A CHARGE-CONSERVATIVE FINITE ELEMENT METHOD FOR INDUCTIONLESS MHD EQUATIONS. PART I: CONVERGENCE

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Abstract. A charge-conservative finite element method is proposed to solve the inductionless and incompressible magnetohydrodynamic (MHD) equations in three dimensions. The method yields an exactly divergence-free current density directly. We prove that, as the spatial mesh size $h \to 0$, the fully discrete solutions converge to the solutions of the semi-continuous problem weakly in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}(\mathrm{div}, \Omega)$ upon an extracted subsequence, and as the time step size $\tau \to 0$, the semi-continuous solutions converge to the solutions of the continuous problem weakly in $\boldsymbol{L}^2(0, T; \boldsymbol{H}^1(\Omega)) \times \boldsymbol{L}^2(0, T; \boldsymbol{H}(\mathrm{div}, \Omega))$ upon an extracted subsequence. This yields the existence of the continuous solutions naturally. Three numerical experiments are presented to show the convergence rate of discrete solutions and the charge-conservation of the method.

Key words. Inductionless MHD equations, conservation of charges, augmented Lagrangian finite element method, block preconditioner, divergence-free constraint.

AMS subject classifications. 65M60, 76W05

1. Introduction.

The incompressible magnetohydrodynamic (MHD) equations describe the dynamic behavior of an electrically conducting fluid under the influence of a magnetic field. They occur in models for, fusion reactor blankets, liquid metal magnetic pumps, aluminum electrolysis among others (see Refs. [1, 19]). To design such liquid metal blankets, numerical simulations of incompressible MHD play an important role in knowing the characteristics of MHD flows at high Hartmann numbers. MHD is a multi-physics phenomenon: the magnetic field changes the momentum of the fluid through the Lorenz force, and conversely, the conducting fluid influences the magnetic field through electric currents. In this way multiple physical fields, such as the velocity, the pressure, and the electromagnetic fields, are coupled. However, for the case that magnetic Reynolds number is small and that the magnetic field tends to be saturated, the time derivative of magnetic induction can be neglected. In this case, the electric field is considered to be quai-static, that is, $\boldsymbol{E} = -\nabla\phi$ where $\phi(\boldsymbol{x}, t)$ stands for the scalar potential (cf. e.g. [20, 21]). With this simplification, the inductionless MHD system reads as follows

$$\rho(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \nabla p - \boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\sigma^{-1}\boldsymbol{J} + \nabla\phi - \boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{0} \quad \text{in } \Omega, \tag{1.1b}$$

$$\mathrm{div}\, \boldsymbol{u} = 0, \quad \mathrm{div}\, \boldsymbol{J} = 0 \quad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega, \tag{1.1d}$$

where Ω is a bounded Lipschitz domain, ρ the density of fluid, ν the dynamic viscosity, σ the electrical conductivity of fluid, $\boldsymbol{f}$ the external force, $\boldsymbol{B}$ the applied magnetic field which is assumed to be given, $\boldsymbol{u}_0 \in \boldsymbol{H}^1(\Omega)$ the initial value of velocity satisfying $\mathrm{div}\, \boldsymbol{u}_0 = 0$, and $\partial_t \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t}$ the time

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derivative of $\boldsymbol{u}$. The unknowns are the velocity $\boldsymbol{u}$, the pressure p, the current density $\boldsymbol{J}$, and the electric scalar potential ϕ. The model combined with appropriate boundary conditions has many important applications in real life, particularly, in modeling liquid lithium-lead blanket of magnetic fusion device TOKAMAK, where the applied magnetic field $\boldsymbol{B}$ is about 2–5 Tesla (cf. e.g. [20,21]).

For the well-posedness of (1.1), we need adequate boundary conditions for various applications. The general form of boundary conditions for hydrodynamic variables reads

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma_{\mathrm{d}}, \tag{1.2a}$$

$$\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - p\boldsymbol{n} = 0 \quad \text{on } \Gamma_{\mathrm{n}} := \Gamma \backslash \bar{\Gamma}_{\mathrm{d}}, \tag{1.2b}$$

where $\mathbb{I}$ is the 3×3 identity matrix, $\Gamma = \partial\Omega$ the boundary of Ω, and $\boldsymbol{n}$ the unit outer normal to Ω. Here we assume $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma_{\mathrm{d}})$ and $\Gamma_{\mathrm{d}} \neq \emptyset$. Since $\operatorname{div} \boldsymbol{u} = 0$, the compatibility of $\boldsymbol{u}$ with its boundary trace requires

$$\boldsymbol{g} \in \left\{ \gamma \boldsymbol{v}|_{\Gamma_{\mathrm{d}}} : \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}(\operatorname{div} 0, \Omega) \right\},$$

where γ: $\boldsymbol{H}^1(\Omega) \rightarrow \boldsymbol{H}^{1/2}(\Gamma)$ stands for the trace operator. Generally, $\Gamma_{\mathrm{d}} = \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{in}}$, where Γ_{wall} stands for fixed wall boundary on which $\boldsymbol{g} = 0$ and Γ_{in} stands for inflow boundary on which $\boldsymbol{g} \cdot \boldsymbol{n} \leq 0$. Moreover, Γ_{n} stands for Neumann boundary or outflow boundary Γ_{out} on which $\boldsymbol{u} \cdot \boldsymbol{n} \geq 0$. In practical applications, a more reasonable boundary condition on Γ_{n} than (1.2b) should be

$$\left[\nu \underline{\underline{\varepsilon}}(\boldsymbol{u}) - p\mathbb{I} \right] \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_{\mathrm{n}},$$

where $\underline{\underline{\varepsilon}}(\boldsymbol{u}) = \dfrac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top} \right)$ is the rate of deformation tensor. For simplicity, we follow [9, page 133] and [8, page 333] to adopt the approximate Neumann boundary given in (1.2b) and restrict our study to homogeneous flows.

For the current density and the electric potential, boundary conditions usually consist of insulating type and conductive type, namely, $\Gamma = \Gamma_{\mathrm{i}} \cup \Gamma_{\mathrm{c}}$. On insulating wall Γ_{i}, electric currents can not penetrate the wall so that $\boldsymbol{J} \cdot \boldsymbol{n} = 0$. On conductive boundaries, the values of electric potential can be measured, meaning that, Dirichlet boundary condition for ϕ can be prescribed. So we impose the boundary conditions for $\boldsymbol{J}$, ϕ as follows

$$\begin{aligned} \boldsymbol{J} \cdot \boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{i}}, \\ \phi &= \xi \quad \text{on } \Gamma_{\mathrm{c}}, \end{aligned}$$

or more generally

$$\boldsymbol{J} \cdot \boldsymbol{n} = J \quad \text{on } \Gamma_{\mathrm{i}}, \tag{1.3a}$$

$$\phi = \xi \quad \text{on } \Gamma_{\mathrm{c}}, \tag{1.3b}$$

where $\xi \in H^{1/2}(\Gamma_{\mathrm{c}})$ and $J \in H^{-1/2}(\Gamma_{\mathrm{i}})$. Let $\gamma_{\boldsymbol{n}}$: $\boldsymbol{H}(\operatorname{div}, \Omega) \rightarrow H^{-1/2}(\Gamma)$ be the normal trace operator. Similarly, the compatibility of J with $\operatorname{div} \boldsymbol{J} = 0$ requires

$$J \in \{ \gamma_{\boldsymbol{n}} \boldsymbol{d}|_{\Gamma_{\mathrm{i}}} : \ \boldsymbol{d} \in \boldsymbol{H}(\operatorname{div} 0, \Omega) \}. \tag{1.4}$$

In this paper, we shall study finite element approximation of (1.1)–(1.3) within a unified framework.

The inductionless MHD model has been widely used and studied in both engineering and mathematical communities (see e.g. [3,9,20,21,25]). Since it replaces Maxwell's equations with Poisson's equation for the electric scalar potential, numerical solution is more economic compared with the full MHD model. In 1988, Perterson proved the existence and uniqueness of weak solutions of stationary inductionless MHD model and studied its finite element approximation [25]. In 2014, Badia et al studied stabilized finite element method for inductionless MHD model [3]. They compute $(\boldsymbol{J}, \phi)$ by a mixed formulation with interior penalties. Block recursive LU preconditioners are proposed to solve the discrete problem. The preconditioners are robust to relatively high Hartmann number, $H_a := \sqrt{\kappa R_e}$, where the coupling number κ and the Reynolds number R_e will be given in the next section.

Recently, divergence-free finite element methods become attractive in the literature. For inductionless MHD model, the divergence-free constraint on the discrete current density $\boldsymbol{J}_h$ can be fulfilled by either post-processing or mixed finite element methods. In 2007, Ni et al developed consistent and charge-conservative schemes for inductionless MHD equations on both structured and unstructured meshes. They solve the Poisson equation of electric potential and compute $\boldsymbol{J}_h$ by post-processing so that $\operatorname{div} \boldsymbol{J}_h = 0$ holds exactly and globally in Ω. The numerical results agree well with experiment results [20, 21]. For full MHD model, the constraint $\operatorname{div} \boldsymbol{J}_h = 0$ is satisfied naturally as a consequence of $\boldsymbol{J}_h := \mathbf{curl}\, \boldsymbol{B}_h$ where the discrete magnetic induction $\boldsymbol{B}_h$ is the primitive variable to be computed. Here we mention some important references in this direction. In 2003, Schneebeli and Schözau proposed a new mixed finite element discretization for stationary full MHD equations where $\boldsymbol{B}$ is discretized by Nédélec's edge elements of the first kind [27]. In 2004, Schötzau proved the optimal error estimates of the finite element method [28]. In 2010, Greif, Li, Schötzau, and Wei extended Schötzau's work to time-dependent MHD equations. They discretized the velocity with $\boldsymbol{H}(\operatorname{div}, \Omega)$–conforming face elements so that $\operatorname{div} \boldsymbol{u}_h = 0$ holds exactly [12]. In 2008, Prohl studied the convergence and error estimates of finite element method for time-dependent MHD equations [26]. We also refer to the work of Gunzburger, Meir, and Peterson which discretizes $\boldsymbol{B}$ by continuous nodal finite elements [11].

The approximate solutions of full MHD equations should satisfy $\operatorname{div} \boldsymbol{B}_h = 0$. In 2017, Hu, Ma, and Xu proposed a stable finite element method which discretizes the magnetic induction and the electric field simultaneously and yields $\operatorname{div} \boldsymbol{B}_h = 0$ exactly [15]. In the same year, Hiptmair et al proposed a fully divergence-free finite element method so that both the discrete velocity and the discrete magnetic induction are divergence-free exactly [14]. We also refer to [17] for the divergence-free central DG method for ideal MHD equations.

The purpose of this paper is to propose a charge-conservative finite element method so that $\operatorname{div} \boldsymbol{J}_h = 0$ holds exactly. We solve $(\boldsymbol{J}, \phi)$ simultaneously within a mixed framework where $\boldsymbol{J}$ is discretized by $\boldsymbol{H}(\operatorname{div}, \Omega)$-conforming face elements and ϕ is discretized by $L^2(\Omega)$-conforming volume elements. For hydrodynamic variables $(\boldsymbol{u}, p)$, we adopt an augmented Lagrangian (AL) formulation of the Navier-Stokes equations. The AL formulation has advantages in both controlling $\operatorname{div} \boldsymbol{u}_h$ and designing robust solvers. AL-stabilized finite element methods have been studied in [16, 23] for for Stokes equations and in [4, 24] for incompressible Navier-Stokes equations. Numerical computations show that, compared with popular preconditioners like pressure-convection-diffusion (PCD) preconditioner and least-squares-commutator (LSC) preconditioner [8], block preconditioners using AL-stabilization are very competitive for Oseen equations or Navier-Stokes equations.

