PML METHOD FOR ELECTROMAGNETIC SCATTERING PROBLEM IN A TWO-LAYER MEDIUM*

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Abstract. The perfectly matched layer (PML) method is well-studied for acoustic scattering problems, electromagnetic scattering problems, and more recently, elastic scattering problems, with homogeneous background media. The purpose of this paper is to present the stability and exponential convergence of the PML method for three-dimensional electromagnetic scattering problem in a two-layer medium. The main contributions of this paper are threefold. Firstly, we establish the well-posedness of the original scattering problem for any Dirichlet boundary value in $\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$ where Γ_D stands for the boundary of the scatterer. Secondly, we propose a new weak formulation for the original problem where the Dirichlet-to-Neumann operator is proposed on a truncation boundary inside PML. This argument is favorable to the analysis for the PML Dirichlet-to-Neumann operator. The inf-sup condition is proved for the bilinear form. Thirdly, we establish the well-posedness of the PML problem and prove that the approximate solution converges to the original scattering solution exponentially as either the PML absorbing coefficient or the thickness of the PML increases.

Key words. Perfectly matched layer, electromagnetic scattering problem, dyadic Green's function, Maxwell's equation, two-layer medium

AMS subject classifications. 35Q60, 65N30

1. Introduction.

We propose and study the perfectly matched layer (PML) method for solving the electromagnetic scattering problem in a two-layer medium:

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - k^2\boldsymbol{E} = 0 \quad \text{in } D_c, \tag{1.1a}$$

$$\boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{g} \quad \text{on } \Gamma_D, \tag{1.1b}$$

$$[\boldsymbol{n} \times \mathbf{curl}\,\boldsymbol{E}] = [\boldsymbol{n} \times \boldsymbol{E}] = 0 \quad \text{on } \Sigma, \tag{1.1c}$$

$$\lim_{\rho\to\infty} \int_{\partial B(\rho)} |\mathbf{curl}\,\boldsymbol{E} \times \boldsymbol{n} - \mathbf{i}k\boldsymbol{E}|^2 = 0, \tag{1.1d}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{g}$ is determined by incoming wave, $D \subset \mathbb{R}^3$ is a bounded domain with Lipschitz-continuous boundary Γ_D, $D_c = \mathbb{R}^3_\pm \setminus \bar{D}$ is the complement of D, $B(\rho) = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < \rho\}$ is the open ball of radius ρ and centering at the origin, and $\boldsymbol{n}$ stands for the unit outer normal to D and $B(\rho)$ on their respective boundaries. We assume that the wave number k is positive and piecewise constant, defined by

$$k(\boldsymbol{x}) = \begin{cases} k_+, & \text{if } \boldsymbol{x} \in \mathbb{R}^3_+, \\ k_-, & \text{if } \boldsymbol{x} \in \mathbb{R}^3_-, \end{cases} \tag{1.2}$$

where $\mathbb{R}^3_\pm = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \pm x_3 > 0\}$. Without loss of generality, we assume in this paper that $k_- > k_+ > 0$. We remark that the boundary condition (1.1b) is not essential for our results. In fact, (1.1b) can be replaced by other boundary conditions such as Neumann or impedance boundary condition on Γ_D. Furthermore, we scale the system such that the diameter of the scatterer satisfies $\mathrm{diam}(D) \geq 1$.

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The basic idea of the PML method is to surround the computational domain by a layer of specially designed model medium that absorbs all waves propagating from inside the computational domain [1]. The convergence of the PML method for homogeneous background materials has drawn considerable attentions in the literature. Lassas and Somersalo [27, 28] and Hohage, Schmidt, and Zschiedrich [25] studied the acoustic scattering problems for circular and smooth PMLs. It is proved that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML tends to infinity. We also refer to the work of Collino and Monk for PML in curvilinear coordinates [21]. In 2003, Chen and Wu proposed the adaptive PML finite element method for grating problems [15]. The adaptive PML method provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the absorbing PMLs. Later on, the adaptive PML finite element method was extended to accoustic scattering problems in [14, 16], to electromagnetic scattering problems in [12, 13], to multiple scattering problems in [26, 32], and to grating problems in [2, 3]. In 2005, Bao and Wu first proved the exponential convergence of PML method for Maxwell's equations [4]. Bramble and Pasciak also studied the stability and exponential convergence of PML method for acoustic and electromagnetic scattering problems in a series of papers [5–8]. They use both circular coordinates and Cartesian coordinates in constructing wave-absorbing materials. We also refer to the recent papers on PML methods for elastic scattering problems [9, 18] and to [11, 17] for exponential convergence of time-domain PMLs.

The studies mentioned above assume homogeneous background materials, namely, wave numbers are constant away from the scatterer. The analysis for scattering problem is very challenging for layered media since scattering waves usually consist of both propagating modes and evanescent modes. For two-layer media, Chen and Zheng proved the stability and exponential convergence of uniaxial PML method for two-dimensional acoustic scattering problem [19]. Their proof is very technical and relies on the Cagniard-de Hoop transform for Green's function. Electromagnetic scattering problems in two-layer media have broad applications in both scientific and engineering areas, such as, near-field imaging, detection of buried objects, and so on. The convergence of PML method is an open issue. In 1998, Cutzach and Hazard proved the existence and uniqueness for electromagnetic scattering problem in a two-layer medium with incident plane waves or incident point source [23] (see also Monk's book [30, Chapter 12]). We also refer to Coyle and Monk [22] and Monk [30, Chapter 12] for the finite element approximation using transparent boundary condition and to [29] for the coupling of finite element method and boundary element method.

For scattering problems in layered media, scattered field becomes much more complicated and high-accuracy approximation of the radiation boundary condition becomes much difficult [24]. It is well-known that numerical method using PML have two advantages compared with that using the Dirichlet-to-Neumann (DtN) operator. Firstly, it does not compute Green's function which is very complicated for layered medium, particularly, in three-dimensional case. Secondly, the numerical method using PML usually yields an algebraic system with sparse matrix. It is favorable in designing effective precondtioners. The purpose of this paper is to investigate the theoretical aspect of the PML method for electromagnetic scattering problem in a two-layer medium. The main theme is threefold.

- We prove the well-posedness of the scattering problem for any Dirichlet boundary data $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$.

- We propose a new weak formulation for the scattering problem where the DtN operator is defined on a truncation boundary inside PML. This formulation is favorite in proving the stability of both the original DtN operator and the PML DtN operator. We also prove the inf-sup condition of the bilinear form which plays the key role in convergence analysis of the PML method.
- We introduce the Cagniard-de Hoop transform to the dyadic Green's function and prove that the Green's function decays exponentially in PML. This is the major novelty of this paper. We prove the well-posedness of the PML problem and the exponential convergence of the PML solution as either the absorbing coefficient or the thickness of the PML increases.

The layout of this paper is organized as follows. In section 2, we derive an explicit form of the dyadic Green's function for the scattering problem in the two-layer medium. The uniqueness and existence of the scattering solution for any Dirichlet boundary data $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$. In section 3, we derive the Cagniard-de Hoop representation of the Green's function. In section 4, we introduce the PML by means of complex coordinate stretching and prove the exponential decay of the modified Green's function. In section 5, we study exterior problems of stretched Maxwell's equation. In section 6, we propose a new weak formulation for the stretched Maxwell equation where the truncation boundary is located inside PML. The inf-sup condition for the bilinear form is proved. In Section 7, we propose the PML approximation to the exterior problem on the truncated domain. The well-posedness and exponential convergence of the PML problem are also proved.

2. The well-posedness of the scattering problem.

The purpose of this section is to study the weak solution of (1.1). First we introduce some Sobolev spaces.

2.1. Sobolev spaces.

For a domain $\Omega \subset \mathbb{R}^3$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$, let $L^2(\Omega)$ be the space of square-integrable functions and $H^1(\Omega)$ be the subspace whose functions have square-integrable gradients, let $\boldsymbol{H}(\mathbf{curl}, \Omega) \subset \boldsymbol{L}^2(\Omega)$ be the subspace whose functions have square-integrable curls. Throughout the paper we denote vector-valued quantities by boldface notations, such as $\boldsymbol{L}^2(\Omega) := L^2(\Omega)^3$.

From [10], we have the surjective mappings

$$\begin{aligned} &\gamma: \ H^1(\Omega) \to H^{1/2}(\Gamma), && \gamma\varphi = \varphi \quad \text{on } \Gamma,\\ &\gamma_t: \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}^{-1/2}(\mathrm{Div}; \Gamma), && \gamma_t \boldsymbol{u} = \boldsymbol{n} \times \boldsymbol{u} \quad \text{on } \Gamma,\\ &\gamma_T: \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}^{-1/2}(\mathrm{Curl}; \Gamma), && \gamma_T \boldsymbol{u} = \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n}) \quad \text{on } \Gamma, \end{aligned}$$

where Div, Curl stand for the surface divergence and surface scalar curl operators respectively. For convenience, we define the equivalent norms on the respective surface Sobolev spaces

$$\|\lambda\|_{H^{1/2}(\Gamma)} = \inf_{\substack{v \in H^1(\Omega)\\ \gamma v = \lambda}} \|v\|_{H^1(\Omega)} \qquad \forall\, \lambda \in H^{1/2}(\Gamma), \tag{2.1}$$

$$\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div};\Gamma)} = \inf_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\Omega)\\ \gamma_t \boldsymbol{v} = \boldsymbol{\lambda}}} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \qquad \forall\, \boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}; \Gamma), \tag{2.2}$$

$$\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl};\Gamma)} = \inf_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\Omega)\\ \gamma_T \boldsymbol{v} = \boldsymbol{\lambda}}} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \qquad \forall\, \boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Curl}; \Gamma). \tag{2.3}$$

For any $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, it holds that

$$\mathrm{Div}(\gamma_t \boldsymbol{u}) = -\,\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n}, \quad \mathrm{Curl}(\gamma_T \boldsymbol{u}) = \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} \quad \text{on } \Gamma.$$

Moreover, the surface gradient operator is defined by, for any $\varphi \in H^1(\Omega)$,

$$\mathrm{Grad}(\gamma_t \varphi) = \gamma_T(\nabla \varphi) \quad \text{on } \Gamma.$$

It is known that $\boldsymbol{H}^{-1/2}(\mathrm{Div};\Gamma)$ and $\boldsymbol{H}^{-1/2}(\mathrm{Curl};\Gamma)$ are dual spaces. For any $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div};\Gamma)$ and $\boldsymbol{\xi} \in \boldsymbol{H}^{-1/2}(\mathrm{Curl};\Gamma)$, the duality paring is defined by

$$\langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle_\Gamma := \int_\Omega (\boldsymbol{u}_{\boldsymbol{\lambda}} \cdot \mathbf{curl}\, \boldsymbol{u}_{\boldsymbol{\xi}} - \mathbf{curl}\, \boldsymbol{u}_{\boldsymbol{\lambda}} \cdot \boldsymbol{u}_{\boldsymbol{\xi}}) \tag{2.4}$$

where $\boldsymbol{u}_{\boldsymbol{\lambda}}, \boldsymbol{u}_{\boldsymbol{\xi}} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ satisfy $\boldsymbol{\lambda} = \gamma_t \boldsymbol{u}_{\boldsymbol{\lambda}}$ and $\boldsymbol{\xi} = \gamma_T \boldsymbol{u}_{\boldsymbol{\xi}}$ on Γ.

For any $S \subset \Gamma$, the subspaces with zero trace and zero tangential trace on S are denoted respectively by

$$\begin{aligned} H^1_S(\Omega) &:= \{v \in H^1(\Omega) : \gamma v = 0 \ \text{ on } S\}, \\ \boldsymbol{H}_S(\mathbf{curl}, \Omega) &:= \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \gamma_t \boldsymbol{v} = 0 \ \text{ on } S\}. \end{aligned}$$

In particular, we use the conventional notations

$$H^1_0(\Omega) := H^1_\Gamma(\Omega), \qquad \boldsymbol{H}_0(\mathbf{curl}, \Omega) := \boldsymbol{H}_\Gamma(\mathbf{curl}, \Omega).$$

2.2. The Dyadic Green's function for the two-layer medium. The dyadic Green's function is the main tool in our analysis for the well-posedness of the scattering problem and for the exponential convergence of the PML method. Throughout the paper, we shall use the convention that for any $z \in \mathbb{C}$, $z^{1/2}$ is the branch of the square root $\sqrt{z}$ such that $\mathrm{Re}(z^{1/2}) \geq 0$. This corresponds to the left half real axis as the branch cut in the complex plane. Then we have, for $z = z_1 + \mathbf{i} z_2$ with $z_1, z_2 \in \mathbb{R}$,

$$z^{1/2} = \sqrt{\frac{|z| + z_1}{2}} + \mathbf{i}\,\mathrm{sgn}(z_2)\sqrt{\frac{|z| - z_1}{2}}\,. \tag{2.5}$$

Let $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ be column vectors of the dyadic Green's function $\mathbb{G}$. Each $\mathbf{g}_j$ satisfies an electromagnetic scattering problem with a polarized dipole source

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{g}_j(k;\boldsymbol{x},\cdot) - k_\pm^2 \mathbf{g}_j(k;\boldsymbol{x},\cdot) = \delta_{\boldsymbol{x}} \mathbf{e}_j \qquad \text{in } \mathbb{R}^3_\pm, \tag{2.6a}$$

$$[\boldsymbol{n} \times \mathbf{curl}\,\mathbf{g}_j(k;\boldsymbol{x},\cdot)] = [\boldsymbol{n} \times \mathbf{g}_j(k;\boldsymbol{x},\cdot)] = 0 \qquad \text{on } \Sigma, \tag{2.6b}$$

$$\lim_{\rho\to\infty} \int_{\partial B(\rho)} |\mathbf{curl}\,\mathbf{g}_j(k;\boldsymbol{x},\cdot) \times \boldsymbol{n} - \mathbf{i}k\,\mathbf{g}_j(k;\boldsymbol{x},\cdot)|^2 = 0, \tag{2.6c}$$

where $\delta_{\boldsymbol{x}}(\boldsymbol{y}) = \delta(|x_1 - y_1|)\delta(|x_2 - y_2|)\delta(|x_3 - y_3|)$ stands for the Dirac source at $\boldsymbol{x} \in \mathbb{R}^3$ and $\mathbf{e}_j$ is the unit vector along the positive direction of the x_j-axis, $j = 1, 2, 3$. Similar to the scattering problem in free space, we write $\mathbf{g}_j$ as

$$\mathbf{g}_j(k;\boldsymbol{x},\cdot) = \mathbf{H}_j(k;\boldsymbol{x},\cdot) + k_\pm^{-2} \nabla\, \mathrm{div}\, \mathbf{H}_j(k;\boldsymbol{x},\cdot) \quad \text{in } \mathbb{R}^3_\pm, \tag{2.7}$$

where $\mathbf{H}_j$ is the Hertz vector for the dipole source polarized in the $\mathbf{e}_j$ direction. From [30, Section 12.4.2], the Hertz vectors satisfy the Helmholtz equations

$$\Delta \mathbf{H}_j(k;\boldsymbol{x},\cdot) + k_\pm^2 \mathbf{H}_j(k;\boldsymbol{x},\cdot) = -\delta_{\boldsymbol{x}} \mathbf{e}_j \quad \text{in } \mathbb{R}^3_\pm, \tag{2.8a}$$

$$[\mathbf{H}_j(k;\boldsymbol{x},\cdot)] = [\boldsymbol{n} \times \mathbf{curl}\,\mathbf{H}_j(k;\boldsymbol{x},\cdot)] = 0, \ \left[k^{-2}\,\mathrm{div}\,\mathbf{H}_j(k;\boldsymbol{x},\cdot)\right] = 0 \quad \text{on } \Sigma, \tag{2.8b}$$

$$\lim_{\rho\to\infty} \int_{\partial B(\rho)\cap\mathbb{R}^3_\pm} \left| \frac{\partial \mathbf{H}_j(k;\boldsymbol{x},\cdot)}{\partial \boldsymbol{n}} - \mathbf{i}k_\pm \mathbf{H}_j(k;\boldsymbol{x},\cdot) \right|^2 = 0. \tag{2.8c}$$

We shall follow Monk [30, Section 12.4.2] to derive an explicit form of $\mathbf{H}_1$. The derivations for $\mathbf{H}_2, \mathbf{H}_3$ are parallel and omitted here.

