientific Computing is known to (almost) everybody.
- Here I will discuss a (minor) part of the role of Mathematics in Scientific Computation
- Within the **M.S.O.** (**M**odelization, **S**imulation, **O**ptimization) paradigm, I will focus on the "**S**" part.
- In particular, I will deal with "basic instruments to compute an approximate solution (as accurate as needed) to a (system of) PDE's".
- I apologize to Numerical Analysts for the first part of this lecture. I hope it will not be too boring.

Basic physical laws

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \wedge \mathbf{E} = 0 \qquad \frac{\partial \mathbf{D}}{\partial t} - \nabla \wedge \mathbf{H} = \mathbf{J}$$

Phenomenological laws (material dependent)

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$$

Compatibility of the right-hand sides

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Incompressible Navier-Stokes Equations

$$\varepsilon = \frac{1}{2}(\nabla + \nabla^T)\,\mathbf{u} \qquad \boldsymbol{\sigma} = (2\mu\varepsilon + \mathbb{I}_{id}\,p)$$

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\cdot\boldsymbol{\sigma} = -\mathbf{f}$$

$$\nabla\cdot\mathbf{u} = 0$$

elast

$\mathbf{u}=$ displacements, $\varepsilon=$ strains, $\boldsymbol{\sigma}=$ stresses, $\mathbf{f}=$ forces,

$$\varepsilon = \frac{1}{2}(\nabla + \nabla^T)\,\mathbf{u} \qquad \boldsymbol{\sigma} = 2\mu\varepsilon + \mathbb{I}_{id}\,trace(\varepsilon)$$

$$\nabla \cdot \boldsymbol{\sigma} = -\mathbf{f}$$

Then one could (should) introduce **geometric** ($\mathbf{u} \rightarrow \varepsilon$) and **constitutive** ($\varepsilon \rightarrow \boldsymbol{\sigma}$) nonlinearities.

VF Darcy

Let us consider the **simplest possible** problem (e.g. : Darcy's flow): *Given a polygon Ω and $f \in L^2(\Omega)$:*

$$\text{find } u \in V \text{ such that } -\Delta u = f \text{ in } \Omega,$$

where $V \equiv H^1_0(\Omega) \equiv \{v |\, v \in L^2(\Omega), \,\mathbf{grad} v \in (L^2(\Omega))^2$ such that $v = 0$ on $\partial\Omega\}$.

The *variational form* of this problem consists in looking for a function $u \in V$ such that:

$$\int_\Omega \mathbf{grad} u \cdot \mathbf{grad} v \mathrm{d}x = \int_\Omega f \, v \mathrm{d}x \qquad \forall\, v \in V.$$

Ga

Galerkin approximations

The Galerkin method consists in choosing a finite dimensional $V_h \subset V$ and looking for $u_h \in V_h$ such that

$$\int_\Omega \mathbf{grad} u_h \cdot \mathbf{grad} v_h \mathrm{d}x = \int_\Omega f\, v_h \mathrm{d}x \qquad \forall\, v_h \in V_h.$$

It is an easy exercise to show that such a u_h exists and is unique in V_h, and satisfies the estimate

$$\int_\Omega |\mathbf{grad}(u - u_h)|^2 \mathrm{d}x \le C \inf_{v_h \in V_h} \int_\Omega |\mathbf{grad}(u - v_h)|^2 \mathrm{d}x$$

bounding the *error* $\|u - u_h\|$ with the best approximation that could be given of u within the subspace V_h.

Math Appr

More generally, the *analysis, from the mathematical point of view,* of these procedures assumes that we are given a *sequence* of subspaces $\{V_h\}_h$ and proves, under suitable assumptions on the subspaces, that the sequence of solutions $\{u_h\}_h$ converges to the exact solution u when h tends to 0.

As far as possible, one also tries to connect the *speed* of this convergence with suitable properties of u and of the sequence $\{V_h\}_h$, and hence to find what are the sequences of subspaces that would provide the best speed, plus possibly other convenient properties (e.g.computability, positivity, conservation of physical quantities, etc.).

FE

Finite Element Methods (FEM)

In the FEM's one decomposes the domain Ω in small pieces and takes V_h as the space of functions that are piece-wise polynomials. The most classical case is that of decompositions in *triangles*

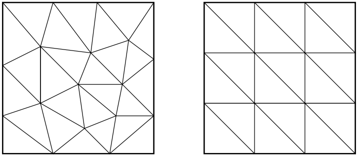


Figure: Triangulations of a square domain: non-uniform or uniform

taking then V_h as the space of functions that are polynomials of degree ≤ 1 in each triangle.

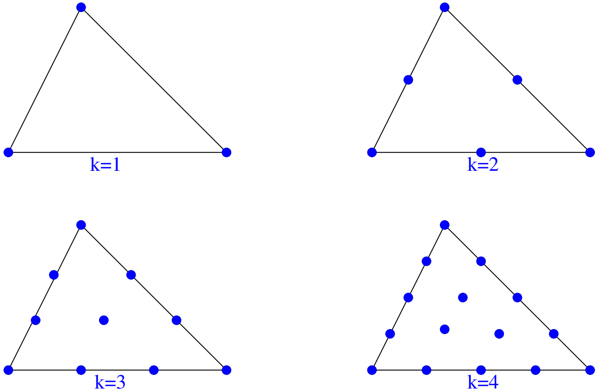
Instead of p.w. polynomials of degree ≤ 1 one can take piecewise polynomials of degree $\leq k$ $(k = 2, 3, ...)$.

For the analysis we consider a sequence of decompositions $\{\mathcal{T}_h\}_h$, and piecewise polynomials of degree $\leq k$, and try to express the speed of convergence (of u_h to u) in terms of k, of h ($=$ biggest diameter among the elements in $\mathcal{T}_h$), and of some additional geometric property θ (e.g. the *minimum angle* of all triangles of all decompositions):

$$\|\mathbf{grad}(u - u_h)\|_{L^2(\Omega)} \;\leq\; C_{\theta,k}\, h^k\, \|D^{k+1}u\|_{L^2(\Omega)}.$$

DOF

Lagrange FEM's - Degrees of freedom



Triangular elements and their degrees of freedom
(=parameters used to identify elements of V_h in each T)

Typical Functional Spaces

Here are the functional spaces most commonly used in variational formulations of PDE problems

$L^2(\Omega)$ (ex. pressures, densities)
$H(\mathrm{div};\Omega)$ (ex. fluxes, **D**, **B**)
$H(\mathbf{curl};\Omega)$ (ex. vector potentials, **E**, **H**)
$H(\mathbf{grad};\Omega)$ (H^1) (ex. displacements, velocities)
$H(\mathbb{D}^2;\Omega)$ (H^2) (ex. in K-L plates, Cahn-Hilliard)

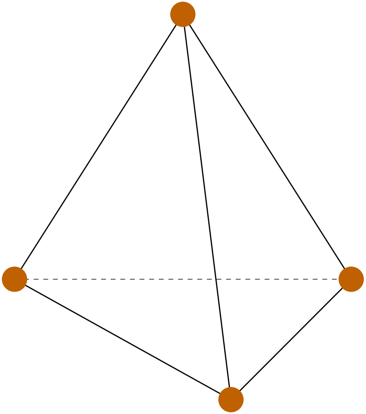
Cont R

For a **piecewise smooth** vector valued function, at the common boundary between two elements,

in order to belong to	**you need to match**
$L^2(\Omega)$	nothing
$H(\mathrm{div};\Omega)$	normal component
$H(\mathbf{curl};\Omega)$	tangential components
$H(\mathbf{grad};\Omega)$	C^0
$H(\mathbb{D}^2;\Omega)$	C^1

