EDDY CURRENT MODEL FOR NONDESTRUCTIVE EVALUATION WITH THIN CRACKS

XUE JIANG*, PEIJUN LI†, AND WEIYING ZHENG‡

Abstract. In this paper, we propose an approximate eddy current model for nondestructive evaluation. Interior cracks of large steel structures are very thin compared with the characteristic length of the system. Numerical methods usually necessitate very fine meshes to characterize the small thickness of cracks, and thus yield very large number of degrees of freedom. The proposed model neglects the thickness of cracks and treats them as interior surfaces. The existence and uniqueness are established for the approximate solution upon introducing proper gauge conditions. The convergence of the approximate solution to the solution of the original eddy current problem is proved as the thickness of cracks tends to zero. And an error estimate is presented for homogeneous conducting materials. A coupled finite element method is proposed to solve the approximate problem. The well-posedness and the error estimate are proved for the discrete solution. Numerical experiments are carried out for engineering benchmark problems to validate the approximate eddy current model and to demonstrate the efficiency of the finite element method.

Key words. Eddy current problem, Maxwell's equations, finite element method, nondestructive evaluation, thin crack

AMS subject classifications. 35Q60, 65N30, 78A25

1. Introduction.

In electrical engineering, the detection of interior cracks or flaws is vital for large metallic structures, such as airfoils, railway tracks of high-speed trains, high-pressure boilers, and so on. Nondestructive evaluation (NDE) of durable devices is active in both scientific research and engineering applications. Among others, eddy current method is one of most popular approaches in NDE and usually yields accurate identification of cracks. In a large metallic structure, the thickness of a crack can be very thin, less than one millimeter in many cases. It makes the numerical solution of Maxwell's equations very difficult and usually necessitates large number of degrees of freedom. The inverse problem for identification of interior cracks appears to be difficult but very interesting. The main task for NDE is to solve the eddy current problem efficiently to locate the cracks and to reconstruct their two-dimensional profiles. In this paper, we study the forward problem for NDE and are going to address the inverse problem in a forthcoming paper.

We propose to study the time-harmonic eddy current problem

$$\mathbf{i}\omega\mu_0 \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} = 0 \quad \text{in } \mathbb{R}^3, \qquad \text{(Farady's law)} \tag{1.1a}$$

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \quad \text{in } \mathbb{R}^3, \qquad \text{(Ampere's law)} \tag{1.1b}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, ω is the angular frequency, and $\boldsymbol{J}$ is the current density defined by:

$$\boldsymbol{J} = \sigma \boldsymbol{E} + \boldsymbol{J}_s \qquad \text{in } \mathbb{R}^3. \tag{1.2}$$

*School of Science, Beijing University of Posts and Telecommunications. The first author was supported in part by China NSF grant 11401040 and by the Fundamental Research Funds for the Central Universities 24820152015RC17. (jxue@lsec.cc.ac.cn)

†Department of Mathematics, Purdue University, West Lafayette, Indiana, 47907. The second author was supported in part by the NSF grant DMS-1151308. (lipeijun@math.purdue.edu)

‡NCMIS, LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, 100190, China. The third author was supported in part by China NSF grants 11171334 and 91430215, by the Funds for Creative Research Groups of China (Grant No. 11321061), and by the National Magnetic Confinement Fusion Science Program 2015GB110003. (zwy@lsec.cc.ac.cn)

Here $\sigma \geq 0$ is the electric conductivity, μ_0 is the magnetic permeability in the empty space, and $\boldsymbol{J}_s \in \boldsymbol{L}^2(\Omega)$ is the source current density carried by some coils. The source current density usually satisfies

$$\operatorname{div} \boldsymbol{J}_s = 0 \qquad \text{in } \mathbb{R}^3.$$

The eddy current problem is a quasi-static approximation to Maxwell's equations at very low frequency by neglecting the displacement currents in Ampere's law (see [2]). For linear eddy current problems, there are many interesting works in the literature on numerical methods (cf. e.g. [1,7,10,18,21,29]) and on the regularity of the solution (cf. e.g. [13]). In [5], Bachinger et al studied the numerical analysis of nonlinear eddy current problem in isotropic materials. Recently, Zheng et al studied the nonlinear eddy current problem for grain-oriented silicon steel laminations in large power transformers [19,22,30]. In [24], Nédélec and Wolf presented the first homogenization result for linear time-harmonic eddy current problem in a transformer core. In [20], Jiang and Zheng studied the homogenization of time-dependent eddy current problem for nonlinear permeability and derived the homogenized Maxwell's equations.

Recently, eddy current model based NDE attracts more and more attentions in numerical analysis and scientific computing. There are plenty of papers arising from the engineering community. In [26], using finite element method, Palanisamy computed remote-field eddy current problems for the nondestructive testing of metal tube. In [27], Philipp et al investigated systematically the finite element method for eddy current NDE, such as the variational formulation, finite element discretization, and boundary conditions for coil-in-air. Rachek et al studied the finite element simulation for eddy current NDE for rotationally symmetric problem [25]. Hamia proposed a finite element analysis for eddy current NDE with an improved giant magnetoresistance magnetometer and a simple single wire as inducer [17]. The thin crack is treated approximately as a nonconducting surface in [6, 11, 14]. In [6], the authors adopt a mixed formulation for $\boldsymbol{A}$ and ϕ, the vector magnetic potential and the scalar electric potential. But the normal component of $\boldsymbol{A} \cdot \boldsymbol{n}$ must be set to zero on the surface. In [11], Choua et al also use $\boldsymbol{A}$ and ϕ as unknown functions. The normal component of the current density is set to zero by duplicating the degrees of freedom of ϕ on the surface. In [14], Dular and Geuzaine proposed a clever decomposition of the magnetic vector potential into a continuous function plus a discontinuous function. The approximate model only solves the discontinuous function on the insulating surface and guarantees that the normal component of the current density vanishes on the surface.

However, rigorous mathematical theories are relatively rare for eddy current NDE. For inverse problem of eddy current model, Ammari et al provided a mathematical analysis and a numerical framework for simulating the imaging of arbitrarily shaped small-volume conductive inclusions from electromagnetic induction data [3]. They derived a small-volume expansion of the eddy current data measured away from the conductive inclusion and proposed a location search algorithm based on the new formula.

In this paper, we study the forward problem of eddy current NDE for large steel structures which comprise interior defects or cracks. To solve the eddy current problem numerically, one has to seek for very fine meshes to characterize the small thickness of crack. But the solution of the inverse problem for NDE requires to solve the forward problem efficiently and accurately. Starting from the conservation of charges, we derive an approximate eddy current model in the variational framework. This model replaces the crack with an interior interface so that the finite element mesh size only needs to be comparable with the width of crack. Since the thickness of crack is usually about one percent of its width, the approximate model is much superior to the original model in terms of computational complexity. It can be shown that the approximate model is equivalent mathematically to the model in [14].

In the theoretical aspects, the paper presents the following results:

1. the existence and uniqueness of the approximate solution,
2. the stability of the solution with respect to the source current,
3. the strong convergence of the approximate solution to the solution of the original problem as the thickness of cracks tends to zero,
4. the error estimate between the approximate solution and the true solution for homogeneous conducting materials.

Our theory can be extended directly to the case when the magnetic permeability is inhomogeneous but linear. For nonlinear eddy current problems, we can still prove the convergence of the approximate solution, but the error estimate will be much more difficult due to the low-regularity of the solution. For simplicity, we only consider the linear and time-harmonic eddy current problem.

The second objective of the paper is to propose a coupled finite element method to solve the approximate problem. We adopt the hybrid of the lowest order edge element method and the lowest nodal element method. By introducing discrete gauge conditions, we proved the well-posedness of the discrete problem. The optimal error estimate is also presented in the sense that the approximation error is bounded by the the interpolation error of the solution. To validate the approximate model, we choose two engineering benchmark problems from the International Compumag Society. The first one is the Team Workshop Problem 21^a-2 whose experimental data are provided [9] and the second one is the Team Workshop Problem 15 for NDE [8]. We carry out numerical experiments for both the original model and the approximate model. The numerical results from the two models agree with each other.

The layout of the paper is as follows. In section 2 we present some notation and Sobolev spaces used in this paper and study the $\boldsymbol{A}$-formulation of (1.1). In section 3 we propose an approximate eddy current model for thin cracks. The uniqueness, existence, and stability of the approximate solution are also presented. In section 4, we prove that the approximate solution converges to the solution of the original eddy current problem as the thickness of crack tends to zero. The error estimate is also proved for homogeneous conducting materials . In section 5, a coupled finite element method is proposed to solve the approximate eddy current model. A Céa-type lemma is proved for the error estimate between the finite element solution and the continuous solution. In this sense, the approximation error of the discrete solution is bounded by the interpolation error of the continuous solution. In section 6 we present two numerical experiments to validate the approximate eddy current model and to demonstrate the efficiency of the finite element method. They are two benchmark problems from the International Compumag Society.

2. The $\boldsymbol{A}$-formulation of the eddy current problem. Let the truncation domain Ω be a cube which encloses all inhomogeneities, such as coils and conductors. We also denote by Ω_c the conducting region and by Ω_{nc} the nonconducting region, that is,

$$\overline{\Omega}_c = \mathrm{supp}(\sigma), \qquad \Omega_{nc} = \Omega\backslash\overline{\Omega}_c\,.$$

For convenience, we only consider one conductor and assume that Ω_c is connected. Our theory can be extend to multiple conductors straightforwardly. Since we are considering interior cracks of Ω_c, the nonconducting region is assumed to be the combination of $M+1$ connected components

$$\Omega_{nc} = \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_M, \qquad \Omega_i \cap \Omega_j = \emptyset,$$

for any $0 \le i, j \le M$ and $i \ne j$, where $\Omega_1, \cdots, \Omega_M$ stand for simply-connected cracks and the open domain Ω_0 stands for the exterior of $\Omega_c \cup \bar{\Omega}_1 \cup \cdots \cup \bar{\Omega}_M$(see Fig. 2.1):

$$\Omega_0 = \Omega \backslash \left(\bar{\Omega}_c \cup \bar{\Omega}_1 \cup \cdots \cup \bar{\Omega}_M \right).$$

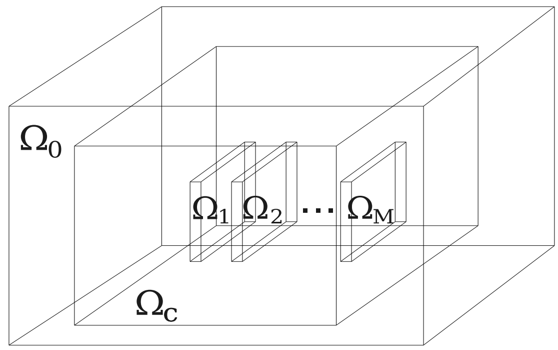

FIG. 2.1. *The conductor Ω_c, the thin cracks $\Omega_1, \cdots, \Omega_M$, and the exterior domain Ω_0.*

Let $L^2(\Omega)$ be the usual Hilbert space of square integrable functions equipped with the following inner product and norm:

$$(u,\, v) := \int_\Omega u\, \bar{v} \qquad \text{and} \qquad \|u\|_{L^2(\Omega)} := (u,\, v)^{1/2},$$

where $\bar{v}$ stands for the complex conjugate of v. Let $\boldsymbol{\xi}$ denote any non-negative triple index and define

$$H^m(\Omega) := \{v \in L^2(\Omega) : D^\xi v \in L^2(\Omega), |\xi| \le m\}$$

Let $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ whose functions have zero traces on $\partial\Omega$. Throughout the paper we denote vector-valued quantities by boldface notation, such as $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^3$.

We define the spaces of functions having square integrable curl by

$$\boldsymbol{H}(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,:\, \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\},$$
$$\boldsymbol{H}_0(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) \,:\, \boldsymbol{n} \times \boldsymbol{v} = 0 \ \text{ on } \partial\Omega\},$$

which are equipped with the following inner product and norm

$$(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{H}(\mathbf{curl},\Omega)} := (\boldsymbol{v}, \boldsymbol{w}) + (\mathbf{curl}\, \boldsymbol{v}, \mathbf{curl}\, \boldsymbol{w}), \quad \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} := (\boldsymbol{v}, \boldsymbol{v})^{1/2}_{\boldsymbol{H}(\mathbf{curl},\Omega)}.$$

Here $\boldsymbol{n}$ denotes the unit outer normal to $\partial\Omega$. We shall also use the spaces of functions having square integrable divergence

$$\boldsymbol{H}(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,:\, \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega)\},$$
$$\boldsymbol{H}_0(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) \,:\, \boldsymbol{n} \cdot \boldsymbol{v} = 0 \ \text{ on } \partial\Omega\},$$

which are equipped with the following inner product and norm

$$(\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{H}(\mathrm{div},\Omega)} := (\boldsymbol{v},\boldsymbol{w}) + (\mathrm{div}\,\boldsymbol{v},\mathrm{div}\,\boldsymbol{w}), \quad \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} := (\boldsymbol{v},\boldsymbol{v})^{1/2}_{\boldsymbol{H}(\mathrm{div},\Omega)}.$$

Denote the boundary of Ω by $\Gamma = \partial\Omega$. We impose an approximate boundary condition on Γ as follows

$$\boldsymbol{B}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{2.1}$$

where $\boldsymbol{B} = \mu_0\boldsymbol{H}$ stands for the magnetic flux density. We remark that the boundary condition is physically reasonable and easy to satisfy by means of vector magnetic potential.