The paper is organized as follows: In Section 2, we derive a mixed weak formulation of the inductionless MHD model. In Section 3, we propose a fully discrete problem by using extrapolations of discrete solutions from previous time steps. The energy stability of discrete solutions is shown. In

Section 4, we prove that the discrete solutions have a subsequence which converges to the solutions of the continuous problem. In Section 5, we present three numerical examples to test the convergence rate and the charge-conservation of discrete solutions. In Section 6, we conclude with the main result of the paper. Throughout the paper, we assume that ρ, ν, μ, σ are positive constants and denote vector-valued quantities by boldface notations, such as $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^3$.

2. Inductionless MHD model. First we introduce some sobolev spaces. Let $L^2(\Omega)$ be the space of square-integrable functions and let $H^1(\Omega)$, $\boldsymbol{H}(\mathrm{div}, \Omega)$ be its subspaces with square integrable gradients and square integrable divergences respectively. Let $H_0^1(\Omega)$, $\boldsymbol{H}_0(\mathrm{div}, \Omega)$ denote their subspaces with vanishing traces and vanishing normal traces on $\Gamma := \partial\Omega$ respectively. We refer to [10, page 26] for their definitions and inner products. We also define

$$\boldsymbol{H}(\mathrm{div}\,0, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) : \ \mathrm{div}\, \boldsymbol{v} = 0\}, \qquad \boldsymbol{H}_0(\mathrm{div}\,0, \Omega) := \boldsymbol{H}(\mathrm{div}\,0, \Omega) \cap \boldsymbol{H}_0(\mathrm{div}, \Omega).$$

For $1 \le p \le +\infty$ and a given Sobolev space X, we introduce the Bochner space

$$L^p(0, T; X) := \{v : T \to X \text{ is Bochner measurable} : \|v\|_X \in L^p(0, T)\},$$

whose norm is defined by

$$\|v\|_{L^p(0,T;X)} := \left[\int_0^T \|v(t)\|_X^p \,\mathrm{d}t\right]^{1/p} \quad \text{for } \ p < +\infty, \qquad \|v\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t \in (0,T)} \|v(t)\|_X .$$

Moreover, for any positive integer m, we define the regular spaces with respect to t by

$$W^{m,p}(0, T; X) := \left\{ \frac{\partial^k v}{\partial t^k} \in L^p(0, T; X) : 0 \le k \le m \right\}.$$

2.1. Dimensionless MHD equations. Let L, t_0, B_0, $u_0 = L/t_0$ be characteristic quantities of length, time, magnetic induction, and fluid velocity respectively. We introduce the dimensionless variables as follows

$$\begin{aligned}
&\boldsymbol{x} \leftarrow \boldsymbol{x}/L, \qquad t \leftarrow t/t_0, \qquad \boldsymbol{u} \leftarrow \boldsymbol{u}/u_0, \qquad p \leftarrow p/(\rho u_0^2), \qquad \phi \leftarrow \phi/(u_0 B_0 L),\\
&\boldsymbol{B} \leftarrow \boldsymbol{B}/B_0, \qquad \boldsymbol{J} \leftarrow \boldsymbol{J}/(\sigma u_0 B_0), \qquad \boldsymbol{f} \leftarrow \boldsymbol{f} t_0/(\rho u_0).
\end{aligned}$$

The dimensionless MHD model with initial and boundary conditions is given by

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \frac{1}{R_e}\Delta \boldsymbol{u} + \nabla p - \kappa \boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1a}$$

$$\boldsymbol{J} + \nabla\phi - \boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{0} \quad \text{in } \Omega, \tag{2.1b}$$

$$\mathrm{div}\, \boldsymbol{u} = 0, \quad \mathrm{div}\, \boldsymbol{J} = 0 \quad \text{in } \Omega, \tag{2.1c}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega, \tag{2.1d}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma_{\mathrm{d}}, \tag{2.1e}$$

$$\frac{1}{R_e}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - p\boldsymbol{n} = 0 \quad \text{on } \Gamma_{\mathrm{n}}, \tag{2.1f}$$

$$\boldsymbol{J} \cdot \boldsymbol{n} = J \quad \text{on } \Gamma_{\mathrm{i}}, \tag{2.1g}$$

$$\phi = \xi \quad \text{on } \Gamma_{\mathrm{c}}, \tag{2.1h}$$

where $R_e = \rho L u_0/\nu$ is the Reynolds number and $\kappa = \sigma L B_0^2/(\rho u_0)$ is the coupling number between the fluid and the magnetic field. Here different boundary conditions are imposed on nonintersecting parts of the whole boundary Γ, namely,

$$\Gamma = \Gamma_{\mathrm{d}} \cup \Gamma_{\mathrm{n}}, \qquad \Gamma = \Gamma_{\mathrm{i}} \cup \Gamma_{\mathrm{c}}, \qquad \Gamma_{\mathrm{d}} \cap \Gamma_{\mathrm{n}} = \emptyset, \qquad \Gamma_{\mathrm{i}} \cap \Gamma_{\mathrm{c}} = \emptyset.$$

2.2. A weak formulation. For convenience, we introduce some notations for function spaces which are, respectively, the spaces of velocity

$$\boldsymbol{V} = \boldsymbol{H}^1(\Omega), \qquad \boldsymbol{V}_{\mathrm{d}} = \{\boldsymbol{v} \in \boldsymbol{V} : \gamma \boldsymbol{v} = 0 \ \text{ on } \Gamma_{\mathrm{d}}\},$$

the space of pressure

$$Q = L^2(\Omega) \quad \text{if } \ \Gamma_{\mathrm{n}} \neq \emptyset, \qquad Q = L_0^2(\Omega) := L^2(\Omega)/\mathbb{R} \quad \text{if } \ \Gamma_{\mathrm{n}} = \emptyset,$$

the spaces of electric current density

$$\boldsymbol{D} = \boldsymbol{H}(\mathrm{div}, \Omega), \qquad \boldsymbol{D}_{\mathrm{i}} = \{\boldsymbol{v} \in \boldsymbol{D} : \gamma_{\boldsymbol{n}} \boldsymbol{v} = 0 \ \text{ on } \Gamma_{\mathrm{i}}\},$$

and the space of electric scalar potential

$$S = L^2(\Omega) \quad \text{if } \ \Gamma_{\mathrm{c}} \neq \emptyset, \qquad S = L_0^2(\Omega) \quad \text{if } \ \Gamma_{\mathrm{c}} = \emptyset.$$

The divergence-free subspaces of $\boldsymbol{V}$ and $\boldsymbol{V}_{\mathrm{d}}$ are defined by

$$\boldsymbol{V}(\mathrm{div}\, 0) = \boldsymbol{V} \cap \boldsymbol{H}(\mathrm{div}\, 0, \Omega), \qquad \boldsymbol{V}_{\mathrm{d}}(\mathrm{div}\, 0) = \boldsymbol{V}_{\mathrm{d}} \cap \boldsymbol{H}(\mathrm{div}\, 0, \Omega).$$

Similarly, the divergence-free subspaces of $\boldsymbol{D}$ and $\boldsymbol{D}_{\mathrm{i}}$ are defined by

$$\boldsymbol{D}(\mathrm{div}\, 0) = \boldsymbol{H}(\mathrm{div}\, 0, \Omega), \qquad \boldsymbol{D}_{\mathrm{i}}(\mathrm{div}\, 0) = \boldsymbol{D}_{\mathrm{i}} \cap \boldsymbol{H}(\mathrm{div}\, 0, \Omega).$$

Multiply both sides of (2.1a) with $\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}(\mathrm{div}\, 0)$. The formula of integration by part yields

$$(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) + \mathscr{A}(\boldsymbol{u}, \boldsymbol{v}) - \kappa(\boldsymbol{J}, \boldsymbol{B} \times \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \tag{2.2}$$

where the bilinear form $\mathscr{A} \colon \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ is defined by

$$\mathscr{A}(\boldsymbol{w}, \boldsymbol{v}) := \frac{1}{R_e}(\nabla \boldsymbol{w}, \nabla \boldsymbol{v}) \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}.$$

Since $\mathrm{div}\, \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} \geq 0$ on Γ_n, the convection term satisfies

$$(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) = \mathscr{O}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}},$$

where the trilinear form $\mathscr{O} \colon \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ is defined by

$$\mathscr{O}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) := \frac{1}{2}\left[(\boldsymbol{w} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{w} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}) + \int_{\Gamma_{\mathrm{n}}} (\gamma_{\boldsymbol{n}}^{\downarrow} \boldsymbol{w})(\boldsymbol{u} \cdot \boldsymbol{v})\right].$$

Here $\gamma_{\boldsymbol{n}}^{\downarrow} \boldsymbol{w} := \max(\gamma_{\boldsymbol{n}} \boldsymbol{w}, 0)$ is the outflow flux on the fixed boundary Γ_{n}.

Multiplying both sides of (2.1b) with $\boldsymbol{d} \in \boldsymbol{D}_{\mathrm{i}}(\mathrm{div}\, 0)$ and using integration by part, we have

$$(\boldsymbol{J}, \boldsymbol{d}) + (\boldsymbol{B} \times \boldsymbol{u}, \boldsymbol{d}) = \langle \boldsymbol{d} \cdot \boldsymbol{n}, \xi \rangle_{\Gamma_{\mathrm{c}}}, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_{\mathrm{c}}}$ denotes the duality between $H^{-1/2}(\Gamma_{\mathrm{c}})$ and $H^{1/2}(\Gamma_{\mathrm{c}})$. Combining (2.2) and (2.3), we obtain a weak formulation of (2.1):

Find $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;\boldsymbol{V}(\operatorname{div}0)) \cap \boldsymbol{W}^{1,4/3}(0,T;\boldsymbol{V}_{\mathrm{d}}(\operatorname{div}0)')$ and $\boldsymbol{J} \in \boldsymbol{L}^2(0,T;\boldsymbol{D}(\operatorname{div}0))$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\gamma\boldsymbol{u} = \boldsymbol{g}$ on Γ_{d}, $\gamma_{\boldsymbol{n}}\boldsymbol{J} = J$ on Γ_{i}, and

$$(\partial_t \boldsymbol{u}, \boldsymbol{v}) + \mathscr{O}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + \mathscr{A}(\boldsymbol{u},\boldsymbol{v}) - \kappa(\boldsymbol{J}\times\boldsymbol{B},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}(\operatorname{div}0), \tag{2.4a}$$

$$(\boldsymbol{J},\boldsymbol{d}) + (\boldsymbol{B}\times\boldsymbol{u},\boldsymbol{d}) = \langle \gamma_{\boldsymbol{n}}\boldsymbol{d}, \xi\rangle_{\Gamma_{\mathrm{c}}} \qquad \forall\, \boldsymbol{d} \in \boldsymbol{D}_{\mathrm{i}}(\operatorname{div}0). \tag{2.4b}$$

3. Finite element approximation. In this section, we study the fully discrete approximation to the inductionless MHD equations. Let $\mathcal{T}_h$ be a quasi-uniform and shape-regular tetrahedral mesh of Ω. The mesh size of $\mathcal{T}_h$ is defined by

$$h = \max_{T\in\mathcal{T}_h} h_T$$

3.1. Finite element spaces. To propose a discrete approximation of (2.4), we introduce the finite element spaces for $\boldsymbol{u}$ and $\boldsymbol{J}$ as follows

$$\boldsymbol{V}^h := \{\boldsymbol{v}\in\boldsymbol{V}:\ \boldsymbol{v}|_K \in \boldsymbol{P}_2(K),\ \ \forall\, K\in\mathcal{T}_h\},$$
$$\boldsymbol{D}^h := \{\boldsymbol{d}\in\boldsymbol{D}:\ \boldsymbol{d}|_K \in \boldsymbol{P}_1(K),\ \forall\, K\in\mathcal{T}_h\},$$

where $P_k(K)$, for an integer $k \geq 0$, is the space of polynomials with degrees no more than k and $\boldsymbol{P}_k(K) = P_k(K)^3$. We refer to [29] for a well-conditioned hierarchical basis of $\boldsymbol{D}^h$ constructed by Xin et al. The subspaces with homogeneous boundary conditions on Γ_{d} or on Γ_{i} are denoted by

$$\boldsymbol{V}^h_{\mathrm{d}} = \boldsymbol{V}_{\mathrm{d}} \cap \boldsymbol{V}^h, \qquad \boldsymbol{D}^h_{\mathrm{i}} = \boldsymbol{D}_{\mathrm{i}} \cap \boldsymbol{D}^h.$$