Note that *the freedom you gain by relaxing the continuity properties can be used to satisfy other properties* *Eleg*

Elegance of FEM spaces: 0-forms

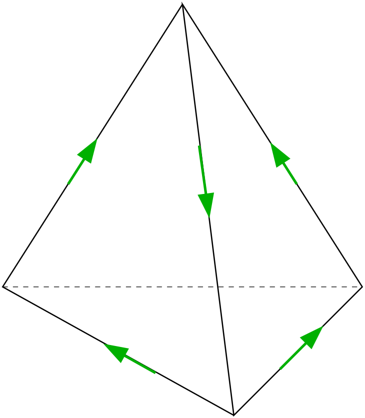


$$\mathbb{P}_1 := \{v = a + \mathbf{c} \cdot \mathbf{x}\} \text{ with } a \in \mathbb{R} \text{ and } \mathbf{c} \in \mathbb{R}^3$$

(d.o.f. = **nodal values**)

$$H(\mathbf{grad}; \Omega) \sim \{v \in H(\mathbf{grad}; \Omega) \text{ s.t. } v_{|T} \in \mathbb{P}_1 \quad \forall T \in \mathcal{T}_h\}.$$

Elegance of FEM spaces: 1-forms

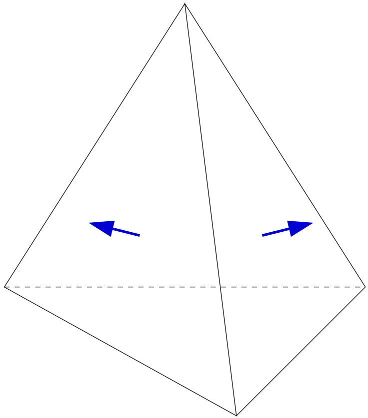


$$N_0 := \{\varphi = \mathbf{a} + \mathbf{c} \wedge \mathbf{x}\} \text{ with } \mathbf{a} \in \mathbb{R}^3 \text{ and } \mathbf{c} \in \mathbb{R}^3$$

(d.o.f. = **edge integrals of tangential component**)

$$H(\mathbf{curl};\Omega) \sim \{\varphi \in H(\mathbf{curl};\Omega) \text{ s.t. } \varphi_{|T} \in N_0 \quad \forall T \in \mathcal{T}_h\}.$$

Elegance of FEM spaces: 2-forms

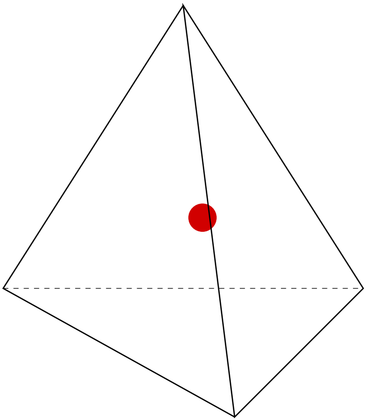


$RT_0 := \{\boldsymbol{\tau} = \mathbf{a} + c\mathbf{x}\}$ with $\mathbf{a} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

(d.o.f. **face integrals of normal component**)

$H(\mathrm{div};\Omega) \sim \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$ s.t. $\boldsymbol{\tau}_{|T} \in \quad \forall T \in \mathcal{T}_h\}$.

Elegance of FEM spaces: 3-forms

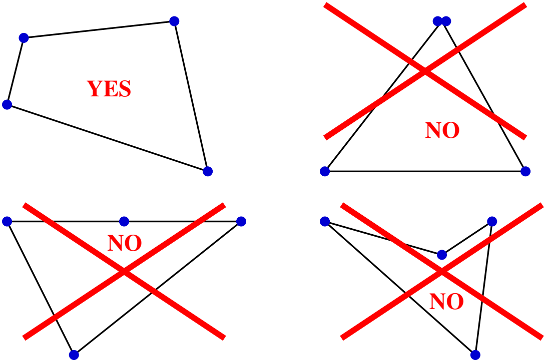


$\mathbb{P}_0 := \{\text{constants}\}$ (d.o.f. $=$ **volume integral**)

$L^2(\Omega) \sim \{q \in L^2(\Omega) \text{ such that } q_{|T} \in \mathbb{P}_0 \quad \forall T \in \mathcal{T}_h\}$.

Dist Quad

Distorted quads can degenerate in many ways:



Loss B

Here is **the most elegant choice** of polynomial spaces for edge elements of degree k on **cubes**

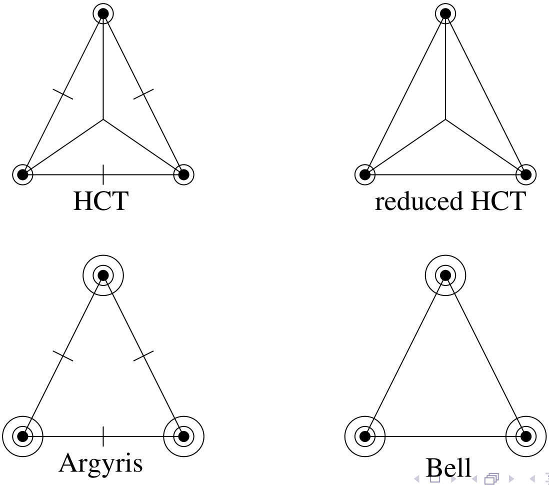
$$\mathit{span}\{yz(w_2(x,z)-w_3(x,y)),\; zx(w_3(x,y)-w_1(y,z)),\; xy(w_1(y,z)-w_2(x,z))\} + (\mathbb{P}_k)^3 + \mathbf{grad}\, s(x,y,z)$$

where each w_i $(i=1,2,3)$ ranges over all polynomials (of 2 variables) of degree $\leq k$ and s ranges over all polynomials of *super linear degree* $\leq k+1$.

N.B. Super linear degree: "ordinary degree ignoring variables that appear linearly".

More difficulties: FE approximations of $H^2(\Omega)$

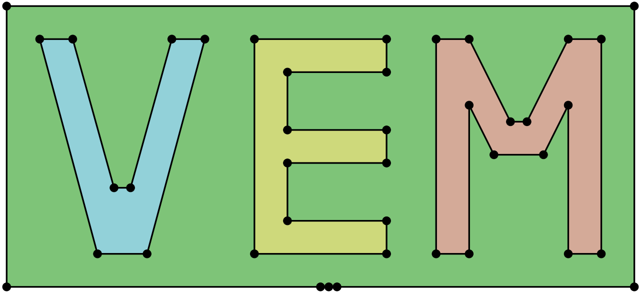
There are relatively few C^1 Finite Elements on the market. Here are some:



Cod liver oil

(Olio di fegato di merluzzo, Huile de foie de morue
Aceite de hígado de bacalao, Dorschlebertran)

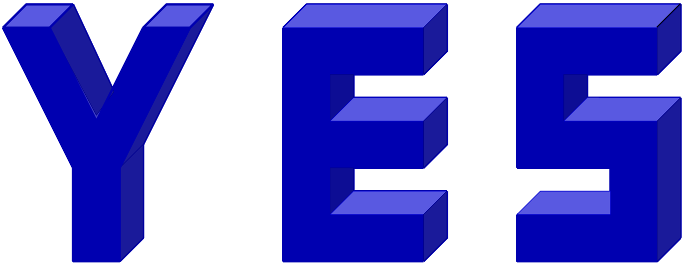
For a decomposition in general sub-polygons, FEM's encounter **considerable** difficulties.
With VEM, instead, you **can** take a decomposition like



having four elements with 8 12 14, and 41 nodes!