Notice that (1.1a) indicates $\mathrm{div}\,\boldsymbol{H} = 0$ in Ω. There exists a magnetic vector potential $\boldsymbol{a}$ such that

$$\mu_0\boldsymbol{H} = \mathbf{curl}\,\boldsymbol{a} \qquad \text{in } \Omega.$$

Then (1.1a) turns into

$$\mathbf{curl}(\mathrm{i}\omega\boldsymbol{a} + \boldsymbol{E}) = 0 \quad \text{in } \Omega.$$

Thus there is a scalar electric potential p such that

$$\mathrm{i}\omega\boldsymbol{a} + \boldsymbol{E} = -\nabla p \quad \text{in } \Omega.$$

Set $\boldsymbol{A} = \boldsymbol{a} - \mathrm{i}\nabla p/\omega$. It follows that

$$\boldsymbol{E} = -\mathrm{i}\omega\big(\boldsymbol{a} - \mathrm{i}\nabla p/\omega\big) = -\mathrm{i}\omega\boldsymbol{A} \qquad \text{and} \qquad \mu_0\boldsymbol{H} = \mathbf{curl}\,\boldsymbol{a} = \mathbf{curl}\,\boldsymbol{A}.$$

Let Div_Γ be the surface divergence operator defined on Γ. It is easy to see that

$$\mu_0\boldsymbol{H}\cdot\boldsymbol{n} = \mathbf{curl}\,\boldsymbol{A}\cdot\boldsymbol{n} = \mathrm{Div}_\Gamma(\boldsymbol{A}\times\boldsymbol{n}) \qquad \text{on } \Gamma.$$

Therefore, (2.1) is easily satisfied by imposing homogeneous Dirichlet boundary condition for the magnetic vector potential

$$\boldsymbol{A}\times\boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Finally, substituting $\mu_0\boldsymbol{H} = \mathbf{curl}\,\boldsymbol{A}$ into (1.1b), we obtain the following boundary value problem

$$\mathrm{i}\omega\sigma\boldsymbol{A} + \nu_0\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A} = \boldsymbol{J}_s \qquad \text{in } \Omega, \tag{2.2a}$$

$$\boldsymbol{A}\times\boldsymbol{n} = 0 \qquad \text{on } \Gamma, \tag{2.2b}$$

where $\nu_0 = \mu_0^{-1}$ stands for the magnetic reluctivity.

A weak formulation equivalent to (2.2) reads: Find $\boldsymbol{A}\in\boldsymbol{H}_0(\mathbf{curl},\Omega)$ such that

$$(\mathrm{i}\omega\sigma\boldsymbol{A},\ \boldsymbol{v}) + \nu_0\,(\mathbf{curl}\,\boldsymbol{A},\ \mathbf{curl}\,\boldsymbol{v}) = (\boldsymbol{J}_s,\ \boldsymbol{v}) \qquad \forall\,\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega). \tag{2.3}$$

It is obvious that the solution of (2.3) is not unique in the insulating region Ω_{nc}. In fact, if $\boldsymbol{A}$ solves (2.3), then $\boldsymbol{A}+\nabla\phi$ also solves (2.3) for any $\phi\in H_0^1(\Omega)$ satisfying $\mathrm{supp}(\phi)\subset\Omega_{nc}$.

To study the wellposedness of the weak solution, we shall impose some gauge condition on the test function space. Define

$$H^1_{\rm c}(\Omega_{nc}) := \left\{\phi \in H^1(\Omega_{nc}):\ \phi = 0 \text{ on } \partial\Omega,\ \ \phi = \alpha_i \text{ on } \partial\Omega_i,\ \ 1 \le i \le M\right\},$$

where $\alpha_1, \cdots, \alpha_M$ are arbitrary constants. It is easy to see that

$$\nabla H^1_{\rm c}(\Omega_{nc}) \subset \boldsymbol{H}_0(\mathbf{curl}, \Omega_{nc}) .$$

We extend each function in $\nabla H^1_{\rm c}(\Omega_{nc})$ by zero to the interior of Ω_c and denote the extension space by

$$\boldsymbol{W}(\Omega;\Omega_{nc}) := \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega):\ \boldsymbol{v}|_{\Omega_{nc}} \in \nabla H^1_{\rm c}(\Omega_{nc}) \ \text{ and } \ \boldsymbol{v} = 0 \ \text{ in } \Omega_c\right\}.$$

For any $\phi \in H^1_{\rm c}(\Omega_{nc})$ and each $1 \le i \le M$, since $\phi = $ Const. on $\partial\Omega_i$, we have $\nabla\phi \times \boldsymbol{n} = 0$ on $\partial\Omega_i$. Therefore, $\boldsymbol{W}(\Omega;\Omega_{nc}) \subset \boldsymbol{H}_0(\mathbf{curl}, \Omega)$.

Define

$$\boldsymbol{X} = \left\{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega):\ (\boldsymbol{v}, \boldsymbol{w}) = 0 \quad \forall\, \boldsymbol{w} \in \boldsymbol{W}(\Omega;\Omega_{nc})\right\}. \tag{2.4}$$

Then $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ admits the orthogonal decomposition

$$\boldsymbol{H}_0(\mathbf{curl}, \Omega) = \boldsymbol{X} \oplus \boldsymbol{W}(\Omega;\Omega_{nc}). \tag{2.5}$$

The following lemma will play an important role in our analysis.

LEMMA 2.1. *$\boldsymbol{X}$ is a Hilbert space endowed with the inner product and norm*

$$(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{X}} = \int_{\Omega_c} \boldsymbol{v} \cdot \bar{\boldsymbol{w}} + \int_{\Omega} \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \bar{\boldsymbol{w}}, \qquad \|\boldsymbol{v}\|_{\boldsymbol{X}} = \sqrt{(\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{X}}}\ . \tag{2.6}$$

And there is a constant C depending only on the diameters of $\Omega_0, \cdots, \Omega_M$ such that

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)} \le C \|\boldsymbol{v}\|_{\boldsymbol{X}} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{X}\ . \tag{2.7}$$

Proof. We need only prove (2.7). For any $\boldsymbol{v} \in \boldsymbol{X}$, we consider the orthogonal decomposition

$$\boldsymbol{v} = \boldsymbol{w} + \nabla\psi, \qquad \psi \in H^1_0(\Omega),\ \ \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \ \text{ satisfying } \ \operatorname{div} \boldsymbol{w} = 0\ . \tag{2.8}$$

The well-known Friedrichs' inequality on $\boldsymbol{H}_0(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\operatorname{div}, \Omega)$ shows that (cf. [4])

$$\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)} \le C \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} = C \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \le C \|\boldsymbol{v}\|_{\boldsymbol{X}}\ , \tag{2.9}$$

where $C > 0$ is a constant depending only on Ω.

Furthermore, the definition of $\boldsymbol{X}$ implies that

$$\Delta\psi = 0 \qquad \text{in } \ \Omega_{nc}\ .$$

Let $\bar\psi \in H^1_0(\Omega)$ be defined by harmonic extension as follows

$$\Delta\bar\psi = 0 \quad \text{in } \ \Omega_{nc} \qquad \text{and} \qquad \bar\psi = \frac{1}{|\Omega_c|}\int_{\Omega_c} \psi \quad \text{in } \ \Omega_c\ .$$

Then $\tilde{\psi} = \psi - \bar{\psi} \in H_0^1(\Omega)$ also satisfies

$$\Delta\tilde{\psi} = 0 \qquad \text{in } \Omega_{nc}\,.$$

Then the stability estimate of elliptic equations shows that

$$\left\|\tilde{\psi}\right\|_{H^1(\Omega_{nc})} \leq C\left\|\tilde{\psi}\right\|_{H^{1/2}(\partial\Omega_{nc})} = C\left\|\tilde{\psi}\right\|_{H^{1/2}(\partial\Omega_c)} \leq C\left\|\tilde{\psi}\right\|_{H^1(\Omega_c)},$$

where we have used the trace theorem in the last inequality and the constant $C > 0$ only depends on Ω_c. An application of Friedrich's inequality yields

$$\left\|\tilde{\psi}\right\|_{H^1(\Omega)} \leq C\left\|\tilde{\psi}\right\|_{H^1(\Omega_c)} \leq C|\psi|_{H^1(\Omega_c)}. \tag{2.10}$$

Since $\nabla\bar{\psi} \in \boldsymbol{W}(\Omega;\Omega_{nc})$, we deduce that

$$(\nabla\bar{\psi},\nabla\bar{\psi}) = (\nabla\psi,\nabla\bar{\psi}) - (\nabla\tilde{\psi},\nabla\bar{\psi}) = (\boldsymbol{v},\nabla\bar{\psi}) - (\nabla\tilde{\psi},\nabla\bar{\psi}) = -(\nabla\tilde{\psi},\nabla\bar{\psi})\,.$$

From (2.10) we obtain

$$|\bar{\psi}|_{H^1(\Omega)} \leq \left|\tilde{\psi}\right|_{H^1(\Omega)} \leq C|\psi|_{H^1(\Omega_c)}\,. \tag{2.11}$$

Combining (2.10) and (2.11) shows that

$$|\psi|_{H^1(\Omega)} \leq \left|\tilde{\psi}\right|_{H^1(\Omega)} + |\bar{\psi}|_{H^1(\Omega)} \leq C|\psi|_{H^1(\Omega_c)} \leq C\left\|\boldsymbol{v}-\boldsymbol{w}\right\|_{\boldsymbol{L}^2(\Omega_c)}. \tag{2.12}$$

Finally using (2.9) and (2.12), we conclude that

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C\left\|\boldsymbol{v}\right\|_{\boldsymbol{L}^2(\Omega_c)} + C\left\|\boldsymbol{w}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C\left\|\boldsymbol{v}\right\|_{\boldsymbol{X}}.$$

This completes the proof. ▯

We end this section with a modified weak formulation on the subspace: Find $\boldsymbol{u} \in \boldsymbol{X}$ such that

$$(\mathbf{i}\omega\sigma\boldsymbol{u},\ \boldsymbol{v}) + \nu_0\,(\mathbf{curl}\,\boldsymbol{u},\ \mathbf{curl}\,\boldsymbol{v}) = (\boldsymbol{J}_s,\ \boldsymbol{v}) \qquad \forall\,\boldsymbol{v} \in \boldsymbol{X}. \tag{2.13}$$

From (2.5), it is easy to see that $\boldsymbol{u}$ also satisfies

$$(\mathbf{i}\omega\sigma\boldsymbol{u},\ \boldsymbol{v}) + \nu_0\,(\mathbf{curl}\,\boldsymbol{u},\ \mathbf{curl}\,\boldsymbol{v}) = (\boldsymbol{J}_s,\ \boldsymbol{v}) \qquad \forall\,\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl},\Omega). \tag{2.14}$$

Here we used the assumption that $\boldsymbol{J}_s \cdot \boldsymbol{n} = 0$ on $\partial\Omega_{nc}$, which is usually satisfied in engineering. This means that $\boldsymbol{u}$ is one solution of (2.3). Although the solution $\boldsymbol{A}$ of (2.3) is not unique, the eddy current density and the magnetic flux density are unique, namely,

$$\mathbf{i}\omega\sigma\boldsymbol{A} = \mathbf{i}\omega\sigma\boldsymbol{u}, \qquad \mathbf{curl}\,\boldsymbol{A} = \mathbf{curl}\,\boldsymbol{u} \qquad \text{in } \Omega.$$

Therefore, we are only interested in $\sigma\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{u}$ throughout this paper.