The weakly divergence-free subspace of $\boldsymbol{V}^h$ and the divergence-free subspaces of $\boldsymbol{D}^h$ are defined by

$$\boldsymbol{V}^h(\operatorname{div}0) := \left\{\boldsymbol{v}_h \in \boldsymbol{V}^h : (\operatorname{div}\boldsymbol{v}_h, q_h) = 0,\ \ \forall\, q_h \in Q^h\right\}, \qquad \boldsymbol{D}^h(\operatorname{div}0) := \boldsymbol{D}^h \cap \boldsymbol{H}(\operatorname{div}0,\Omega),$$

where Q^h is the continuous and piecewise linear finite element space for the pressure

$$Q^h := \left\{q \in Q\cap H^1(\Omega):\ q|_K \in P_1(K),\ \ \forall\, K\in\mathcal{T}_h\right\}.$$

For convenience, we also write

$$\boldsymbol{V}^h_{\mathrm{d}}(\operatorname{div}0) := \boldsymbol{V}^h(\operatorname{div}0)\cap\boldsymbol{V}_{\mathrm{d}}, \qquad \boldsymbol{D}^h_{\mathrm{i}}(\operatorname{div}0) := \boldsymbol{D}^h(\operatorname{div}0)\cap\boldsymbol{D}_{\mathrm{i}}.$$

Clearly the finite element subspaces $\boldsymbol{V}^h(\operatorname{div}0)$ and $\boldsymbol{D}^h(\operatorname{div}0)$ are defined upon global constraints on Ω. Each $\boldsymbol{v}_h \in \boldsymbol{V}^h(\operatorname{div}0)$ is continuous and each $\boldsymbol{d}_h \in \boldsymbol{D}^h(\operatorname{div}0)$ is normally continuous across inner element faces of $\mathcal{T}_h$. If Γ is connected, by the de Rham diagram for finite element spaces [13], we have $\boldsymbol{D}^h(\operatorname{div}0) = \mathbf{curl}\,\boldsymbol{C}^h$, where

$$\boldsymbol{C}^h := \{\boldsymbol{w}_h \in \boldsymbol{H}(\mathbf{curl},\Omega) : \boldsymbol{w}_h|_K \in \boldsymbol{P}_2(K),\ \ \forall\, K\in\mathcal{T}_h\}$$

is the second-order edge element space and $\boldsymbol{H}(\mathbf{curl},\Omega) = \{\boldsymbol{w}\in\boldsymbol{L}^2(\Omega) : \mathbf{curl}\,\boldsymbol{w}\in\boldsymbol{L}^2(\Omega)\}$. It is difficult to write out the basis functions of $\boldsymbol{V}^h(\operatorname{div}0)$ and $\boldsymbol{D}^h(\operatorname{div}0)$ explicitly. The two subspaces are only used for theoretical analysis, not for practical computations, throughout the paper.

It is well-known that the well-posedness of discrete Stokes equations or discrete Navier-Stokes equations depends greatly on the discrete inf-sup condition on the pair of finite element spaces

$\boldsymbol{V}_{\mathrm{d}}^h \times Q^h$. For $\Gamma_{\mathrm{n}} = \emptyset$, there are plentiful studies on the inf-sup condition in the literature. In 1997, Boffi proved the inf-sup condition for three-dimensional Taylor-Hood finite elements [5]. Recently, Zhang et al studied more general finite element pairs which satisfy inf-sup conditions (see [30,32,33] and the references therein). Here we assume that, with a constant $C_{\inf} > 0$ independent of h,

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{\mathrm{d}}^h} \frac{\mathscr{B}(q_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{H}^1(\Omega)}} \geq C_{\inf} \|q_h\|_{L^2(\Omega)} \qquad \forall\, q_h \in Q^h. \tag{3.1}$$

3.2. An augmented Lagrangian finite element approximation. Without loss of generality, we assume that the initial and boundary values $\boldsymbol{u}_0$, $\boldsymbol{g}$, J can be approximated by finite element functions

$$\boldsymbol{u}_0^h \in \boldsymbol{V}^h, \quad \boldsymbol{g}^h \in \left\{ (\gamma \boldsymbol{v}_h)|_{\Gamma_{\mathrm{d}}} : \forall\, \boldsymbol{v}_h \in \boldsymbol{V}^h \right\}, \quad J^h \in \left\{ (\gamma_{\boldsymbol{n}} \boldsymbol{\xi}_h)|_{\Gamma_{\mathrm{i}}} : \forall\, \boldsymbol{\xi}_h \in \boldsymbol{D}^h \right\},$$

which satisfy

$$\lim_{h\to 0} \left(\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{g} - \boldsymbol{g}^h\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}^{1/2}(\Gamma_{\mathrm{d}}))} + \|J - J^h\|_{L^2(0,T;H^{-1/2}(\Gamma_{\mathrm{i}}))} \right) = 0. \tag{3.2}$$

For convenience, we drop the superscripts and simply write them into $\boldsymbol{u}_0$, $\boldsymbol{g}$, and J.

Let $\{t_n = n\tau :\ n = 0, 1, \cdots, N\}$, $\tau = T/N$, be an equidistant partition of $[0, T]$. For a sequence of functions $\{\boldsymbol{v}_n\}$, we define the finite difference operator and the mean values by

$$\delta_t \boldsymbol{v}_n := \frac{1}{\tau}(\boldsymbol{v}_n - \boldsymbol{v}_{n-1}), \qquad \bar{\boldsymbol{v}}_n := \frac{1}{2}(\boldsymbol{v}_n + \boldsymbol{v}_{n-1}). \tag{3.3}$$

Define the time averages of known data over $[t_{n-1}, t_n]$ by

$$\Psi_n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \Psi(t)\mathrm{d}t, \qquad \Psi = \boldsymbol{B},\ \boldsymbol{f},\ J,\ \xi. \tag{3.4}$$

The approximation of $\boldsymbol{g}(t_n)$ is defined by

$$\boldsymbol{g}_n = \frac{1}{\tau} \int_{t_{n-1/2}}^{t_{n+1/2}} \boldsymbol{g}(t)\mathrm{d}t. \tag{3.5}$$

Inspired by [23], we introduce an augmented Lagrangian stabilization to control the divergence of discrete velocity. Define the bilinear form with AL–stabilization by

$$\mathscr{A}_{\mathrm{AL}}(\boldsymbol{w}, \boldsymbol{v}) := \mathscr{A}(\boldsymbol{w}, \boldsymbol{v}) + \alpha(\operatorname{div} \boldsymbol{w}, \operatorname{div} \boldsymbol{v}),$$

where $\alpha > 0$ is the stabilization parameter. Since the exact solution satisfies $\operatorname{div} \boldsymbol{u} = 0$, we have

$$\mathscr{A}(\boldsymbol{u}, \boldsymbol{v}) = \mathscr{A}_{\mathrm{AL}}(\boldsymbol{u}, \boldsymbol{v}).$$

Therefore, (2.4a) has an equivalent form which is given by replacing $\mathscr{A}(\boldsymbol{u}, \boldsymbol{v})$ with $\mathscr{A}_{\mathrm{AL}}(\boldsymbol{u}, \boldsymbol{v})$. Using the AL–stabilization, this provides us with the Crank-Nicolson scheme of (2.4):

Find $\boldsymbol{u}_n \in \boldsymbol{V}^h(\operatorname{div} 0)$, $\boldsymbol{J}_n \in \boldsymbol{D}^h(\operatorname{div} 0)$ such that $\gamma \boldsymbol{u}_n = \boldsymbol{g}_n$ on Γ_{d}, $\gamma_{\boldsymbol{n}} \boldsymbol{J}_n = J_n$ on Γ_{i}, and

$$(\delta_t \boldsymbol{u}_n, \boldsymbol{v}) + \mathscr{O}(\bar{\boldsymbol{u}}_n; \bar{\boldsymbol{u}}_n, \boldsymbol{v}) + \mathscr{A}_{\mathrm{AL}}(\bar{\boldsymbol{u}}_n, \boldsymbol{v}) - \kappa(\boldsymbol{J}_n \times \boldsymbol{B}_n, \boldsymbol{v}) = (\boldsymbol{f}_n, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}^h(\operatorname{div} 0), \tag{3.6a}$$

$$(\boldsymbol{J}_n, \boldsymbol{d}) + (\boldsymbol{B}_n \times \bar{\boldsymbol{u}}_n, \boldsymbol{d}) = \langle \gamma_{\boldsymbol{n}} \boldsymbol{d}, \xi_n \rangle_{\Gamma_{\mathrm{c}}} \qquad \forall\, \boldsymbol{d} \in \boldsymbol{D}_{\mathrm{i}}^h(\operatorname{div} 0). \tag{3.6b}$$

Numerical experiments show that the AL-stabilization plays an important role in local mass conservation [7]. The technique has been studied extensively in the literature for Stokes equations [16, 23] and Navier-Stokes equations (cf. [4, 7, 22, 24]). To obtain a higher-order-of-accuracy for control over $\operatorname{div}\boldsymbol{u}_n$, the parameter α can not be much smaller than 1. However, for $\alpha \gg 1$, the discrete problem becomes difficult to solve. Therefore, we recommend to choose $\alpha = 1$ in practice.

Remark 3.1. *The approximations of Ψ_n to $\Psi(t_{n-1/2})$ in (3.4) and of $\boldsymbol{g}_n$ to $\boldsymbol{g}(t_n)$ in (3.5) are of second-order if the functions are C^2–smooth in time. In fact, Taylor's expansion shows*

$$\Psi(t) = \Psi(t_{n-1/2}) + \Psi'(t_{n-1/2}) \cdot (t - t_{n-1/2}) + \frac{1}{2}\Psi''(\theta_n) \cdot (t - t_{n-1/2})^2 \qquad \forall\, t \in [t_{n-1}, t_n],$$

where $t_{n-1/2} = t_{n-1} + \tau/2$ and $\theta_n \in (t_{n-1}, t_n)$. Clearly we have

$$\Psi_n - \Psi(t_{n-1/2}) = \frac{1}{2\tau}\Psi''(\theta_n)\int_{t_{n-1}}^{t_n} (t - t_{n-1/2})^2 \mathrm{d}t = O(\tau^2). \tag{3.7}$$

3.3. An extrapolated finite element approximation. The discrete problem is nonlinear and expensive to solve at each time step. Thus, inspired by [2, 14, 31], we propose to linearize the discrete problem (3.6) with extrapolated solutions

$$\boldsymbol{u}_n^* = \frac{1}{2}(3\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-2}), \qquad n \ge 2. \tag{3.8}$$

Linearizing the convection term in (3.6) with $\boldsymbol{u}_n^*$, we get a semi-implicit time-stepping scheme:

Find $\boldsymbol{u}_n \in \boldsymbol{V}^h(\operatorname{div} 0)$ and $\boldsymbol{J}_n \in \boldsymbol{D}^h(\operatorname{div} 0)$ such that $\gamma\boldsymbol{u}_n = \boldsymbol{g}_n$ on Γ_{d}, $\gamma_{\boldsymbol{n}}\boldsymbol{J}_n = J_n$ on Γ_{i}, and

$$(\delta_t\boldsymbol{u}_n, \boldsymbol{v}) + \mathscr{O}(\boldsymbol{u}_n^*; \bar{\boldsymbol{u}}_n, \boldsymbol{v}) + \mathscr{A}_{\mathrm{AL}}(\bar{\boldsymbol{u}}_n, \boldsymbol{v}) - \kappa(\boldsymbol{J}_n \times \boldsymbol{B}_n, \boldsymbol{v}) = (\boldsymbol{f}_n, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}^h(\operatorname{div} 0), \tag{3.9a}$$

$$(\boldsymbol{J}_n, \boldsymbol{d}) + (\boldsymbol{B}_n \times \bar{\boldsymbol{u}}_n, \boldsymbol{d}) = \langle \gamma_{\boldsymbol{n}}\boldsymbol{d}, \xi_n \rangle_{\Gamma_{\mathrm{c}}} \qquad \forall\, \boldsymbol{d} \in \boldsymbol{D}_{\mathrm{i}}^h(\operatorname{div} 0). \tag{3.9b}$$

For $n = 1$, we use $\boldsymbol{u}_1^* = \boldsymbol{u}_1$ in (3.9a) and solve a nonlinear problem. For $n > 1$, a linear system of equations results from this approximation.