Can we work in 3D as well?

3D



WE CAN !! These are three possible 3D elements

Polygonal and Polyhedral elements

There is a wide literature on Polygonal and Polyhedral Elements (Polytopes)

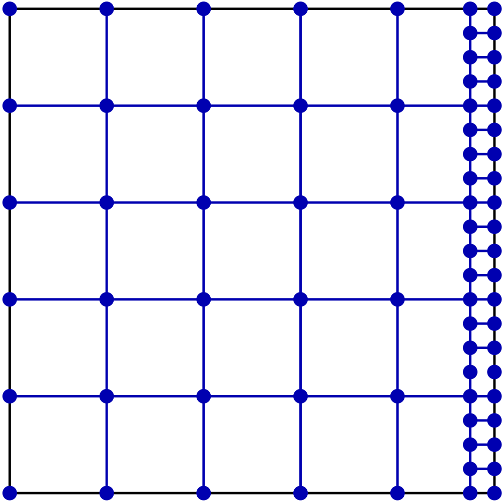
- Rational Polynomials (Wachspress, 1975, 2010)
- Voronoi tassellations (Sibson, 1980; Hiyoshi-Sugihara, 1999; Sukumar et als, 2001)
- Mean Value Coordinates (Floater, 2003)
- Metric Coordinates (Malsch-Lin-Dasgupta, 2005)
- Maximum Entropy (Arroyo-Ortiz, 2006; Hormann-Sukumar, 2008)
- Harmonic Coordinates (Joshi et als 2007; Martin et als, 2008; Bishop 2013)

Why Polygonal/Polyhedral Elements?

There are several types of problems where Polygonal and Polyhedral elements are used:

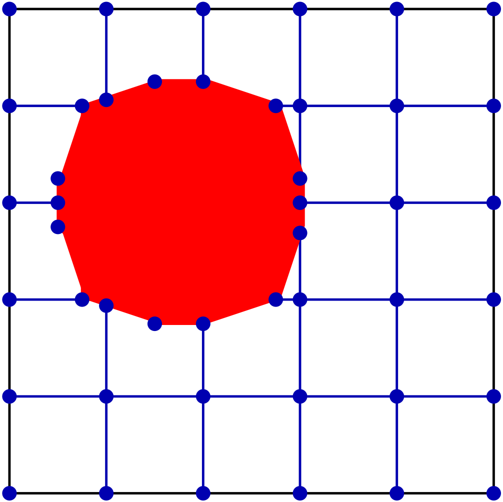
- Crack propagation and Fractured materials (e.g. T. Belytschko, N. Sukumar)
- Topology Optimization (e.g. O. Sigmund, G.H. Paulino)
- Computer Graphics (e.g. M.S. Floater)
- Fluid-Structure Interaction (e.g. W.A. Wall)
- Complex Micro structures (e.g. N. Moes)
- Two-phase flows (e.g. J. Chessa)
- Contact Problems (e.g. P. Wriggers, B.D. Reddy)

Boundary layers



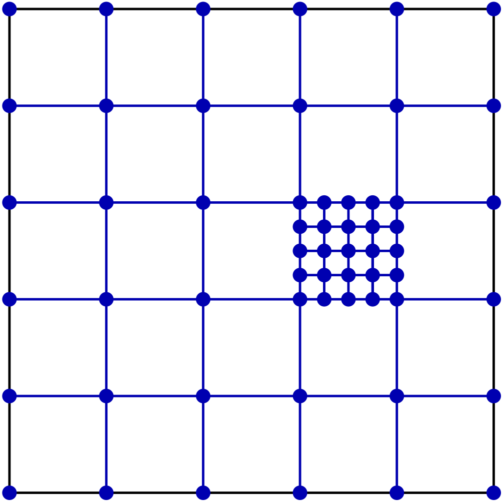
The "interface" elements are treated as epta-gons.

Moving Objects



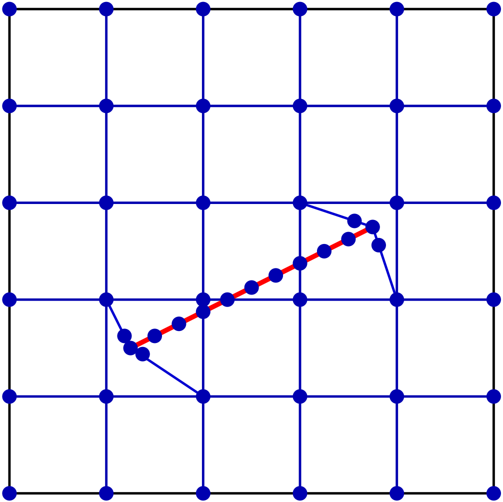
At each time step, the mesh is adapted to the object

Local Refinement



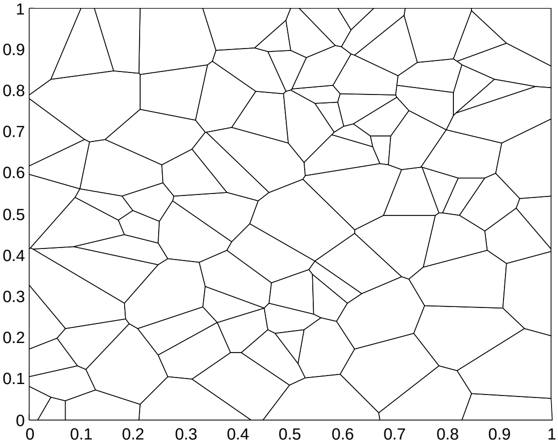
Combining a fine mesh with a coarse one

Something going on there...



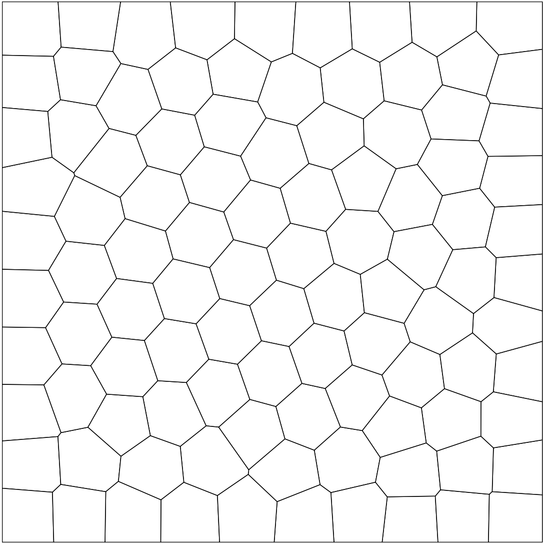
A fracture, or a 1-d intrusion

Voronoi Meshes



LL

Lloyd Meshes



Example: piecewise linear FEMs

Given a *triangulation* $\mathcal{T}_h$ of Ω, with N internal nodes, we set V_h = continuous piecewise linear functions vanishing on $\partial\Omega$, and we look for u_h in V_h such that