THEOREM 2.2. *Assume* $\boldsymbol{J}_s \in \boldsymbol{L}^2(\Omega)$, $\operatorname{div}\boldsymbol{J}_s = 0$, *and* $\operatorname{supp}(\boldsymbol{J}_s) \subset \Omega_0$. *Then* (2.13) *has a unique solution and there exists a constant* $C > 0$ *depending only on* Ω, σ, *and* ν_0 *such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{X}} \leq C\left\|\boldsymbol{J}_s\right\|_{\boldsymbol{L}^2(\Omega)}.$$

Proof. The theorem is a direct consequence of Lemma 2.1. ▯

3. An approximate eddy current model for interior cracks. In this section, we shall propose an approximate weak formulation of the eddy current problem which omits the thickness of thin cracks. Denote the union of the conducting region and thin cracks by D_c , whose closure satisfies

$$\bar{D}_c = \bar{\Omega}_c \cup \bar{\Omega}_1 \cup \cdots \cup \bar{\Omega}_M = \Omega \backslash \Omega_0 \,.$$

To simplify the setting, we assume

$$D_c := (X_0, X_1) \times (Y_0, Y_1) \times (Z_0, Z_1), \quad \Omega_i := (x_i, x_i + d) \times (y_0, y_1) \times (z_0, z_1),$$

where $Y_0 \le y_0 < y_1 \le Y_1$, $Z_0 \le z_0 < z_1 \le Z_1$, and d denotes the thickness of thin cracks (see Fig. 3.1 (left)). Write $x_0 = X_0$ and $x_{M+1} = X_1$. For non-destructive evaluation, we usually have

$$0 < d \ll \min_{0 \le i \le M} (x_{i+1} - x_i, y_1 - y_0, z_1 - z_0).$$

We remark that our theory also applies to more general cases when $\Omega_0, \cdots, \Omega_M$ are simply-connected and have Lipschitz continuous boundaries. To avoid tedious descriptions, we do not elaborate on the details.

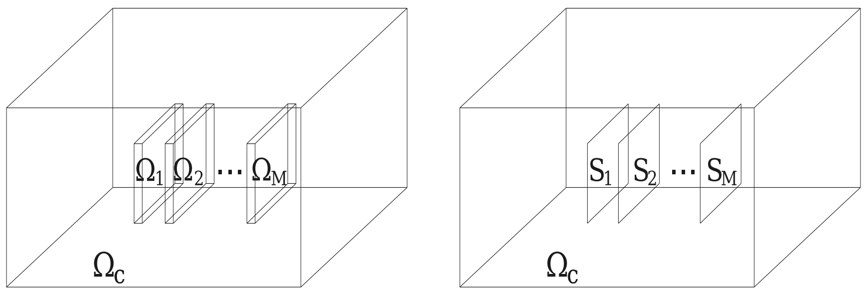

FIG. 3.1. *Left: original conductor with thin cracks $\Omega_1, \cdots, \Omega_M$. Right: extended conductor where the think cracks are replaced by interfaces $S_1, \cdots, S_M$.*

3.1. An approximate eddy current model. Recall that $\operatorname{div} \boldsymbol{J}_s = 0$ in Ω. Taking $\boldsymbol{v} = \nabla\varphi$ in (2.3), we find that

$$\mathbf{i}\omega \int_{\Omega_c} \sigma \boldsymbol{A} \cdot \nabla\varphi = 0 \qquad \forall\, \varphi \in H_0^1(\Omega). \tag{3.1}$$

This implies the conservation of charges in Ω_c, namely, the eddy current density $\boldsymbol{J} = \mathbf{i}\omega\sigma\boldsymbol{A}$ satisfies

$$\operatorname{div} \boldsymbol{J} = 0 \quad \text{in } \Omega_c \qquad \text{and} \qquad \boldsymbol{J} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega_c \,. \tag{3.2}$$

Now we consider the case that the thickness of cracks tends to zero. As $d \to 0$, each Ω_i will degenerate to the surface (see Fig. 3.1 (right))

$$S_i = x_i \times (y_0, y_1) \times (z_0, z_1), \qquad 1 \le i \le M.$$

We shall propose an eddy current model which does not allow the eddy current flowing across each surface S_i.

First we define the modified conductivity $\tilde\sigma$ by extending $\sigma|_{\Omega_c}$ continuously to the cracks such that $\tilde\sigma = \sigma$ in $\Omega_0 \cup \Omega_c$, $\tilde\sigma > 0$ in D_c, and

$$\|\tilde\sigma\|_{L^\infty(D_c)} \le \|\sigma\|_{L^\infty(\Omega_c)} \,. \tag{3.3}$$

For $d = 0$, the modified current density $\tilde{\boldsymbol{J}}$, which will be defined later, should satisfy the conservation of charges, that is,

$$\operatorname{div} \tilde{\boldsymbol{J}} = 0 \quad \text{in } \; D_c \qquad \text{and} \qquad \tilde{\boldsymbol{J}} \cdot \boldsymbol{n} = 0 \quad \text{on } \; \partial D_c \cup S, \tag{3.4}$$

where $S = \cup_{i=1}^M S_i$.

To realize (3.4), we are going to introduce a local domain D_i with S_i being a part of its boundary. Let $H > 0$ be the thickness parameter satisfying

$$d \ll H < \frac{1}{2} \min_{1\le i\le M} (x_i - x_{i-1}) \,.$$

Define

$$D_i = (x_i - H, x_i) \times (y_0, y_1) \times (z_0, z_1), \qquad 1 \le i \le M.$$

Clearly Ω_i and D_i share S_i as the common boundary and are located at its opposite sides respectively (see Fig. 3.2). Then (3.4) can be equivalently written as follows

$$\int_{D_c} \tilde{\boldsymbol{J}} \cdot \nabla\varphi = 0 \qquad \forall\, \varphi \in H_0^1(\Omega), \tag{3.5}$$

$$\int_{D_i} \tilde{\boldsymbol{J}} \cdot \nabla\varphi = 0 \qquad \forall\, \varphi \in H^1_{\partial D_i \backslash \bar S_i}(D_i), \quad 1 \le i \le M, \tag{3.6}$$

where

$$H^1_{\partial D_i \backslash \bar S_i}(D_i) := \left\{ \varphi \in H^1(D_i) : \; \varphi = 0 \; \text{ on } \; \partial D_i \backslash \overline{S}_i \right\}.$$

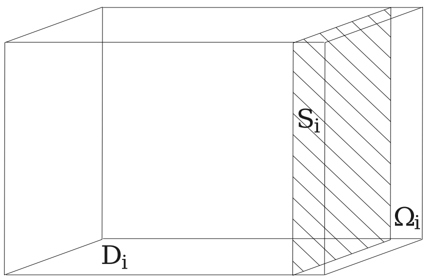

FIG. 3.2. *The illustration for Ω_i and D_i sharing the interface S_i, and $O_i = \Omega_i \cup S_i \cup D_i$.*

A comparison of (3.5)–(3.6) with (3.1) inspires us to enlarge the test function space from $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ to $\boldsymbol{H}_0(\mathbf{curl}, \Omega) + \sum_{i=1}^M \boldsymbol{U}_i$ where $\boldsymbol{U}_i$ consists of zero extensions of functions in $\nabla H^1_{\partial D_i \backslash \bar S_i}(D_i)$, namely,

$$\boldsymbol{U}_i := \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \; \boldsymbol{v}|_{D_i} \in \nabla H^1_{\partial D_i \backslash \bar S_i}(D_i) \; \text{ and } \; \boldsymbol{v} = 0 \; \text{ in } \; \Omega \backslash \bar D_i \right\}.$$

We define a modified curl operator by

$$\widetilde{\mathbf{curl}}\,(\boldsymbol{v}+\boldsymbol{\xi}) := \mathbf{curl}\,\boldsymbol{v} \qquad \forall\,\boldsymbol{v}\in \boldsymbol{H}(\mathbf{curl},\Omega),\ \ \boldsymbol{\xi}\in \sum_{i=1}^{M}\boldsymbol{U}_i. \tag{3.7}$$

It is clear that $\widetilde{\mathbf{curl}}$ is just the normal curl operator on $\boldsymbol{H}(\mathbf{curl},\Omega)$:

$$\widetilde{\mathbf{curl}}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{v} \qquad \forall\,\boldsymbol{v}\in \boldsymbol{H}(\mathbf{curl},\Omega). \tag{3.8}$$

An approximate problem to (2.3) reads: Find $\tilde{\boldsymbol{A}}\in \boldsymbol{H}_0(\mathbf{curl},\Omega)+\sum_{i=1}^{M}\boldsymbol{U}_i$ such that

$$a(\tilde{\boldsymbol{A}},\boldsymbol{v}) = (\boldsymbol{J}_s,\boldsymbol{v}) \qquad \forall\,\boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl},\Omega)+\sum_{i=1}^{M}\boldsymbol{U}_i. \tag{3.9}$$

where $a(\cdot,\cdot)$ is a sesquilinear form defined as follows

$$a(\boldsymbol{v},\boldsymbol{w}) = \mathbf{i}\omega(\tilde{\sigma}\boldsymbol{v},\boldsymbol{w}) + \nu_0(\widetilde{\mathbf{curl}}\,\boldsymbol{v},\widetilde{\mathbf{curl}}\,\boldsymbol{w}). \tag{3.10}$$

Similar to (2.3), the solution of (3.9) is not unique.

To study the wellposedness of (3.9), we define

$$\boldsymbol{U} := \left\{\boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl},\Omega):\ (\boldsymbol{v},\nabla\varphi)=0\ \ \forall\,\varphi\in H_0^1(\Omega),\ \varphi=\text{Const. in }\ \overline{D}_c\right\}. \tag{3.11}$$

The modified text function space is defined as follows

$$\tilde{\boldsymbol{X}} := \boldsymbol{U} + \sum_{i=1}^{M}\boldsymbol{U}_i. \tag{3.12}$$

A modified problem of (3.9) reads: Find $\tilde{\boldsymbol{u}}\in\tilde{\boldsymbol{X}}$ such that

$$a(\tilde{\boldsymbol{u}},\boldsymbol{v}) = (\boldsymbol{J}_s,\boldsymbol{v}) \qquad \forall\,\boldsymbol{v}\in\tilde{\boldsymbol{X}}. \tag{3.13}$$

Lemma 3.1. *The space $\tilde{\boldsymbol{X}}$ admits the decomposition in a direct sum*

$$\tilde{\boldsymbol{X}} = \tilde{\boldsymbol{U}} + \sum_{i=1}^{M}\boldsymbol{U}_i, \qquad \tilde{\boldsymbol{U}} = \left\{\boldsymbol{v}\in\boldsymbol{U}:\ \mathrm{div}\,\boldsymbol{v}=0 \quad \text{in }\ D_1\cup\cdots\cup D_M\right\}.$$

Proof. Clearly $\tilde{\boldsymbol{U}}+\sum_{i=1}^{M}\boldsymbol{U}_i\subset\tilde{\boldsymbol{X}}$. The inverse inclusion only necessitates to show $\boldsymbol{U}\subset\tilde{\boldsymbol{U}}+\sum_{i=1}^{M}\boldsymbol{U}_i$. For any $\boldsymbol{v}\in\boldsymbol{U}$ and any $1\le i\le M$, let $\phi_i\in H_0^1(D_i)$ solve the elliptic problem

$$\int_{D_i}\nabla\phi_i\cdot\nabla\varphi = \int_{D_i}\boldsymbol{v}\cdot\nabla\varphi \qquad \forall\,\varphi\in H_0^1(D_i).$$

We extend ϕ_i by zero to the exterior of D_i. Since $\bigcup_{i=1}^{M}\overline{D}_i\subset D_c$, we have $\tilde{\boldsymbol{v}} = \boldsymbol{v}-\sum_{i=1}^{M}\nabla\phi_i\in\tilde{\boldsymbol{U}}$ and thus $\boldsymbol{v}\in\tilde{\boldsymbol{U}}+\sum_{i=1}^{M}\boldsymbol{U}_i$.