Remark 3.2. *The truncation error for (3.8) is of second order. In fact, for a smooth function v of t, let $\bar{v}_n = [v(t_n) + v(t_{n-1})]/2$ and $v_n^* = [3v(t_{n-1}) - v(t_{n-2})]/2$. Then*

$$v_n^* - \bar{v}_n = \frac{1}{2}[v(t_{n-2}) + v(t_n) - 2v(t_{n-1})] = v_{tt}(t_{n-1})\tau^2 + O(\tau^3).$$

Theorem 3.3. *Problem (3.9) has a unique solution in each time step. Moreover, the discrete solutions satisfy*

$$\delta_t\mathsf{E}_n + \mathsf{P}_n = (\boldsymbol{f}_n, \bar{\boldsymbol{u}}_n) + \kappa\langle \gamma_{\boldsymbol{n}}\boldsymbol{J}_n, \xi_n \rangle_{\Gamma_{\mathrm{c}}} \qquad \forall\, n \ge 1, \tag{3.10}$$

where

$$\mathsf{E}_n := \frac{1}{2}\left\|\boldsymbol{u}_n\right\|_{\boldsymbol{L}^2(\Omega)}^2,$$

$$\mathsf{P}_n := \mathscr{A}_{\mathrm{AL}}(\bar{\boldsymbol{u}}_n, \bar{\boldsymbol{u}}_n) + \frac{1}{2}\int_{\Gamma_{\mathrm{n}}} \left(\gamma_{\boldsymbol{n}}^{\downarrow}\boldsymbol{u}_n^*\right)\left|\bar{\boldsymbol{u}}_n\right|^2 + \kappa\left\|\boldsymbol{J}_n\right\|_{\boldsymbol{L}^2(\Omega)}^2.$$

Proof. We first prove (3.10). Taking $\boldsymbol{v} = \bar{\boldsymbol{u}}_n$ in (3.9a) and $\boldsymbol{d} = \kappa \boldsymbol{J}_n$ in (3.9b) yields

$$
\begin{aligned}
&(\delta_t \boldsymbol{u}_n, \bar{\boldsymbol{u}}_n) + \int_{\Gamma_{\mathrm{n}}} \left(\gamma_{\boldsymbol{n}}^{\downarrow} \boldsymbol{u}_n^*\right) |\bar{\boldsymbol{u}}_n|^2 + \mathscr{A}_{\mathrm{AL}}(\bar{\boldsymbol{u}}_n, \bar{\boldsymbol{u}}_n) - \kappa(\boldsymbol{J}_n, \boldsymbol{B}_n \times \bar{\boldsymbol{u}}_n) = (\boldsymbol{f}_n, \bar{\boldsymbol{u}}_n), \\
&\kappa \|\boldsymbol{J}_n\|_{\boldsymbol{L}^2(\Omega)}^2 + \kappa(\boldsymbol{J}_n, \boldsymbol{B}_n \times \bar{\boldsymbol{u}}_n) = \kappa \langle \gamma_{\boldsymbol{n}} \boldsymbol{J}_n, \xi_n \rangle_{\Gamma_{\mathrm{c}}}.
\end{aligned}
$$

Adding the two equalities and using $(\delta_t \boldsymbol{u}_n, \bar{\boldsymbol{u}}_n) = \delta_t \mathsf{E}_n$, we obtain (3.10).

Now we prove the well-posedness of (3.9). Write $\boldsymbol{\Psi}_n = (\bar{\boldsymbol{u}}_n, \boldsymbol{J}_n)$ and $\boldsymbol{\Phi} = (\boldsymbol{v}, \boldsymbol{d})$. Then (3.9) can be written into a reduced form: Find $\boldsymbol{\Psi}_n \in \boldsymbol{V}^h(\mathrm{div}\, 0) \times \boldsymbol{D}^h(\mathrm{div}\, 0)$ such that

$$
a(\boldsymbol{\Psi}_n, \boldsymbol{\Phi}) = f(\boldsymbol{\Phi}) \qquad \forall\, \boldsymbol{\Phi} \in \boldsymbol{V}_{\mathrm{d}}^h(\mathrm{div}\, 0) \times \boldsymbol{D}_{\mathrm{i}}^h(\mathrm{div}\, 0). \tag{3.11}
$$

Given $\boldsymbol{B}_n$ and $\boldsymbol{u}_n^*$, the bilinear form $a(\cdot,\cdot)$ and the linear form $f(\cdot)$ are defined as follows

$$
\begin{aligned}
a(\boldsymbol{\Psi}_n, \boldsymbol{\Phi}) &:= \frac{2}{\tau}(\bar{\boldsymbol{u}}_n, \boldsymbol{v}) + \mathscr{O}(\boldsymbol{u}_n^*; \bar{\boldsymbol{u}}_n, \boldsymbol{v}) + \mathscr{A}_{\mathrm{AL}}(\bar{\boldsymbol{u}}_n, \boldsymbol{v}) + \kappa \left[(\boldsymbol{J}_n, \boldsymbol{d}) + (\boldsymbol{B}_n \times \bar{\boldsymbol{u}}_n, \boldsymbol{d}) - (\boldsymbol{J}_n, \boldsymbol{B}_n \times \boldsymbol{v})\right], \\
f(\boldsymbol{\Phi}) &:= \left(\boldsymbol{f}_n + \frac{2}{\tau} \boldsymbol{u}_{n-1}, \boldsymbol{v}\right) + \kappa \langle \gamma_{\boldsymbol{n}} \boldsymbol{d}, \xi_n \rangle_{\Gamma_{\mathrm{c}}}.
\end{aligned}
$$

It is easy to verify that, for any $\boldsymbol{\Phi} = (\boldsymbol{v}, \boldsymbol{d}) \in \boldsymbol{V}_{\mathrm{d}}^h(\mathrm{div}\, 0) \times \boldsymbol{D}_{\mathrm{i}}^h(\mathrm{div}\, 0)$

$$
\begin{aligned}
a(\boldsymbol{\Phi}, \boldsymbol{\Phi}) &= \frac{2}{\tau} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + \int_{\Gamma_{\mathrm{n}}} \left(\gamma_{\boldsymbol{n}}^{\downarrow} \boldsymbol{u}_n^*\right) |\boldsymbol{v}|^2 + \mathscr{A}_{\mathrm{AL}}(\boldsymbol{v}, \boldsymbol{v}) + \kappa \|\boldsymbol{d}\|_{\boldsymbol{L}^2(\Omega)}^2 \\
&\geq \min(2\tau^{-1}, R_e^{-1}) \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}^2 + \kappa \|\boldsymbol{d}\|_{\boldsymbol{H}(\mathrm{div}, \Omega)}^2 .
\end{aligned}
$$

Therefore, $a(\cdot,\cdot)$ is coercive on $\boldsymbol{V}_{\mathrm{d}}^h(\mathrm{div}\, 0) \times \boldsymbol{D}_{\mathrm{i}}^h(\mathrm{div}\, 0)$. The continuity of $a(\cdot,\cdot)$ can also be proven easily. We do not elaborate on the details.

By the trace theorem on $\boldsymbol{D}$, we have

$$
\begin{aligned}
|f(\boldsymbol{\Phi})| &\leq \left\|2\tau^{-1} \boldsymbol{u}_{n-1} + \boldsymbol{f}_n\right\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \kappa \|\gamma_{\boldsymbol{n}} \boldsymbol{d}\|_{H^{-1/2}(\Gamma_{\mathrm{c}})} \|\xi_n\|_{H^{1/2}(\Gamma_{\mathrm{c}})} \\
&\leq \left\|2\tau^{-1} \boldsymbol{u}_{n-1} + \boldsymbol{f}_n\right\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \kappa \|\xi_n\|_{H^{-1/2}(\Gamma_{\mathrm{c}})} \|\boldsymbol{d}\|_{\boldsymbol{H}(\mathrm{div}, \Omega)} .
\end{aligned}
$$

So f provides a bounded functional on $\boldsymbol{V}_{\mathrm{d}}^h(\mathrm{div}\, 0) \times \boldsymbol{D}_{\mathrm{i}}^h(\mathrm{div}\, 0)$. By the Lax-Milgram theorem, problem (3.11) has a unique solution. ∎

COROLLARY 3.4. *Define* $M := \mathsf{E}_0 + \|\boldsymbol{f}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}^2 + \|\xi\|_{L^2(0,T;H^{1/2}(\Gamma_c))}^2$ *and let* $\boldsymbol{u}_n$, $\boldsymbol{J}_n$ *be the solutions of problem* (3.9). *There is a constant* $C > 0$ *independent of* τ, h *such that*

$$
\max_{0 \leq n \leq N} \mathsf{E}_n + \sum_{n=0}^{N} \tau \mathsf{P}_n \leq CM.
$$

Proof. From Theorem 3.3 we know that

$$
\mathsf{E}_n - \mathsf{E}_{n-1} + \tau \mathsf{P}_n = \tau(\boldsymbol{f}_n, \bar{\boldsymbol{u}}_n) + \kappa\tau \langle \gamma_{\boldsymbol{n}} \boldsymbol{J}_n, \xi_n \rangle_{\Gamma_c} \qquad \forall\, n > 0.
$$

For any $1 \leq m \leq N$, summing up the equalities for $n = 1, \cdots, m$ leads to

$$
\mathsf{E}_m + \sum_{n=1}^{m} \tau \mathsf{P}_n = \mathsf{E}_0 + \sum_{n=1}^{m} \tau(\boldsymbol{f}_n, \bar{\boldsymbol{u}}_n) + \kappa \sum_{n=1}^{m} \tau \langle \gamma_{\boldsymbol{n}} \boldsymbol{J}_n, \xi_n \rangle_{\Gamma_c}. \tag{3.12}
$$

By Poincáre's inequality, there is a constant C_p depending only on Ω such that

$$\|\bar{\boldsymbol{u}}_n\|_{\boldsymbol{L}^2(\Omega)} \le C_p \|\nabla \bar{\boldsymbol{u}}_n\|_{\boldsymbol{L}^2(\Omega)} .$$

Then using Jensen's inequality, we have

$$\sum_{n=1}^{m} \tau(\boldsymbol{f}_n, \bar{\boldsymbol{u}}_n) \le \frac{1}{2} R_e C_p^2 \sum_{n=1}^{m} \tau \|\boldsymbol{f}_n\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2} \sum_{n=1}^{m} \tau \mathsf{P}_n. \tag{3.13}$$

Similarly, the third term on the right-hand side of (3.12) can be estimated as follows

$$\begin{aligned} \left| \kappa \sum_{n=1}^{m} \tau \langle \gamma_{\boldsymbol{n}} \boldsymbol{J}_n, \xi_n \rangle_{\Gamma_c} \right| &\le \kappa \sum_{n=1}^{m} \tau \|\gamma_{\boldsymbol{n}} \boldsymbol{J}_n\|_{H^{-1/2}(\Gamma_c)} \|\xi_n\|_{H^{1/2}(\Gamma_c)} \\ &\le \frac{\kappa}{2} \sum_{n=1}^{m} \tau \|\boldsymbol{J}_n\|^2_{\boldsymbol{H}(\mathrm{div},\Omega)} + C \sum_{n=1}^{m} \tau \|\xi_n\|^2_{H^{1/2}(\Gamma_c)} . \end{aligned} \tag{3.14}$$

Substituting (3.13)–(3.14) into (3.12) and using $\mathrm{div}\, \boldsymbol{J}_n = 0$, we get

$$\mathsf{E}_m + \frac{1}{2} \sum_{n=1}^{m} \tau \mathsf{P}_n \le C \sum_{n=1}^{m} \tau \left(\|\boldsymbol{f}_n\|^2_{\boldsymbol{L}^2(\Omega)} + \|\xi_n\|^2_{H^{1/2}(\Gamma_c)} \right) \le CM \qquad \forall\, 1 \le m \le N.$$

The proof is completed. □

4. The convergence of discrete solutions. The purpose of this section is to prove the convergence of the discrete solutions for $\Gamma_{\mathrm{n}} = \emptyset$ as $\tau, h \to 0$. We only prove that, upon an extracted subsequence, the discrete solutions converge weakly to the solutions of the continuous MHD problem (2.4). For simplicity, we fix τ and let $h \to 0$ first to study the semi-continuous limits of discrete solutions. Next we will let $\tau \to 0$ and study the limits of the semi-continuous solutions.