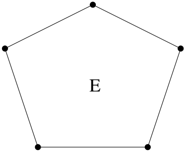
$$a(u_h, v_h) := \int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega = \int_\Omega f \, v_h \, d\Omega \quad \forall \, v_h \in V_h.$$

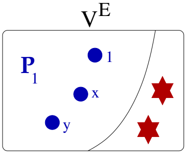
In practice, the $N \times N$ matrix associated to $a(u_h, v_h)$ is computed as the sum of the contributions of the single elements:

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, \mathrm{d}\Omega = \sum_{E \in \mathcal{T}_h} a^E(u_h, v_h) \equiv \sum_{E \in \mathcal{T}_h} \int_E \nabla u_h \cdot \nabla v_h \, dE.$$

$V_h := \{v \in V : \; v \text{ linear on each edge}, -\Delta v = 0 \; \text{ in } E \, \forall E\}$

On a single element E





if u is in $\mathbb{P}_1(E)$, then $a^E(u, v)$ can be computed *exactly*.

$$a^E(p_1, v) = \int_E \nabla p_1 \cdot \nabla v \, dE = \int_{\partial E} \frac{\partial p_1}{\partial n} v \, d\ell =: a_h^E(p_1, v)$$

But we cannot compute $a(u_h, v_h)$ for general u_h and v_h.

Hence we must look for some *approximate form* $a_h(u_h, v_h)$

Manipulating VEM's

When dealing with VEM, we cannot manipulate them as we please. As we don't want to use approximate solutions of the PDE problems in each element, we have to use only the *degrees of freedom* and all the information that you can deduce *exactly* from the degrees of freedom.

In a sense, is like doing *Robotic Surgery*

Approx Pb

Guidelines for choosing a_h

We consider again the continuous model problem:
Find $u \in V \equiv H^1_0(\Omega)$ *such that*

$$a(u,v) \equiv \int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f \, v \, d\Omega \quad \forall \, v \in V.$$

Given $V_h \subset V$ we want to construct a discretized version:

Find $u_h \in V_h$ *such that*
$$a_h(u_h, v_h) = (f_h \, , v_h) \quad \forall \, v_h \in V_h.$$

We look for *sufficient conditions* on a_h (and on f_h) that ensure *all the good properties that you would have with standard Finite Elements*.

H1 $a_h^E(p_1, v) = a^E(p_1, v) \quad \forall E, \forall v \in V^E, \forall p_1 \in \mathbb{P}_1(E).$

H2 $\exists\, \alpha^*,\ \alpha_* > 0$ such that $\forall E,\ \forall\, v \in V^E$:

$$\alpha_*\, a^E(v, v) \le a_h^E(v, v) \le \alpha^*\, a^E(v, v).$$

Under Assumptions **H1** and **H2** the discrete problem has a unique solution. Moreover the **Patch Test** of order 1 is **satisfied**: *on any patch of elements, if the exact solution is a global polynomial of degree* 1, *then the exact solution and the approximate solution* **coincide**.

$$\|u - u_h\|_1 \le C \left(\|u - u_I\|_1 + \|u - u_\pi\|_{1,h} + \|f - f_h\|_{V_h'} \right) \le Ch.$$

how

We saw already that knowing v on ∂E we can compute $a^E(v, p_1)$ for every p_1 in $\mathbb{P}_1(E)$. This allows to construct in each E a **computable projection operator** Π_1^∇ **from** V^E **into** $\mathbb{P}_1(E)$ **defined by**

$$a^E(v - \Pi_1^\nabla v, p_1) = 0 \;\; \forall\, p_1 \quad \text{and} \quad \int_{\partial E} (v - \Pi_1^\nabla v)\, d\ell.$$

Note that $\Pi_1^\nabla p_1 = p_1$ for all p_1 in $\mathbb{P}_1(E)$.

Then we set, for all u and v in V^E

$$a_h^E(u, v) := a^E(\Pi_1^\nabla u, \Pi_1^\nabla v) + S(u - \Pi_1^\nabla u, v - \Pi_1^\nabla v)$$

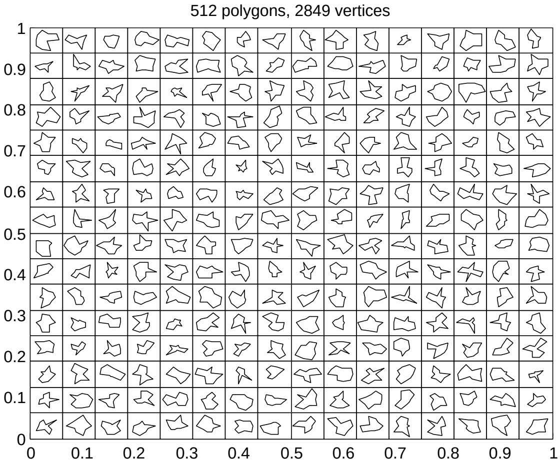
where the *stabilizing* bilinear form S is (**for instance**) the Euclidean inner product in $\mathbb{R}^5$.

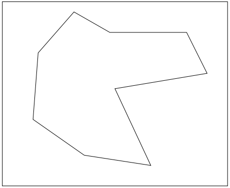
Loc Mat

Structure of the Local Matrix in a different basis

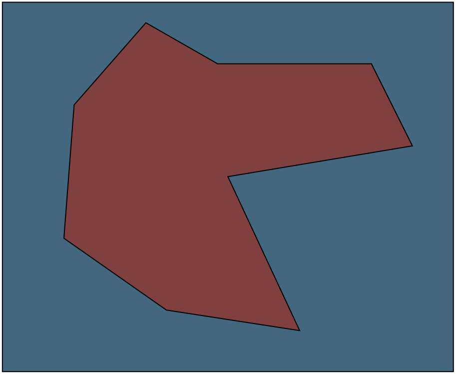
	Polynomials	Others
Polynomials	$a_h = a$	$a_h = a$
Others	$a_h = a$	S

Does it work?





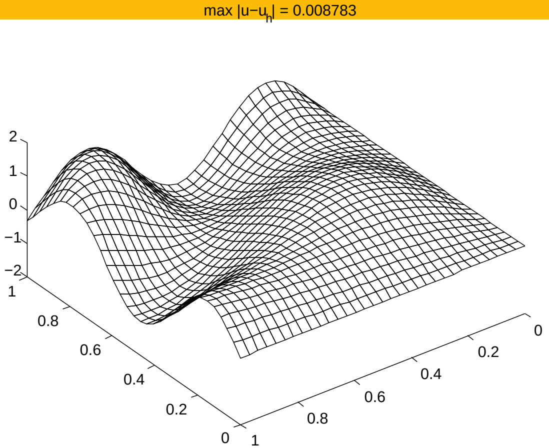
1 square



Note that the pink element is a polygon with 9 edges, while the blue element is a polygon (not simply connected) with 13 edges. We are exact on linears...