To prove the direct sum, we take any $\tilde{\boldsymbol{v}}\in\tilde{\boldsymbol{U}}$ and $\boldsymbol{v}_i\in\boldsymbol{U}_i$, $1\le i\le M$ satisfying

$$\tilde{\boldsymbol{v}}+\sum_{i=1}^{M}\boldsymbol{v}_i = 0.$$

Then for each $1 \le i \le M$, there exists a $\phi_i \in H^1_{\partial D_i \backslash \overline{S}_i}(D_i)$ such that

$$\tilde{\boldsymbol{v}} = 0 \quad \text{in } \ \Omega \backslash \left(\overline{D}_1 \cup \cdots \cup \overline{D}_M\right) \qquad \text{and} \qquad \tilde{\boldsymbol{v}} = \nabla \phi_i \quad \text{in } \ D_i \ .$$

Since $\tilde{\boldsymbol{v}} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$, the second equality implies that

$$\nabla \phi_i \times \boldsymbol{n} = 0 \quad \text{in } \ \partial D_i.$$

This shows $\phi_i \in H^1_0(D_i)$ for each $1 \le i \le M$. And the definition of $\tilde{\boldsymbol{U}}$ implies $\Delta \phi_i = 0$ in D_i and thus $\phi_i = 0$ in D_i. Therefore, $\tilde{\boldsymbol{v}} \equiv 0$ in Ω. So (3.11) is a direct sum. □

LEMMA 3.2. *The space $\tilde{\boldsymbol{X}}$ is a Hilbert space under the inner product and norm*

$$\begin{aligned} \|\boldsymbol{v}\|_{\tilde{\boldsymbol{X}}} &:= \sqrt{(\boldsymbol{v}, \boldsymbol{v})_{\tilde{\boldsymbol{X}}}}, \\ (\boldsymbol{v}, \boldsymbol{w})_{\tilde{\boldsymbol{X}}} &:= \int_{D_c} \boldsymbol{v} \cdot \boldsymbol{w} + \int_\Omega \tilde{\mathbf{curl}}\, \boldsymbol{v} \cdot \tilde{\mathbf{curl}}\, \boldsymbol{w} \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in \tilde{\boldsymbol{X}}. \end{aligned}$$

Proof. First we prove the completeness of $\tilde{\boldsymbol{X}}$. It is clear that $\boldsymbol{U}_i$ is complete by the isomorphism to $\nabla H^1_{\partial D_i \backslash \overline{S}_i}(D_i)$. By similar arguments as in Lemma 2.1, $\|\cdot\|_{\tilde{\boldsymbol{X}}}$ is an equivalent norm to $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)}$ on $\boldsymbol{U}$. Let $\{\boldsymbol{v}_n\}_{n=1}^\infty \subset \boldsymbol{U}$ be a Cauchy sequence under $\|\cdot\|_{\tilde{\boldsymbol{X}}}$. Then there exists a $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ such that

$$\begin{aligned} &\lim_{n\to\infty} \|\boldsymbol{v}_n - \boldsymbol{v}\|_{\tilde{\boldsymbol{X}}} = 0, \\ &(\boldsymbol{v}, \nabla\varphi) = \lim_{n\to\infty} (\boldsymbol{v}_n, \nabla\varphi) = 0 \quad \forall\, \varphi \in H^1_0(\Omega) \ \text{ satisfying } \ \varphi = \text{Const. in } \ D_c \ . \end{aligned}$$

Thus $\boldsymbol{v} \in \boldsymbol{U}$. Then $\boldsymbol{U}$ is complete, and so does $\tilde{\boldsymbol{X}}$.

Next we prove that $\|\cdot\|_{\tilde{\boldsymbol{X}}}$ is a norm. It is sufficient to show that $\boldsymbol{v} \in \tilde{\boldsymbol{X}}$ and $\|\boldsymbol{v}\|_{\tilde{\boldsymbol{X}}} = 0$ yield $\boldsymbol{v} = 0$. Write $\boldsymbol{v} = \tilde{\boldsymbol{v}} + \sum_{i=1}^M \boldsymbol{v}_i$ with $\tilde{\boldsymbol{v}} \in \tilde{\boldsymbol{U}}$ and $\boldsymbol{v}_i \in \boldsymbol{U}_i$, $1 \le i \le M$. Then from (3.7) we have

$$\mathbf{curl}\, \tilde{\boldsymbol{v}} = 0 \quad \text{in } \ \Omega, \qquad \tilde{\boldsymbol{v}} = 0 \quad \text{in } \ D_c \backslash \left(\overline{D}_1 \cup \cdots \cup \overline{D}_M\right), \qquad \tilde{\boldsymbol{v}} + \boldsymbol{v}_i = 0 \quad \text{in } \ D_i \ .$$

The first equality indicates that $\tilde{\boldsymbol{v}} = \nabla\phi$ for some $\phi \in H^1_0(\Omega)$. From Lemma 3.1, the decomposition of $\boldsymbol{v}$ is a direct sum. Then $\tilde{\boldsymbol{v}} \equiv 0$ in D_c and thus $\phi = \text{Const.}$ in D_c. The definition of $\tilde{\boldsymbol{U}}$ shows $\tilde{\boldsymbol{v}} \equiv 0$ in Ω. Therefore, $\|\cdot\|_{\tilde{\boldsymbol{X}}}$ is a norm on $\tilde{\boldsymbol{X}}$. And equipped with this norm, $\tilde{\boldsymbol{X}}$ is a Hilbert space. □

REMARK 3.3. *The choice of $\{D_i,\ 1 \le i \le M\}$ is not essential. Actually $D_i \subset D_c$ can be any Lipschitz domain satisfying $\partial D_i \supset S_i$ and $\overline{D}_i \cap S_j = \emptyset$ for $i \ne j$.*

We end up this section with the following theorem on the well-posedness of problem (3.13).

THEOREM 3.4. *Assume $\boldsymbol{J}_s \in \boldsymbol{L}^2(\Omega)$, $\operatorname{div} \boldsymbol{J}_s = 0$, and $\operatorname{supp}(\boldsymbol{J}_s) \cap D_c = \emptyset$. Then problem (3.13) has a unique solution $\tilde{\boldsymbol{u}} \in \tilde{\boldsymbol{X}}$, and there exists a constant C only depending on Ω, $\tilde{\sigma}$, and ν_0 such that*

$$\|\tilde{\boldsymbol{u}}\|_{\tilde{\boldsymbol{X}}} \le C \, \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)} \, . \tag{3.14}$$

Proof. The theorem is a direct consequence of Lemma 2.1 and Lemma 3.2. □

THEOREM 3.5. *Assume* $\boldsymbol{J}_s \in \boldsymbol{L}^2(\Omega)$, $\operatorname{div} \boldsymbol{J}_s = 0$, *and* $\operatorname{supp}(\boldsymbol{J}_s) \cap D_c = \emptyset$. *Then the solution of problem* (3.13) *satisfies*

$$\operatorname{div}(\tilde{\sigma}\tilde{\boldsymbol{u}}) = 0 \quad \text{in } D_c, \qquad \tilde{\sigma}\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial D_c \cup S_1 \cup \cdots \cup S_M \,.$$

Proof. From (3.11), any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ admits an orthogonal decomposition

$$\boldsymbol{v} = \boldsymbol{v}_\perp + \nabla\varphi,$$

where $\boldsymbol{v}_\perp \in \boldsymbol{U}$ and $\varphi \in H_0^1(\Omega)$ satisfying $\varphi = \text{Const}$ in $\overline{D}_c$. Since $\operatorname{supp}(\boldsymbol{J}_s) \cap D_c = \emptyset$ and $\operatorname{supp}(\tilde{\sigma}) = \overline{D}_c$, we have

$$\mathbf{i}\omega(\tilde{\sigma}\tilde{\boldsymbol{u}}, \nabla\varphi) + \nu_0(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}, \widetilde{\mathbf{curl}}\,\nabla\varphi) = (\boldsymbol{J}_s, \nabla\varphi) \,.$$

Thanks to (3.13) and $\boldsymbol{v}_\perp \in \tilde{\boldsymbol{X}}$, we have

$$\mathbf{i}\omega(\tilde{\sigma}\tilde{\boldsymbol{u}}, \boldsymbol{v}_\perp) + \nu_0(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}, \widetilde{\mathbf{curl}}\,\boldsymbol{v}_\perp) = (\boldsymbol{J}_s, \boldsymbol{v}_\perp).$$

Adding up the above two equalities yields

$$\mathbf{i}\omega(\tilde{\sigma}\tilde{\boldsymbol{u}}, \boldsymbol{v}) + \nu_0(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}, \widetilde{\mathbf{curl}}\,\boldsymbol{v}) = (\boldsymbol{J}_s, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \,.$$

It implies that (3.13) holds for a larger test function space, namely,

$$\mathbf{i}\omega(\tilde{\sigma}\tilde{\boldsymbol{u}}, \boldsymbol{v}) + \nu_0(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}, \widetilde{\mathbf{curl}}\,\boldsymbol{v}) = (\boldsymbol{J}_s, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) + \sum_{i=1}^{M} \boldsymbol{U}_i \,. \tag{3.15}$$

Now taking $\boldsymbol{v} = \nabla\varphi$ for all $\varphi \in H_0^1(\Omega)$ shows that

$$\operatorname{div}(\tilde{\sigma}\tilde{\boldsymbol{u}}) = 0 \quad \text{in } D_c, \qquad \tilde{\sigma}\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial D_c \,.$$

And it follows that $[\tilde{\sigma}\tilde{\boldsymbol{u}} \cdot \boldsymbol{n}]_{S_i} = 0$ for $1 \le i \le M$. Furthermore, since (3.15) holds for all $\boldsymbol{v} \in \boldsymbol{U}_i$, we also have

$$(\tilde{\sigma}\tilde{\boldsymbol{u}})_{D_i} \cdot \boldsymbol{n} = 0 \quad \text{on } S_i \,,$$

where $(\tilde{\sigma}\tilde{\boldsymbol{u}})_{D_i}$ is understood to take limit of $\tilde{\sigma}\tilde{\boldsymbol{u}}$ from inside D_i. This means $\tilde{\sigma}\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0$ on S_i for all $1 \le i \le M$. ◻

4. Convergence of the approximate solution. The purpose of this section is to study the convergence of the exact solution as $d \to 0$, where d denotes the thickness of cracks. For convenience in notation, we append the solution of (2.13) with a subscript d, namely, $\boldsymbol{u}_d \in \boldsymbol{X}$ denotes the solution of (2.13). We are actually interested in the current density $\mathbf{i}\omega\sigma\boldsymbol{u}_d$ and the magnetic flux $\mathbf{curl}\,\boldsymbol{u}_d$ that are important in non-destructive evaluation. Throughout this section we shall make the following assumption

$$\boldsymbol{J}_s \in \boldsymbol{L}^2(\Omega), \qquad \operatorname{supp}(\boldsymbol{J}_s) \cap D_c = \emptyset, \qquad \operatorname{div} \boldsymbol{J}_s = 0 \quad \text{in } \Omega \,. \tag{4.1}$$

4.1. Convergence for general conductivities. First we present the convergence of the solution for general conductivities. Remember that D_i and Ω_i share the common boundary S_i. Their union constructs a Lipschitz domain (a rectangular domain here, see Fig. 3.2):

$$O_i := D_i \cup S_i \cup \Omega_i = (x_i - H, x_i + d) \times (y_0, y_1) \times (z_0, z_1), \qquad 1 \le i \le M.$$

THEOREM 4.1. *Let* $\boldsymbol{u}_d \in \boldsymbol{X}$ *and* $\tilde{\boldsymbol{u}} \in \tilde{\boldsymbol{X}}$ *be the solutions of* (2.13) *and* (3.13) *respectively. Then*

$$\lim_{d\to 0} \left\{ \omega \left\| \sigma^{\frac{1}{2}} (\boldsymbol{u}_d - \tilde{\boldsymbol{u}}) \right\|^2_{\boldsymbol{L}^2(\Omega)} + \nu_0 \left\| \widetilde{\mathbf{curl}}\, (\boldsymbol{u}_d - \tilde{\boldsymbol{u}}) \right\|^2_{\boldsymbol{L}^2(\Omega)} \right\} = 0. \tag{4.2}$$

Proof. For any $\varphi \in H^1_{\partial D_i \setminus \overline{S}_i}(D_i)$, by the extension theorem [16, Theorem 1.4.3.1, p.25], there exists an extension of φ denoted by $\tilde{\varphi} \in H^1_0(O_i)$ such that

$$\tilde{\varphi} = \varphi \quad \text{in } D_i, \qquad \|\tilde{\varphi}\|_{H^1(O_i)} \le C \|\varphi\|_{H^1(D_i)},$$

where the constant $C > 0$ only depends on D_i and Ω_i. Then we extend $\tilde{\varphi}$ by zero to the exterior of O_i such that the extension $\tilde{\varphi} \in H^1_0(\Omega)$. Since $\mathrm{supp}(\boldsymbol{J}_s) \cap D_c = \emptyset$ and $\sigma = 0$ in Ω_i, taking $\boldsymbol{v} = \nabla \tilde{\varphi}$ in (2.14) leads to

$$\int_{D_i} \sigma \boldsymbol{u}_d \cdot \nabla \varphi = \int_{\Omega} \sigma \boldsymbol{u}_d \cdot \nabla \tilde{\varphi} = 0 \qquad \forall\, \varphi \in H^1_{\partial D_i \setminus \overline{S}_i}(D_i). \tag{4.3}$$

Adding (4.3) to (2.14) for all $1 \le i \le M$, we have

$$\mathbf{i}\omega(\sigma \boldsymbol{u}_d, \boldsymbol{v}) + \nu_0 (\widetilde{\mathbf{curl}}\, \boldsymbol{u}_d, \widetilde{\mathbf{curl}}\, \boldsymbol{v}) = (\boldsymbol{J}_s, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) + \sum_{i=1}^{M} \boldsymbol{U}_i, \tag{4.4}$$

where we have used the fact that $\mathbf{curl}\, \boldsymbol{u}_d = \widetilde{\mathbf{curl}}\, \boldsymbol{u}_d$.