4.1. The semi-continuous limits. Without loss of generality, let $\mathcal{T}_1 \prec \mathcal{T}_2 \prec \cdots \prec \mathcal{T}_k \prec \cdots$ be a quasi-uniform and shape-regular sequence of meshes of Ω such that $\lim_{k\to\infty} h_k = 0$ and $\mathcal{T}_{k+1}$ is a refinement of $\mathcal{T}_k$. To specify the dependency of discrete functions on $\mathcal{T}_k$, we endow them with a superscript and write

$$\boldsymbol{u}_n^{(k)} := \boldsymbol{u}_n, \quad \bar{\boldsymbol{u}}_n^{(k)} := \bar{\boldsymbol{u}}_n, \quad \boldsymbol{u}_n^{(k,*)} := \boldsymbol{u}_n^*, \quad \boldsymbol{g}_n^{(k)} := \boldsymbol{g}_n, \quad \boldsymbol{J}_n^{(k)} := \boldsymbol{J}_n, \quad J_n^{(k)} := J_n.$$

Without specifications, we also use $\boldsymbol{V}^{(k)}$, $\boldsymbol{D}^{(k)}$, $Q^{(k)}$ to denote finite element spaces on $\mathcal{T}_k$. Throughout this section, in the case of sequences such as $\boldsymbol{u}_n^{(k)}$ and $\boldsymbol{J}_n^{(k)}$, we retain the same notation even after extracting subsequences.

LEMMA 4.1. *Let $\mathcal{P}\colon \boldsymbol{V}_{\mathrm{d}} \to \boldsymbol{V}_{\mathrm{d}}(\mathrm{div}\, 0)$ and $\mathcal{P}_k\colon \boldsymbol{V}_{\mathrm{d}} \to \boldsymbol{V}^{(k)}_{\mathrm{d}}(\mathrm{div}\, 0)$ be the projection operators under the semi-norm $|\cdot|_{\boldsymbol{H}^1(\Omega)}$. Then*

$$\lim_{k\to\infty} |\mathcal{P}\boldsymbol{w} - \mathcal{P}_k \boldsymbol{w}|_{\boldsymbol{H}^1(\Omega)} = 0 \qquad \forall\, \boldsymbol{w} \in \boldsymbol{V}_{\mathrm{d}}. \tag{4.1}$$

Proof. It is easy to see that $(\mathcal{P}\boldsymbol{w}, \vartheta = 0) \in \boldsymbol{V}_{\mathrm{d}} \times Q$ solve the continuous Stokes problem

$$(\nabla(\mathcal{P}\boldsymbol{w}), \nabla \boldsymbol{v}) - (\vartheta, \mathrm{div}\, \boldsymbol{v}) = (\nabla \boldsymbol{w}, \nabla \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}, \tag{4.2a}$$

$$(\mathrm{div}(\mathcal{P}\boldsymbol{w}), q) = 0 \quad \forall\, q \in Q, \tag{4.2b}$$

and $(\mathcal{P}_k\boldsymbol{w}, \vartheta_k = 0) \in \boldsymbol{V}_{\mathrm{d}}^{(k)} \times Q^{(k)}$ solve the discrete Stokes problem

$$(\nabla(\mathcal{P}_k\boldsymbol{w}), \nabla\boldsymbol{v}) - (\vartheta_k, \operatorname{div}\boldsymbol{v}) = (\nabla\boldsymbol{w}, \nabla\boldsymbol{v}) \quad \forall\,\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}^{(k)}, \tag{4.3a}$$
$$(q, \operatorname{div}(\mathcal{P}_k\boldsymbol{w})) = 0 \quad \forall\,q \in Q^{(k)}. \tag{4.3b}$$

By the inf-sup condition on $\boldsymbol{H}_0^1(\Omega) \times Q$ [10, Chapter 2], there exists a constant $C > 0$ such that

$$\inf_{\boldsymbol{w}\in\boldsymbol{V}_{\mathrm{d}}} \frac{(q, \operatorname{div}\boldsymbol{w})}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}} \geq \inf_{\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega)} \frac{(q, \operatorname{div}\boldsymbol{w})}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}} \geq C\,\|q\|_Q \qquad \forall\,q \in Q.$$

So (4.2) has unique solutions. Similarly, the discrete inf-sup condition in (3.1) shows that (4.3) has unique solutions. By [10, Theorem 1.1], there is a constant $C > 0$ independent of h_k such that

$$\|\mathcal{P}\boldsymbol{w} - \mathcal{P}_k\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \leq C\left(\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_{\mathrm{d}}^{(k)}} \|\mathcal{P}\boldsymbol{w} - \boldsymbol{v}_h\|_{\boldsymbol{H}^1(\Omega)} + \inf_{q_h\in Q^{(k)}} \|\vartheta - q_h\|_{L^2(\Omega)}\right). \tag{4.4}$$

So the denseness of $\bigcup_{k=1}^{\infty} \boldsymbol{V}_{\mathrm{d}}^{(k)}$ in $\boldsymbol{V}_{\mathrm{d}}$ yields (4.1). $\square$

THEOREM 4.2. *Suppose* $\boldsymbol{B} \in \boldsymbol{L}^1(0,T;\boldsymbol{L}^3(\Omega))$ *and*

$$\lim_{k\to\infty} \left\|\boldsymbol{g}_n^{(k)} - \boldsymbol{g}_n\right\|_{\boldsymbol{H}^{1/2}(\Gamma_{\mathrm{d}})} = \lim_{k\to\infty} \left\|J_n^{(k)} - J_n\right\|_{\boldsymbol{H}^{-1/2}(\Gamma_{\mathrm{i}})} = 0.$$

There exist a $\boldsymbol{u}_n \in \boldsymbol{V}(\operatorname{div}0)$*, a* $\boldsymbol{J}_n \in \boldsymbol{D}(\operatorname{div}0)$*, and some subsequences of* $\left\{\boldsymbol{u}_n^{(k)}\right\}$, $\left\{\bar{\boldsymbol{u}}_n^{(k)}\right\}$, $\left\{\boldsymbol{J}_n^{(k)}\right\}$ *such that, as* $k \to \infty$*,*

$$\boldsymbol{J}_n^{(k)} \rightharpoonup \boldsymbol{J}_n \ \text{ weakly in } \boldsymbol{D}, \qquad \boldsymbol{u}_n^{(k)},\ \bar{\boldsymbol{u}}_n^{(k)} \rightharpoonup \boldsymbol{u}_n \ \text{ weakly in } \boldsymbol{V}. \tag{4.5}$$

Define $\bar{\boldsymbol{u}}_n = (\boldsymbol{u}_n + \boldsymbol{u}_{n-1})/2$ *and* $\boldsymbol{u}_n^* = (3\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-2})/2$*. Then the limits satisfy* $\gamma\boldsymbol{u}_n = \boldsymbol{g}_n$ *on* Γ_{d}*,* $\gamma_{\boldsymbol{n}}\boldsymbol{J}_n = J_n$ *on* Γ_{i}*, and*

$$(\delta_t\boldsymbol{u}_n, \boldsymbol{v}) + \mathscr{O}(\boldsymbol{u}_n^*; \bar{\boldsymbol{u}}_n, \boldsymbol{v}) + \mathscr{A}(\bar{\boldsymbol{u}}_n, \boldsymbol{v}) - \kappa(\boldsymbol{J}_n \times \boldsymbol{B}_n, \boldsymbol{v}) = (\boldsymbol{f}_n, \boldsymbol{v}) \quad \forall\,\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}(\operatorname{div}0), \tag{4.6a}$$
$$(\boldsymbol{J}_n, \boldsymbol{d}) + (\boldsymbol{B}_n \times \bar{\boldsymbol{u}}_n, \boldsymbol{d}) = \langle\gamma_{\boldsymbol{n}}\boldsymbol{d}, \xi_n\rangle_{\Gamma_{\mathrm{c}}} \qquad \forall\,\boldsymbol{d} \in \boldsymbol{D}_{\mathrm{i}}(\operatorname{div}0). \tag{4.6b}$$

Proof. By Corollary 3.4, $\left\{\boldsymbol{u}_n^{(k)}\right\}$ and $\left\{\bar{\boldsymbol{u}}_n^{(k)}\right\}$ are bounded in $\boldsymbol{V}$ and $\left\{\boldsymbol{J}_n^{(k)}\right\}$ is bounded in $\boldsymbol{D}$. They have subsequences satisfying (4.5). Since $\boldsymbol{J}_n^{(k)} \in \boldsymbol{D}(\operatorname{div}0)$, we have $\boldsymbol{J}_n \in \boldsymbol{D}(\operatorname{div}0)$. Moreover, since $Q^{(l)} \subset Q^{(k)}$ for any $0 < l < k$, the definition of $\boldsymbol{V}_{\mathrm{d}}^{(k)}(\operatorname{div}0)$ indicates

$$(\operatorname{div}\boldsymbol{u}_n, q_l) = \lim_{k\to\infty} (\operatorname{div}\boldsymbol{u}_n^{(k)}, q_l) = 0 \qquad \forall\,q_l \in Q^{(l)}.$$

The denseness of $\bigcup_{l=1}^{\infty} Q^{(l)}$ in Q shows $\operatorname{div}\boldsymbol{u}_n = 0$.

Since γ: $\boldsymbol{V} \to \boldsymbol{H}^{1/2}(\Gamma_{\mathrm{d}})$ and $\gamma_{\boldsymbol{n}}$: $\boldsymbol{D} \to H^{-1/2}(\Gamma_{\mathrm{i}})$ are surjective, (4.5) implies

$$\boldsymbol{g}_n^{(k)} = \gamma\boldsymbol{u}_n^{(k)} \rightharpoonup \gamma\boldsymbol{u}_n \ \text{ weakly in } \boldsymbol{H}^{1/2}(\Gamma_{\mathrm{d}}), \qquad J_n^{(k)} = \gamma_{\boldsymbol{n}}\boldsymbol{J}_n^{(k)} \rightharpoonup \gamma_{\boldsymbol{n}}\boldsymbol{J}_n \ \text{ weakly in } \boldsymbol{H}^{-1/2}(\Gamma_{\mathrm{i}}).$$

The assumptions of the theorem show $\gamma\boldsymbol{u}_n = \boldsymbol{g}_n$ and $\gamma_{\boldsymbol{n}}\boldsymbol{J}_n = J_n$.