Sol

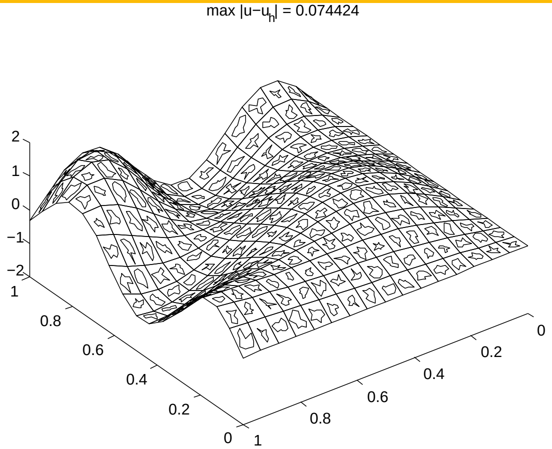
The exact solution of the PDE



For reasons of "*glastnost*", we take as exact solution

$$w = x(x-0.3)^3(2-y)^2\sin(2\pi x)\sin(2\pi y) + \sin(10xy)$$

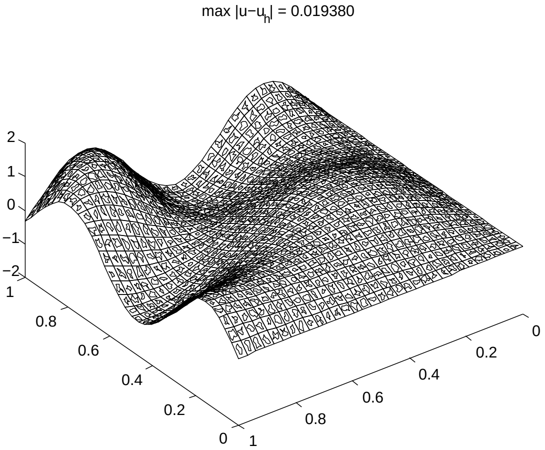
It works!



Mesh of 512 ($16 \times 16 \times 2$) elements. Max-Err$=0.074$

ris 2048

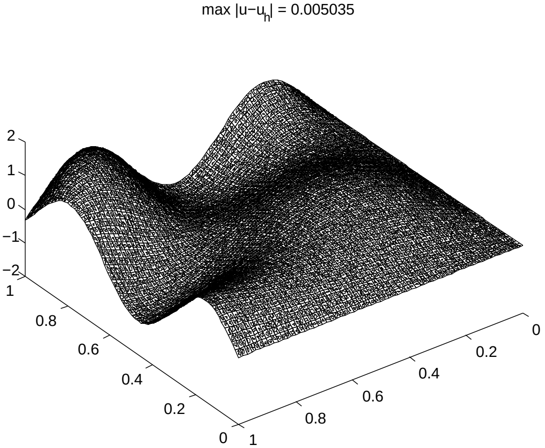
Finer grids



Mesh of 2048 ($32 \times 32 \times 2$) elements. Max-Err$=$0.019

ris 8192

An even finer grid



Mesh of 8192 ($64 \times 64 \times 2$) elements. Max-Err=0.005

Note the $O(h^2)$ convergence in L^∞ !!.

Esch

The next steps? (by M.C. Escher)

What about a mesh like that?

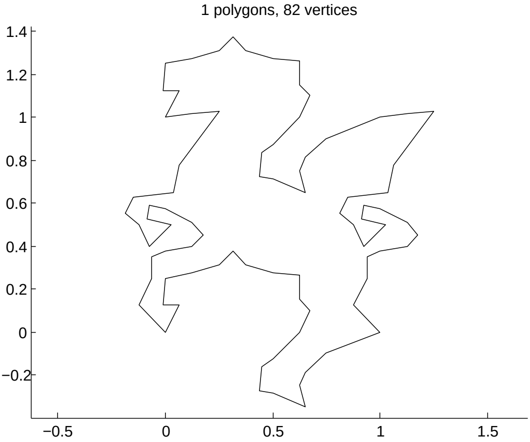
cavalli

The next steps? (by M.C. Escher)

Or possibly like this one?

1p

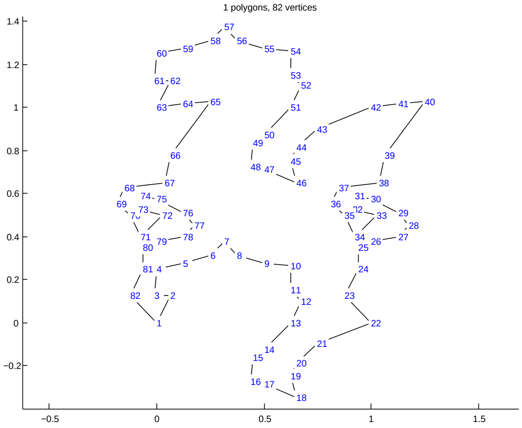
Going berserk



The first step: a pegasus-shaped polygon with **82 edges**.

num

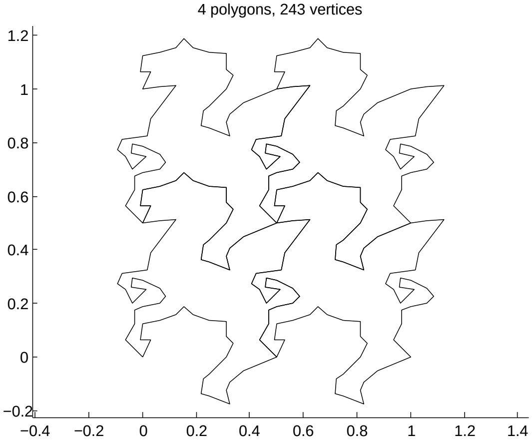
Going berserk



The second step: local numbering of the **82 nodes**.

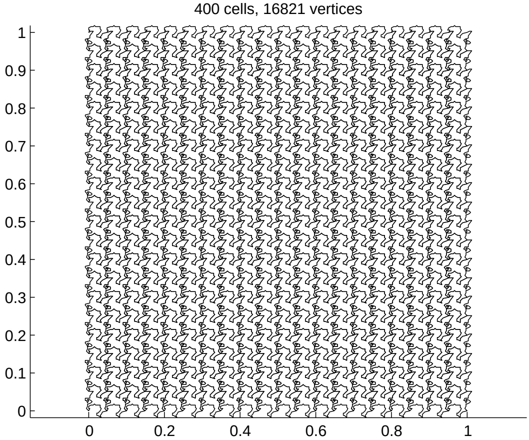
4

Going berserk



The third step: a mesh of 2×2 pegasus.

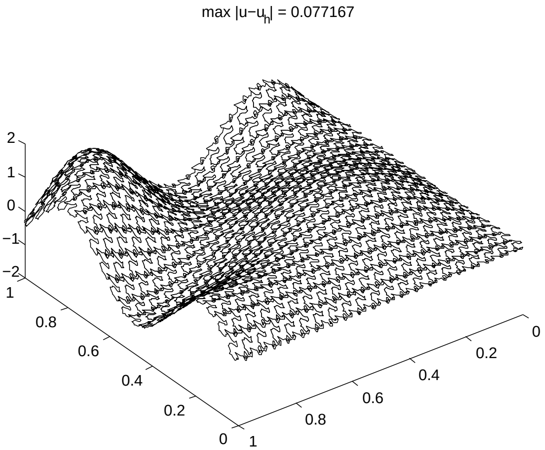
Going totally berserk



A mesh of 20×20 pegasus.