Subtracting (4.4) from (3.15) shows that, for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) + \sum_{i=1}^{M} \boldsymbol{U}_i$,

$$\mathbf{i}\omega \int_{\Omega_c} \sigma(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d) \cdot \boldsymbol{v} + \nu_0 \int_{\Omega} \widetilde{\mathbf{curl}}\, (\tilde{\boldsymbol{u}} - \boldsymbol{u}_d) \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v} = -\mathbf{i}\omega \sum_{i=1}^{M} \int_{\Omega_i} \tilde{\sigma} \tilde{\boldsymbol{u}} \cdot \boldsymbol{v}\,.$$

Taking $\boldsymbol{v} = \tilde{\boldsymbol{u}} - \boldsymbol{u}_d$ and using $\mathrm{supp}(\sigma) = \overline{\Omega}_c$, we find that

$$\left\| \sigma^{\frac{1}{2}} (\tilde{\boldsymbol{u}} - \boldsymbol{u}_d) \right\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\nu_0}{\omega} \left\| \widetilde{\mathbf{curl}}\, (\tilde{\boldsymbol{u}} - \boldsymbol{u}_d) \right\|^2_{\boldsymbol{L}^2(\Omega)} \le \sum_{i=1}^{M} \left| \int_{\Omega_i} \tilde{\sigma} \tilde{\boldsymbol{u}} \cdot (\tilde{\boldsymbol{u}} - \boldsymbol{u}_d) \right|. \tag{4.5}$$

By Theorem 2.2 and 3.4, both $\|\boldsymbol{u}_d\|_{\boldsymbol{L}^2(\Omega)}$ and $\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}$ are uniformly bounded with respect to d. Thus

$$\lim_{d \to 0} \int_{\Omega_i} \tilde{\sigma} \tilde{\boldsymbol{u}} \cdot (\tilde{\boldsymbol{u}} - \boldsymbol{u}_d) = \lim_{|\Omega_i| \to 0} \int_{\Omega_i} \tilde{\sigma} \tilde{\boldsymbol{u}} \cdot (\tilde{\boldsymbol{u}} - \boldsymbol{u}_d) = 0,$$

where $|\Omega_i|$ stands for the volume of Ω_i for any $1 \le i \le M$. This proves (4.2). ◻

4.2. Error estimate for constant conductivity. Theorem 4.1 only gives the convergence of the solution $\boldsymbol{u}_d$. Since we can not expect $d \to 0$ in practice, an error estimate for $\tilde{\boldsymbol{u}} - \boldsymbol{u}_d$ in terms of d will help to evaluate the approximate solution better. The proof for the error estimate depends on the assumption that $\sigma \equiv \sigma_0$ in Ω_c. In this case, the modified conductivity is defined as follows

$$\tilde{\sigma} = \sigma_0 \quad \text{in } \; D_c \; , \qquad \tilde{\sigma} = 0 \quad \text{elsewhere} \; .$$

For convenience in notation, we write

$$O_c = D_c \backslash \left(\bar{O}_1 \cup \cdots \cup \bar{O}_M \right) , \qquad O_{nc} = \Omega_0 \cup O_1 \cup \cdots \cup O_M \; .$$

Clearly O_c and O_{nc} are Lipschitz domains and satisfy

$$O_c \subset \Omega_c \subset D_c, \qquad \Omega_{nc} \subset O_{nc} \; .$$

And similar to (2.4), we define

$$H_{\mathrm{c}}^1(O_{nc}) := \left\{ \phi \in H^1(O_{nc}) : \; \phi = 0 \text{ on } \partial\Omega, \;\; \phi = \alpha_i \text{ on } \partial O_i, \;\; 1 \le i \le M \right\} ,$$

where $\alpha_1, \cdots, \alpha_M$ are arbitrary constants. Since $\nabla H_{\mathrm{c}}^1(O_{nc}) \subset \boldsymbol{H}_0(\mathbf{curl}, O_{nc})$, the extension space

$$\boldsymbol{W}(\Omega; O_{nc}) := \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \; \boldsymbol{v}|_{O_{nc}} \in \nabla H_{\mathrm{c}}^1(O_{nc}) \;\text{ and }\; \boldsymbol{v} = 0 \;\text{ in } O_c \right\}$$

is a subspace of $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$. Let the orthogonal complement of $\boldsymbol{W}(\Omega; O_{nc})$ in $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ be denoted by

$$\boldsymbol{X}_1 := \{ \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) : \; (\boldsymbol{v}, \boldsymbol{w}) = 0 \quad \forall \, \boldsymbol{w} \in \boldsymbol{W}(\Omega; O_{nc}) \} \, .$$

By similar arguments as in the proof of Lemma 2.1, $\boldsymbol{X}_1$ is a Hilbert subspace equipped with the inner product and norm

$$(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{X}_1} = \int_{O_c} \boldsymbol{v} \cdot \bar{\boldsymbol{w}} + \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \bar{\boldsymbol{w}}, \qquad \|\boldsymbol{v}\|_{\boldsymbol{X}_1} = \sqrt{(\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{X}_1}} \; . \tag{4.6}$$

And there is a constant C depending on the diameters of $O_1, \cdots, O_M$, but independent of d, such that

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)} \le C \, \|\boldsymbol{v}\|_{\boldsymbol{X}_1} \qquad \forall \, \boldsymbol{v} \in \boldsymbol{X}_1 \; . \tag{4.7}$$

LEMMA 4.2. *There exists a constant $C > 0$ independent of d such that*

$$\sum_{i=1}^M \left| \int_{\Omega_i} \tilde{\boldsymbol{u}} \cdot (\tilde{\boldsymbol{u}} - \boldsymbol{u}_d) \right| \le C \, \|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_1 \cup \cdots \cup \Omega_M)} \, \|\tilde{\boldsymbol{u}} - \boldsymbol{u}_d\|_{\boldsymbol{X}_1}$$

Proof. First we consider the decomposition of the approximate solution

$$\tilde{\boldsymbol{u}} = \hat{\boldsymbol{u}} + \sum_{i=1}^M \boldsymbol{u}_i, \qquad \hat{\boldsymbol{u}} \in \boldsymbol{U}, \quad \boldsymbol{u}_i \in \boldsymbol{U}_i \; .$$

We further split $\hat{\boldsymbol{u}} - \boldsymbol{u}_d$ orthogonally into

$$\hat{\boldsymbol{u}} - \boldsymbol{u}_d = \hat{\boldsymbol{u}}_\perp + \boldsymbol{w}, \qquad \hat{\boldsymbol{u}}_\perp \in \boldsymbol{X}_1, \quad \boldsymbol{w} \in \boldsymbol{W}(\Omega; O_{nc}).$$

And (4.7) shows that

$$\|\hat{\boldsymbol{u}}_\perp\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\,\|\hat{\boldsymbol{u}}_\perp\|_{\boldsymbol{X}_1} = C\,\|\hat{\boldsymbol{u}} - \boldsymbol{u}_d\|_{\boldsymbol{X}_1} = C\,\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_d\|_{\boldsymbol{X}_1}\ .$$

Let $\boldsymbol{w} = \nabla\phi$ in O_{nc} for some $\phi \in H^1_{\mathrm{c}}(O_{nc})$. Then there exists a constant α_i such that $\phi = \alpha_i$ on ∂O_i. And the conservation property in Theorem 3.5 yields

$$\operatorname{div}\tilde{\boldsymbol{u}} = 0 \quad \text{in } \ \Omega_i, \qquad \tilde{\boldsymbol{u}}\cdot\boldsymbol{n} = 0 \quad \text{on } \ S_i, \qquad 1 \le i \le M\ .$$

We deduce that

$$\int_{\Omega_i} \tilde{\boldsymbol{u}}\cdot\boldsymbol{w} = \int_{\Omega_i} \tilde{\boldsymbol{u}}\cdot\nabla\phi = \int_{\partial\Omega_i} (\tilde{\boldsymbol{u}}\cdot\boldsymbol{n})\phi = \alpha_i \int_{\partial\Omega_i\setminus\overline{S}_i} \tilde{\boldsymbol{u}}\cdot\boldsymbol{n} = \alpha_i \int_{\partial\Omega_i} \tilde{\boldsymbol{u}}\cdot\boldsymbol{n} = 0\ .$$

An application of the Cauchy-Schwarz inequality leads to

$$\begin{aligned}\sum_{i=1}^{M}\left|\int_{\Omega_i} \tilde{\boldsymbol{u}}\cdot(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d)\right| = \sum_{i=1}^{M}\left|\int_{\Omega_i} \tilde{\boldsymbol{u}}\cdot\hat{\boldsymbol{u}}_\perp\right| &\le \|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_1\cup\cdots\cup\Omega_M)}\,\|\hat{\boldsymbol{u}}_\perp\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\\ &\le C\,\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_1\cup\cdots\cup\Omega_M)}\,\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_d\|_{\boldsymbol{X}_1}\ .\end{aligned}$$

The proof is completed. □

THEOREM 4.3. *Assume $\tilde{\boldsymbol{u}} \in \boldsymbol{H}^1(D_c\setminus\cup_{i=1}^M \overline{D}_i)$. Then there exists a constant $C > 0$ depending on ω, σ_0, and ν_0, but independent of d such that*

$$\|\boldsymbol{u}_d - \tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_c)} + \|\widetilde{\mathbf{curl}}\,(\boldsymbol{u}_d - \tilde{\boldsymbol{u}})\|_{\boldsymbol{L}^2(\Omega)} \le C d^{\frac{1}{3}}.$$

Proof. From (4.5) we know that

$$\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_d\|^2_{\boldsymbol{L}^2(\Omega_c)} + \frac{\nu_0}{\omega\sigma_0}\,\|\widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d)\|^2_{\boldsymbol{L}^2(\Omega)} \le \sum_{i=1}^{M}\left|\int_{\Omega_i} \tilde{\boldsymbol{u}}\cdot(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d)\right|\ . \tag{4.8}$$

Since $O_c \subset \Omega_c$, it is clear that

$$\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_d\|_{\boldsymbol{X}_1} \le \left(\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_d\|^2_{\boldsymbol{L}^2(\Omega_c)} + \|\widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d)\|^2_{\boldsymbol{L}^2(\Omega)}\right)^{\frac{1}{2}}\ .$$

Recall the imbedding $\boldsymbol{L}^6(D) \subset \boldsymbol{H}^1(D)$ for any Lipschitz domain D. There exists a constant C independent of d such that

$$\begin{aligned}\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_i)} &\le |\Omega_i|^{\frac{1}{3}}\,\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^6(D_c\setminus\cup_{i=1}^M\overline{D}_i)} \le C d^{\frac{1}{3}}\,\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^6(D_c\setminus\cup_{i=1}^M\overline{D}_i)}\\ &\le C d^{\frac{1}{3}}\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{H}^1(D_c\setminus\cup_{i=1}^M\overline{D}_i)} \le C d^{\frac{1}{3}}.\end{aligned}$$

An application of Lemma 4.2 shows that

$$\sum_{i=1}^{M}\left|\int_{\Omega_i} \tilde{\boldsymbol{u}}\cdot(\tilde{\boldsymbol{u}} - \boldsymbol{u})\right| \le C d^{\frac{1}{3}}\left(\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_d\|^2_{\boldsymbol{L}^2(\Omega_c)} + \|\widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d)\|^2_{\boldsymbol{L}^2(\Omega)}\right)^{\frac{1}{2}}. \tag{4.9}$$

The proof is completed upon combining (4.8) and (4.9). □

REMARK 4.4. *The assumption $\tilde{\boldsymbol{u}} \in \boldsymbol{H}^1(D_c\setminus\cup_{i=1}^M \overline{D}_i)$ in Theorem 4.3 is about the regularity of the approximate solution. We are not able to prove the assumption at present. It seems reasonable since $\tilde{\boldsymbol{u}} \in \boldsymbol{H}(\mathbf{curl}, D_c\setminus\cup_{i=1}^M \overline{D}_i)$ and*