It is left to show that $(\boldsymbol{u}_n, \boldsymbol{J}_n)$ satisfy (4.6). For any $\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}(\operatorname{div} 0)$, let $\mathcal{P}_k \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}^{(k)}(\operatorname{div} 0)$ be its projection given in Lemma 4.1. From (3.9a) we know that

$$(\delta_t \boldsymbol{u}_n^{(k)}, \mathcal{P}_k \boldsymbol{v}) + \mathscr{O}(\boldsymbol{u}_n^{(k,*)}; \bar{\boldsymbol{u}}_n^{(k)}, \mathcal{P}_k \boldsymbol{v}) + \mathscr{A}_{\mathrm{AL}}(\bar{\boldsymbol{u}}_n^{(k)}, \mathcal{P}_k \boldsymbol{v}) - \kappa(\boldsymbol{J}_n^{(k)} \times \boldsymbol{B}_n, \mathcal{P}_k \boldsymbol{v}) = (\boldsymbol{f}_n, \mathcal{P}_k \boldsymbol{v}).$$

By the compact injection $\boldsymbol{V} \hookrightarrow\hookrightarrow \boldsymbol{L}^3(\Omega)$, $\boldsymbol{u}_n^{(k,*)}$ converges to $\boldsymbol{u}_n^*$ strongly in $\boldsymbol{L}^3(\Omega)$. Using (4.5) and Lemma 4.1, it is easy to see

$$\begin{aligned}
&\lim_{k\to\infty} \mathscr{O}(\boldsymbol{u}_n^{(k,*)}; \bar{\boldsymbol{u}}_n^{(k)}, \mathcal{P}_k \boldsymbol{v}) = \lim_{k\to\infty} \mathscr{O}(\boldsymbol{u}_n^*; \bar{\boldsymbol{u}}_n^{(k)}, \mathcal{P}_k \boldsymbol{v}) = \lim_{k\to\infty} \mathscr{O}(\boldsymbol{u}_n^*; \bar{\boldsymbol{u}}_n^{(k)}, \boldsymbol{v}) = \mathscr{O}(\boldsymbol{u}_n^*; \bar{\boldsymbol{u}}_n, \boldsymbol{v}), \\
&\lim_{k\to\infty} (\delta_t \boldsymbol{u}_n^{(k)} - \boldsymbol{f}_n, \mathcal{P}_k \boldsymbol{v}) = \lim_{k\to\infty} (\delta_t \boldsymbol{u}_n^{(k)} - \boldsymbol{f}_n, \boldsymbol{v}) = (\delta_t \boldsymbol{u}_n - \boldsymbol{f}_n, \boldsymbol{v}), \\
&\lim_{k\to\infty} \mathscr{A}_{\mathrm{AL}}(\bar{\boldsymbol{u}}_n^{(k)}, \mathcal{P}_k \boldsymbol{v}) = \lim_{k\to\infty} \mathscr{A}(\bar{\boldsymbol{u}}_n^{(k)}, \boldsymbol{v}) = \mathscr{A}(\boldsymbol{u}_n, \boldsymbol{v}),
\end{aligned}$$

Since $\boldsymbol{B}_n \in \boldsymbol{L}^3(\Omega)$ by the assumption, we also have

$$\lim_{k\to\infty} (\boldsymbol{J}_n^{(k)} \times \boldsymbol{B}_n, \mathcal{P}_k \boldsymbol{v}) = \lim_{k\to\infty} (\boldsymbol{J}_n^{(k)} \times \boldsymbol{B}_n, \boldsymbol{v}) = \lim_{k\to\infty} (\boldsymbol{J}_n^{(k)}, \boldsymbol{B}_n \times \boldsymbol{v}) = (\boldsymbol{J}_n, \boldsymbol{B}_n \times \boldsymbol{v}).$$

So we get (4.6a). The proof of (4.6b) is similar and omitted here. ▯

4.2. The continuous limits. Now we prove the convergence of the semi-continuous limits as $\tau \to 0$. Define the piecewise linear interpolations of $\boldsymbol{u}_n, \boldsymbol{g}_n$ with respect to t by

$$\begin{aligned}
\boldsymbol{u}_\tau(t) &= l(t)\boldsymbol{u}_{n-1} + [1 - l(t)]\boldsymbol{u}_n \\
\boldsymbol{g}_\tau(t) &= l(t)\boldsymbol{g}_{n-1} + [1 - l(t)]\boldsymbol{g}_n
\end{aligned} \qquad \forall\, t \in [t_{n-1}, t_n), \quad n > 0,$$

where $l(t) = (t_n - t)/\tau$. The piecewise constant interpolants are defined by

$$\Psi_\tau(t) = \Psi_n \qquad \forall\, t \in [t_{n-1}, t_n), \quad n > 0 \quad \text{for } \ \Psi = \bar{\boldsymbol{u}}, \boldsymbol{u}^*, \boldsymbol{J}, J, \boldsymbol{f}, \xi.$$

The discrete problem (3.9) can be written into an equivalent form: Find $\boldsymbol{u}_\tau \in \boldsymbol{V}(\operatorname{div} 0)$, $\boldsymbol{J}_\tau \in \boldsymbol{D}(\operatorname{div} 0)$ such that $\gamma \boldsymbol{u}_\tau = \boldsymbol{g}_\tau$ on Γ, $\gamma_{\boldsymbol{n}} \boldsymbol{J}_\tau = J_\tau$ on Γ, and

$$(\partial_t \boldsymbol{u}_\tau, \boldsymbol{v}) + \mathscr{O}(\boldsymbol{u}_\tau^*; \bar{\boldsymbol{u}}_\tau, \boldsymbol{v}) + \mathscr{A}(\bar{\boldsymbol{u}}_\tau, \boldsymbol{v}) - \kappa(\boldsymbol{J}_\tau \times \boldsymbol{B}_\tau, \boldsymbol{v}) = (\boldsymbol{f}_\tau, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}(\operatorname{div} 0), \tag{4.7a}$$

$$(\boldsymbol{J}_\tau, \boldsymbol{d}) + (\boldsymbol{B}_\tau \times \bar{\boldsymbol{u}}_\tau, \boldsymbol{d}) = \langle \gamma_{\boldsymbol{n}} \boldsymbol{d}, \xi_n \rangle_{\Gamma_{\mathrm{c}}} \qquad \forall\, \boldsymbol{d} \in \boldsymbol{D}_{\mathrm{i}}(\operatorname{div} 0). \tag{4.7b}$$

LEMMA 4.3. *Let M be given in Corollary 3.4. There is a constant C independent of τ such that*

$$\|\boldsymbol{u}_\tau\|^2_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{u}_\tau\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{V})} + \|\boldsymbol{J}_\tau\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{D})} \le CM. \tag{4.8}$$

Proof. From Corollary 3.4 and the weak convergence of $\boldsymbol{u}_n^{(k)}$ and $\boldsymbol{J}_n^{(k)}$, we easily get

$$\max_{0\le n\le N} \|\boldsymbol{u}_n\|^2_{\boldsymbol{L}^2(\Omega)} + \sum_{n=0}^{N} \tau \left(\|\boldsymbol{u}_n\|^2_{\boldsymbol{V}} + \|\boldsymbol{J}_n\|^2_{\boldsymbol{L}^2(\Omega)} \right) \le CM.$$

This yields (4.8). The proof is completed. ▯

LEMMA 4.4. *Suppose $\boldsymbol{B} \in \boldsymbol{L}^\infty(0,T;\boldsymbol{L}^3(\Omega))$. There is a constant C independent of τ such that*

$$\|\partial_t \boldsymbol{u}_\tau\|_{\boldsymbol{L}^{4/3}(0,T;\boldsymbol{V}_{\mathrm{d}}(\operatorname{div} 0)')} \le C.$$

Proof. For any $\boldsymbol{v} \in \boldsymbol{L}^4\big(0,T;\boldsymbol{V}_{\mathrm{d}}(\operatorname{div} 0)\big)$, equation (4.7a) indicates

$$\int_0^T (\partial_t \boldsymbol{u}_\tau, \boldsymbol{v}) = \int_0^T \left[(\boldsymbol{f}_\tau + \kappa \boldsymbol{J}_\tau \times \boldsymbol{B}_\tau, \boldsymbol{v}) - \mathscr{O}(\boldsymbol{u}_\tau^*; \bar{\boldsymbol{u}}_\tau, \boldsymbol{v}) - \mathscr{A}(\bar{\boldsymbol{u}}_\tau, \boldsymbol{v})\right].$$

By Cauchy-Schwarz inequality and (4.8), we have

$$\begin{aligned}
\int_0^T |\mathscr{O}(\boldsymbol{u}_\tau^*; \bar{\boldsymbol{u}}_\tau, \boldsymbol{v})| &\le \int_0^T \|\boldsymbol{u}_\tau^*\|_{\boldsymbol{L}^2(\Omega)}^{1/2} \|\boldsymbol{u}_\tau^*\|_{\boldsymbol{L}^6(\Omega)}^{1/2} \|\bar{\boldsymbol{u}}_\tau\|_{\boldsymbol{V}} \|\boldsymbol{v}\|_{\boldsymbol{V}} \\
&\le \|\boldsymbol{u}_\tau^*\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))}^{1/2} \|\boldsymbol{u}_\tau^*\|_{\boldsymbol{L}^2(0,T;\boldsymbol{V})}^{1/2} \|\bar{\boldsymbol{u}}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{V})} \|\boldsymbol{v}\|_{\boldsymbol{L}^4(0,T;\boldsymbol{V})} \\
&\le C \|\boldsymbol{v}\|_{\boldsymbol{L}^4(0,T;\boldsymbol{V})}.
\end{aligned}$$

The other terms are estimated as follows

$$\begin{aligned}
&\int_0^T |(\boldsymbol{f}_\tau, \boldsymbol{v})| \le C \|\boldsymbol{f}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \le C \|\boldsymbol{v}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{V})}, \\
&\int_0^T |\mathscr{A}(\bar{\boldsymbol{u}}_\tau, \boldsymbol{v})| \le C \|\bar{\boldsymbol{u}}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{V})} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{V})} \le C \|\boldsymbol{v}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{V})}, \\
&\int_0^T |(\boldsymbol{J}_\tau \times \boldsymbol{B}_\tau, \boldsymbol{v})| \le \|\boldsymbol{B}_\tau\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^3(\Omega))} \|\boldsymbol{J}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^6(\Omega))} \le C \|\boldsymbol{v}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{V})}.
\end{aligned}$$

Collecting all inequalities above, we have

$$\left|\int_0^T (\partial_t \boldsymbol{u}_\tau, \boldsymbol{v})\right| \le C \|\boldsymbol{v}\|_{\boldsymbol{L}^4(0,T;\boldsymbol{V})} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{L}^4\big(0,T;\boldsymbol{V}_{\mathrm{d}}(\operatorname{div} 0)\big). \tag{4.9}$$

The proof is completed. □

Now we state the main theorem of this section.

THEOREM 4.5. *Suppose the given functions satisfy*

$$\begin{aligned}
&\lim_{\tau\to 0} \Big(\|\boldsymbol{B}_\tau - \boldsymbol{B}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^3(\Omega))} + \|\boldsymbol{f}_\tau - \boldsymbol{f}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\Big) = 0, \\
&\lim_{\tau\to 0} \Big(\|\boldsymbol{g}_\tau - \boldsymbol{g}\|_{L^2(0,T;\boldsymbol{H}^{1/2}(\Gamma))} + \|\xi_\tau - \xi\|_{L^2(0,T;H^{1/2}(\Gamma))} + \|J_\tau - J\|_{L^2(0,T;H^{-1/2}(\Gamma))}\Big) = 0.
\end{aligned}$$

There exist a $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;\boldsymbol{V}(\operatorname{div} 0))$, a $\boldsymbol{J} \in \boldsymbol{L}^2(0,T;\boldsymbol{D}(\operatorname{div} 0))$, and subsequences of $\{\boldsymbol{u}_\tau\}$, $\{\bar{\boldsymbol{u}}_\tau\}$, $\{\boldsymbol{J}_\tau\}$ such that

$$\begin{aligned}
&\boldsymbol{J}_\tau \rightharpoonup \boldsymbol{J} && \textit{weakly in } \boldsymbol{L}^2(0,T;\boldsymbol{D}), \\
&\boldsymbol{u}_\tau,\ \bar{\boldsymbol{u}}_\tau \to \boldsymbol{u} && \textit{strongly in } \boldsymbol{L}^2(0,T;\boldsymbol{L}^3(\Omega)), \\
&\boldsymbol{u}_\tau,\ \bar{\boldsymbol{u}}_\tau \rightharpoonup \boldsymbol{u} && \textit{weakly in } \boldsymbol{L}^2(0,T;\boldsymbol{V}) \textit{ and weakly* in } \boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega)), \\
&\partial_t \boldsymbol{u}_\tau \rightharpoonup \partial_t \boldsymbol{u} && \textit{weakly in } \boldsymbol{L}^{4/3}\big(0,T;\boldsymbol{V}_{\mathrm{d}}'\big).
\end{aligned} \tag{4.10}$$

Moreover, the limits $(\boldsymbol{u}, \boldsymbol{J})$ satisfy the continuous weak problem (2.4).