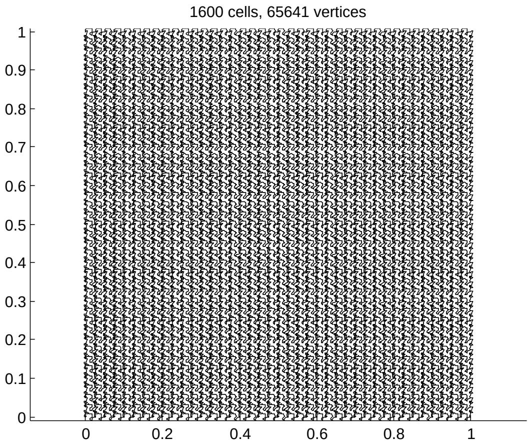
sol

Going totally berserk



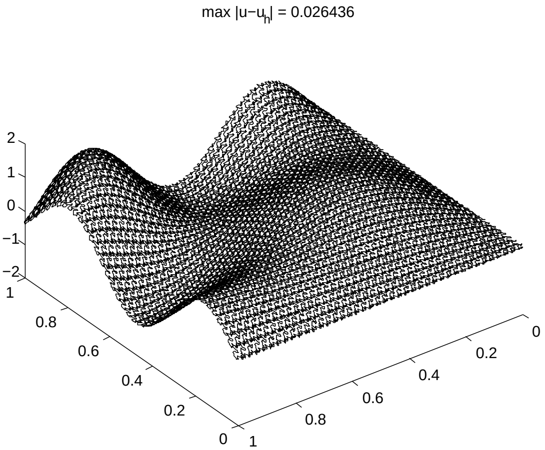
Solution on a 20 $\times$ 20-pegasus mesh. Max-Err=0.077

mesh 40



A mesh of 40×40 pegasus.

Going **totally** berserk !!



Solution on a 40×40-pegasus mesh. Max-Err=0.026

Summarizing the main features of VEM

As for other methods on **polytopal** elements

- the trial and test functions inside each element are rather complicated (e.g. solutions of suitable PDE's or systems of PDE's).

Contrary to other methods on polytopal elements,

- they **do not** require the approximate evaluation of trial and test functions at the integration points.
- In most cases they satisfy the *patch test* **exactly** (up to the computer accuracy).
- We have now a full family of spaces.

gen ph

The general philosophy

In every element, to *define* the generic (scalar or vector valued) element v of our VEM space:

- You start from the **boundary** d.o.f. and use a 1D edge-by edge reconstruction
- Then you define v **inside** as the solution of a (system of) PDE's, typically with a polynomial right-hand side.
- The construction is such that **all polynomials** of a certain degree belong to the local space. In general the local space also contains some additional elements.

Let us see some examples.

vem nod sp k

We take, for every integer $k \geq 1$

$$V_h^E = \{v|\ v_{|e} \in \mathbb{P}_k(e)\, \forall \text{ edge } e \text{ and } \Delta v \in \mathbb{P}_{k-2}(E)\}$$

It is easy to see that the local space will contain all $\mathbb{P}_k$.
As degrees of freedom we take:

- the values of v at the vertices,
- the moments $\int_e v\, p_{k-2} \mathrm{d}e$ on each edge,
- the moments $\int_E v\, p_{k-2} \mathrm{d}E$ inside.

It is easy to see that these d.o.f. are *unisolvent*.

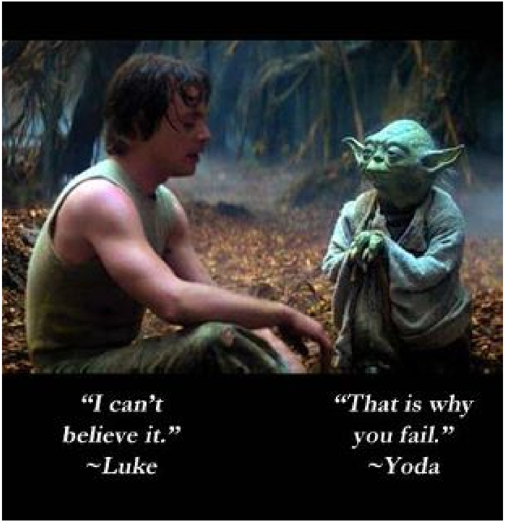
3 Card M

The L^2-projection

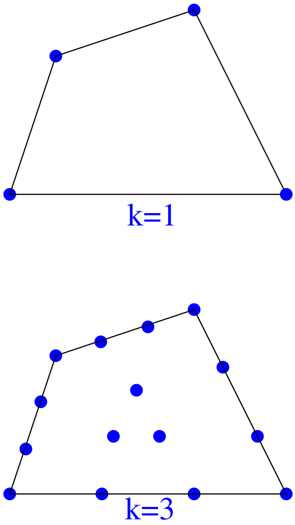
A fantastic trick (sometimes called *The Three Card Monte trick*), often allows the *exact* computation of the moments of order $k-1$ and k of every $v \in V_h^E$.

This is very useful for dealing with the 3D case.

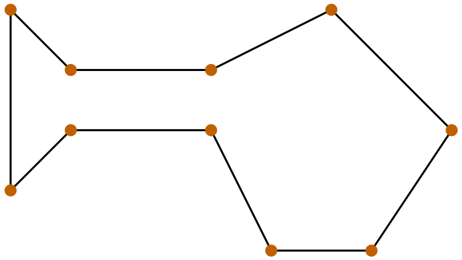
The *Three Card Monte Trick* is hard to believe



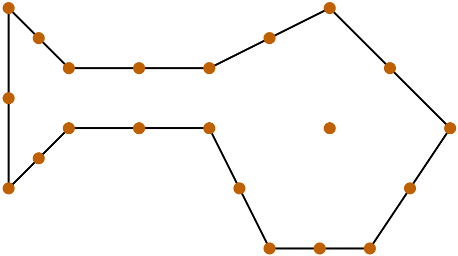
Example: Degrees of freedom of nodal VEM's in 2D



More general geometries $k = 1$



More general geometries $k = 2$



3Dim

For a given integer $k \geq 1$, and for every element E, we set

$$V_h^E = \{v \in H^1(E) | \, v_{|e} \in \mathbb{P}_k(e) \, \forall \text{ edge } e,$$
$$v_{|f} \in V_h^f \, \forall \text{ face } f, \text{ and } \Delta v \in \mathbb{P}_{k-2}(E)\}$$

with the degrees of freedom:

- values of v at the vertices,
- moments $\int_e v \, p_{k-2}(e)$ on each edge e,
- moments $\int_f v \, p_{k-2}(f)$ on each face f, and
- moments $\int_E v \, p_{k-2}(E)$ on E.

Ex: for $k = 3$ the number of degrees of freedom would be: the number of vertices, plus $2\times$ the number of edges, plus $3\times$ the number of faces, plus 4. On a cube this makes $8 + 24 + 18 + 4 = 54$ against 64 for $\mathbb{Q}_3$.

Oth Sp

Along the same lines (more or less...), one can build approximations of the other spaces discussed above, and thus have

- $VEM, nodal \subseteq H^1(\Omega)$
- $VEM, edge \subseteq H(\mathbf{curl};\Omega)$
- $VEM, face \subseteq H(\mathrm{div};\Omega)$
- $VEM, volume \subseteq L^2(\Omega)$
- $VEM, nodal - C^1 \subseteq H^2(\Omega)$

obviously with different degrees of accuracy k.