Let the delta source be located at $\boldsymbol{x} \in \mathbb{R}^3_+$ and write $\boldsymbol{\Pi}^\pm(\boldsymbol{y}) = \mathbf{H}_1(k;\boldsymbol{x},\boldsymbol{y})$ for any $\boldsymbol{y} \in \mathbb{R}^3_\pm$. From [30, Section 12.4.2], $\boldsymbol{\Pi}^\pm = (\Pi_1^\pm, 0, \Pi_3^\pm)$ and it can be solved by the coupled Helmholtz equations in the two-layer medium

$$\Delta\Pi_1^\pm + k_\pm\Pi_1^\pm = -\delta_{\boldsymbol{x}}, \quad \Delta\Pi_3^\pm + k_\pm\Pi_3^\pm = 0 \quad \text{in } \mathbb{R}^3_\pm, \tag{2.9a}$$

$$\boldsymbol{\Pi}^+ = \boldsymbol{\Pi}^-, \quad \frac{1}{k_+^2}\operatorname{div}\boldsymbol{\Pi}^+ = \frac{1}{k_-^2}\operatorname{div}\boldsymbol{\Pi}^-, \quad \frac{\partial\Pi_1^+}{\partial y_3} = \frac{\partial\Pi_1^-}{\partial y_3} \quad \text{on } \Sigma, \tag{2.9b}$$

$$\lim_{\rho\to\infty}\int_{\partial B(\rho)\cap\mathbb{R}^3_\pm}\left|\frac{\partial\boldsymbol{\Pi}^\pm}{\partial\boldsymbol{n}} - \mathbf{i}k_\pm\boldsymbol{\Pi}^\pm\right|^2 = 0. \tag{2.9c}$$

Let $\Phi(\omega;\boldsymbol{x},\boldsymbol{y}) = \frac{e^{\mathbf{i}\omega|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$ be the fundamental solution of the three-dimensional Helmholtz equation with constant number ω. Write

$$\Pi_1^+(\boldsymbol{y}) = \hat{\Pi}_1^+(\boldsymbol{y}) + \Phi(k_+;\boldsymbol{x},\boldsymbol{y}) \quad \forall\, \boldsymbol{y} \in \mathbb{R}^3_+.$$

From (2.9), it is easy to see that $\hat{\Pi}_1^+$ satisfies

$$\Delta\hat{\Pi}_1^+ + k_+\hat{\Pi}_1^+ = 0 \quad \text{in } \mathbb{R}^3_+, \qquad \lim_{\rho\to\infty}\int_{\partial B(\rho)\cap\mathbb{R}^3_+}\left|\frac{\partial\hat{\Pi}_1^+}{\partial\boldsymbol{n}} - \mathbf{i}k_+\hat{\Pi}_1^+\right|^2 = 0. \tag{2.10}$$

Applying Fourier transform to (2.10) with respect to y_1 and y_2, the solution can be represented as follows

$$\hat{\Pi}_1^+ = \frac{\mathbf{i}}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a_+(\lambda_1,\lambda_2)e^{\mathbf{i}[(x_1-y_1)\lambda_1+(x_2-y_2)\lambda_2+(x_3+y_3)\mu_+]}\mathrm{d}\lambda_1\mathrm{d}\lambda_2, \tag{2.11}$$

where a_+ is the Fourier coefficient which depends on $\boldsymbol{x}$ but is independent of $\boldsymbol{y}$. Here $\mu_\pm$ are square roots defined by the limiting absorption principle

$$\mu_\pm(\lambda_1,\lambda_2) = \lim_{\varepsilon\to0^+}\left[(k_\pm + \mathbf{i}\varepsilon)^2 - \lambda_1^2 - \lambda_2^2\right]^{1/2} \qquad \forall\,(\lambda_1,\lambda_2) \in \mathbb{R}^2. \tag{2.12}$$

Similarly, we have

$$\Pi_3^+ = \frac{\mathbf{i}}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} b_+(\lambda_1,\lambda_2)e^{\mathbf{i}[(x_1-y_1)\lambda_1+(x_2-y_2)\lambda_2+(x_3+y_3)\mu_+]}\mathrm{d}\lambda_1\mathrm{d}\lambda_2, \tag{2.13}$$

$$\Pi_1^- = \frac{\mathbf{i}}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a_-(\lambda_1,\lambda_2)e^{\mathbf{i}[(x_1-y_1)\lambda_1+(x_2-y_2)\lambda_2+x_3\mu_+-y_3\mu_-]}\mathrm{d}\lambda_1\mathrm{d}\lambda_2, \tag{2.14}$$

$$\Pi_3^- = \frac{\mathbf{i}}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} b_-(\lambda_1,\lambda_2)e^{\mathbf{i}[(x_1-y_1)\lambda_1+(x_2-y_2)\lambda_2+x_3\mu_+-y_3\mu_-]}\mathrm{d}\lambda_1\mathrm{d}\lambda_2. \tag{2.15}$$

From equation (2.2.26) of [20, Page 64], the fundamental solution has the form

$$\Phi(k_+;\boldsymbol{x},\boldsymbol{y}) = \frac{\mathbf{i}}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\mu_+}e^{\mathbf{i}[(x_1-y_1)\lambda_1+(x_2-y_2)\lambda_2+|x_3-y_3|\mu_+]}\mathrm{d}\lambda_1\mathrm{d}\lambda_2, \tag{2.16}$$

Inserting (2.11)–(2.15) into (2.9b) and matching the Fourier modes at $y_3 = 0$, we get

$$a_- - a_+ = \frac{1}{2\mu_+}, \quad b_+ = b_-, \quad a_+\mu_+ + a_-\mu_- = \frac{1}{2},$$
$$\frac{1}{k_+^2}\left(\lambda_1 a_- - \mu_+ b_+\right) = \frac{1}{k_-^2}\left(\lambda_1 a_- + \mu_- b_-\right).$$

The Fourier coefficients are given by

$$a_+ = h_1 - \frac{1}{2\mu_+}, \quad a_- = h_1, \quad b_+ = b_- = \lambda_1 h_3,$$

where we have used the notations

$$h_1 = \frac{1}{\mu_+ + \mu_-}, \qquad h_2 = \frac{1}{k_-^2\mu_+ + k_+^2\mu_-}, \qquad h_3 = \frac{k_-^2 - k_+^2}{k_-^2\mu_+ + k_+^2\mu_-}h_1. \tag{2.17}$$

The second and third Hertz vectors can be obtained similarly.

We split the Hertz tensor $\mathbb{H} = (\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3)$ into $\mathbb{H} = \mathbb{S} - \mathbb{P}$ where $\mathbb{S}$ is the double source tensor standing for the singular part and $\mathbb{P}$ is the perturbation tensor standing for the regular part. The double source tensor is given by

$$\mathbb{S}(k; \boldsymbol{x}, \boldsymbol{y}) = \mathbb{I} \times \begin{cases} \Phi(k_+; \boldsymbol{x}, \boldsymbol{y}) - \Phi(k_+; \boldsymbol{x}', \boldsymbol{y}) & \text{if } x_3 > 0,\ y_3 > 0, \\ \Phi(k_-; \boldsymbol{x}, \boldsymbol{y}) - \Phi(k_-; \boldsymbol{x}', \boldsymbol{y}) & \text{if } x_3 < 0,\ y_3 < 0, \\ 0 & \text{elsewhere}, \end{cases} \tag{2.18}$$

where $\mathbb{I}$ is the identity matrix and $\boldsymbol{x}' = (x_1, x_2, -x_3)$ is the image of $\boldsymbol{x} = (x_1, x_2, x_3)$ with respect to Σ. The perturbation tensor $\mathbb{P}$ has the form

$$\mathbb{P} = \begin{pmatrix} \mathrm{P}_{11} & 0 & 0 \\ 0 & \mathrm{P}_{22} & 0 \\ \mathrm{P}_{13} & \mathrm{P}_{23} & \mathrm{P}_{33} \end{pmatrix}. \tag{2.19}$$

For given any function f, we define

$$J(f; \boldsymbol{x}, \boldsymbol{y}) := \frac{\mathbf{i}}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\lambda_1, \lambda_2) e^{\mathbf{i}[(x_1 - y_1)\lambda_1 + (x_2 - y_2)\lambda_2 + (|x_3| + |y_3|)\mu_+]} \mathrm{d}\lambda_1 \mathrm{d}\lambda_2.$$

From (2.5), the above integral is convergent absolutely for any function satisfying

$$|f(\lambda_1, \lambda_2)| \le C(1 + \lambda_1^2 + \lambda_2^2)^m \qquad \forall\, m \in \mathbb{R}.$$

The entries of $\mathbb{P}$ are defined respectively as follows: for $j = 1, 2$,

$$\mathrm{P}_{jj}(k; \boldsymbol{x}, \boldsymbol{y}) = \begin{cases} J(h_1; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(h_1 e^{\mathbf{i}(\mu_+ - \mu_-)y_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_+^3,\ \boldsymbol{y} \in \mathbb{R}_-^3, \\ J(h_1 e^{\mathbf{i}(\mu_+ - \mu_-)x_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_-^3,\ \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(h_1 e^{\mathbf{i}(\mu_+ - \mu_-)(x_3 + y_3)}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_-^3, \end{cases} \tag{2.20}$$

$$\mathrm{P}_{j3}(k; \boldsymbol{x}, \boldsymbol{y}) = \begin{cases} J(\lambda_j h_3; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(\lambda_j h_3 e^{\mathbf{i}(\mu_+ - \mu_-)y_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_+^3,\ \boldsymbol{y} \in \mathbb{R}_-^3, \\ J(\lambda_j h_3 e^{\mathbf{i}(\mu_+ - \mu_-)x_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_-^3,\ \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(\lambda_j h_3 e^{\mathbf{i}(\mu_+ - \mu_-)(x_3 + y_3)}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_-^3, \end{cases} \tag{2.21}$$

$$\mathrm{P}_{33}(k; \boldsymbol{x}, \boldsymbol{y}) = \begin{cases} J(k_-^2 h_2; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(k_-^2 h_2 e^{\mathbf{i}(\mu_+ - \mu_-)y_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_+^3,\ \boldsymbol{y} \in \mathbb{R}_-^3, \\ J(k_+^2 h_2 e^{\mathbf{i}(\mu_+ - \mu_-)x_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x} \in \mathbb{R}_-^3,\ \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(k_+^2 h_2 e^{\mathbf{i}(\mu_+ - \mu_-)(x_3 + y_3)}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_-^3. \end{cases} \tag{2.22}$$

To end this subsection, we study the singularity of the perturbation tensor.

LEMMA 2.1. *There exists a constant $C > 0$ depending only on k such that, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_\pm$, $i = 0, 1$, and $j = 1, 2, 3$,*

$$\left|\frac{\partial^i}{\partial x_j}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| + \left|\frac{\partial^i}{\partial y_j}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| \le C\left(1 + |\boldsymbol{x}-\boldsymbol{y}|^{-i-1}\right) \quad \text{if} \quad x_3 y_3 < 0, \tag{2.23}$$

$$\left|\frac{\partial^i}{\partial x_j}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| + \left|\frac{\partial^i}{\partial y_j}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| \le C\left(1 + |\boldsymbol{x}-\boldsymbol{y}'|^{-i-1}\right) \quad \text{if} \quad x_3 y_3 > 0. \tag{2.24}$$

Proof. Without loss of generality, we only consider $\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})$ for $\boldsymbol{x} \in \mathbb{R}^3_+$ and $\boldsymbol{y} \in \mathbb{R}^3_-$ and write $\rho = |\boldsymbol{x} - \mathbf{y}|$ for convenience. The proofs for other cases are similar.

Write $\xi = \sqrt{\lambda_1^2 + \lambda_2^2}$ for convenience. Then (2.12) indicates $\operatorname{Im}\mu_\pm = \sqrt{\xi^2 - k_\pm^2}$ for $\xi \ge k_\pm$ and $\operatorname{Im}\mu_\pm = 0$ otherwise. We have $\operatorname{Im}\mu_+ \ge \operatorname{Im}\mu_-$, $\left|e^{\mathbf{i}(\mu_+ - \mu_-)y_3}\right| \le 1$, and

$$|\mu_+ - \mu_-| = \frac{k_-^2 - k_+^2}{|\mu_- + \mu_+|} = O\left(\xi^{-1}\right) \qquad \text{as} \quad \xi \to \infty. \tag{2.25}$$

Write $z = x_3 - y_3$ for convenience. From (2.22), we deduce that

$$\left|\frac{\partial^i}{\partial x_j}\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})\right| \le C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mu_+|^{i-1}\, e^{-z \operatorname{Im}\mu_+}\mathrm{d}\lambda_1\mathrm{d}\lambda_2 \le C(1 + z^{-1-i}), \tag{2.26}$$

for any $i = 0, 1$ and $j = 1, 2, 3$. It suffices to prove the lemma for $z < 1$.

Recall that $\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y}) = k_-^2 J(h_2 e^{\mathbf{i}(\mu_+ - \mu_-)y_3};\boldsymbol{x},\boldsymbol{y})$. By Taylor's expansion,

$$\begin{cases} e^{\mathbf{i}(\mu_+ - \mu_-)y_3} = 1 + \mathbf{i}(\mu_+ - \mu_-)y_3 - \dfrac{1}{2}(\mu_+ - \mu_-)^2 y_3^2 + O\left(\xi^{-3}\right), \\ h_2 = \dfrac{1}{(k_+^2 + k_-^2)\mu_+}\left[1 + \dfrac{k_+^2(\mu_+ - \mu_-)}{(k_+^2 + k_-^2)\mu_+}\right] + O(\xi^{-4}), \end{cases} \quad \text{as} \quad \xi \to \infty.$$

We can write $h_2 e^{\mathbf{i}(\mu_+ - \mu_-)y_3} = \sum_{j=1}^3 R_j$ where

$$R_1 = \frac{1}{k_+^2 + k_-^2}\frac{1}{\mu_+}, \qquad R_2 = \frac{\mathbf{i}(k_+^2 - k_-^2)y_3}{2(k_+^2 + k_-^2)}\frac{1}{\mu_+^2},$$
$$R_3 = \frac{k_+^2 - k_-^2}{(k_+^2 + k_-^2)^2}\left[\frac{k_+^2}{2} + \frac{y_3^2}{8}(k_-^4 - k_+^4)\right]\frac{1}{\mu_+^3} + O\left(\xi^{-4}\right).$$

Then we have $\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y}) = k_-^2 \sum_{j=1}^3 J(R_j;\boldsymbol{x},\boldsymbol{y})$.