Proof. The proof consists of four steps. Step 1 proves (4.10). Step 2 proves that $(\boldsymbol{u}, \boldsymbol{J})$ satisfy (2.4). Step 3 proves the initial condition for $\boldsymbol{u}$. Step 4 proves the boundary conditions for $(\boldsymbol{u}, \boldsymbol{J})$.

Step 1. Since $\{\boldsymbol{J}_\tau\}_{\tau>0}$ is uniformly bounded in $\boldsymbol{L}^2(0,T;\boldsymbol{D}(\mathrm{div}\,0))$, it has a subsequence which converges weakly to some $\boldsymbol{J} \in \boldsymbol{L}^2(0,T;\boldsymbol{D}(\mathrm{div}\,0))$. From Lemma 4.3, $\{\boldsymbol{u}_\tau\}_{\tau>0}$ is uniformly bounded in both $\boldsymbol{L}^2(0,T;\boldsymbol{V}(\mathrm{div}\,0))$ and $\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))$. It has a subsequence converging to $\boldsymbol{u}$ weakly in $\boldsymbol{L}^2(0,T;\boldsymbol{V}(\mathrm{div}\,0))$ and weakly* in $\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))$, and so does $\{\bar{\boldsymbol{u}}_\tau\}_{\tau>0}$. By Lemma 4.4, $\{\partial_t \boldsymbol{u}_\tau\}_{\tau>0}$ is uniformly bounded in $\boldsymbol{L}^{4/3}(0,T;\boldsymbol{V}_{\mathrm{d}}(\mathrm{div}\,0)')$ and has a subsequence converging weakly to $\partial_t \boldsymbol{u}$ in $\boldsymbol{L}^{4/3}(0,T;\boldsymbol{V}_{\mathrm{d}}(\mathrm{div}\,0)')$. By the compact injection in [9, Lemma 2.8], $\{\boldsymbol{u}_\tau\}$ contains a subsequence converging strongly to $\boldsymbol{u}$ in $\boldsymbol{L}^2(0,T;\boldsymbol{L}^3(\Omega))$, and so does $\{\bar{\boldsymbol{u}}_\tau\}$.

Step 2. Integrating both sides of (4.7a) over $[0,T]$ shows

$$\int_0^T [(\partial_t \boldsymbol{u}_\tau, \boldsymbol{v}) + \mathscr{O}(\boldsymbol{u}_\tau^*; \bar{\boldsymbol{u}}_\tau, \boldsymbol{v}) + \mathscr{A}(\bar{\boldsymbol{u}}_\tau, \boldsymbol{v}) - \kappa(\boldsymbol{J}_\tau \times \boldsymbol{B}_\tau, \boldsymbol{v})] = \int_0^T (\boldsymbol{f}_\tau, \boldsymbol{v}),$$

for any $\boldsymbol{v} \in \boldsymbol{L}^4(0,T;\boldsymbol{V}_{\mathrm{d}}(\mathrm{div}\,0))$. By the assumption and the convergence in (4.10), we easily get

$$\begin{aligned}
&\lim_{\tau\to 0} \int_0^T [(\partial_t \boldsymbol{u}_\tau - \boldsymbol{f}_\tau, \boldsymbol{v}) + \mathscr{A}(\boldsymbol{u}_\tau, \boldsymbol{v})] = \int_0^T [(\partial_t \boldsymbol{u} - \boldsymbol{f}, \boldsymbol{v}) + \mathscr{A}(\boldsymbol{u}, \boldsymbol{v})]\,, \\
&\lim_{\tau\to 0} \int_0^T (\boldsymbol{J}_\tau \times \boldsymbol{B}_\tau, \boldsymbol{v}) = \lim_{\tau\to 0} \int_0^T (\boldsymbol{J}_\tau \times \boldsymbol{B}, \boldsymbol{v}) = \int_0^T (\boldsymbol{J} \times \boldsymbol{B}, \boldsymbol{v}), \\
&\lim_{\tau\to 0} \int_0^T \mathscr{O}(\boldsymbol{u}_\tau^*; \bar{\boldsymbol{u}}_\tau, \boldsymbol{v}) = \lim_{\tau\to 0} \int_0^T \mathscr{O}(\boldsymbol{u}; \bar{\boldsymbol{u}}_\tau, \boldsymbol{v}) = \int_0^T \mathscr{O}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}).
\end{aligned}$$

Combining these equalities yields

$$\int_0^T [(\partial_t \boldsymbol{u}, \boldsymbol{v}) + \mathscr{O}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + \mathscr{A}(\boldsymbol{u}, \boldsymbol{v}) - \kappa(\boldsymbol{J} \times \boldsymbol{B}, \boldsymbol{v})] = \int_0^T (\boldsymbol{f}, \boldsymbol{v}).$$

The arbitrariness of $\boldsymbol{v}$ shows that (2.4a) holds in the distributional sense. Equation (2.4b) can be proven similarly. We do not elaborate on the details here.

Step 3. Write $\xi(t) = 1 - t/T$. For any $\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega)$, we have

$$\begin{aligned}
\big(\boldsymbol{u}(0), \boldsymbol{v}\big) = \big(\boldsymbol{u}(0), \boldsymbol{v}\xi(0)\big) &= -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\big(\boldsymbol{u}, \boldsymbol{v}\xi\big) = -\int_0^T [(\partial_t \boldsymbol{u}, \boldsymbol{v}\xi) + (\boldsymbol{u}, \boldsymbol{v}\xi')] \\
&= -\lim_{\tau\to 0} \int_0^T [(\partial_t \boldsymbol{u}_\tau, \boldsymbol{v}\xi) + (\boldsymbol{u}_\tau, \boldsymbol{v}\xi')] = \lim_{\tau\to 0} (\boldsymbol{u}_\tau(0), \boldsymbol{v}) = (\boldsymbol{u}_0, \boldsymbol{v})\,.
\end{aligned}$$

The arbitrariness of $\boldsymbol{v}$ implies $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

Step 4. Since γ: $\boldsymbol{V} \to \boldsymbol{H}^{1/2}(\Gamma_{\mathrm{d}})$ and $\gamma_{\boldsymbol{n}}$: $\boldsymbol{D} \to H^{-1/2}(\Gamma_{\mathrm{i}})$ are surjective, (4.10) shows

$$\begin{aligned}
&\gamma_{\boldsymbol{n}} \boldsymbol{J}_\tau \rightharpoonup \gamma_{\boldsymbol{n}} \boldsymbol{J} \quad \text{weakly in } \boldsymbol{L}^2(0,T;H^{-1/2}(\Gamma_{\mathrm{i}})), \\
&\gamma \boldsymbol{u}_\tau \rightharpoonup \gamma \boldsymbol{u} \qquad \text{weakly in } \boldsymbol{L}^2(0,T;\boldsymbol{H}^{1/2}(\Gamma_{\mathrm{d}})).
\end{aligned}$$

By the assumptions of the theorem, we also have

$$\begin{aligned}
&\gamma_{\boldsymbol{n}} \boldsymbol{J}_\tau = J_\tau \to J \quad \text{strongly in } \boldsymbol{L}^2(0,T;H^{-1/2}(\Gamma_{\mathrm{i}})), \\
&\gamma \boldsymbol{u}_\tau = \boldsymbol{g}_\tau \to \boldsymbol{g} \qquad \text{strongly in } \boldsymbol{L}^2(0,T;\boldsymbol{H}^{1/2}(\Gamma_{\mathrm{d}})).
\end{aligned}$$

We conclude $\gamma \boldsymbol{u} = \boldsymbol{g}$ on $\Gamma_{\rm d}$ and $\gamma_{\boldsymbol{n}} \boldsymbol{J} = J$ on $\Gamma_{\rm i}$. □

REMARK 4.6. *By Theorem 4.2, the fully discrete solutions* $\big(\boldsymbol{u}_n^{(k)}, \boldsymbol{J}_n^{(k)}\big)$ *converge weakly to the solutions of the semi-continuous problem* (4.6) *in* $\boldsymbol{V} \times \boldsymbol{D}$ *upon an extracted subsequence. Moreover, by Theorem 4.5, the semi-continuous solutions* $(\boldsymbol{u}_\tau, \boldsymbol{J}_\tau)$ (*or* $(\boldsymbol{u}_n, \boldsymbol{J}_n)$ *equivalently*) *converge weakly to the solutions of the continuous problem* (2.4) *in* $\boldsymbol{L}^2(0,T;\boldsymbol{V}) \times \boldsymbol{L}^2(0,T;\boldsymbol{D})$ *upon an extracted subsequence. This proves the existence of the solutions of problem* (2.4).

5. A saddle-point formulation. The discrete problem (3.9) is difficult to solve due to the global constraints in $\boldsymbol{V}_{\rm d}^h(\operatorname{div} 0)$ and $\boldsymbol{D}_{\rm i}^h(\operatorname{div} 0)$. To resolve the issue, we introduce the discrete pressure $p_n \in Q^h$ and the discrete potential $\phi_n \in S^h$ where

$$S^h := \begin{cases} \big\{ s \in L^2(\Omega) : \ s|_K \in P_0(K), \ \ \forall K \in \mathcal{T}_h \big\} & \text{if } \Gamma_{\rm c} \neq \emptyset, \\ \big\{ s \in L_0^2(\Omega) : \ s|_K \in P_0(K), \ \ \forall K \in \mathcal{T}_h \big\} & \text{if } \Gamma_{\rm c} = \emptyset. \end{cases}$$

An unconstrained saddle-point problem for (3.9) reads

Find $(\boldsymbol{u}_n, p_n, \boldsymbol{J}_n, \phi_n) \in \boldsymbol{V}^h \times Q^h \times \boldsymbol{D}^h \times S^h$ such that $\gamma \boldsymbol{u}_n = \boldsymbol{g}_n$ on $\Gamma_{\rm d}$, $\gamma_{\boldsymbol{n}} \boldsymbol{J}_n = J_n$ on $\Gamma_{\rm i}$, and

$$(\delta_t \boldsymbol{u}_n, \boldsymbol{v}) + \mathscr{O}(\boldsymbol{u}_n^*; \bar{\boldsymbol{u}}_n, \boldsymbol{v}) + \mathscr{A}_{\rm AL}(\bar{\boldsymbol{u}}_n, \boldsymbol{v}) - (p_n, \operatorname{div} \boldsymbol{v}) - \kappa(\boldsymbol{J}_n \times \boldsymbol{B}_n, \boldsymbol{v}) = (\boldsymbol{f}_n, \boldsymbol{v}), \tag{5.1a}$$
$$(\boldsymbol{J}_n, \boldsymbol{d}) + (\boldsymbol{B}_n \times \bar{\boldsymbol{u}}_n, \boldsymbol{d}) - (\phi_n, \operatorname{div} \boldsymbol{d}) = \langle \gamma_{\boldsymbol{n}} \boldsymbol{d}, \xi_n \rangle_{\Gamma_{\rm c}}, \tag{5.1b}$$
$$(q, \operatorname{div} \bar{\boldsymbol{u}}_n) = 0, \tag{5.1c}$$
$$(\varphi, \operatorname{div} \boldsymbol{J}_n) = 0, \tag{5.1d}$$

for all $(\boldsymbol{v}, q, \boldsymbol{d}, \varphi) \in \boldsymbol{V}_{\rm d}^h \times Q^h \times \boldsymbol{D}_{\rm i}^h \times S^h$.