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \ D_c, \qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \ \partial D_c\cup S_1\cup\cdots\cup S_M.$$

5. Finite element approximation. The purpose of this section is to study the finite element approximation to problem (3.13). Let $\mathcal{T}_h$ be a tetrahedral triangulation of Ω which also subdivides D_c and D_i, $1 \le i \le M$ into the union of tetrahedra. Now we introduce the lowest-order Lagrange finite element space [12] and Nédélec's edge element space of the first class [23] as follows

$$H(\mathrm{grad}_h, \Omega) = \left\{ v \in H^1(\Omega) : v|_K = \boldsymbol{a} \cdot \boldsymbol{x} + b \ \text{ with } \boldsymbol{a} \in \mathbb{R}^3, b \in \mathbb{R}^1 \text{ for any } K \in \mathcal{T}_h \right\},$$
$$\boldsymbol{H}(\mathbf{curl}_h, \Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \boldsymbol{v}|_K = \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b} \ \text{ with } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3 \text{ for any } K \in \mathcal{T}_h \right\}.$$

The finite element spaces satisfying homogeneous boundary conditions are defined by

$$H_0(\mathrm{grad}_h, \Omega) = H(\mathrm{grad}_h, \Omega) \cap H_0^1(\Omega),$$
$$\boldsymbol{H}_0(\mathbf{curl}_h, \Omega) = \boldsymbol{H}(\mathbf{curl}_h, \Omega) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega).$$

Similar to (3.12) and (3.11), we define the finite element spaces

$$\boldsymbol{U}_h = \left\{ \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}_h, \Omega) : (\boldsymbol{v}, \nabla\varphi) = 0 \ \ \forall \varphi \in H_0(\mathrm{grad}_h, \Omega), \ \varphi|_{\overline{D}_c} \equiv \text{Const.} \right\},$$
$$\boldsymbol{X}_h = \boldsymbol{U}_h + \sum_{i=1}^{M} \boldsymbol{U}_{i,h}, \qquad \boldsymbol{U}_{i,h} = \{ \boldsymbol{v} \in \boldsymbol{U}_i : \ \boldsymbol{v}|_{D_i} \in \nabla H(\mathrm{grad}_h, D_i) \}.$$

Lemma 5.1. *The space $\boldsymbol{X}_h$ admits the decomposition in a direct sum*

$$\boldsymbol{X}_h = \tilde{\boldsymbol{U}}_h + \sum_{i=1}^{M} \boldsymbol{U}_{i,h}, \qquad \tilde{\boldsymbol{U}}_h = \left\{ \boldsymbol{v} \in \boldsymbol{U}_h : \ (\boldsymbol{v}, \boldsymbol{\xi}) = 0 \quad \forall \boldsymbol{\xi} \in \sum_{i=1}^{M} \boldsymbol{U}_{i,h} \right\}. \tag{5.1}$$

And $\|\cdot\|_{\tilde{\boldsymbol{X}}}$ is a norm on $\boldsymbol{X}_h$.

Proof. We first prove $\boldsymbol{X}_h = \tilde{\boldsymbol{U}}_h + \sum_{i=1}^M \boldsymbol{U}_{i,h}$. It suffices to prove

$$\boldsymbol{U}_h \subset \tilde{\boldsymbol{U}}_h + \sum_{i=1}^{M} \boldsymbol{U}_{i,h}.$$

Define

$$H_{\partial D_i \setminus \overline{S}_i}(\mathrm{grad}_h, D_i) := \left\{ v \in H(\mathrm{grad}_h, D_i) : \ v = 0 \ \text{ on } \partial D_i \setminus \overline{S}_i \right\}.$$

For any $\boldsymbol{v} \in \boldsymbol{U}_h$ and $1 \le i \le M$, let $\phi_i \in H_{\partial D_i \setminus \overline{S}_i}(\mathrm{grad}_h, D_i)$ be the unique solution of the discrete problem

$$\int_{D_i} \nabla\phi_i \cdot \nabla\varphi = \int_{D_i} \boldsymbol{v} \cdot \nabla\varphi \qquad \forall \varphi \in H_{\partial D_i \setminus \overline{S}_i}(\mathrm{grad}_h, D_i).$$

We extend $\nabla\phi_i$ by zero to the exterior of D_i and denote the extension by $\boldsymbol{v}_i$. Then $\boldsymbol{v}_i \in \boldsymbol{U}_{i,h}$ and $\hat{\boldsymbol{v}} = \boldsymbol{v} - \sum_{i=1}^M \boldsymbol{v}_i \in \tilde{\boldsymbol{U}}_h$. This indicates $\boldsymbol{v} \in \tilde{\boldsymbol{U}}_h + \sum_{i=1}^M \boldsymbol{U}_{i,h}$. So $\boldsymbol{U}_h \subset \tilde{\boldsymbol{U}}_h + \sum_{i=1}^M \boldsymbol{U}_{i,h}$.

Now we prove that $\|\cdot\|_{\tilde{\boldsymbol{X}}}$ is a norm on $\boldsymbol{X}_h$. Take any $\boldsymbol{v} \in \boldsymbol{X}_h$ satisfying $\|\boldsymbol{v}\|_{\tilde{\boldsymbol{X}}} = 0$. Write $\boldsymbol{v} = \hat{\boldsymbol{v}} + \sum_{i=1}^M \boldsymbol{v}_i$ with $\hat{\boldsymbol{v}} \in \tilde{\boldsymbol{U}}_h$ and $\boldsymbol{v}_i \in \boldsymbol{U}_{i,h}$. Then (3.7) and the definition of $\tilde{\boldsymbol{U}}_h$ show that

$$\mathbf{curl}\, \hat{\boldsymbol{v}} = 0 \ \text{ in } \Omega, \qquad \hat{\boldsymbol{v}} + \boldsymbol{v}_i = 0 \ \text{ in } D_i, \qquad \hat{\boldsymbol{v}} = 0 \ \text{ in } D_c \setminus \left(\overline{D}_1 \cup \cdots \cup \overline{D}_M \right).$$

The first equality indicates that $\hat{\boldsymbol{v}} = \nabla\phi$ for some $\phi \in H_0(\mathrm{grad}_h, \Omega)$. The second equality and the definition of $\tilde{\boldsymbol{U}}_h$ yield

$$\|\hat{\boldsymbol{v}}\|^2_{\boldsymbol{L}^2(D_i)} = -\int_{D_i} \hat{\boldsymbol{v}} \cdot \boldsymbol{v}_i = 0, \qquad 1 \le i \le M.$$

Together with the third equality, it yields $\hat{\boldsymbol{v}} = 0$ in D_c. Thus $\phi = \text{Const.}$ in D_c. The definition of $\boldsymbol{U}_h$ shows that

$$(\hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}) = (\hat{\boldsymbol{v}}, \nabla\phi) = 0.$$

This shows $\hat{\boldsymbol{v}} \equiv 0$ and thus $\boldsymbol{v}_i \equiv 0$ for all $1 \le i \le M$. And $\|\cdot\|_{\tilde{\boldsymbol{X}}}$ is a norm on $\boldsymbol{X}_h$.

To prove the direct sum, we assume that $\hat{\boldsymbol{v}} + \sum_{i=1}^M \boldsymbol{v}_i = 0$ with $\hat{\boldsymbol{v}} \in \tilde{\boldsymbol{U}}_h$ and $\boldsymbol{v}_i \in \boldsymbol{U}_{i,h}$. By the arguments in the previous paragraph, we know that $\hat{\boldsymbol{v}} \equiv 0$ and $\boldsymbol{v}_i \equiv 0$ for all $1 \le i \le M$. So (5.1) is a direct sum. □

The finite element approximation to (3.13) reads: Find $\boldsymbol{u}_h \in \boldsymbol{X}_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{J}_s, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{X}_h. \tag{5.2}$$

REMARK 5.2. *The orthogonality in $\tilde{\boldsymbol{U}}_h$ and $\boldsymbol{U}_h$ is only used in theoretical analysis, not in practical computations. Notice that*

$$\boldsymbol{X}_h = \tilde{\boldsymbol{U}}_h + \sum_{i=1}^M \boldsymbol{U}_{i,h} = \boldsymbol{H}_0(\mathbf{curl}_h, \Omega) + \sum_{i=1}^M \boldsymbol{U}_{i,h}\,.$$

We only solve an $\boldsymbol{a}_h \in \boldsymbol{H}_0(\mathbf{curl}_h, \Omega)$ and a $\phi_{i,h} \in H_{\partial D_i \setminus \bar{S}_i}(\mathrm{grad}_h, D_i)$ locally in each D_i such that $\boldsymbol{u}_h = \boldsymbol{a}_h + \sum_{i=1}^M \nabla\phi_{i,h}$ by alternating iteration method (see Algorithm 5.3). Although $\boldsymbol{a}_h$ and $\phi_{i,h}$ are not unique, the sum $\boldsymbol{a}_h + \sum_{i=1}^M \nabla\phi_{i,h}$ is unique. Thus the magnetic flux $\boldsymbol{B}_h = \widetilde{\mathbf{curl}}\,\boldsymbol{u}_h = \mathbf{curl}\,\boldsymbol{a}_h$ and the current density $\boldsymbol{J}_h = \tilde{\sigma}\boldsymbol{u}_h$ are unique.

ALGORITHM 5.3 (Alternating Iteration Method). Given the tolerance $\epsilon = 10^{-4}$ and the maximal iteration number $N > 0$. Set the initial guess by $\boldsymbol{a}^{(0)} = 0$ and $\phi_1^{(0)} = 0$, $\phi_2^{(0)} = 0$, $\cdots$, $\phi_M^{(0)} = 0$. Set $k = 0$ and $r = 1$.
While ($r > \epsilon$ and $k < N$) do

1. Solve the Maxwell equation: Find $\boldsymbol{a}^{(k+1)} \in \boldsymbol{H}_0(\mathbf{curl}_h, \Omega)$ such that

$$a\big(\boldsymbol{a}^{(k+1)}, \boldsymbol{v}\big) = (\boldsymbol{J}_s, \boldsymbol{v}) - \mathbf{i}\omega \sum_{i=1}^M \int_{D_i} \tilde{\sigma}\nabla\phi_i^{(k)} \cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}_h, \Omega).$$

2. Solve the Poisson equations: Find $\phi_i^{(k+1)} \in H_{\partial D_i \setminus \bar{S}_i}(\mathrm{grad}_h, D_i)$ such that

$$\int_{D_i} \tilde{\sigma}\nabla\phi_i^{(k+1)} \cdot \nabla\varphi = -\int_{D_i} \tilde{\sigma}\boldsymbol{a}^{(k+1)} \cdot \nabla\varphi \qquad \forall\, \varphi \in H_{\partial D_i \setminus \bar{S}_i}(\mathrm{grad}_h, D_i),$$

for all $i = 1, 2, \cdots, M$.

3. Compute the relative error

$$r = \left\|\boldsymbol{a}^{(k)}\right\|^{-1}_{\boldsymbol{H}(\mathbf{curl},\Omega)} \left\|\boldsymbol{a}^{(k+1)} - \boldsymbol{a}^{(k)}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \sum_{i=1}^M \left\|\phi_i^{(k)}\right\|^{-1}_{H^1(D_i)} \left\|\phi_i^{(k+1)} - \phi_i^{(k)}\right\|_{H^1(D_i)}.$$

End while.