From (2.16), the fundamental solution has the Fourier integral form

$$\Phi(k_+;\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}J(\mu_+^{-1};\boldsymbol{x},\boldsymbol{y}).$$

This shows that

$$J(R_1;\boldsymbol{x},\boldsymbol{y}) = \frac{2}{k_+^2 + k_-^2}\Phi(k_+;\boldsymbol{x},\boldsymbol{y}), \qquad |J(R_1;\boldsymbol{x},\boldsymbol{y})| \le C\rho^{-1}. \tag{2.27}$$

For the second term, since

$$\frac{\partial}{\partial x_3}J(R_2;\boldsymbol{x},\boldsymbol{y}) = \frac{(k_-^2 - k_+^2)y_3}{k_+^2 + k_-^2}\Phi(k_+;\boldsymbol{x},\boldsymbol{y}), \tag{2.28}$$

this yields

$$
\begin{aligned}
|J(R_2;\boldsymbol{x},\boldsymbol{y})| &= \left|J(R_1;(x_1,x_2,\rho),\boldsymbol{y}) - \frac{(k_-^2-k_+^2)y_3}{k_+^2+k_-^2}\int_{x_3}^{\rho}\Phi(k_+;(x_1,x_2,t),\boldsymbol{y})\mathrm{d}t\right| \\
&\le C(\rho-y_3)^{-1} + C(\rho-x_3)\max_{x_3\le t\le\rho}|\Phi(k_+;(x_1,x_2,t),\boldsymbol{y})| \\
&\le C\left(1+\rho^{-1}\right). \qquad (2.29)
\end{aligned}
$$

The third term is easy to be estimated as follows

$$
|J(R_3;\boldsymbol{x},\boldsymbol{y})| \le C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\lambda_1^2+\lambda_2^2)^{-3/2}\mathrm{d}\lambda_1\mathrm{d}\lambda_2 \le C.
$$

Combining the above inequalities, we get

$$
|\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})| \le C\left(1+\rho^{-1}\right).
$$

The derivatives of P_{33} can be estimated by using (2.27)–(2.28) and similar arguments as in the proof of (2.29). We do not elaborate on the details here. □

2.3. Existence and uniqueness of the scattering solution. Now we study the well-posedness of the scattering problem. The idea is inspired by [23] and [30, Chapter 12] where incident point sources and incident plane waves are considered. Let $B_0 = B(R_0)$, $R_0 \ge 1$, be an open ball of radius R_0 such that $\bar{D} \subset B_0$. Write $\Gamma_0 = \partial B_0$ and let $\Omega_0 = B_0\backslash\bar{D}$ denote the domain where the scattering field is interested. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be the cut-off function satisfying $\mathrm{supp}(\chi) \subset B_0$ and $\chi \equiv 1$ on $\bar{D}$.

We introduce the modified Green's function $\mathbb{G}_\chi(\boldsymbol{x},\boldsymbol{y}) = \chi(\boldsymbol{y})\mathbb{G}(k;\boldsymbol{x},\boldsymbol{y})$ and define the wave propagation operator by, for any $\boldsymbol{u} \in \boldsymbol{L}^1(\Omega_0)$,

$$
\mathcal{P}(\boldsymbol{u}) := \int_{\Omega_0}\left[\mathbf{curl}_{\boldsymbol{y}}\,\mathbf{curl}_{\boldsymbol{y}}\,\mathbb{G}_\chi(\cdot,\boldsymbol{y}) - k^2\mathbb{G}_\chi(\cdot,\boldsymbol{y})\right]^\top \boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\,. \qquad (2.30)
$$

From [30, Section 12.4.3], the scattering solution $\boldsymbol{E}$ of (1.1) satisfies

$$
\boldsymbol{E}(\boldsymbol{x}) = \mathcal{P}(\boldsymbol{E})(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \mathbb{R}^3_\pm\backslash B_0\,.
$$

THEOREM 2.2. *For any* $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)$, (1.1) *has a unique solution* $\boldsymbol{E}$. *Moreover, for any bounded domain* $\Omega \subset D_c$, *there exists a constant* $C > 0$ *depending only on* k *and* Ω *such that*

$$
\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\,\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}\,. \qquad (2.31)
$$

Proof. The uniqueness of the solution is proved by Cutzach and Hazard in [23]. It is left to prove the existence. Let $B_1 = B(R_1)$ be a ball containing B_0 and write $\Omega_1 = B_1\backslash\bar{D}$ and $\Gamma_1 = \partial B_1$. Define

$$
\boldsymbol{U} = \left\{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\Omega_1) : \gamma_T\boldsymbol{v} \in \boldsymbol{L}^2(\Gamma_1)\right\}, \quad \boldsymbol{U}_0 = \{\boldsymbol{v} \in \boldsymbol{U} : \gamma_t\boldsymbol{v} = 0 \text{ on } \Gamma_D\}\,.
$$

By [30, Theorem 4.1], $\boldsymbol{U}$ forms a Hilbert space under the inner product and norm

$$
(\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{U}} = \int_{\Omega_1}(\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} + \boldsymbol{u}\cdot\bar{\boldsymbol{v}}) + \int_{\Gamma_1}\gamma_T\boldsymbol{u}\cdot\gamma_T\bar{\boldsymbol{v}}, \qquad \|\boldsymbol{u}\|_{\boldsymbol{U}} := (\boldsymbol{u},\boldsymbol{u})_{\boldsymbol{U}}^{1/2}\,.
$$

By [30, Theorem 4.7], we have the direct-sum decomposition $\boldsymbol{U}_0 = \hat{\boldsymbol{U}}_0 + \nabla S_0$ where

$$\hat{\boldsymbol{U}}_0 = \left\{\boldsymbol{v} \in \boldsymbol{U}_0 :\ \operatorname{div}(k^2\boldsymbol{v}) = 0\right\}, \quad S_0 = \left\{v \in H^1_{\Gamma_D}(\Omega_1) :\ v|_{\Gamma_1} = \text{Const.}\right\}. \tag{2.32}$$

Moreover, $\hat{\boldsymbol{U}}_0$ is embedded compactly into $\boldsymbol{L}^2(\Omega_1)$.

Let $s_+\colon \boldsymbol{U}\times\boldsymbol{U} \to \mathbb{C}$ be the sesquilinear form defined by

$$s_+(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega_1} \left(\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} + k^2\boldsymbol{u}\cdot\bar{\boldsymbol{v}}\right) - \mathbf{i}\int_{\Gamma_1} \gamma_T\boldsymbol{u}\cdot\gamma_T\bar{\boldsymbol{v}}. \tag{2.33}$$

Clearly s_+ is continuous and coercive on $\boldsymbol{U}$. There is a unique $\boldsymbol{E}_g \in \boldsymbol{U}$ satisfying

$$s_+(\boldsymbol{E}_g,\boldsymbol{v}) = 0 \quad \forall\,\boldsymbol{v}\in\boldsymbol{U}_0, \qquad \gamma_t\boldsymbol{E}_g = \boldsymbol{g} \quad \text{on } \Gamma_D. \tag{2.34}$$

Furthermore, there is a constant $C>0$ depending only on k and Ω_1 such that

$$\|\boldsymbol{E}_g\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \le C\,\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}. \tag{2.35}$$

Therefore, a weak formulation for (1.1) can be proposed as follows: Find $\boldsymbol{E}_1 := \boldsymbol{E} - \boldsymbol{E}_g \in \boldsymbol{U}_0$ such that

$$s(\boldsymbol{E}_1,\boldsymbol{v}) + s_1(\boldsymbol{E}_1,\boldsymbol{v}) = -s(\boldsymbol{E}_g,\boldsymbol{v}) - s_1(\boldsymbol{E}_g,\boldsymbol{v}) \qquad \forall\,\boldsymbol{v}\in\boldsymbol{U}_0, \tag{2.36}$$

where the sesquilinear forms are defined by

$$s(\boldsymbol{u},\boldsymbol{v}) = s_+(\boldsymbol{u},\boldsymbol{v}) - \int_{\Omega_1} 2k^2\boldsymbol{u}\cdot\bar{\boldsymbol{v}}, \quad s_1(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_1} \left[\gamma_t\,\mathbf{curl}\,\mathcal{P}(\boldsymbol{u}) + \mathbf{i}\gamma_T\mathcal{P}(\boldsymbol{u})\right]\cdot\gamma_T\bar{\boldsymbol{v}}.$$

Clearly s, s_1 are continuous on $\boldsymbol{U}$. It suffices to show that (2.36) has a solution.

From (1.1a) and (1.1c), we have $\mathbf{curl}\,\boldsymbol{E}\in\boldsymbol{H}(\mathbf{curl},\Omega_1)$ and $\operatorname{div}(k^2\boldsymbol{E}) = 0$ in Ω_1. Taking $\boldsymbol{v} = \nabla\varphi$, $\varphi\in S_0$ in (2.34), we also have $\operatorname{div}(k^2\boldsymbol{E}_g) = 0$ in Ω_1. This implies $\boldsymbol{E}_1 \in \hat{\boldsymbol{U}}_0$. Using (2.34), $\boldsymbol{E}_1$ can be solved in the subspace: Find $\boldsymbol{E}_1\in\hat{\boldsymbol{U}}_0$ such that

$$s(\boldsymbol{E}_1,\boldsymbol{v}) + s_1(\boldsymbol{E}_1,\boldsymbol{v}) = 2(k^2\boldsymbol{E}_g,\boldsymbol{v})_{\Omega_1} - s_1(\boldsymbol{E}_g,\boldsymbol{v}) \qquad \forall\,\boldsymbol{v}\in\hat{\boldsymbol{U}}_0. \tag{2.37}$$

Let $K_1, K_2\colon \boldsymbol{L}^2(\Omega_1)\to\hat{\boldsymbol{U}}_0$ be the linear operators defined by

$$s_+(K_1(\boldsymbol{u}),\boldsymbol{v}) = 2(k^2\boldsymbol{u},\boldsymbol{v})_{\Omega_1}, \quad s_+(K_2(\boldsymbol{u}),\boldsymbol{v}) = s_1(\boldsymbol{u},\boldsymbol{v}) \qquad \forall\,\boldsymbol{v}\in\hat{\boldsymbol{U}}_0. \tag{2.38}$$

Since s_+ is coercive, K_1, K_2 are well-defined and $\|K_1(\boldsymbol{u})\|_{\boldsymbol{U}} \le C\,\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_1)}$. By the compact embedding of $\hat{\boldsymbol{U}}_0$ into $\boldsymbol{L}^2(\Omega_1)$, K_1 is a compact operator.

By Lemma 2.1, $\mathbb{G}$ and its partial derivatives are bounded as follows

$$\|\nabla_{\boldsymbol{x}}\mathbb{G}(k;\boldsymbol{x},\cdot)\|_{W^{2,\infty}(B_0)} + \|\mathbb{G}(k;\boldsymbol{x},\cdot)\|_{W^{2,\infty}(B_0)} \le C \qquad \forall\,\boldsymbol{x}\in\mathbb{R}^3_\pm\backslash B_1,\ \boldsymbol{y}\in\bar{B}_0.$$

By the definition of $\mathcal{P}$ and the Cauchy-Schwarz inequality, we have

$$\|K_2(\boldsymbol{u})\|_{\boldsymbol{U}} \le C\,\|\boldsymbol{n}\times\mathbf{curl}\,\mathcal{P}(\boldsymbol{u}) + \mathbf{i}\gamma_T\mathcal{P}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Gamma_1)} \le C\,\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_1)}.$$

Therefore, K_2 is also a compact mapping from $\hat{\boldsymbol{U}}_0$ to $\boldsymbol{L}^2(\Omega_1)$.

Now we can write (2.37) into an operator equation

$$\boldsymbol{E}_1 - K_1(\boldsymbol{E}_1) + K_2(\boldsymbol{E}_1) = K_1(\boldsymbol{E}_g) - K_2(\boldsymbol{E}_g)\,. \tag{2.39}$$

This is a Fredholm equation on $\boldsymbol{L}^2(\Omega_1)$. Since the scattering solution $\boldsymbol{E}$ is unique, the solution $\boldsymbol{E}_1$ of (2.37) is also unique. By the Fredholm alternative, we conclude that (2.39) attains a unique solution $\boldsymbol{E}_1 \in \boldsymbol{L}^2(\Omega_1)$. From (2.39) we know that $\boldsymbol{E}_1 \in \hat{\boldsymbol{U}}_0$. Therefore, the weak problem (2.37) or (2.36) has a unique solution.

For the stability of the solution, from (2.39) and (2.35) we know that

$$\|\boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega_1)} = \left\|(I - K_1 + K_2)^{-1}(K_1 - K_2)\boldsymbol{E}_g\right\|_{\boldsymbol{L}^2(\Omega_1)} \leq C \|\boldsymbol{E}_g\|_{\boldsymbol{L}^2(\Omega_1)}.$$

By (2.35), this shows $\|\boldsymbol{E}\|_{\boldsymbol{L}^2(\Omega_1)} \leq C \|\boldsymbol{E}_g\|_{\boldsymbol{L}^2(\Omega_1)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}$. Finally, taking $\boldsymbol{v} = \boldsymbol{E}_1$ in (2.36) leads to

$$\|\mathbf{curl}\,\boldsymbol{E}_1\|^2_{\boldsymbol{L}^2(\Omega_1)} \leq k_-^2 \left|(\boldsymbol{E}, \boldsymbol{E}_1)_{\Omega_1}\right| + |s_1(\boldsymbol{E}, \boldsymbol{E}_1)| + \left|(\mathbf{curl}\,\boldsymbol{E}_g, \mathbf{curl}\,\boldsymbol{E}_1)_{\Omega_1}\right|.$$

We conclude that $\|\mathbf{curl}\,\boldsymbol{E}_1\|_{\boldsymbol{L}^2(\Omega_1)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}$. By (2.35), this yields

$$\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.$$

Finally, for any bounded domain Ω, we need only choose R_1 large enough such that $\Omega \subset \Omega_1$. Then (2.31) follows clearly from the above estimate. □

3. The Cagniard-de Hoop transform. In this section, we shall derive a new integral form of $\mathbb{P}$ by the Cagniard de-Hoop transform [20, Page 215]. It plays the key role in proving the exponential decay of the solution in PML. Without loss of generality, we only consider P_{33} for $\boldsymbol{x} \in \mathbb{R}^3_+$ and $\boldsymbol{y} \in \mathbb{R}^3_-$. The results can be extended straightforwardly to other cases of $\boldsymbol{x}, \boldsymbol{y}$ and to other entries of $\mathbb{P}$.

LEMMA 3.1. *For any* $\boldsymbol{x}_3 \in \mathbb{R}^3_+$ *and* $\boldsymbol{y} \in \mathbb{R}^3_-$, *write* $x_1 - y_1 = r\cos\phi$, $x_2 - y_2 = r\sin\phi$ *with* $r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$ *and* $\phi \in [0, 2\pi]$. *Then, for any* $\varepsilon > 0$,

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})}{\partial x_1^l \partial x_2^m \partial x_3^n} = \mathbf{i}^{l+m+n}\frac{1+\mathbf{i}\varepsilon}{2\pi^2}\int_0^\infty \int_{-\infty}^\infty \frac{\lambda_1^l \lambda_2^m \mu_+^n e^{\mathbf{i}(r\xi + x_3\mu_+ - y_3\mu_-)}}{\mu_+ + k_-^{-2}k_+^2\mu_-}\mathrm{d}\xi\mathrm{d}q,$$

for any integers $l, m, n \geq 0$, *where* $\mu_\pm = \mu_\pm(\xi, (\varepsilon - \mathbf{i})q)$ *and*

$$\lambda_1 = \xi\cos\phi + (\varepsilon - \mathbf{i})q\sin\phi, \qquad \lambda_2 = \xi\sin\phi - (\varepsilon - \mathbf{i})q\cos\phi.$$

Proof. We consider the rotational transform $\xi = \lambda_1\cos\phi + \lambda_2\sin\phi$, $\eta = \lambda_1\sin\phi - \lambda_2\cos\phi$. The definitions of $\mu_\pm$ indicate

$$\mu_\pm(\lambda_1, \lambda_2) = \left(k_\pm^2 - \lambda_1^2 - \lambda_2^2\right)^{1/2} = \left(k_\pm^2 - \xi^2 - \eta^2\right)^{1/2} = \mu_\pm(\xi, \eta).$$

Write $N = l + m + n$. Then from (2.22) we have

$$\frac{\partial^N \mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})}{\partial x_1^l \partial x_2^m \partial x_3^n} = \frac{\mathbf{i}^{N+1}}{2\pi^2}\int_0^\infty F(\eta)\mathrm{d}\eta, \quad F(\eta) = \int_{-\infty}^\infty \frac{\lambda_1^l \lambda_2^m \mu_+^n e^{\mathbf{i}(r\xi + x_3\mu_+ - y_3\mu_-)}}{\mu_+ + k_-^{-2}k_+^2\mu_-}\mathrm{d}\xi.$$

For any fixed $\eta \in \mathbb{C}$, when $|\xi| \gg k_- + |\eta|$, we have

$$\operatorname{Im}\mu_\pm(\xi,\eta) = \sqrt{\frac{|\mu_\pm(\xi,\eta)^2| - \operatorname{Re}\mu_\pm(\xi,\eta)^2}{2}} \geq \sqrt{\xi^2 - k_\pm^2 - |\eta|^2} \geq \frac{1}{2}|\xi|.$$

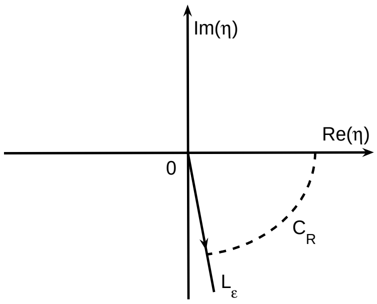

FIG. 3.1. *The deformation of integral path from the real axis to L_ε.*

Thus the dξ-integration in $F(\eta)$ converges for any $|x_3|+|y_3|>0$. Since $\operatorname{Re}\mu_\pm>0$ and $(\operatorname{Im}\mu_+)(\operatorname{Im}\mu_-)>0$, we have $\mu_+ + k_-^{-2}k_+^2\mu_- \neq 0$. Therefore, F defines an analytic function of η.