THEOREM 5.1. *Problem* (5.1) *has unique solutions. Moreover, its solutions* $(\boldsymbol{u}_n, \boldsymbol{J}_n)$ *also satisfy the constrained problem* (3.9).

Proof. Clearly (5.1c) implies $\boldsymbol{u}_n \in \boldsymbol{V}^h(\operatorname{div} 0)$ and (5.1d) implies $\boldsymbol{J}_n \in \boldsymbol{D}^h(\operatorname{div} 0)$. So the solutions of (5.1) satisfy (3.9). By Theorem 3.3, $(\boldsymbol{u}_n, \boldsymbol{J}_n)$ exist and are unique.

The existence and uniqueness of p_n follow directly from (5.1a) and the discrete inf-sup condition (3.1). Moreover, since Ω is connected, we have $\operatorname{div} \boldsymbol{D}_{\rm i}^h = Q^h$ from [6, III.3.4]. There exists a constant $C_0 > 0$ independent of h such that

$$\sup_{\boldsymbol{d}_h \in \boldsymbol{D}_{\rm i}^h} \frac{(\varphi_h, \operatorname{div} \boldsymbol{d}_h)}{\|\boldsymbol{d}_h\|_{\boldsymbol{H}(\operatorname{div},\Omega)}} \geq C_0 \|\varphi_h\|_{L^2(\Omega)} \quad \forall \varphi_h \in S^h. \tag{5.2}$$

Therefore, there exists a unique $\phi_n \in S^h$ which satisfies (3.9b). □

6. Numerical results. In this section, we report three numerical experiments to show the convergence rate of the extrapolated finite element method. Let $\mathcal{T}_1, \cdots, \mathcal{T}_4$ be four successively refined meshes listed in Table 6.1. The approximation errors at $t_N = T$ are denoted by

$$e_u := \|\boldsymbol{u}(T) - \boldsymbol{u}_N\|_{H^1(\Omega)}, \qquad e_p := \|p(T) - p_N\|_{L^2(\Omega)},$$
$$e_J := \|\boldsymbol{J}(T) - \boldsymbol{J}_N\|_{\boldsymbol{H}(\operatorname{div},\Omega)}, \qquad e_\phi := \|\phi(T) - \phi_N\|_{L^2(\Omega)}.$$

The relative tolerance for solving algebraic systems is set by $\varepsilon = 10^{-10}$. For all examples, we set

$$\Omega = (0,1)^3, \qquad \boldsymbol{B} = (1,0,0)^\top, \qquad R_e = \kappa = 1,$$

TABLE 6.1
Numbers of DOFs on four successively refined meshes. (Example 6.1)

Meshes (h)	DOFs of $\boldsymbol{u}_h$	DOFs of $\boldsymbol{J}_h$	DOFs of p_h	DOFs of ϕ_h
$\mathcal{T}_1$ (0.866)	375	360	27	48
$\mathcal{T}_2$ (0.433)	2,187	2,592	125	384
$\mathcal{T}_3$ (0.217)	14,739	19,584	729	3,072
$\mathcal{T}_4$ (0.108)	107,811	152,064	4,913	24,576

and impose Dirichlet boundary conditions to $\boldsymbol{u}$ on the whole boundary Γ. Moverover, Dirichlet boundary conditions are also imposed to $\boldsymbol{J}$ in Example 6.1–6.2 and to ϕ in Example 6.3.

EXAMPLE 6.1 (Convergence rate for time discretization). *This example tests the convergence rate of time discretization. The true solutions are so chosen that spatial errors are zero*

$$\boldsymbol{u} = \left(\sin(10t)z^2, x, y^2 e^{-t}\right)^\top, \quad \boldsymbol{J} = \left(\cos(10t), t^2 x, y\right)^\top, \quad p = \sin t\,(x+y+z), \quad \phi = 0.$$

We set the terminal time by $T = 0.4$ and solve the problem on $\mathcal{T}_3$. Table 6.2 shows that the discrete solutions have the asymptotic behaviors

$$e_u \sim O(\tau^2), \qquad e_p \sim O(\tau^2), \qquad e_J \sim O(\tau^2), \qquad e_\phi \sim O(\tau^2).$$

This indicates that the extrapolated finite element method is second-order in time.

TABLE 6.2
Convergence rate of time discretization at $t = 0.4$. (Example 6.1)

τ	e_u (order)	e_p (order)	e_J (order)	e_ϕ (order)
0.1	8.27e-03 (—)	4.27e-02 (—)	1.33e-01 (—)	1.22e-05 (—)
0.05	1.72e-03 (2.27)	9.01e-03 (2.24)	3.06e-02 (2.12)	1.24e-05 (-0.02)
0.025	4.41e-04 (1.96)	2.43e-03 (1.89)	7.51e-03 (2.03)	4.03e-06 (1.62)
0.0125	1.13e-04 (1.96)	6.48e-04 (1.91)	1.87e-03 (2.01)	1.12e-06 (1.85)
0.00625	2.87e-05 (1.98)	1.68e-04 (1.95)	4.66e-04 (2.00)	2.90e-07 (1.95)

EXAMPLE 6.2 (Convergence rate of the extrapolated method). *This example tests the convergence rate for both time and space approximations. The true solutions are*

$$\boldsymbol{u} = (\sin(t+y), 0, \cos x)^\top, \qquad \boldsymbol{J} = (e^{-t}\sin z, \sin t\cos x, 0), \qquad p = \sin x, \qquad \phi = x.$$

We set the terminal time by $T = 1.0$ and the initial timestep by $\tau_0 = 0.2$. Table 6.3 shows that the discrete solutions have the asymptotic behaviors

$$e_u \sim O(\tau^2 + h^2), \quad e_p \sim O(\tau^2 + h^2), \quad e_J \sim O(\tau^2 + h^2), \quad e_\phi \sim O(\tau^2 + h).$$

Clearly the optimal convergence rates are obtained for both time and space variables.

From Table 6.4, we find that the convergence rate for $\|\operatorname{div} \boldsymbol{u}_N\|_{L^2(\Omega)}$ is approximately second-order. In view of Theorem 5.1, the discrete current density is exactly divergence-free. However, since

TABLE 6.3
Convergence rate of the extrapolated finite element method at $t = 1$. (Example 6.2)

$(\tau, \mathcal{T}_h)$	e_u (order)	e_p (order)	e_J (order)	e_ϕ (order)
$(\tau_0, \mathcal{T}_1)$	5.40e-03 (—)	1.08e-02 (—)	7.37e-03 (—)	1.02e-01 (—)
$(\tau_0/2, \mathcal{T}_2)$	1.34e-03 (2.01)	2.11e-03 (2.36)	1.63e-03 (2.18)	5.10e-02 (1.00)
$(\tau_0/4, \mathcal{T}_3)$	3.35e-04 (2.00)	5.11e-04 (2.05)	4.07e-04 (2.00)	2.55e-02 (1.00)
$(\tau_0/8, \mathcal{T}_4)$	8.36e-05 (1.84)	1.26e-04 (2.02)	1.02e-04 (2.00)	1.28e-02 (0.99)

TABLE 6.4
Convergence of $\operatorname{div} \boldsymbol{u}_N$ *and* $\operatorname{div} \boldsymbol{J}_N$ *for* $\alpha = 1$ (Example 6.2)

$(\tau, \mathcal{T}_h)$	$\|\operatorname{div} \boldsymbol{u}_N\|_{L^2(\Omega)}$ (order)	$\|\operatorname{div} \boldsymbol{J}_N\|_{L^2(\Omega)}$
$(\tau_0, \mathcal{T}_1)$	1.68e-03 (—)	9.51e-12
$(\tau_0/2, \mathcal{T}_2)$	1.61e-04 (5.22)	8.47e-12
$(\tau_0/4, \mathcal{T}_3)$	4.54e-05 (1.77)	2.25e-12
$(\tau_0/8, \mathcal{T}_4)$	1.21e-05 (1.87)	5.42e-12

the system of algebraic equations resulting from the discrete problem is only solved approximately, the approximate solution yields $\|\operatorname{div} \boldsymbol{J}_N\|_{L^2(\Omega)}$ in the order of 10^{-12} due to the solution error. In fact, the relative residual is reduced to less than 10^{-10} in solving the algebraic system.

EXAMPLE 6.3 (Conservation of charges). *This example verifies the condition* $\operatorname{div} \boldsymbol{J}_n = 0$ *and investigates how the Reynolds number and the AL stabilization parameter influence* $\operatorname{div} \boldsymbol{u}_n$ *by computing a driven cavity flow. The external force is set to* $\boldsymbol{f} = 0$. *We set the Dirichelt boundary conditions by* $\phi = 0$ *and* $\boldsymbol{u} = (v, 0, 0)^\top$ *on* Γ *where* $v \in C^1(\bar{\Omega})$ *satisfies*

$$v(x, y, 1) = 1 \quad \textit{and} \quad v(x, y, z) = 0 \quad \forall\, z \in [0, 1 - h].$$

We solve the discrete problem on $\mathcal{T}_4$. The terminal time is $T = 1.0$ and the time step size is $\tau = 0.1$. For fixed α, Table 6.5 shows that both $\|\operatorname{div} \boldsymbol{u}_h\|_{L^2(\Omega)}$ and $\|\operatorname{div} \boldsymbol{J}_h\|_{L^2(\Omega)}$ decreases when R_e increases. For fixed R_e and increasing α, Table 6.6 shows that $\|\operatorname{div} \boldsymbol{u}_h\|_{L^2(\Omega)}$ decreases, while $\|\operatorname{div} \boldsymbol{J}_h\|_{L^2(\Omega)}$ increases slightly. This verifies the control of the AL-stabilization term over $\operatorname{div} \boldsymbol{u}_n$. In fact, $\|\boldsymbol{J}_N\|_{L^2(\Omega)}$ is in the order of tolerance for solving linear algebraic systems.

TABLE 6.5
Values of $\|\operatorname{div} \boldsymbol{u}_N\|_{L^2(\Omega)}$, $\|\operatorname{div} \boldsymbol{J}_N\|_{L^2(\Omega)}$ *for* $\alpha = 1$ *and variant* R_e (Example 6.3)

R_e	$\|\operatorname{div} \boldsymbol{u}_N\|_{L^2(\Omega)}$	$\|\operatorname{div} \boldsymbol{J}_N\|_{L^2(\Omega)}$
1	3.61e-02	8.42e-11
10^2	2.18e-02	2.27e-12
10^4	1.27e-03	9.65e-13

7. Conclusions. In this paper, we propose a charge-conservative finite element method for inductionless MHD equations. The discrete current density is divergence-free exactly in the domain. The convergence of discrete solutions is proven in the sense of extracted subsequences. This yields the existence of the continuous solutions.

TABLE 6.6
Values of $\|\mathrm{div}\, \boldsymbol{u}_N\|_{L^2(\Omega)}$, $\|\mathrm{div}\, \boldsymbol{J}_N\|_{L^2(\Omega)}$ *for* $R_e = 10^4$ *and variant* α (Example 6.3)

α	$\|\mathrm{div}\, \boldsymbol{u}_N\|_{L^2(\Omega)}$	$\|\mathrm{div}\, \boldsymbol{J}_N\|_{L^2(\Omega)}$
0	6.16e-01	2.45e-13
1/4	5.14e-03	2.33e-13
1	1.27e-03	9.65e-13

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