Seq Ex

A very useful property

Moreover, the classical differential operators *grad*, *curl*, and *div* send these VEM spaces one into the other (up to the obvious adjustments for the polynomial degrees). Indeed:

$$\textbf{grad}(\mathit{VEM}, \mathit{nodal}) \subseteq \mathit{VEM}, \mathit{edge}$$

$$\textbf{curl}(\mathit{VEM}, \mathit{edge}) \subseteq \mathit{VEM}, \mathit{face}$$

$$\mathrm{div}(\mathit{VEM}, \mathit{face}) \subseteq \mathit{VEM}, \mathit{volume}$$

$$\mathbb{R} \xrightarrow{i} V_k^{\mathrm{nod}}(\Omega) \xrightarrow{\textbf{grad}} V_{k-1}^{\mathrm{edg}}(\Omega) \xrightarrow{\textbf{curl}} V_{k-2}^{\mathrm{fac}}(\Omega) \xrightarrow{\mathrm{div}} V_{k-3}^{\mathrm{vol}}(\Omega) \xrightarrow{o} 0$$

and the corresponding d.o.f.s are **computable**. Scal Pr 1

The crucial feature common to all these choices is the possibility **to construct** (starting from the degrees of freedom, and without solving approximate problems in the element) **a symmetric bilinear form** $[\mathbf{u},\mathbf{v}]_h$ such that, on each element E, we have

$$[\mathbf{p}_k,\mathbf{v}]_h^E=\int_E \mathbf{p}_k\cdot\mathbf{v}\,\mathrm{d}E\;\forall\mathbf{p}_k\in(\mathbb{P}_k(E))^d,\;\forall\mathbf{v}\text{ in the VEM space}$$

and $\exists\,\alpha^*\geq\alpha_*>0$ independent of h such that

$$\alpha_*\|\mathbf{v}\|^2_{L^2(E)}\leq[\mathbf{v},\mathbf{v}]_h^E\leq\alpha^*\|\mathbf{v}\|^2_{L^2(E)},\qquad\forall\mathbf{v}\text{ in the VEM space}$$

Scal Pr 2

In other words: In each VEM space (nodal, edge, face, volume) we have a corresponding **inner product**

$$\left[\cdot,\cdot\right]_{VEM,nodal}, \left[\cdot,\cdot\right]_{VEM,edge}, \left[\cdot,\cdot\right]_{VEM,face}, \left[\cdot,\cdot\right]_{VEM,volume}$$

that **scales properly**, and reproduces **exactly** the L^2 inner product when **at least one of the two entries** is a **polynomial** of degree $\leq k$.

This can be applied to the discretization of PDE

Strong formulation of Darcy's law

- p = pressure
- $\mathbf{u}$ = velocities *(volumetric flow per unit area)*
- f = source
- $\mathbb{K}$ = material-depending (full) tensor

- $\mathbf{u} = -\mathbb{K}\nabla p$ (Constitutive Equation)
- $\mathrm{div}\,\mathbf{u} = f$ (Conservation Equation)

$$-\mathrm{div}(\mathbb{K}\nabla p) = f \qquad \text{in } \Omega,$$
$$p = 0 \qquad \text{on } \partial\Omega, \qquad \text{for simplicity.}$$

The **variational formulation** of Darcy problem is:
find $p \in H^1_0(\Omega)$ *such that*

$$\int_\Omega \mathbb{K}\nabla p \cdot \nabla q \,\mathrm{d}x = \int_\Omega f\, q \mathrm{d}x \qquad \forall q \in H^1_0(\Omega).$$

and as **VEM approximate problem** we can take:
find $p_h \in$ *VEM,nodal such that:*

$$[\mathbb{K}\nabla p_h, \nabla q_h]_{VEM,edge} = [f,\ q_h]_{VEM,nodal} \ \forall q_h \in \text{VEM,nodal}$$

D Mix F

Darcy problem, in *mixed form*, is instead:
find $p \in L^2(\Omega)$ *and* $\mathbf{u} \in H(\mathrm{div};\Omega)$ *such that:*

$$\int_\Omega \mathbb{K}^{-1}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}V = \int_\Omega p\,\mathrm{div}\mathbf{v}\,\mathrm{d}V \quad \forall \mathbf{v} \in H(\mathrm{div};\Omega)$$

and

$$\int_\Omega \mathrm{div}\mathbf{u}\, q\,\mathrm{d}V = \int_\Omega f\, q\,\mathrm{d}V \qquad \forall q \in L^2(\Omega).$$

The approximate mixed formulation can be written as:

find $p_h \in VEM, volume$ *and* $\mathbf{u}_h \in VEM, face$ *such that*

$$[\mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v}_h]_{VEM,face} = [p_h, \mathrm{div}\mathbf{v}_h]_{VEM,volume}\ \forall \mathbf{v}_h \in VEM,face$$

and

$$[\mathrm{div}\mathbf{u}_h, q_h]_{VEM,volume} = [f,\ q_h]_{VEM,volume}\ \forall q_h \in VEM,volume.$$

Magn

- $\mathbf{j}$ = *divergence free* current density
- μ = magnetic permeability
- $\mathbf{B}$ = magnetic induction
- $\mathbf{H} = \mu^{-1}\mathbf{B}$ =magnetic field
- $\mathbf{curl}\,\mathbf{H} = \mathbf{j}$

The classical magnetostatic equations become now

$$\mathbf{curl}\,\mathbf{H} = \mathbf{j} \text{ and } \mathrm{div}(\mu\mathbf{H}) = 0 \text{ in } \Omega,$$

$$\mathbf{H} \times \mathbf{n} = 0 \text{ on } \partial\Omega.$$

Magn VF Dis

A variational formulation of the magnetostatic problem is:

$$\begin{cases} \textit{Find } \mathbf{H} \in H_0(\mathbf{curl}, \Omega) \textit{ and } p \in H_0^1(\Omega) \textit{ such that :} \\ (\mathbf{curl}\,\mathbf{H}, \mathbf{curl}\,\mathbf{v}) - (\nabla p, \mu \mathbf{v}) = (\mathbf{j}, \mathbf{curl}\mathbf{H})\;\forall\, \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \\ \qquad\qquad (\mu \mathbf{H}, \nabla q) = 0 \qquad \forall\, q \in H_0^1(\Omega), \end{cases}$$

and the **VEM approximation** can be chosen as:

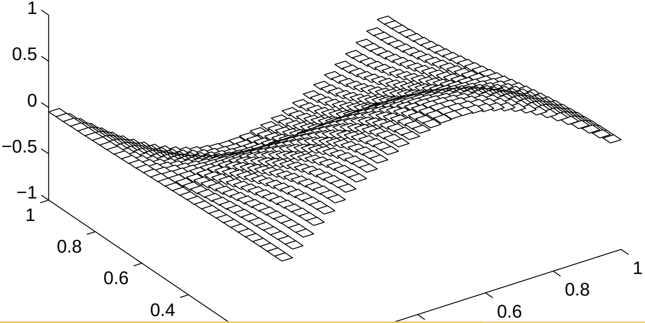
$$\begin{cases} \textit{Find } \mathbf{H}_h \in V_e := VEM, edges \textit{ and } p_h \in V_n := VEM, nodal \textit{ s.t.} \\ [\mathbf{curl}\mathbf{H}_h, \mathbf{curl}\,\mathbf{v}]_{h,f} - [\mathbf{grad}\, p_h, \mathbf{v}]_{h,e} = [\mathbf{j}, \mathbf{curl}\mathbf{v}]_{h,f}\;\forall \mathbf{v} \in V_e, \\ [\mathbf{H}_h, \mathbf{grad}\, q]_{h,e} = 0 \quad \forall\, q \in V_n. \end{cases}$$