Now we define a half line in the fourth quadrant of the complex η-plane

$$L_\varepsilon = \{(\varepsilon - \mathbf{i})q : q \geq 0\} = \left\{te^{-\mathbf{i}\theta_\varepsilon} : t \geq 0\right\}, \qquad \theta_\varepsilon = \arcsin\frac{1}{\sqrt{1+\varepsilon^2}}.$$

Let $C_R = \{Re^{\mathbf{i}\theta} : 0 < \theta < \theta_\varepsilon\}$ be the arc of radius R which is bounded by L_ε and the real axis (See Fig. 3.1). We orient L_ε to the downward direction. Suppose that

$$\lim_{R\to\infty}\int_{C_R} F(\eta)\mathrm{d}\eta = 0. \tag{3.1}$$

Then the result follows from Cauchy's theorem and the fact that

$$\mathbf{i}\int_0^\infty F(\eta)\mathrm{d}\eta = \mathbf{i}\int_{L_\varepsilon} F(\eta)\mathrm{d}\eta = (1+\mathbf{i}\varepsilon)\int_0^\infty\int_{-\infty}^\infty \frac{\lambda_1^l\lambda_2^m\mu_+^n e^{\mathbf{i}(r\xi + x_3\mu_+ - y_3\mu_-)}}{\mu_+ + k_-^{-2}k_+^2\mu_-}\mathrm{d}\xi\mathrm{d}q.$$

It is left to show (3.1). Assume $R \gg 2(1+k_-)$ and recall

$$\operatorname{Im}\mu_\pm^2 = \operatorname{Im}\left(k_\pm^2 - \xi^2 - R^2e^{-2\mathbf{i}\theta}\right) = R^2\sin 2\theta \geq 0 \qquad \forall\,\theta\in(0,\theta_\varepsilon).$$

We have $\operatorname{Re}\mu_\pm \geq 0$, $\operatorname{Im}\mu_\pm \geq 0$. There is a constant C independent of ξ,η such that

$$\left|F(Re^{\mathbf{i}\theta})\right| \leq CR^N\int_0^\infty (1+\xi)^N e^{-x_3\operatorname{Im}\mu_+}\mathrm{d}\xi = CR^N[F_1(R,\theta)+F_2(R,\theta)], \tag{3.2}$$

where

$$F_1(R,\theta) = \int_0^{2R}(1+\xi)^N e^{-x_3\operatorname{Im}\mu_+}\mathrm{d}\xi, \quad F_2(R,\theta) = \int_{2R}^\infty (1+\xi)^N e^{-x_3\operatorname{Im}\mu_+}\mathrm{d}\xi.$$

We consider $F_1(R,\theta)$ first. For any $0\leq\theta\leq\theta_\varepsilon/2$, we have

$$\operatorname{Im}\mu_\pm \geq \frac{1}{2}\sqrt{\left|\mu_\pm^2\right| - \operatorname{Re}\mu_\pm^2} \geq \frac{1}{2}\sqrt{\left|\mu_\pm^2\right| + \xi^2 + R^2\cos 2\theta - k_\pm^2} \geq \frac{\varepsilon R}{4} - k_-.$$

For any $\theta_\varepsilon/2 < \theta < \theta_\varepsilon$, from (2.5) we know that

$$\operatorname{Im}\mu_+ = \frac{\operatorname{Im}\mu_+^2}{2\operatorname{Re}\mu_+} \geq \frac{R^2\sin 2\theta}{2(|\xi| + k_- + R)} \geq \frac{R}{8}\min(\sin 2\theta_\varepsilon, \sin\theta_\varepsilon).$$

This shows

$$\lim_{R\to\infty} R^{N+1}\int_0^{\theta_\varepsilon} F_1(R,\theta)=0. \tag{3.3}$$

As for $F_2(R,\theta)$, since $\xi\geq 2R$, we deduce that

$$(\operatorname{Im}\mu_\pm)^2\geq \frac{1}{2}\left(\left|\mu_\pm^2\right|+\xi^2+R^2\cos 2\theta-k_\pm^2\right)\geq \xi^2+R^2\cos 2\theta-k_\pm^2\geq \frac{1}{4}\xi^2.$$

This indicates

$$F_2(R,\theta)\leq \int_{2R}^{+\infty}(1+\xi)^N e^{-\frac{1}{2}x_3\xi}\mathrm{d}\xi\leq Cx_3^{-N-1}e^{-x_3R}.$$

We conclude that

$$\lim_{R\to\infty} R^{N+1}\int_0^{\theta_\varepsilon} F_2(R,\theta)=0. \tag{3.4}$$

Finally, the proof is finished by combining (3.3)–(3.4) with (3.2). □

The integral form in Lemma 3.1 is still unfavorable to the PML analysis. We are going to derive the Cagniard-de Hoop representation of $\mathbb{P}$. This will be fulfilled by deforming the dξ-integration from the real axis to a hyperbolic integral path.

For convenience in notation, we write

$$\kappa_\pm(\varepsilon,q):=\left[k_\pm^2-(\varepsilon-\mathbf{i})^2q^2\right]^{1/2}=\left[k_\pm^2+(1+\mathbf{i}\varepsilon)^2q^2\right]^{1/2}\qquad \forall\, q>0.$$

We shall always abbreviate the notations to $\kappa_\pm:=\kappa_\pm(\varepsilon,q)$ without specifying their dependency on ε and q in this and next sections. From (2.5), we know that $\operatorname{Re}\kappa_\pm$, $\operatorname{Im}\kappa_\pm$ are positive and satisfy

$$\operatorname{Im}\kappa_-\leq \operatorname{Im}\kappa_+\leq \frac{\varepsilon q^2}{\sqrt{k_+^2+(1-\varepsilon^2)q^2}}, \tag{3.5}$$

$$\operatorname{Re}\kappa_-\geq \operatorname{Re}\kappa_+\geq \sqrt{k_+^2+(1-\varepsilon^2)q^2}. \tag{3.6}$$

For convenience, we also use

$$\mu_1(\xi)=\left(\kappa_+^2-\xi^2\right)^{1/2}=\mu_+(\xi,(\varepsilon-\mathbf{i})q),\quad \mu_2(\xi)=\left(\kappa_-^2-\xi^2\right)^{1/2}=\mu_-(\xi,(\varepsilon-\mathbf{i})q),$$

without specifying their dependency on ε and q. By (2.5), the branch cuts for the square roots $\mu_{1,2}=(\kappa_\pm+\xi)^{1/2}(\kappa_\pm-\xi)^{1/2}$ are given by four half-lines

$$C_\pm^R=\{\xi:\,\xi=\kappa_\pm+t,\ t\geq 0\}\,,\qquad C_\pm^L=\{\xi:\,\xi=-\kappa_\pm-t,\ t\geq 0\}\,.$$

For any fixed $q\geq 0$, μ_1,μ_2 are analytic functions of ξ in $\mathbb{C}\backslash(C_+^L\cup C_+^R)$ and $\mathbb{C}\backslash(C_-^L\cup C_-^R)$ respectively.

Given a function h in the complex ξ-plane, define

$$I(h;r,z)=\int_{-\infty}^{+\infty}h(\xi)e^{\mathbf{i}(r\xi+z\mu_1)}\mathrm{d}\xi\qquad \forall\, r>0,\ z>0.$$

We shall rewrite the integral by the Cagniard-de Hoop transform. The theory will be applied to the Perturbation tensor later.

LEMMA 3.2. *For any $q > 0$ and $0 < \varepsilon \ll 1$, let h be an analytic function in $\mathbb{C}\backslash(C_+^L \cup C_+^R \cup C_-^L \cup C_-^R)$ and satisfy $|h(\xi)| \leq C(1+|\xi|)^m$ for some integer m and some constant $C > 0$. Then for any r, z satisfying $z \geq 2\varepsilon r > 0$,*

$$I(h; r, z) = -\mathbf{i}\int_1^\infty [h(\xi_+(t))\Lambda_+(t) + h(\xi_-(t))\Lambda_-(t)]\,\frac{e^{\mathbf{i}\kappa_+\rho t}}{\sqrt{t^2-1}}\mathrm{d}t, \tag{3.7}$$

where $\rho = \sqrt{r^2+z^2}$ and $\xi_\pm, \Lambda_\pm$ are defined by the Cagniard-de Hoop transform

$$\xi_\pm(t) = \frac{\kappa_+}{\rho}\big(rt \pm \mathbf{i}z\sqrt{t^2-1}\big), \qquad \Lambda_\pm(t) = \frac{\kappa_+}{\rho}\big(zt \mp \mathbf{i}r\sqrt{t^2-1}\big). \tag{3.8}$$

Proof. First we define a hyperbolic integral path $\Gamma = \Gamma_+ \cup \Gamma_-$ where

$$\Gamma_\pm = \{\xi_\pm(t) :\ t \geq 1\}\ .$$

Notice that $\Lambda_\pm^2(t) = \kappa_+^2 - \xi_\pm^2(t)$ for any $\xi_\pm(t) \in \Gamma$. From (3.5)–(3.6), we have

$$\rho\,\mathrm{Re}\,\Lambda_\pm(t) \geq zt\,\mathrm{Re}\,\kappa_+ - r\sqrt{t^2-1}\,\mathrm{Im}\,\kappa_+ \geq \frac{\varepsilon rt}{\mathrm{Re}\,\kappa_+}\left[2(1-\varepsilon^2)q^2 - q^2\right] \geq 0.$$

By the convention in (2.5), we have $\Lambda_\pm(t) = \mu_1(\xi_\pm(t))$.

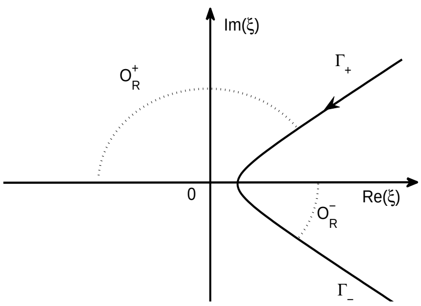

FIG. 3.2. *The Cagniard-de Hoop transform from the real axis to $\Gamma_+ \cup \Gamma_-$.*

For any $R > 0$, let O_R^+, O_R^- be the parts of the circle $\{\xi : |\xi| = R\}$ that are bounded by the real axis and $\Gamma_\pm$ respectively (see Fig. 3.2). Suppose that

$$\lim_{R\to\infty} F_\pm(R) = 0, \qquad F_\pm(R) := \int_{O_R^\pm} h(\xi)e^{\mathbf{i}(r\xi+z\mu_1)}\mathrm{d}\xi. \tag{3.9}$$

By Cauchy's theorem, (3.7) follows from the fact that

$$I(h; r, z) = \int_\Gamma h(\xi)e^{\mathbf{i}(r\xi+z\mu_1)}\mathrm{d}\xi.$$

It is left to show (3.9). We only prove the limit for $F_+(R)$. The proof for $F_-(R)$ is similar and omitted here. Let $\xi_R = Re^{\mathbf{i}\theta_R}$ be the intersection of O_R^+ and Γ_+. Then $F_+(R)$ can be written into

$$F_+(R) = \mathbf{i}\int_{\theta_R}^{\pi} h(Re^{\mathbf{i}\theta})e^{-rR\sin\theta}e^{\mathbf{i}(rR\cos\theta+\mu_1 z)}Re^{\mathbf{i}\theta}\mathrm{d}\theta,$$

where $\mu_1 = \mu_1(Re^{\mathbf{i}\theta})$. Since $\left|h(Re^{\mathbf{i}\theta})\right| \le C(1+R)^m$, we have

$$|F_+(R)| \le CR^{m+1} \int_{\theta_R}^{\pi} e^{-(rR\sin\theta + z\,\mathrm{Im}\,\mu_1)} \mathrm{d}\theta \,. \tag{3.10}$$

Without loss of generality, we assume that $R \gg 2(k_- + q)$ and define

$$\theta_0 := \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{2\varepsilon q^2}{R^2}. \tag{3.11}$$

Clearly $\pi/4 < \theta_0 < \pi/2$. Since $\mathrm{Im}\,\mu_1^2 = 2\varepsilon q^2 - R^2\sin 2\theta$, by (2.5), we have

$$\mathrm{Im}\,\mu_1 > 0 \quad \forall\,\theta \in (\theta_0, \pi); \qquad \mathrm{Im}\,\mu_1 < 0 \quad \forall\,\theta \in (\theta_R, \theta_0). \tag{3.12}$$

From (2.5), we also have $\mathrm{Im}\,\mu_1 \ge R/8$ for any $5\pi/6 \le \theta \le \pi$. Then

$$\int_{\theta_0}^{\pi} e^{-(rR\sin\theta + z\,\mathrm{Im}\,\mu_1)} \mathrm{d}\theta \le \int_{\theta_0}^{\frac{5\pi}{6}} e^{-rR\sin\theta}\mathrm{d}\theta + \int_{\frac{5\pi}{6}}^{\pi} e^{-z\,\mathrm{Im}\,\mu_1}\mathrm{d}\theta \le \pi e^{-\frac{rR}{4}} + \pi e^{-\frac{zR}{8}} \,.$$

This shows that

$$\lim_{R\to\infty} R^{m+1} \int_{\theta_0}^{\pi} e^{-(rR\sin\theta + z\,\mathrm{Im}\,\mu_1)}\mathrm{d}\theta = 0. \tag{3.13}$$

Now we consider the case of $\theta \in (\theta_R, \theta_0)$. For $R \gg \left|\kappa_+^2\right|$, careful calculations yield

$$\frac{\partial}{\partial\theta}\,\mathrm{Im}\left(\kappa_+^2 - R^2 e^{2\mathbf{i}\theta}\right)^{1/2} \ge 0 \qquad \forall\,\theta \in (\theta_R, \theta_0).$$

Thus $\mathrm{Im}\,\mu_1(Re^{\mathbf{i}\theta})$ is increasing with respect to $\theta \in (\theta_R, \theta_0)$. Let $r = \rho\cos\phi$, $z = \rho\sin\phi$ with $\phi \in (0, \pi/2)$. Since $\xi_R = \xi_+(t_R)$ for some $t_R \ge 1$, we have

$$\xi_R = \kappa_+\left(\cos\phi\, t_R + \mathbf{i}\sin\phi\sqrt{t_R^2 - 1}\right), \quad \mu_1(\xi_R) = \kappa_+\left(\sin\phi\, t_R - \mathbf{i}\cos\phi\sqrt{t_R^2 - 1}\right).$$