THEOREM 5.4. *The discrete problem* (5.2) *has a unique solution* $\boldsymbol{u}_h \in \boldsymbol{U}_h$ *and there exists a generic constant* $C > 0$ *independent of* $\mathcal{T}_h$ *and* d *such that*

$$\|\boldsymbol{u}_h\|_{\tilde{\boldsymbol{X}}} \le C \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)} . \tag{5.3}$$

Let $\tilde{\boldsymbol{u}}$ *be the solution of* (3.13)*. Then*

$$\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h\|_{\tilde{\boldsymbol{X}}} \le C \inf_{\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}_h,\Omega)+\sum_{i=1}^M \boldsymbol{U}_{i,h}} \|\tilde{\boldsymbol{u}} - \boldsymbol{v}_h\|_{\tilde{\boldsymbol{X}}} . \tag{5.4}$$

Proof. First we write $\boldsymbol{u}_h = \hat{\boldsymbol{u}}_h + \sum_{i=1} \boldsymbol{u}_{i,h}$ with $\hat{\boldsymbol{u}} \in \tilde{\boldsymbol{U}}_h$ and $\boldsymbol{u}_{i,h} \in \boldsymbol{U}_{i,h}$. The Helmholtz decomposition of $\hat{\boldsymbol{u}}_h$ yields

$$\hat{\boldsymbol{u}}_h = \nabla\phi + \hat{\boldsymbol{u}}_\perp, \qquad \phi \in H_0^1(\Omega), \quad \hat{\boldsymbol{u}}_\perp \in \boldsymbol{H}_0(\mathbf{curl},\Omega) \text{ satisfying } \operatorname{div}\hat{\boldsymbol{u}}_\perp = 0.$$

By the imbedding theorem in [4], there is a constant C depending only on Ω such that

$$\begin{aligned}\|\hat{\boldsymbol{u}}_\perp\|_{\boldsymbol{H}^1(\Omega)} &\le C\left[\|\mathbf{curl}\,\hat{\boldsymbol{u}}_\perp\|_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div}\hat{\boldsymbol{u}}_\perp\|_{L^2(\Omega)}\right] = C\|\mathbf{curl}\,\hat{\boldsymbol{u}}_\perp\|_{\boldsymbol{L}^2(\Omega)} \\ &= C\|\mathbf{curl}\,\hat{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)} = C\|\widetilde{\mathbf{curl}}\,\boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)} \le C\|\boldsymbol{u}_h\|_{\tilde{\boldsymbol{X}}} .\end{aligned}$$

Recall that $\operatorname{supp}(\boldsymbol{J}_s) \cap D_c = \emptyset$ and $\operatorname{div}\boldsymbol{J}_s = 0$. Then for any $\boldsymbol{v} \in \boldsymbol{X}_h$,

$$|(\boldsymbol{J}_s, \boldsymbol{u}_h)| = |(\boldsymbol{J}_s, \hat{\boldsymbol{u}}_\perp)| \le \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)} \|\hat{\boldsymbol{u}}_\perp\|_{\boldsymbol{L}^2(\Omega)} \le C\|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}_h\|_{\tilde{\boldsymbol{X}}} .$$

Then (5.3) is proved upon taking $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (5.2) and using the coercivity of a with respect to $\|\cdot\|_{\tilde{\boldsymbol{X}}}$. The uniqueness and existence of $\boldsymbol{u}_h$ also follows.

Now we are going to prove the error estimate. Remember from (3.15) that

$$a(\tilde{\boldsymbol{u}}, \boldsymbol{v}) = (\boldsymbol{J}_s, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl},\Omega) + \sum_{i=1}^M \boldsymbol{U}_i .$$

By the definitions of $\boldsymbol{X}_h$ and similar arguments as in deriving (3.15), it is easy to verify that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{J}_s, \boldsymbol{v}_h) \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}_h,\Omega) + \sum_{i=1}^M \boldsymbol{U}_{i,h} .$$

Since $\boldsymbol{H}_0(\mathbf{curl}_h,\Omega) \subset \boldsymbol{H}_0(\mathbf{curl},\Omega)$ and $\boldsymbol{U}_{i,h} \subset \boldsymbol{U}_i$, the above two equalities yield the Galerkin orthogonality:

$$a(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h, \boldsymbol{v}_h) = 0 \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}_h,\Omega) + \sum_{i=1}^M \boldsymbol{U}_{i,h} . \tag{5.5}$$

Then for any $\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}_h,\Omega) + \sum_{i=1}^M \boldsymbol{U}_{i,h}$, it follows that

$$|a(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h, \tilde{\boldsymbol{u}} - \boldsymbol{u}_h)| = |a(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h, \tilde{\boldsymbol{u}} - \boldsymbol{v}_h)| \le C\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h\|_{\tilde{\boldsymbol{X}}} \|\tilde{\boldsymbol{u}} - \boldsymbol{v}_h\|_{\tilde{\boldsymbol{X}}} .$$

The proof is completed by the coercivity of $a(\cdot,\cdot)$ with respect to $\|\cdot\|_{\tilde{\boldsymbol{X}}}$. □

In the next theorem, we shall assume that $\tilde{\boldsymbol{u}}$ is smooth enough to obtain the optimal error estimate. The assumption may be too strong to hold for singular sources. But nevertheless, the theorem justifies the space approximation of $\boldsymbol{X}_h$ to $\tilde{\boldsymbol{X}}$.

THEOREM 5.5. *Let $\tilde{\boldsymbol{u}}$, $\boldsymbol{u}_h$ be the solution of* (3.13) *and* (5.2) *respectively. Suppose there exists a decomposition of $\tilde{\boldsymbol{u}}$ such that*

$$\tilde{\boldsymbol{u}} = \boldsymbol{u}_\perp + \sum_{i=1}^{M} \boldsymbol{u}_i, \qquad \boldsymbol{u}_\perp \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega), \quad \boldsymbol{u}_i \in \boldsymbol{H}^1(D_i) \cap \boldsymbol{U}_i.$$

Then with a constant C independent of h and d, the error estimate holds

$$\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h\|_{\tilde{\boldsymbol{X}}} \le Ch. \tag{5.6}$$

Proof. Let $\boldsymbol{\Pi}_h$: $\boldsymbol{H}^2(\Omega) \mapsto \boldsymbol{H}(\mathbf{curl}_h, \Omega)$ be the interpolation operator of Nédélec's edge elements [23] and let $\pi_{i,h}$: $H^2(D_i) \mapsto H(\mathrm{grad}_h, D_i)$ be the interpolation operator of Lagrange nodal elements [12]. Suppose $\boldsymbol{u}_i = \nabla\psi_i$ in D_i for some $\psi_i \in H^2(D_i) \cap H^1_{\partial D_i \setminus \overline{S}_i}(D_i)$. It follows that

$$\boldsymbol{\Pi}_h \boldsymbol{u}_\perp + \sum_{i=1}^{M} \nabla(\pi_{i,h}\psi_i) \in \boldsymbol{H}_0(\mathbf{curl}_h, \Omega) + \sum_{i=1}^{M} \boldsymbol{U}_{i,h}\ .$$

By the Galerkin orthogonality in (5.5) we deduce that

$$\begin{aligned} \|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h\|_{\tilde{\boldsymbol{X}}} &\le C\,\|\boldsymbol{u}_\perp - \boldsymbol{\Pi}_h \boldsymbol{u}_\perp\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + C\sum_{i=1}^{M} |\psi_i - \pi_{i,h}\psi_i|_{H^1(D_i)} \\ &\le Ch\left[|\boldsymbol{u}_\perp|_{\boldsymbol{H}^2(\Omega)} + \sum_{i=1}^{M} |\boldsymbol{u}_i|_{\boldsymbol{H}^1(D_i)} \right]. \end{aligned}$$

The proof is completed. ◻

6. Numerical experiments. The purpose of this section is to validate the approximation of the approximate eddy current model (3.13) to the original eddy current problem (2.13) numerically.

EXAMPLE 6.1. *We consider the TEAM Workshop Problem* 21^a-2 *which is one benchmark problem from the International Compumag Society. The system consists of one nonmagnetic plate with two narrow slits and two coils which carry the source currents in opposite directions. The source current is 3000 Ampere/Turn and its frequency is* $\omega = 50$ *Hertz. The geometry of the model is illustrated in Fig. 6.1. We refer to [9] for more details of this benchmark problem.*

Our implement is based on the adaptive finite element package "Parallel Hierarchical Grid" (PHG) [28] and the computations are carried out on the cluster LSEC-III of the State Key Laboratory on Scientific and Engineering Computing, Chinese Academy of Sciences.

Fig. 6.2 shows the values of B_x along the line

$$\{(x, y, z) :\ x = 5.76\,\mathrm{mm}, \ \ y = 0\,\mathrm{mm}\}.$$

The three curves stand for the calculated values using the original model (2.3), the calculated values using the approximate model (3.13), and the experimental values.

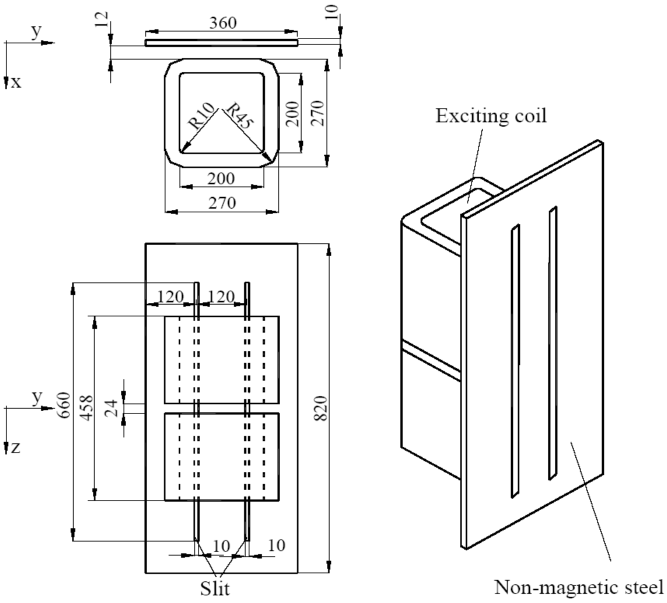

FIG. 6.1. *Schematic diagram of Team Workshop Problem* 21^a-2.

It is clear that the calculated magnetic flux density using both eddy current models agree very well with the measured data. Thus we conclude that the approximate model (3.13) provides an accurate approximation to the original problem (2.3).

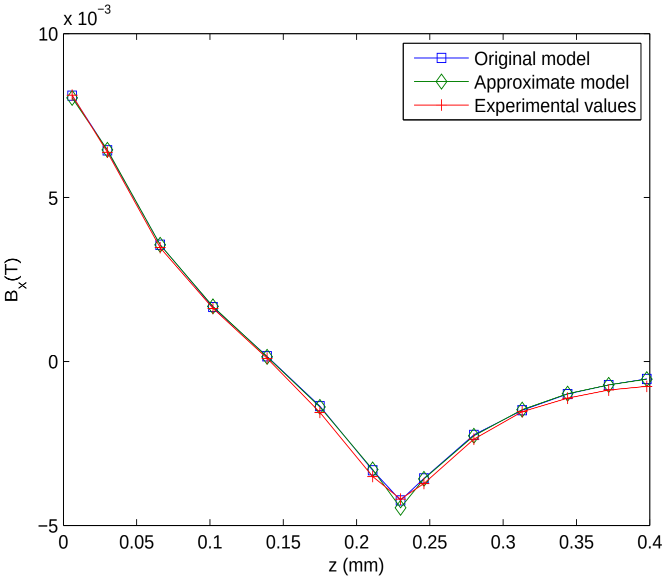

FIG. 6.2. *Magnetic flux density* B_x *evaluated along the line* $\{(x, y, z) : x = 5.76\,\text{mm},\ y = 0\,\text{mm}\}$. *The three curves represent respectively:* (1) *calculated values using the original model* (2.3)*;* (2) *calculated values using the approximate model* (3.13)*;* (3) *experimental values.*

Fig. 6.3 shows the distributions of the eddy current density on a parallel intersection of the steel plate. The left figure represents the eddy current density calculated from the original problem (2.3). It shows that the eddy currents are prevented by the slits and do not flow across them. While the right one represents the eddy current density calculated from the approximate problem (3.13). In this case, the two slits Ω_1, Ω_2 are replaced by two interfaces S_1 and S_2. Clearly the eddy currents do not flow across the two interfaces. This validates the conservation property of approximate solution (see Lemma 3.5)

$$\tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad S_1 \cup S_2 \, .$$

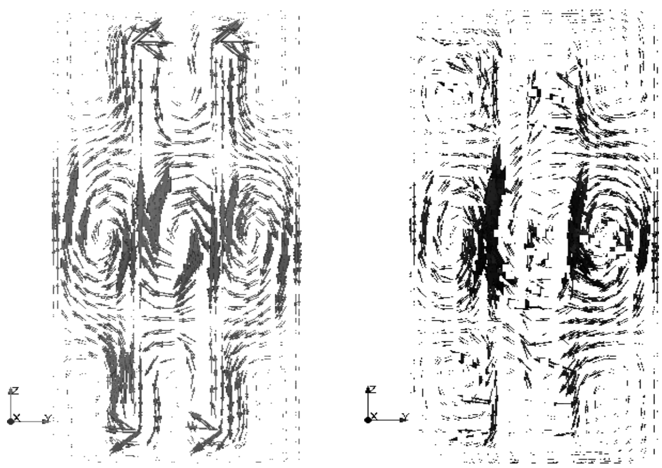

FIG. 6.3. *Eddy current distributions. Left: the original eddy current problem. Right: the approximate eddy current problem.*

EXAMPLE 6.2. *We consider the TEAM Workshop Problem* 15 *from the International Compumag Society. The system consists of one thick conducting plate with a rectangular slot in the plate and a single air-cored AC coil. The source current is* 1 *Ampere/Turn and the frequency is* $\omega = 50$ *Hertz. The geometry of the model is illustrated in Fig. 6.4. The parameters for this test experiment are listed in Table 6.1. This problem is completely described in [8].*

This benchmark problem is used to test numerical methods for nondestructive evaluation. Here we just use the setting of the problem and validate the approximation of the approximate model (3.13) to the original model (2.3). Fig. 6.5 shows the values of B_z along the line

$$\{(x, y, z) : \; y = 0\,\text{mm}, \;\; z = 0.5\,\text{mm}\}.$$

The two curves stand for the calculated values using the original model (2.3) and the approximate model (3.13). We find that the values computed by the approximate model agree with those computed by the original model.