Ris Mix

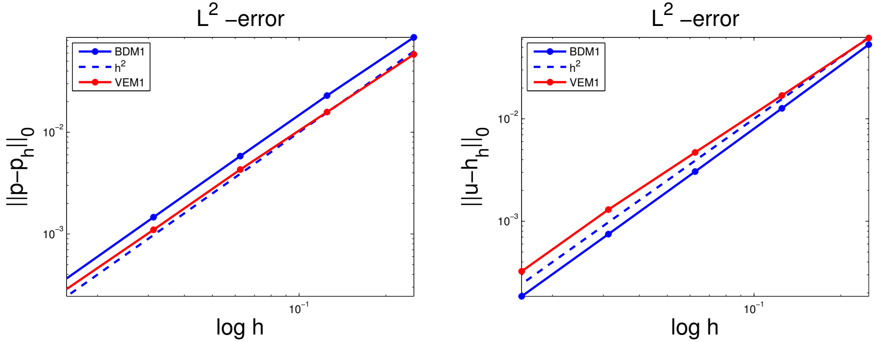
Mesh of squares 4x4, 8x8, ...,64x64

Exact solution p=sin(2x)cos(3y)

Here below the pw constant approximate solution

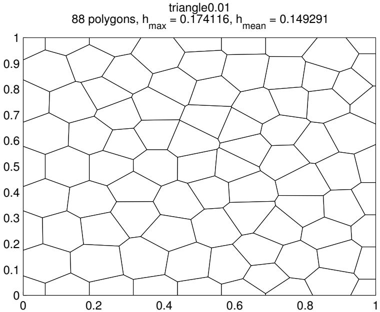


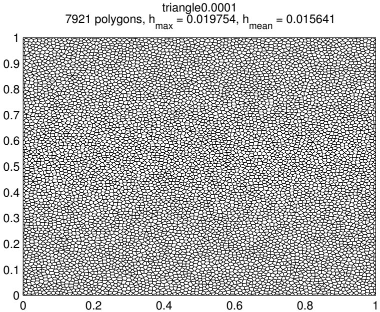
Numerical results-Squares



D Vo M

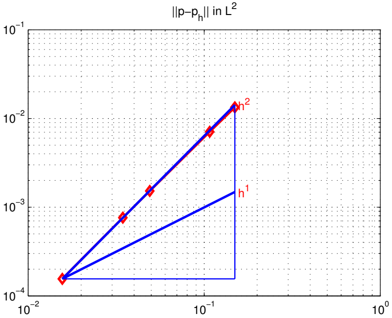
Voronoi polygons 88,...,7921
Exact solution p=sin(2x)cos(3y)

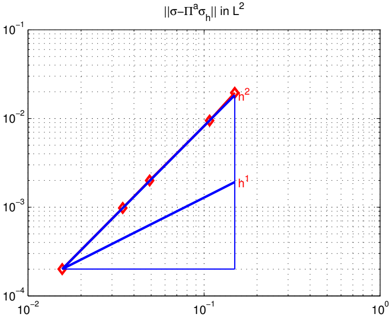




D Vo R

Numerical results-Voronoi



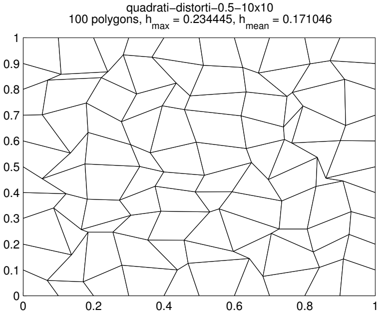


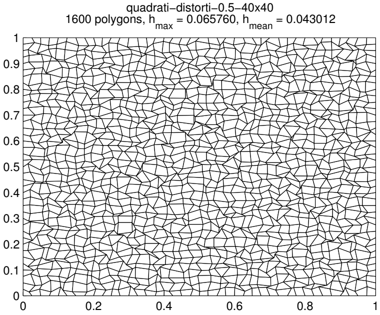
Dist M

Numerical results-Distorted Quads meshes

Mesh of distorted quads: 10x10; 20x20; 40x40

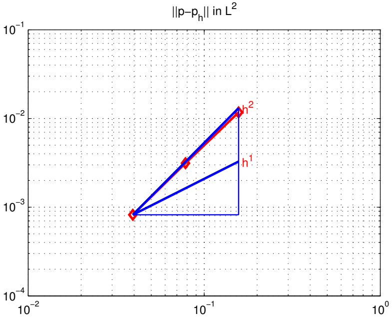
Exact solution: $p = \sin(2x)\cos(3y)$

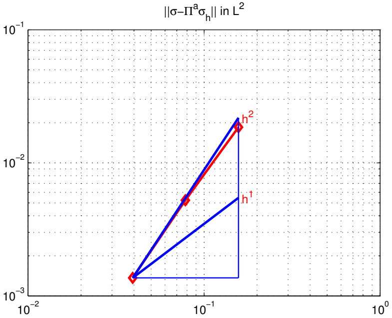




Dist M

Numerical results–Distorted Quads

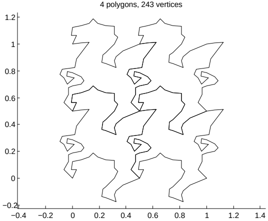


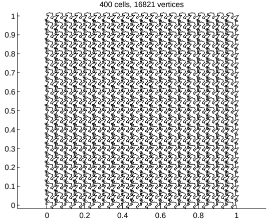


Horses Mesh

Mesh of horses: 4x4; 8x8; 10x10; 16x16

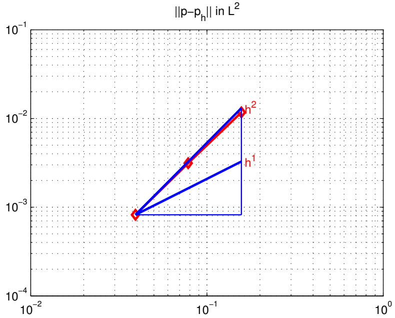
Exact solution: $p = \sin(2x)\cos(3y)$

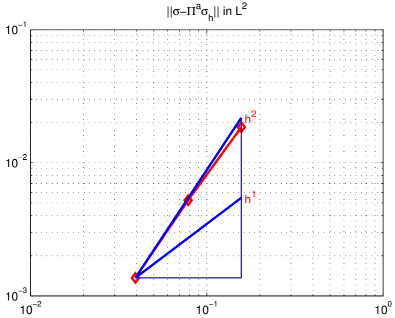




Horses Res

Numerical results–Winged horses





Conc

- Virtual Elements are a new method, and a lot of work is needed to assess their *pros* and *cons*.
- Their major interest is on polygonal and polyhedral elements, but their use on distorted quads, hexa, and the like, is also quite promising.
- For triangles and tetrahedra the interest seems to be concentrated in higher order continuity (e.g. plates).
- The use of VEM mixed methods seems to be quite interesting, in particular for their connections with other methods for polygonal/polyhedral elements.