Note that $R = |\kappa_+|\sqrt{t_R^2\cos^2\phi + (t_R^2 - 1)\sin^2\phi} \le |\kappa_+|\,t_R$. We get

$$rR\sin\theta + z\,\mathrm{Im}\,\mu_1 \ge r\,\mathrm{Im}\,\xi_R + z\,\mathrm{Im}\,\mu_1(\xi_R) = \rho t_R\,\mathrm{Im}\,\kappa_+ \ge \frac{\mathrm{Im}\,\kappa_+}{|\kappa_+|}\rho R\,.$$

This yields

$$\lim_{R\to\infty} R^{m+1} \int_{\theta_R}^{\theta_0} e^{-(rR\sin\theta + z\,\mathrm{Im}\,\mu_1)}\mathrm{d}\theta = 0. \tag{3.14}$$

Finally, we obtain (3.9) by substituting (3.13) and (3.14) into (3.10). □

Now we apply Lemma 3.2 to the perturbation tensor and its derivatives. For convenience, we write $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, $z = |x_3| + |y_3|$, and $\rho = \sqrt{r^2 + z^2}$. Let the Cagniard-de Hoop transform be defined in (3.8). Since $\mathrm{Re}\,\Lambda_\pm \ge 0$, it is easy to see that

$$\mu_1(\xi_\pm) = \Lambda_\pm, \qquad \mu_2(\xi_\pm) = \left(k_-^2 - k_+^2 + \Lambda_\pm^2\right)^{1/2}.$$

For a function $f(\xi)$, we define

$$J_{\rm cdh}(f;\boldsymbol{x},\boldsymbol{y})=\frac{\varepsilon-\mathbf{i}}{2\pi^2}\int_0^\infty\int_1^\infty[\Lambda_+(t)f(\xi_+(t))+\Lambda_-(t)f(\xi_-(t))]\,\frac{e^{\mathbf{i}\kappa_+\rho t}}{\sqrt{t^2-1}}\mathrm{d}t\mathrm{d}q.$$

Let h_1,h_2,h_3 be defined by (2.17) with μ_+,μ_- replaced by μ_1,μ_2 respectively. The diagonal entries of the perturbation tensor $\mathbb{P}$ are given by, for $j=1,2$ and $z\geq 2\varepsilon r$,

$$\mathrm{P}_{jj}(k;\boldsymbol{x},\boldsymbol{y})=\begin{cases}J_{\rm cdh}(h_1;\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_+,\\ J_{\rm cdh}(h_1e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_+,\ \boldsymbol{y}\in\mathbb{R}^3_-,\\ J_{\rm cdh}(h_1e^{\mathbf{i}(\mu_1-\mu_2)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_-,\ \boldsymbol{y}\in\mathbb{R}^3_+,\\ J_{\rm cdh}(h_1e^{\mathbf{i}(\mu_1-\mu_2)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_-,\end{cases}\tag{3.15}$$

$$\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})=\begin{cases}k_-^2J_{\rm cdh}(h_2;\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_+,\\ k_-^2J_{\rm cdh}(h_2e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_+,\ \boldsymbol{y}\in\mathbb{R}^3_-,\\ k_+^2J_{\rm cdh}(h_2e^{\mathbf{i}(\mu_1-\mu_2)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_-,\ \boldsymbol{y}\in\mathbb{R}^3_+,\\ k_+^2J_{\rm cdh}(h_2e^{\mathbf{i}(\mu_1-\mu_2)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_-.\end{cases}\tag{3.16}$$

Write $x_1-y_1=r\cos\phi$, $x_2-y_2=r\sin\phi$. Then $\mathrm{P}_{23}=\tan\phi\,\mathrm{P}_{13}$ where

$$\frac{\mathrm{P}_{13}(k;\boldsymbol{x},\boldsymbol{y})}{\cos\phi}=\begin{cases}J_{\rm cdh}(\xi h_3;\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_+,\\ J_{\rm cdh}(\xi h_3e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_+,\ \boldsymbol{y}\in\mathbb{R}^3_-,\\ J_{\rm cdh}(\xi h_3e^{\mathbf{i}(\mu_1-\mu_2)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x}\in\mathbb{R}^3_-,\ \boldsymbol{y}\in\mathbb{R}^3_+,\\ J_{\rm cdh}(\xi h_3e^{\mathbf{i}(\mu_1-\mu_2)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if } \boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_-.\end{cases}\tag{3.17}$$

Similarly, we can obtain the Cagniard-de Hoop representations for derivatives of $\mathbb{P}$. We only give the derivatives of $\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})$ for $\boldsymbol{x}\in\mathbb{R}^3_+$ and $\boldsymbol{y}\in\mathbb{R}^3_-$. The other cases are similar. By Lemma 3.2, we have, for any $z\geq 2\varepsilon r$,

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})}{\partial x_1^l\partial x_2^m\partial x_3^n}=\mathbf{i}^{l+m+n}k_-^2J_{\rm cdh}(\lambda_1^l\lambda_2^m\mu_1^nh_2e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}),\tag{3.18}$$

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})}{\partial y_1^l\partial y_2^m\partial y_3^n}=(-\mathbf{i})^{l+m+n}k_-^2J_{\rm cdh}(\lambda_1^l\lambda_2^m\mu_2^nh_2e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}),\tag{3.19}$$

where $\lambda_1=\xi\cos\phi+(\varepsilon-\mathbf{i})q\sin\phi$ and $\lambda_2=\xi\sin\phi-(\varepsilon-\mathbf{i})q\cos\phi$.

4. Perfectly matched layer. Now we introduce the wave-absorbing material, or, the perfectly matched layer. To make the main theme more focused on the layered medium, we only consider spherical PML in this paper.

4.1. Complex coordinate stretching. Let $B_0:=B(R_0)$ be the ball of radius $R_0\geq 1$ which contains D and where the scattering field is interested. For any $\boldsymbol{x}\in\mathbb{R}^3$, let $\rho=|\boldsymbol{x}|$ and $\hat{\boldsymbol{x}}=\boldsymbol{x}/\rho$. The complex stretching is defined by $\tilde{\boldsymbol{x}}=\tilde{\rho}\hat{\boldsymbol{x}}$ where

$$\tilde{\rho}=\rho\alpha(\rho),\qquad \alpha(\rho)=1+\frac{\mathbf{i}}{\rho}\int_0^\rho\sigma(t)\mathrm{d}t.\tag{4.1}$$

The PML medium property σ is defined piecewise by

$$\sigma(R_0+sR_0/2)=\sigma_0\hat{\sigma}(s),\qquad \hat{\sigma}(s)=\begin{cases}0 & \text{if } s\leq 0,\\ 6s^2-4s^3 & \text{if } 0<s<1,\\ 2 & \text{if } s\geq 1.\end{cases}\tag{4.2}$$

Clearly σ is C^1-smooth and satisfies $\sigma(t) = 0$ for all $t \le R_0$ and $\sigma(t) = 2\sigma_0$ for all $t \ge 1.5R_0$. Here $\sigma_0 \ge 1$ is the medium property parameter. It is well-known that larger value of σ_0 means faster decay of the scattering solution in the PML. For theoretical analysis, we assume $\sigma_0 \ge 4$ in the rest of the paper. The theory allows more general definitions of σ. Here we do not elaborate on the details (see [6,12]).

Write the complex stretching by $\boldsymbol{F}(\boldsymbol{x}) := \tilde{\boldsymbol{x}}$. Then $\boldsymbol{F}$ is C^2-smooth. In the rest, both $\boldsymbol{F}(\boldsymbol{x})$ and $\tilde{\boldsymbol{x}}$ will denote the same complex vector. It is easy to see that the Jacobi matrix of $\mathbb{F}$ is given by

$$\mathbb{B} := D\boldsymbol{F} = \alpha(\rho)\mathbb{I} + \rho\alpha'(\rho)\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top. \tag{4.3}$$

Clearly $\mathbb{B}$ is symmetric and C^1-smooth. Its determinant is given by

$$J = \det(\mathbb{B}) = \alpha^2(\alpha + \rho\alpha'). \tag{4.4}$$

The analytic continuations of 2D and 3D distance functions are defined by

$$r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \left[(\tilde{x}_1 - \tilde{y}_2)^2 + (\tilde{x}_2 - \tilde{y}_2)^2\right]^{1/2}, \qquad d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \left[(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{y}}) \cdot (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{y}})\right]^{1/2}.$$

By direct calculations, we have $\operatorname{Im} r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \ge 0$ and

$$|\boldsymbol{x} - \boldsymbol{y}| \le |d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \le \sqrt{1 + 16\sigma_0^2}\, |\boldsymbol{x} - \boldsymbol{y}|, \tag{4.5}$$

$$|r(\boldsymbol{x}, \boldsymbol{y})| \le |r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \le \sqrt{1 + 16\sigma_0^2}\, |r(\boldsymbol{x}, \boldsymbol{y})|. \tag{4.6}$$

Moreover, if $\max(|\boldsymbol{x}|, |\boldsymbol{y}|) \ge 2R_0$, we also have

$$\operatorname{Im} d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \ge \frac{1}{2}\sigma_0 |\boldsymbol{x} - \boldsymbol{y}|. \tag{4.7}$$

LEMMA 4.1. *For any* $\boldsymbol{x} \in \mathbb{R}^3_+$ *and* $\boldsymbol{y} \in \mathbb{R}^3_-$*, suppose* $\max(|\boldsymbol{x}|, |\boldsymbol{y}|) \ge 2R_0$ *and* $\operatorname{Im}(\tilde{x}_3 - \tilde{y}_3) \ge x_3 - y_3$*. Then*

$$|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \ge |r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})|.$$

Proof. For convenience, we write $\tilde{r} = r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = r_1 + \mathbf{i}r_2$, $\tilde{z} = \tilde{x}_3 - \tilde{y}_3 = z_1 + \mathbf{i}z_2$ with $r_1, r_2 \ge 0$ and $z_2 \ge z_1 \ge 0$. If $r_2 \ge r_1$, we find that

$$|d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})|^4 = |\tilde{r}|^4 + |\tilde{z}|^4 + 8r_1r_2z_1z_2 + 2(z_2^2 - z_1^2)(r_2^2 - r_1^2) \ge |\tilde{r}|^4 + |\tilde{z}|^4.$$

If $r_2 < r_1$, we have $\operatorname{Re} \tilde{r}^2 \ge 0$ and $|\tilde{r}|^2 \le 2|\boldsymbol{x} - \boldsymbol{y}|^2$. The proof is finished by (4.7). ☐

4.2. PML extension of the Cagniard de-Hoop transform. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_\pm$, let $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ be their complex stretching respectively. The analytic continuation of the dyadic Green's function is defined by

$$\mathbb{G}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) + \frac{1}{k^2}\nabla_{\tilde{\boldsymbol{y}}} \operatorname{div}_{\tilde{\boldsymbol{y}}} \mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}), \qquad \tilde{\boldsymbol{x}} \ne \tilde{\boldsymbol{y}}.$$

where $\mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \mathbb{S}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) - \mathbb{P}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is the analytic continuation of the Hertz tensor for $\tilde{\boldsymbol{x}} \ne \tilde{\boldsymbol{y}}$, $\mathbb{S}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is defined by replacing $\boldsymbol{x}, \boldsymbol{y}$ with $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ in (2.18), and $\mathbb{P}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is defined by replacing $\boldsymbol{x}, \boldsymbol{y}$ with $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ in (3.15)–(3.17). We extend the Cagniard-de

Hoop transform from real coordinates to complex coordinates and prove some useful estimates.

LEMMA 4.2. *For any* $\boldsymbol{x} \in \mathbb{R}^3_+$ *and* $\boldsymbol{y} \in \mathbb{R}^3_-$, *let* $\tilde{r} = r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$, $\tilde{z} = \tilde{x}_3 - \tilde{y}_3$, *and* $\tilde{d} = d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$. *Assume* $\max(|\boldsymbol{x}|, |\boldsymbol{y}|) \geq 2R_0$ *and* $\operatorname{Im} \tilde{z} \geq \operatorname{Re} \tilde{z}$. *For any* $\kappa > 0$, *if* $\xi_\pm(t) = \kappa \tilde{d}^{-1}\big(\tilde{r}t \pm \mathbf{i}\tilde{z}\sqrt{t^2-1}\big)$, *then*

$$\big(\kappa^2 - \xi_\pm^2(t)\big)^{1/2} = \kappa \tilde{d}^{-1}\left(\tilde{z}t \mp \mathbf{i}\tilde{r}\sqrt{t^2-1}\right) \qquad \forall\, t \geq 1.$$

Proof. Write $\Lambda_\pm = \kappa\tilde{d}^{-1}(\tilde{z}t \mp \mathbf{i}\tilde{r}t_1)$ with $t_1 = \sqrt{t^2-1}$. It is clear that $\Lambda_\pm^2 + \xi_\pm^2 = \kappa^2$. By the convention in (2.5), it suffices to show $\operatorname{Re}\Lambda_\pm \geq 0$. We only prove $\operatorname{Re}\Lambda_+ \geq 0$ here. The proof for $\operatorname{Re}\Lambda_- \geq 0$ is similar.

Without loss of generality, we assume $|\boldsymbol{x}| \geq |\boldsymbol{y}|$. Let $\tilde{r} = r_1 + \mathbf{i}r_2$, $\tilde{z} = z_1 + \mathbf{i}z_2$, $\tilde{d} = d_1 + \mathbf{i}d_2$ with $r_i, d_i \geq 0$ and $z_2 \geq z_1 \geq 0$. Then

$$\kappa^{-1}\big|\tilde{d}\big|^2 \operatorname{Re}\Lambda_+ = d_1(z_1t + r_2t_1) + d_2(z_2t - r_1t_1) \geq t_1(M-N),$$

where $M = d_1z_1 + d_1r_2 + d_2z_2$ and $N = d_2r_1$. Using Lemma 4.1 and $z_2 \geq z_1$, we know that $\operatorname{Re}\tilde{z}^2 \leq 0$ and $\big|\tilde{d}\big| \geq |\tilde{r}|$. From (4.5) and (4.7), we have $\operatorname{Re}\tilde{d}^2 \leq 0$. Then the convention in (2.5) shows that

$$\begin{aligned}
\frac{1}{2}\left(M^2 - N^2\right) &\geq \frac{1}{2}\left(d_1^2z_1^2 + d_2^2z_2^2 + d_1^2r_2^2 - d_2^2r_1^2\right) \\
&= \big|\tilde{d}^2\big|\,\big|\tilde{z}^2\big| + \operatorname{Re}\tilde{d}^2\operatorname{Re}\tilde{z}^2 + \operatorname{Re}\tilde{d}^2\,\big|\tilde{r}^2\big| - \big|\tilde{d}^2\big|\operatorname{Re}\tilde{r}^2 \\
&\geq \big|\tilde{d}^2\big|\big(\big|\tilde{z}^2\big| + \operatorname{Re}\tilde{z}^2\big) + \operatorname{Re}\tilde{d}^2\big(\big|\tilde{r}^2\big| - \big|\tilde{d}^2\big|\big) \geq \big|\tilde{d}^2\big|\big(\big|\tilde{z}^2\big| + \operatorname{Re}\tilde{z}^2\big).
\end{aligned}$$

Therefore, $M \geq N$, that is, $\operatorname{Re}\Lambda_+ \geq 0$. ☐

LEMMA 4.3. *Let* $\xi_\pm$ *and the assumptions be same to those in Lemma 4.2. For either* $\xi = \xi_+$ *or* $\xi = \xi_-$, *define* $\mu_j = \left(\kappa_j^2 - \xi^2\right)^{1/2}$, $j = 1, 2$ *with* $\kappa_2 \geq \kappa_1 = \kappa$. *Then*

$$\operatorname{Im}\left[(\mu_1 - \mu_2)(a + \mathbf{i}b)\right] \leq 0 \qquad \forall\, b \geq a \geq 0.$$

Proof. We only prove the lemma for $\xi = \kappa_1\tilde{d}^{-1}(\tilde{r}t + \mathbf{i}\tilde{z}t_1)$ where $t_1 = \sqrt{t^2-1}$. The proof for $\xi = \kappa_1\tilde{d}^{-1}(\tilde{r}t - \mathbf{i}\tilde{z}t_1)$ is similar and omitted here.