Fig. 6.6 shows the distribution of the eddy current density on the intersection, $z = -2.5$mm, of the steel plate. Clearly the intersection plane is orthogonal to the crack whose normal direction is parallel to the y-direction. Fig. 6.7 shows the distribution of J_y which is the component of the current density in the normal direction to the crack. The left figures of Fig. 6.6 and Fig. 6.7 represent the eddy current density calculated

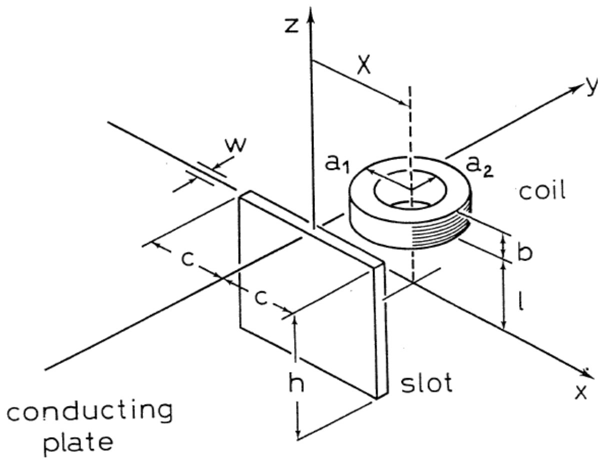

FIG. 6.4. *Schematic diagram of Team Workshop Problem* 15.

TABLE 6.1
Geometric and physical parameters for Example 6.2. (see Fig. 6.4)

The coil	
Inner radius	$a_2 = 6.15$ mm
Outer radius	$a_1 = 12.4$ mm
Length	$b = 6.15$ mm
Number of turns	$N = 3790$
Lift-off	$l = 0.88$ mm
The test specimen	
Conductivity	$\sigma = 3.06 \times 10^7$ S/m
Thickness	12.22 mm
The defect	
Length	$2c = 12.60$ mm
Depth	$h = 5.00$ mm
Width	$w = 0.34$ mm

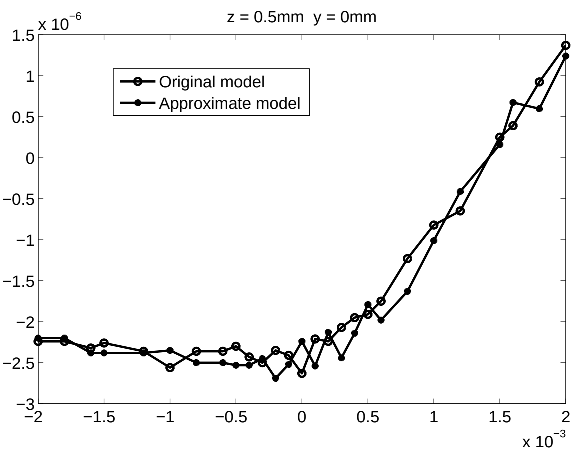

FIG. 6.5. *The values of B_z at a series of points on the line $y = 0$mm, $z = 0.5$mm.*

with the original problem (2.3). We find that the eddy currents are prevented by the crack. While the right figures of Fig. 6.6 and Fig. 6.7 represent the eddy current density calculated with the approximate problem (3.13). In this case, the crack is replaced by an interface. Clearly the eddy currents do not flow across the interface. This validates the conservation property of the approximate solution, that is, $\tilde{\boldsymbol{u}}\cdot\boldsymbol{n} = 0$ on the interface.

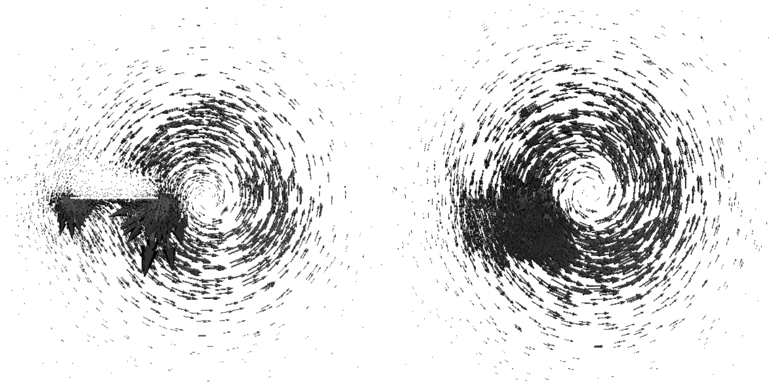

FIG. 6.6. *The illustrations of the current density on the cross-section* $z = -2.5\,\mathrm{mm}$. *Left: computed with the original model. Right: computed with the approximate model.*

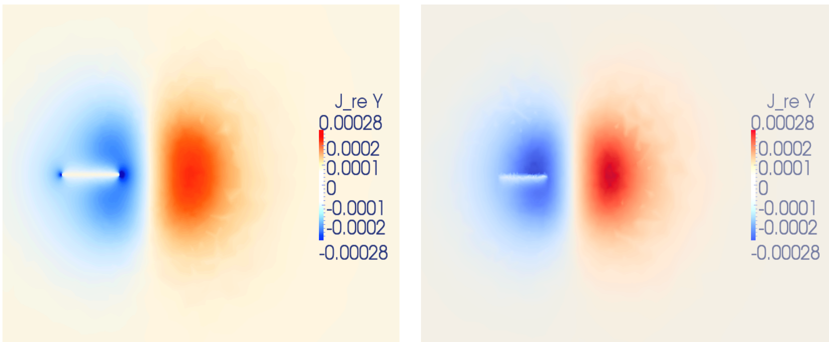

FIG. 6.7. *The distribution of* J_y*, the component of the current density in the direction perpendicular to the crack, on the cross-section* $z = -2.5\,\mathrm{mm}$. *Left: computed with the original model. Right: computed with the approximate model.*

Acknowledgement. The authors thank the anonymous referees very much for their valuable suggestions and comments on this paper, particularly, for the suggestions on Example 6.2 and boundary condition (2.1).

REFERENCES

[1] R. ACEVEDO, S. MEDDAHI, R. RODRÍGUEZ, *An E-based mixed formulation for a time-dependent eddy current problem*, Math. Comp. 78 (2009), pp. 1929-1949.

[2] H. AMMARI, A. BUFFA, AND J. NÉDÉLEC, *A justification of eddy current model for the maxwell equations*, SIAM J. Appl. Math., 60 (2000), pp. 1805–1823.

[3] H. AMMARI, J. CHEN, Z. CHEN, J. GARNIER, AND D. VOLKOV, *Target detection and characterization from electromagnetic induction data*, J. Math. Pures Appl., 101 (2014) pp. 5475.

[4] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potentials in three-dimensional non-smooth domains*, Math. Meth. Appl. Sci., 21 (1998), pp. 823–864.
[5] F. Bachinger, U. Langer, J. Schöberl, *Numerical Analysis of Nonlinear Multiharmonic Eddy Current Problems*, Numer. Math., 100 (2005), pp. 593-616.
[6] Zsolt Badics, Hidenobu Komatsu, Yoshihiro Matsumoto, Kazuhiko Aoki, Fumio Nakayasu, *A Thin Sheet Finite Element Crack Model in Eddy Current NDE*, IEEE Transactions on Magnetics, 30 (1994), pp. 3080-3083.
[7] R. Beck, R. Hiptmair, R. Hoppe and B. Wohlmuth, *Residual based a posteriori error estimators for eddy current computation*, Math. Model. Numer. Anal. 34 (2000), pp. 159–182.
[8] S.K. Burke, *A benchmark problem for computation of ΔZ in eddy-current nondestructive evaluation (NDE)*, J. Nondestructive Evaluation, 7 (1988), No. 1-2, pp. 35–41. TEAM Workshop Problem 15: http://www.compumag.org/jsite/team.html
[9] Z. Cheng, N. Takahashi, and B. Forghani, *TEAM Problem 21 Family (V. 2009)*, approved by the International Compumag Society Board at Compumag-2009, Florianópolis, Brazil, http://www.compumag.org/jsite/team.html .
[10] J. Chen, Z. Chen, T. Cui and L. Zhang, *An adaptive finite element method for the eddy current model with circuit/field couplings*, SIAM J. Sci. Comput., 32 (2010), pp. 1020-1042.
[11] Y. Choua, L. Santandrea, Y. Le Bihan, and C. Marchand, *Thin crack modeling in ECT with combined potential formulations*, IEEE Transactions on Magnetics, 43 (2007), pp. 1789-1792.
[12] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Vol. 4 of Studies in Mathematics and its Applications, North-Holland, Amsterdam, 1978.
[13] M. Costabel, M. Dauge, and S. Nicaise, *Singularities of eddy current problems*, ESAIM: Mathematical Modelling and Numerical Analysis, 37 (2003), pp. 807–831.
[14] Patrick Dular and Christophe Geuzaine, *Modeling of thin insulating layers with dual 3-D magnetodynamic formulations*, IEEE Transactions on Magnetics, 39 (2003), pp. 1139-1142.
[15] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, Heidelberg, 1986.
[16] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.
[17] R. Hamia, C. Cordier, S. Saez, and C. Dolabdjian, *Eddy-current nondestructive testing using an improved GMR magnetometer and a singleWire as Inducer: a FEM performance analysis*, IEEE Trans. Magn., 46 (2010), no. 10, pp. 3731-3737.
[18] R. Hiptmair, *Analysis of multilevel methods for eddy current problems*, Math. Comp., 72 (2002), pp. 1281–1303.
[19] X. Jiang and W. Zheng, *An efficient eddy current model for nonlinear Maxwell equations with laminated conductors*, SIAM Journal on Applied Mathematics, 72 (2012), pp. 1021-1040.
[20] X. Jiang and W. Zheng, *Homogenization of quasi-static Maxwell's equations*, Multiscal Modeling and Simulation: A SIAM Interdisciplinary Journal, 12 (2014). pp. 152-180.
[21] P.D. Ledger, and S. Zaglmayr, *hp-Finite element simulation of three-dimensional eddy current problems on multiply connected domains*, Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 3386-3401.
[22] P. Li and W. Zheng, *An H-ψ formulation for the three-dimensional eddy current problem in laminated structures*, Journal of Differential Equations, 254 (2013), pp. 3476-3500.
[23] J.C. Nédélec, *Mixed finite elements in $\mathbb{R}^3$*, Numer. Math. 35 (1980), pp. 315-341.
[24] J.C. Nédélec and S. Wolf, *Homogenization of the problem of eddy currents in a transformer core*, SIAM J. Numer. Anal., 26 (1989), pp. 1407-1424.
[25] M. Rachek, M. Zaouia, H. Denoun, and C. Birouche, *Finite element computational model for defect simulation and detection by eddy currents non destructive testing*, CD-ROM Proceedings of the 9th WSEAS International Conference on Circuit and Systems, Athens, 2005, pp. 11-13.
[26] R. Palanisamy, *Finite Element Modeling of Remote-Field Eddy Current Problems for the Nondestructive Testing of Metal Tube*, Journal of Offshore Mechanics and Arctic Engineering, 111 (1989), pp. 101–108.
[27] L.D. Philipp, Q.H. Nguyen, D.D. Derkacht, D.J. Lynch, and A. Mahmood, *Exact first order finite element modeling for eddy current NDE*, Res. Nondestr. Eval., 3 (1991), pp. 235-253.
[28] L. Zhang, *A Parallel Algorithm for Adaptive Local Refinement of Tetrahedral Meshes Using Bisection*, Numer. Math.: Theor. Method Appl., 2 (2009) 65C89.
[29] W. Zheng, Z. Chen, and L. Wang, *An adaptive finite element method for the $\boldsymbol{H}$–ψ formulation of time-dependent eddy current problems*, Numer. Math., 103 (2006), pp. 667–689.
[30] W. Zheng and Z. Cheng, *An inner-constrained separation technique for 3D finite element*

modeling of GO silicon steel laminations, IEEE Transactions on Magnetics, 48 (2012), no. 8, pp. 2277-2283.