Write $\mu_j = \alpha_j + \mathbf{i}\beta_j$ with $\alpha_j, \beta_j \in \mathbb{R}$, $j = 1, 2$. Since $\mu_2^2 - \mu_1^2 = \kappa_2^2 - \kappa_1^2$, we have

$$\alpha_2^2 - \beta_2^2 = \kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2, \qquad \alpha_1\beta_1 = \alpha_2\beta_2.$$

We recall (2.5) and deduce that

$$\sqrt{2}\alpha_2 = \left[\sqrt{\left(\kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2\right)^2 + 4\alpha_1^2\beta_1^2} + \left(\kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2\right)\right]^{1/2}, \tag{4.8}$$

$$\sqrt{2}\,|\beta_2| = \left[\sqrt{\left(\kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2\right)^2 + 4\alpha_1^2\beta_1^2} - \left(\kappa_2^2 - \kappa_1^2 + \alpha_1^2 - \beta_1^2\right)\right]^{1/2}. \tag{4.9}$$

Since $\alpha_1 \geq 0$ by (2.5), direct calculations show that

$$\alpha_2 \geq \alpha_1 \geq 0, \qquad |\beta_2| \leq |\beta_1|, \qquad \operatorname{sign}(\beta_1) = \operatorname{sign}(\beta_2). \tag{4.10}$$

Since

$$\operatorname{Im}\left[(\mu_1-\mu_2)(a+\mathbf{i}b)\right]=a(\beta_1-\beta_2)+b(\alpha_1-\alpha_2), \tag{4.11}$$

the lemma now follows obviously for $\beta_1 \le 0$.

Now we assume $\beta_1>0$. By Lemma 4.2, we have $\mu_1=\kappa_1\tilde{d}^{-1}\left(\tilde{z}t-\mathbf{i}\tilde{r}t_1\right)$. Similar to the proof of Lemma 4.2, we write $\tilde{d}=d_1+\mathbf{i}d_2$, $\tilde{r}=r_1+\mathbf{i}r_2$, $\tilde{z}=z_1+\mathbf{i}z_2$ with $r_1,r_2\ge 0$, $d_2\ge d_1\ge 0$, and $z_2\ge z_1\ge 0$. Then

$$\begin{aligned}\kappa_1^{-1}|\tilde{d}|^2\alpha_1&=t(d_1z_1+d_2z_2)+t_1(d_1r_2-d_2r_1),\\ \kappa_1^{-1}|\tilde{d}|^2\beta_1&=t(d_1z_2-d_2z_1)-t_1(d_1r_1+d_2r_2).\end{aligned}$$

We deduce that

$$\kappa_1^{-1}|\tilde{d}|^2(\alpha_1-\beta_1)\ge(d_2-d_1)(z_1+z_2+r_2-r_1)\ge 0.$$

This means $\alpha_2\ge\alpha_1\ge\beta_1$. Since $\beta_1>0$ and $\alpha_1\beta_1=\alpha_2\beta_2$, by (4.11), we have

$$\operatorname{Im}\left[(\mu_1-\mu_2)(a+\mathbf{i}b)\right]=\frac{1}{\beta_1}(\beta_2-\beta_1)(b\alpha_2-a\beta_1)\le\frac{a}{\beta_1}(\beta_2-\beta_1)(\alpha_2-\beta_1)\le 0,$$

where we have used $0\le a\le b$ and (4.10). The proof is completed. □

4.3. Exponential decay of the Green's function in PML. Now we prove the exponential decay of $\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ as $|\boldsymbol{x}-\boldsymbol{y}|\to+\infty$. It depends greatly on the estimate in (4.7). Therefore, we assume through this subsection that

$$\max(|\boldsymbol{x}|,|\boldsymbol{y}|)\ge 2R_0\,.$$

From Lemma 3.2 and (3.15)–(3.19), we know that $\mathbb{P}$ and its derivatives are defined with an arbitrary parameter $0<\varepsilon\ll 1$ through

$$\kappa_\pm(\varepsilon,q)=\left[k_\pm^2+(1+\mathbf{i}\varepsilon)^2q^2\right]^{1/2}\qquad\forall\, q\ge 0.$$

LEMMA 4.4. *Suppose* $|\boldsymbol{x}-\boldsymbol{y}|\ge 1$ *and* $\max(|\boldsymbol{x}|,|\boldsymbol{y}|)\ge 2R_0$. *Then* $\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ *and its derivatives are given by setting* $\varepsilon=0$ *and replacing* $\boldsymbol{x},\boldsymbol{y}$ *with* $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ *in* (3.15)–(3.19).

Proof. Without loss of generality, we only prove the lemma for $\frac{\partial^m \mathrm{P}_{33}}{\partial\tilde{x}_1^m}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ and for the case of $\boldsymbol{x}\in\mathbb{R}^3_+$ and $\boldsymbol{y}\in\mathbb{R}^3_-$. The results can be extended straightforwardly to other cases of $\boldsymbol{x},\boldsymbol{y}$, to other entries of $\mathbb{P}$, and to other derivatives of $\mathbb{P}$. The details are omitted here.

From (3.18)–(3.19), P_{33} and its derivatives depend on the parameter $\varepsilon>0$. We write $\mathrm{P}_{33}(k,\varepsilon;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}):=\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ to specify the dependency of P_{33} on ε. It suffices to show

$$\frac{\partial^m \mathrm{P}_{33}}{\partial\tilde{x}_1^m}(k,0;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})=\lim_{\varepsilon\to 0^+}\frac{\partial^m \mathrm{P}_{33}}{\partial\tilde{x}_1^m}(k,\varepsilon;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}),\qquad m\ge 0, \tag{4.12}$$

where $\frac{\partial^m \mathrm{P}_{33}}{\partial\tilde{x}_1^m}(k,0;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ is given by setting $\varepsilon=0$ and replacing $\boldsymbol{x},\boldsymbol{y}$ with $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ in (3.18).

For convenience, we write $\tilde{r}=r(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$, $\tilde{z}=\tilde{x}_3-\tilde{y}_3$, and $\tilde{d}=d(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$. The PML extension of the Cagniard-de Hoop transform in (3.8) is defined by

$$\xi_\pm(t)=\frac{\kappa_+(\varepsilon,q)}{\tilde{d}}\left(\tilde{r}t\pm\mathbf{i}\tilde{z}\sqrt{t^2-1}\right),\quad \Lambda_\pm(t)=\frac{\kappa_+(\varepsilon,q)}{\tilde{d}}\left(\tilde{z}t\mp\mathbf{i}\tilde{r}\sqrt{t^2-1}\right)\quad\forall\, t\ge 1.$$

Let C denote the generic constant which is independent of $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{y}}$, ε, q, and t. By Lemma 4.1, we have

$$|\xi_\pm(t)| + |\Lambda_\pm(t)| \le C\frac{|\tilde{r}| + |\tilde{z}|}{|\tilde{d}|}(k_+ + q)t \le C(k_+ + q)t. \tag{4.13}$$

Define $\mu_1(\xi_\pm) = \left[\kappa_+(\varepsilon, q)^2 - \xi_\pm^2\right]^{1/2}$ and $\mu_2(\xi_\pm) = \left[\kappa_-(\varepsilon, q)^2 - \xi_\pm^2\right]^{1/2}$. From (2.5), we have

$$\operatorname{Re}\mu_1 \ge 0, \qquad \operatorname{Re}\mu_2 \ge 0, \qquad \operatorname{sign}(\operatorname{Im}\mu_1) = \operatorname{sign}(\operatorname{Im}\mu_2).$$

This shows $|\mu_1 - \mu_2| \le |\mu_1 + \mu_2|$. Since $\mu_1^2 - \mu_2^2 = k_+^2 - k_-^2$, we have

$$\left|k_-^2\mu_1 + k_+^2\mu_2\right| \ge k_+^2\,|\mu_1 + \mu_2| \ge k_+^2\left|\mu_1^2 - \mu_2^2\right|^{1/2} \ge k_+^2(k_- - k_+). \tag{4.14}$$

Let $z = x_3 - y_3$, $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, and $d = |\boldsymbol{x} - \boldsymbol{y}|$. Replacing $\boldsymbol{x}, \boldsymbol{y}$ with $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ in (3.18), we find that, for any $z \ge \varepsilon r$,

$$\left|\frac{\partial^m \mathrm{P}_{33}}{\partial \tilde{x}_1^m}(k, \varepsilon; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})\right| \le Ce^{k_-|y_3|}\int_0^\infty\int_1^\infty [(k_+ + q)t]^{m+1}\frac{\left|e^{\mathbf{i}\kappa_+(\varepsilon,q)\tilde{d}t}\right|}{\sqrt{t^2 - 1}}\mathrm{d}t\mathrm{d}q. \tag{4.15}$$

From (4.5) and (4.7), we have $\operatorname{Im}\tilde{d} \ge \frac{1}{2}\sigma_0 d \ge 2d$. This indicates $\operatorname{Re}\tilde{d}^2 \le 0$ and $\operatorname{Im}\tilde{d} \ge \operatorname{Re}\tilde{d}$. Then using (3.5)–(3.6), we have $\left|e^{\mathbf{i}\kappa_+(\varepsilon,q)\tilde{d}t}\right| \le e^{-\frac{1}{8}\sigma_0 d(k_+ + q)t}$. Inserting the estimates into (4.15), we get

$$\left|\frac{\partial^m \mathrm{P}_{33}}{\partial \tilde{x}_1^m}(k, \varepsilon; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})\right| \le Ce^{k_-|y_3|}\int_0^\infty\int_1^\infty \frac{[(k_+ + q)t]^{m+1}}{\sqrt{t^2 - 1}}e^{-\frac{1}{8}\sigma_0 d(k_+ + q)t}\mathrm{d}t\mathrm{d}q.$$

The integral on the righthand side is convergent and independent of ε. Then (4.12) is obtained by using the dominated convergence theorem. □

Lemma 4.5. *Let $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ be the complex stretching of $\boldsymbol{x}, \boldsymbol{y}$ and write $\boldsymbol{\zeta} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$. There is a constant C depending only on k such that*

$$\left|\frac{\partial^{m+n}}{\partial \zeta_i^m \partial \zeta_j^n}\mathbb{S}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})\right| \le C \times \begin{cases} |\boldsymbol{x} - \boldsymbol{y}|^{-m-n-1}, & \text{if } \max(|\boldsymbol{x}|, |\boldsymbol{y}|) < 2R_0,\ |\boldsymbol{x} - \boldsymbol{y}| \le 1, \\ e^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x} - \boldsymbol{y}|}, & \text{otherwise}, \end{cases}$$

for any $1 \le i, j \le 6$ and $m, n \ge 0$.

Proof. Since $\Phi(k_\pm; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = [4\pi d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})]^{-1}e^{ik_\pm d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})}$, the lemma comes directly from (2.18), (4.5), and (4.7). □

Lemma 4.6. *For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_\pm$, let $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ be their complex stretching and assume*

$$\max(|\boldsymbol{x}|, |\boldsymbol{y}|) \ge 2R_0, \qquad (x_1 - y_1)^2 + (x_2 - y_3)^2 + (|x_3| + |y_3|)^2 \ge 1.$$

Write $\boldsymbol{\zeta} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$. There is a constant C depending only on k such that

$$\left|\frac{\partial^{m+n}}{\partial \zeta_i^m \partial \zeta_j^n}\mathbb{P}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})\right| \le Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x} - \boldsymbol{y}|}, \qquad 1 \le i, j \le 6, \quad m, n \ge 0.$$

Proof. Without loss of generality, we assume $\boldsymbol{x} \in \mathbb{R}^3_+$, $\boldsymbol{y} \in \mathbb{R}^3_-$ and only consider the derivatives of $\mathrm{P}_{33}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ with respect to $\tilde{\boldsymbol{x}}$. The results can be extended straightforwardly to other entries of $\mathbb{P}$, to other derivatives, and to other cases of $\boldsymbol{x}, \boldsymbol{y}$. Furthermore, by Lemma 4.4, it suffices to consider the case of $\varepsilon = 0$. This means

$$\kappa_\pm = \kappa_\pm(0, q) = (k_\pm^2 + q^2)^{1/2} > 0.$$

Write $\tilde{r} = r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$, $\tilde{z} = \tilde{x}_3 - \tilde{y}_3$, and $\tilde{d} = d(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ for convenience. Define the PML extension of the Cagniard-de Hoop transform by

$$\xi_\pm(t) = \kappa_+ \tilde{d}^{-1}\left(\tilde{r}t \pm \mathbf{i}\tilde{z}\sqrt{t^2-1}\right), \quad \Lambda_\pm(t) = \kappa_+ \tilde{d}^{-1}\left(\tilde{z}t \mp \mathbf{i}\tilde{r}\sqrt{t^2-1}\right), \quad \forall\, t \geq 1.$$

Write $\mu_1(\xi) = \left(\kappa_+^2 - \xi^2\right)^{1/2}$ and $\mu_2(\xi) = \left(\kappa_-^2 - \xi^2\right)^{1/2}$. From Lemma 4.2 we have

$$\mu_1(\xi_\pm) = \Lambda_\pm, \qquad \mu_2(\xi_\pm) = \left(k_-^2 - k_+^2 + \Lambda_\pm^2\right)^{1/2}.$$

Replacing $\boldsymbol{x}, \boldsymbol{y}$ with $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ and setting $\varepsilon = 0$ in (3.18), we find that

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})}{\partial\tilde{x}_1^l \partial\tilde{x}_2^m \partial\tilde{x}_3^n} = \mathbf{i}^{l+m+n-1}\frac{k_-^2}{2\pi^2}\left[F_+(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) + F_-(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})\right], \tag{4.16}$$

where

$$F_\pm(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \int_0^\infty \int_1^\infty \lambda_1^l(\xi_\pm)\lambda_2^m(\xi_\pm)\Lambda_\pm^{n+1}\frac{e^{\mathbf{i}[\mu_1(\xi_\pm)-\mu_2(\xi_\pm)]\tilde{y}_3}}{k_-^2\mu_1(\xi_\pm) + k_+^2\mu_2(\xi_\pm)}\frac{e^{\mathbf{i}\kappa_+\tilde{d}t}}{\sqrt{t^2-1}}\mathrm{d}t\mathrm{d}q.$$

Here $\lambda_1(\xi) = \xi\cos\phi - \mathbf{i}q\sin\phi$, $\lambda_2(\xi) = \xi\sin\phi + \mathbf{i}q\cos\phi$, and the polar angle satisfies

$$x_1 - y_1 = r\cos\phi, \qquad x_2 - y_2 = r\sin\phi, \qquad r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}.$$

It suffices to estimate $F_\pm(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$. Similar to (4.13), there is a generic constant C independent of $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ such that

$$|\lambda_1(\xi_\pm)| + |\lambda_2(\xi_\pm)| + |\xi_\pm| + |\Lambda_\pm| \leq C(k_+ + q)t,$$

By Lemma 4.3 and (4.14), we have $\left|\frac{e^{\mathbf{i}[\mu_1(\xi_\pm)-\mu_2(\xi_\pm)]\tilde{y}_3}}{k_-^2\mu_1(\xi_\pm)+k_+^2\mu_2(\xi_\pm)}\right| \leq C$. Together with (4.7), this shows $|F_\pm(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})| \leq Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}-\boldsymbol{y}|}$. The proof is completed by using (4.16). □

LEMMA 4.7. *Let $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ be given in Lemma 4.6 and $\boldsymbol{\zeta} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$. Then there is a constant C depending only on k such that*

$$\left|\frac{\partial^{m+n}}{\partial\zeta_i^m \partial\zeta_j^n}\mathbb{G}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})\right| \leq Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}-\boldsymbol{y}|}, \qquad 1 \leq i, j \leq 6, \quad m, n \geq 0.$$

Proof. Since $\mathbb{G} = \mathbb{H} + k_\pm^{-2}\nabla_{\tilde{\boldsymbol{y}}}\operatorname{div}_{\tilde{\boldsymbol{y}}}\mathbb{H}$ and $\mathbb{H} = \mathbb{S} - \mathbb{P}$, the lemma is a direct consequence of Lemma 4.5 and Lemma 4.6. □

4.4. Exponential decay of the scattering solution in PML. We present the main result of this section, that is, the exponential decay of the scattering solution in PML. From [30, Section 12.4.3], the solution $\boldsymbol{E}$ of (1.1) admits the integral representation

$$\boldsymbol{E} = \boldsymbol{\Psi}_{\mathrm{SL}}(\boldsymbol{\mu}) + \boldsymbol{\Psi}_{\mathrm{DL}}(\boldsymbol{g}) \qquad \text{in } D_c, \tag{4.17}$$

where $\boldsymbol{g} = \gamma_t \boldsymbol{E}$ and $\boldsymbol{\mu} = \gamma_t(\mathbf{curl}\, \boldsymbol{E})$ are the Dirichlet trace and the Neumann trace of the solution on Γ_D. The Maxwell single and double layer potentials are defined by

$$\boldsymbol{\Psi}_{\rm SL}(\boldsymbol{\mu}) = \int_{\Gamma_D} \mathbb{G}^\top(k;\boldsymbol{x},\boldsymbol{y})\boldsymbol{\mu}(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}, \quad \boldsymbol{\Psi}_{\rm DL}(\boldsymbol{g}) = \int_{\Gamma_D} (\mathbf{curl}_{\boldsymbol{y}}\, \mathbb{G})^\top(k;\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{g}(\boldsymbol{y})\mathrm{d}S_{\boldsymbol{y}}.$$

The analytic continuation of the scattering solution is defined by

$$\boldsymbol{E}(\tilde{\boldsymbol{x}}) = \boldsymbol{\Psi}_{\rm SL}(\boldsymbol{\mu})(\tilde{\boldsymbol{x}}) + \boldsymbol{\Psi}_{\rm DL}(\boldsymbol{g})(\tilde{\boldsymbol{x}}) \tag{4.18}$$

THEOREM 4.8. *There is a constant $C > 0$ depending only on k and R_0 such that, for any $\boldsymbol{x} \in \mathbb{R}^3_\pm$ satisfying $|\boldsymbol{x}| \geq 2R_0$,*

$$|\boldsymbol{E}(\tilde{\boldsymbol{x}})| + |\mathbf{curl}_{\tilde{\boldsymbol{x}}}\, \boldsymbol{E}(\tilde{\boldsymbol{x}})| \leq Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}|} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.$$

Proof. Since γ_t, γ_T are bounded operators, using (1.1a) and Theorem 2.2, we have

$$\begin{aligned}
|\boldsymbol{\Psi}_{\rm SL}(\boldsymbol{\mu})(\tilde{\boldsymbol{x}})| &\leq \|\boldsymbol{\mu}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)} \|\gamma_T \mathbb{G}(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_D)} \\
&\leq \|\mathbf{curl}\, \boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \|\mathbb{G}(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \\
&\leq C\, \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \|\mathbb{G}(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \\
&\leq Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}|} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)},
\end{aligned}$$

where we have used $\tilde{\boldsymbol{y}} = \boldsymbol{y}$ in $\Omega_0 = B_0 \backslash \bar{D}$. Similarly, the double layer potential can be estimated as follows

$$\begin{aligned}
|\boldsymbol{\Psi}_{\rm DL}(\boldsymbol{g})(\tilde{\boldsymbol{x}})| &\leq \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)} \|\gamma_T(\mathbf{curl}\, \mathbb{G})(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_D)} \\
&\leq \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)} \|\mathbf{curl}\, \mathbb{G}(\tilde{\boldsymbol{x}},\cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \\
&\leq Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}|} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.
\end{aligned}$$

This yields $|\boldsymbol{E}(\tilde{\boldsymbol{x}})| \leq Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}|} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}$. The estimate for $\mathbf{curl}_{\tilde{\boldsymbol{x}}}\, \boldsymbol{E}(\tilde{\boldsymbol{x}})$ is similar and omitted here. ◻

5. Exterior Maxwell problems. In this section, we shall study exterior problems of the stretched Maxwell's equation on unbounded domains. It is a key step for proving the stability of the truncated PML problem. Without specifications, $C > 0$ denotes the generic constant which depends only on k, R_0, D in the rest of the paper.

We introduce the stretched gradient, curl, divergence, and Laplace operators

$$\begin{aligned}
&\tilde{\nabla} v := \mathbb{B}^{-\top}\nabla v, && \tilde{\nabla} \times \boldsymbol{u} := J^{-1}\mathbb{B}\, \mathbf{curl}(\mathbb{B}^\top \boldsymbol{u}), \\
&\tilde{\nabla} \cdot \boldsymbol{u} := J^{-1}\, \mathrm{div}(J\mathbb{B}^{-1}\boldsymbol{u}), && \tilde{\Delta} v := J^{-1}\, \mathrm{div}(\mathbb{A}^{-1}\nabla v),
\end{aligned} \tag{5.1}$$

where $\mathbb{A} = J^{-1}\mathbb{B}\mathbb{B}^\top$. It is easy to see

$$\tilde{\nabla} \times \tilde{\nabla} \times \boldsymbol{u} = J^{-1}\mathbb{B}\, \mathbf{curl}\left[\mathbb{A}\, \mathbf{curl}(\mathbb{B}^\top \boldsymbol{u})\right]. \tag{5.2}$$

We define the stretched dyadic Green's function by

$$\tilde{\mathbb{G}}(\tilde{\boldsymbol{x}},\boldsymbol{y}) = \mathbb{B}^\top(\boldsymbol{y})\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}).$$

Using (2.6) and the argument in the proof of [28, Theorem 2.8], we have

$$\mathbf{curl}\left[\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(\tilde{\boldsymbol{x}},\cdot)\right]-k_{\pm}^{2}\mathbb{A}^{-1}\tilde{\mathbb{G}}(\tilde{\boldsymbol{x}},\cdot)=\delta_{\boldsymbol{x}}\mathbb{B}^{-1}\qquad\text{in }\ \mathbb{R}^{3}_{\pm}.\tag{5.3}$$

Since the complex stretching $\boldsymbol{F}$ is C^2-smooth, from (2.6b) we have

$$\left[\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(\tilde{\boldsymbol{x}},\cdot)\times\boldsymbol{n}\right]=\left[\tilde{\mathbb{G}}(\tilde{\boldsymbol{x}},\cdot)\times\boldsymbol{n}\right]=0\qquad\text{on }\ \Sigma.\tag{5.4}$$

LEMMA 5.1. *Let $\boldsymbol{f}\in\boldsymbol{L}^2(\mathbb{R}^3)$. The function*

$$\boldsymbol{u}(\boldsymbol{x})=\int_{\mathbb{R}^3}\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})\boldsymbol{f}(\boldsymbol{y})J(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.\tag{5.5}$$

satisfies the stretched Helmholtz equation and the stability estimate

$$\tilde{\Delta}\boldsymbol{u}+k_{\pm}^2\boldsymbol{u}=-\boldsymbol{f}\quad\text{in }\ \mathbb{R}^3_{\pm},\qquad\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\mathbb{R}^3)}\le C\sigma_0^4\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}\,.\tag{5.6}$$

Proof. By (2.8a) and using the argument in the proof of [28, Theorem 2.8], we get the stretched Helmholtz equation for the Hertz tensor

$$\tilde{\Delta}_{\boldsymbol{x}}\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})+k_{\pm}^2\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})=-J^{-1}(\boldsymbol{x})\delta_{\boldsymbol{y}}(\boldsymbol{x})\mathbb{I}\quad\text{in }\ \mathbb{R}^3_{\pm}.\tag{5.7}$$

Combining (5.5) and (5.7) yields the stretched Helmholtz equation in (5.6). It is left to prove the stability.

From (2.8b), $\mathbb{H}(k;\boldsymbol{y},\boldsymbol{x})$ is continuous with respect to $\boldsymbol{x}$ when $\boldsymbol{x}\neq\boldsymbol{y}$. So (5.5) implies that $\boldsymbol{u}$ is also continuous across Σ. From (5.1), we find that

$$\nabla\boldsymbol{u}(\boldsymbol{x})=\mathbb{B}^{\top}(\boldsymbol{x})\int_{\mathbb{R}^3}\nabla_{\tilde{\boldsymbol{x}}}\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})\boldsymbol{f}(\boldsymbol{y})J(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

From Lemma 4.5 and Lemma 4.6, the stretched Hertz tensor satisfies

$$|\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})|+|\nabla_{\tilde{\boldsymbol{x}}}\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})|\le Ce^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}-\boldsymbol{y}|}\qquad\text{for }\ |\boldsymbol{x}-\boldsymbol{y}|\ge 2R_0.$$

From Lemma 2.1 and Lemma 4.5, we know that

$$|\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})|+|\nabla_{\tilde{\boldsymbol{x}}}\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})|\le C\left(1+|\boldsymbol{x}-\boldsymbol{y}|^{-2}\right)\qquad\text{for }\ |\boldsymbol{x}-\boldsymbol{y}|<2R_0.$$

So in general, the stretched Hertz tensor satisfies

$$|\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})|+|\nabla_{\tilde{\boldsymbol{x}}}\mathbb{H}(k;\tilde{\boldsymbol{y}},\tilde{\boldsymbol{x}})|\le C\left(1+|\boldsymbol{x}-\boldsymbol{y}|^{-2}\right)e^{-\frac{1}{2}k_+\sigma_0|\boldsymbol{x}-\boldsymbol{y}|}\qquad\forall\,\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_{\pm}.$$

Write $w(\boldsymbol{x},\boldsymbol{y})=(1+|\boldsymbol{x}-\boldsymbol{y}|^{-2})e^{-\frac{1}{4}k_+\sigma_0|\boldsymbol{x}-\boldsymbol{y}|}$ for convenience. We find that

$$\begin{aligned}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\mathbb{R}^3)}&\le C\sigma_0^8\int_{\mathbb{R}^3}\left[\int_{\mathbb{R}^3}w(\boldsymbol{x},\boldsymbol{y})\,|\boldsymbol{f}(\boldsymbol{y})|^2\,\mathrm{d}\boldsymbol{y}\right]\left[\int_{\mathbb{R}^3}w(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right]\mathrm{d}\boldsymbol{x}\\&\le C\sigma_0^8\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}w(\boldsymbol{x},\boldsymbol{y})\,|\boldsymbol{f}(\boldsymbol{y})|^2\,\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\le C\sigma_0^8\,\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}\,.\end{aligned}$$

The proof is completed. □

LEMMA 5.2. *For any* $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^3)$ *satisfying* $\operatorname{div} \boldsymbol{f} = 0$*, there exists a function* $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$ *which satisfies the stretched Maxwell's equation*

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) - k_\pm^2 \mathbb{A}^{-1}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^3_\pm, \tag{5.8a}$$

$$\left[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n}\right] = \left[\boldsymbol{u} \times \boldsymbol{n}\right] = 0 \quad \text{on } \Sigma, \tag{5.8b}$$

and the stability estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)} \le C\sigma_0^4 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

Proof. Let $\boldsymbol{w}(\boldsymbol{x}) = \displaystyle\int_{\mathbb{R}^3} \mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})\mathbb{B}(\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$. From Lemma 5.1,

$$\tilde{\Delta}\boldsymbol{w} + k_\pm^2 \boldsymbol{w} = -J^{-1}\mathbb{B}\boldsymbol{f}.$$

Since $\mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})$ is continuous with respect to $\boldsymbol{x}$, $\boldsymbol{w}$ is also a continuous function. Similar to the proof of Lemma 5.1, we can prove

$$\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathbb{R}^3)} \le C\sigma_0^2 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{5.9}$$

Define the weighted Sobolev space for exterior elliptic problem by

$$W_1(\mathbb{R}^3) = \left\{ u : u(1+r^2)^{-1/2} \in L^2(\mathbb{R}^3),\ \nabla u \in \boldsymbol{L}^2(\mathbb{R}^3) \right\}.$$

From [31, Theorem 2.5.13], $\|\nabla u\|_{\boldsymbol{L}^2(\mathbb{R}^3)}$ provides an equivalent norm on $W_1(\mathbb{R}^3)$. Write $\tilde{\boldsymbol{w}} = \mathbb{B}^\top \boldsymbol{w}$ and consider the weak formulation: Find $\psi \in W_1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \mathbb{A}^{-1}\nabla\psi \cdot \nabla\varphi = -\int_{\mathbb{R}^3} \mathbb{A}^{-1}\tilde{\boldsymbol{w}} \cdot \nabla\varphi \qquad \forall\, \varphi \in W_1(\mathbb{R}^3). \tag{5.10}$$

From (4.4), we have $\operatorname{Re}(\mathbb{A}^{-1}\nabla\varphi \cdot \nabla\bar{\varphi}) \ge |\nabla\varphi|^2/4$. The Lax-Milgram lemma shows that (5.10) has a unique solution which satisfies

$$\tilde{\nabla} \cdot (\tilde{\nabla}\psi + \boldsymbol{w}) = 0, \qquad \|\nabla\psi\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \le C\sigma_0 \|\tilde{\boldsymbol{w}}\|_{L^2(\mathbb{R}^3)}. \tag{5.11}$$

Since $\operatorname{div} \boldsymbol{f} = 0$, we find that $\phi := k^{-2}\tilde{\nabla} \cdot \boldsymbol{w} \in \boldsymbol{L}^2(\mathbb{R}^3)$ satisfies

$$\tilde{\Delta}\phi = k_\pm^{-2}\tilde{\nabla} \cdot (\tilde{\Delta}\boldsymbol{w}) = -k_\pm^{-2}\tilde{\nabla} \cdot (J^{-1}\mathbb{B}\boldsymbol{f} + k_\pm^2\boldsymbol{w}) = -\tilde{\nabla} \cdot \boldsymbol{w} = -k_\pm^2\phi \quad \text{in } \mathbb{R}^3_\pm. \tag{5.12}$$

Combining (5.11) and (5.12) yields $\tilde{\Delta}\psi = \tilde{\Delta}\phi$ in $\mathbb{R}^3_\pm$.

For any bounded convex domain $\Omega \subset \mathbb{R}^3_\pm$ and any $\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega)$, let $\theta \in H_0^1(\Omega)$ be the unique solution of the problem

$$\int_\Omega J\tilde{\nabla}\theta \cdot \tilde{\nabla}\varphi = \int_\Omega J\boldsymbol{v} \cdot \tilde{\nabla}\varphi \qquad \forall\, \varphi \in H_0^1(\Omega).$$

This shows that $\tilde{\nabla} \cdot (\boldsymbol{v} - \tilde{\nabla}\theta) = 0$. We have $\boldsymbol{v} - \tilde{\nabla}\theta = \tilde{\nabla} \times \boldsymbol{\Theta}$ where $\boldsymbol{\Theta} \in \boldsymbol{L}^2(\Omega)$ satisfies $\tilde{\nabla} \times \boldsymbol{\Theta} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{\Theta} \times \tilde{\boldsymbol{n}} = 0$ on $\partial\Omega$. Here $\tilde{\boldsymbol{n}} = \mathbb{B}^{-\top}\boldsymbol{n}$ and $\boldsymbol{n}$ is the unit outer normal to Ω. By the boundary conditions of θ and $\boldsymbol{\Theta}$, we also know that

$$\tilde{\nabla}\theta \times \tilde{\boldsymbol{n}} = 0, \quad (\tilde{\nabla} \times \boldsymbol{\Theta}) \cdot \tilde{\boldsymbol{n}} = 0 \quad \text{on } \partial\Omega.$$

Since $\boldsymbol{v}=0$ on $\partial\Omega$, we conclude that $\tilde{\nabla}\theta=0$, $(\tilde{\nabla}\times\boldsymbol{\Theta})\cdot\tilde{\boldsymbol{n}}=0$ on $\partial\Omega$.

Since $\psi-\phi\in L^2(\Omega)$, $\tilde{\nabla}(\psi-\phi)$ provides a linear functional on the subspace

$$\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega):\ \tilde{\nabla}\cdot\boldsymbol{v}\in L^2(\Omega),\ \ \tilde{\nabla}\times\boldsymbol{v}\in\boldsymbol{L}^2(\Omega),\ \ \boldsymbol{v}=0\ \text{ on }\partial\Omega\}.$$

Therefore, for any $\boldsymbol{v}\in\boldsymbol{C}_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega J\tilde{\nabla}(\psi-\phi)\cdot\boldsymbol{v}&=\int_\Omega J\tilde{\nabla}(\psi-\phi)\cdot(\tilde{\nabla}\theta+\tilde{\nabla}\times\boldsymbol{\Theta})=\int_\Omega J\tilde{\nabla}(\psi-\phi)\cdot\tilde{\nabla}\theta\\&=\int_\Omega J\tilde{\Delta}(\psi-\phi)\,\theta=0.\end{aligned}$$

This implies $\tilde{\nabla}\psi=\tilde{\nabla}\phi$ in Ω. By the arbitrariness of Ω, we have $\tilde{\nabla}\phi\in\boldsymbol{L}^2(\mathbb{R}^3)$. Moreover, since $\phi\in L^2(\mathbb{R}^3)$ and $\psi(1+r^2)^{-1/2}\in L^2(\mathbb{R}^3)$, $\psi-\phi$ can not be constant. We conclude that $\psi=\phi$ and ψ satisfies (5.12).

Define $\boldsymbol{u}_1=\boldsymbol{w}+\tilde{\nabla}\psi$. Clearly (5.11) shows $\tilde{\nabla}\cdot\boldsymbol{u}_1=0$ in $\mathbb{R}^3$. Together with (5.12), the well-known identity $-\tilde{\Delta}=\tilde{\nabla}\times\tilde{\nabla}\times-\tilde{\nabla}\tilde{\nabla}\cdot$ yields

$$\tilde{\nabla}\times\tilde{\nabla}\times\boldsymbol{u}_1=-\tilde{\Delta}\boldsymbol{u}_1=J^{-1}\mathbb{B}\boldsymbol{f}+k_\pm^2\boldsymbol{w}+k_\pm^2\tilde{\nabla}\psi=J^{-1}\mathbb{B}\boldsymbol{f}+k_\pm^2\boldsymbol{u}_1\quad\text{in }\ \mathbb{R}^3_\pm.$$

Define $\boldsymbol{u}=\mathbb{B}^\top\boldsymbol{u}_1=\tilde{\boldsymbol{w}}+\nabla\psi$. We obtain

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u})=\boldsymbol{f}+k_\pm^2\mathbb{A}^{-1}\boldsymbol{u}\quad\text{in }\ \mathbb{R}^3_\pm.$$

The stability of $\boldsymbol{u}$ follows from (5.9) and (5.11).

For the continuities in (5.8b), we recall (5.4) and get

$$\left[\mathbb{A}\,\mathbf{curl}(\mathbb{B}^\top\mathbb{H})(k;\tilde{\boldsymbol{y}},\boldsymbol{F}(\cdot))\times\boldsymbol{n}\right]=\left[\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(\tilde{\boldsymbol{y}},\cdot)\times\boldsymbol{n}\right]=0.$$

This shows $\left[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}\right]=\left[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{n}\right]=0$ on Σ. Moreover, since both $\boldsymbol{w}$ and ψ are continuous across Σ, we conclude that

$$[\boldsymbol{u}\times\boldsymbol{n}]=[\mathbb{B}^\top\boldsymbol{w}\times\boldsymbol{n}]+[\nabla\psi\times\boldsymbol{n}]=0\quad\text{on }\ \Sigma.$$

The proof is completed. □

LEMMA 5.3. *For any $\boldsymbol{f}\in\boldsymbol{L}^2(\mathbb{R}^3)$, there exists a function $\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)$ which satisfies the stretched Maxwell's equation*

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u})-k_\pm^2\mathbb{A}^{-1}\boldsymbol{u}=\boldsymbol{f}\quad\textit{in }\ \mathbb{R}^3_\pm,\tag{5.13a}$$

$$\left[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}\right]=\left[\boldsymbol{u}\times\boldsymbol{n}\right]=0\quad\textit{on }\ \Sigma,\tag{5.13b}$$

and the stability estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)}\le C\sigma_0^5\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}\,.\tag{5.14}$$

Proof. First we consider the weak formulation: Find $\psi\in W_1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}k^2\mathbb{A}^{-1}\nabla\psi\cdot\nabla v=-\int_{\mathbb{R}^3}\boldsymbol{f}\cdot\nabla v\qquad\forall\,v\in W_1(\mathbb{R}^3).$$

Similar to (5.10), the problem has a unique solution which satisfies

$$\left\|\nabla\psi\right\|_{\boldsymbol{L}^2(\mathbb{R}^3)}\le C\,\|\boldsymbol{f}\|_{L^2(\mathbb{R}^3)}\,,\tag{5.15}$$

So $\boldsymbol{f}_1 := \boldsymbol{f} + k^2\mathbb{A}^{-1}\nabla\psi$ satisfies $\operatorname{div}\boldsymbol{f}_1 = 0$. By Lemma 5.2, there is a solution to

$$\begin{aligned} \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_1) - k_\pm^2\mathbb{A}^{-1}\boldsymbol{u}_1 &= \boldsymbol{f}_1 \quad \text{in } \mathbb{R}^3_\pm, \\ [\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_1 \times \boldsymbol{n}] = [\boldsymbol{u}_1 \times \boldsymbol{n}] &= 0 \quad \text{on } \Sigma, \end{aligned}$$

and there is a constant $C > 0$ depending only on k, R_0 such that

$$\|\boldsymbol{u}_1\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)} \le C\sigma_0^4 \|\boldsymbol{f}_1\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \le C\sigma_0^5 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{5.17}$$

Define $\boldsymbol{u} = \boldsymbol{u}_1 + \nabla\psi$. Then (5.13a) is deduced as follows

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) = \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_1) = \boldsymbol{f}_1 + k_\pm^2\mathbb{A}^{-1}\boldsymbol{u}_1 = \boldsymbol{f} + k_\pm^2\mathbb{A}^{-1}\boldsymbol{u} \quad \text{in } \mathbb{R}^3_\pm.$$

Furthermore, (5.14) comes from (5.15) and (5.17). Finally, we have

$$[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_1 \times \boldsymbol{n}] = [\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n}] = 0, \qquad [\boldsymbol{u} \times \boldsymbol{n}] = [\boldsymbol{u}_1 \times \boldsymbol{n}] + [\nabla\psi \times \boldsymbol{n}] = 0.$$

This completes the proof. ◻

LEMMA 5.4. *Suppose $\boldsymbol{f} \in \boldsymbol{L}^2(D_c)$. There exists a function $\boldsymbol{u} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ which satisfies*

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) - k_\pm^2\mathbb{A}^{-1}\boldsymbol{u} = \boldsymbol{f} \qquad \text{in } \mathbb{R}^3_\pm \cap D_c, \tag{5.18a}$$

$$[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n}] = [\boldsymbol{u} \times \boldsymbol{n}] = 0 \qquad \text{on } \Sigma, \tag{5.18b}$$

and the stability estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},D_c)} \le C\sigma_0^5 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(D_c)}.$$

Proof. First we extend $\boldsymbol{f}$ by zero to the interior of D and denote the extension still by $\boldsymbol{f}$. By Lemma 5.3, there exists a $\boldsymbol{u}_0 \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$ satisfying

$$\begin{aligned} \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_0) - k_\pm^2\mathbb{A}^{-1}\boldsymbol{u}_0 &= \boldsymbol{f} \quad \text{in } \mathbb{R}^3_\pm, \\ [\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_0 \times \boldsymbol{n}] = [\boldsymbol{u}_0 \times \boldsymbol{n}] &= 0 \quad \text{on } \Sigma. \end{aligned}$$

Furthermore, there exists a constant $C > 0$ depending only on k, R_0 such that

$$\|\boldsymbol{u}_0\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)} \le C\sigma_0^5 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)} = C\sigma_0^5 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(D_c)}.$$

From Theorem 2.2, the scattering problem

$$\begin{aligned} &\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_1 - k_\pm^2\boldsymbol{u}_1 = 0 \quad \text{in } \mathbb{R}^3_\pm\backslash\bar{D}, \\ &[\mathbf{curl}\,\boldsymbol{u}_1 \times \boldsymbol{n}] = [\boldsymbol{u}_1 \times \boldsymbol{n}] = 0 \quad \text{on } \Sigma, \\ &\gamma_t\boldsymbol{u}_1 = \gamma_t\boldsymbol{u}_0 \quad \text{on } \Gamma_D, \\ &\lim_{\rho\to\infty}\int_{\partial B(\rho)} |\mathbf{curl}\,\boldsymbol{u}_1 \times \boldsymbol{n} - \mathbf{i}k\boldsymbol{u}_1|^2 = 0, \end{aligned}$$

has a unique solution which satisfies

$$\|\boldsymbol{u}_1\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \le C\|\gamma_t\boldsymbol{u}_0\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)} \le C\|\boldsymbol{u}_0\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \le C\sigma_0^5 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(D_c)}.$$

Similar to (4.17)–(4.18), we define the analytic continuation of $\boldsymbol{u}_1$ by

$$\boldsymbol{u}_1(\tilde{\boldsymbol{x}}) = \boldsymbol{\Psi}_{\mathrm{SL}}(\gamma_t\,\mathbf{curl}\,\boldsymbol{u}_1)(\tilde{\boldsymbol{x}}) + \boldsymbol{\Psi}_{\mathrm{DL}}(\gamma_t\boldsymbol{u}_1)(\tilde{\boldsymbol{x}}) \qquad \forall\,\boldsymbol{x} \in D_c.$$

By (4.18), $\tilde{\boldsymbol{u}}_1(\boldsymbol{x}) = \mathbb{B}^\top(\boldsymbol{x})\boldsymbol{u}_1(\tilde{\boldsymbol{x}})$ decays exponentially as $|\boldsymbol{x}| \to \infty$ so that

$$\|\tilde{\boldsymbol{u}}_1\|_{\boldsymbol{H}(\mathbf{curl},D_c)} \le C\,\|\boldsymbol{u}_1\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \le C\sigma_0^5\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D_c)}\,,$$

and satisfies the exterior problem

$$\begin{aligned} \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{u}}_1) - k_\pm^2\mathbb{A}^{-1}\tilde{\boldsymbol{u}}_1 = 0 \quad &\text{in } \mathbb{R}^3_\pm\backslash\bar{D}, \\ [\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{u}}_1 \times \boldsymbol{n}] = [\tilde{\boldsymbol{u}}_1 \times \boldsymbol{n}] = 0 \quad &\text{on } \Sigma, \\ \gamma_t\tilde{\boldsymbol{u}}_1 = \gamma_t\boldsymbol{u}_0 \quad &\text{on } \Gamma_D. \end{aligned}$$

Clearly $\boldsymbol{u} = \boldsymbol{u}_0 - \tilde{\boldsymbol{u}}_1 \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ and satisfies (5.18). ▯

6. The weak solution of the stretched Maxwell equation. The purpose of this section is to study the weak formulation for the analytic continuation of the scattering solution. Let $\boldsymbol{E}(\tilde{\boldsymbol{x}})$ be given in (4.18) and define

$$\tilde{\boldsymbol{E}}(\boldsymbol{x}) = \mathbb{B}^\top(\boldsymbol{x})\boldsymbol{E}(\tilde{\boldsymbol{x}}) \qquad \forall\,\boldsymbol{x} \in D_c\,. \tag{6.1}$$

By arguments similar to (5.3)–(5.4), we know that

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{E}}) - k^2\mathbb{A}^{-1}\tilde{\boldsymbol{E}} = 0 \quad \text{in } D_c, \tag{6.2a}$$

$$\big[\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{E}} \times \boldsymbol{n}\big] = \big[\tilde{\boldsymbol{E}} \times \boldsymbol{n}\big] = 0 \quad \text{on } \Sigma, \tag{6.2b}$$

$$\gamma_t\tilde{\boldsymbol{E}} = \boldsymbol{g} \quad \text{on } \Gamma_D. \tag{6.2c}$$

To introduce the Dirichlet-to-Neumann (DtN) operator, we define $B_1 = B(R_1)$, $\Omega_1 = B_1\backslash\bar{D}$, and $\Gamma_1 = \partial B_1$, where $R_1 = R_0 + s_1R_0/2$ and $0 < s_1 < 1$ is the solution of the equation

$$s^4 - 2s^3 + 3/\sigma_0 = 0.$$

From (4.1)–(4.2), it is easy to see

$$\alpha(R_1) = 1 + \frac{\mathbf{i}}{R_1}\int_0^{R_1}\sigma(t)\mathrm{d}t = 1 + \mathbf{i}\frac{\sigma_0R_0}{2R_1}\int_0^{s_1}(6s^2 - 4s^3)\mathrm{d}s = 1 + \mathbf{i}\frac{3R_0}{2R_1}.$$

This means that

$$1 < \operatorname{Im}\alpha(R_1) < 1.5, \qquad |\alpha(R_1)| < 2. \tag{6.3}$$

Moreover, let $s_2 \in (s_1, 1)$ solve the algebraic equation

$$s_2^4 - 2s_2^3 + 6/\sigma_0 = 0.$$

Let $\hat{B}_1 = B(R_0 + s_2R_0/2)$ be a larger ball containing B_1 and define

$$\hat{\Omega}_1 = \hat{B}_1\backslash\bar{B}_1. \tag{6.4}$$

Similar to (6.3), we have

$$1 < \operatorname{Im}\alpha(|\boldsymbol{x}|) < 3, \quad |\alpha(|\boldsymbol{x}|)| < 4 \qquad \forall\,\boldsymbol{x} \in \hat{\Omega}_1. \tag{6.5}$$

6.1. The Dirichlet-to-Neumann operator. For any $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$, the DtN operator $\mathcal{G}$: $\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1) \to \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$ is defined by

$$\mathcal{G}(\boldsymbol{\lambda}) := \gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) \quad \text{on } \Gamma_1, \tag{6.6}$$

where $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3\backslash\bar{B}_1)$ is the solution of the exterior problem

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) - k^2\mathbb{A}^{-1}\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^3_\pm \setminus \bar{B}_1, \tag{6.7a}$$

$$\big[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n}\big] = \big[\boldsymbol{u} \times \boldsymbol{n}\big] = 0 \quad \text{on } \Sigma, \tag{6.7b}$$

$$\gamma_t \boldsymbol{u} = \boldsymbol{\lambda} \quad \text{on } \Gamma_1. \tag{6.7c}$$

LEMMA 6.1. *The exterior problem* (6.7) *has a unique solution which satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3\backslash\bar{B}_1)} \le C\sigma_0 \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)}.$$

Moreover, $\boldsymbol{u}$ *admits the integral representation, for any* $\boldsymbol{x} \in \mathbb{R}^3\backslash\bar{B}_1$*,*

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbb{B}(\boldsymbol{x}) \int_{\Gamma_1} \Big[\tilde{\mathbb{G}}^\top(\tilde{\boldsymbol{x}}, \cdot)\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) + (\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}})^\top(\tilde{\boldsymbol{x}}, \cdot)\gamma_t \boldsymbol{u}\Big]. \tag{6.8}$$

Proof. Similar to (6.20), we have an equivalent weak formulation of (6.7): Find $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3\backslash\bar{B}_1)$ such that $\gamma_t \boldsymbol{u} = \boldsymbol{\lambda}$ on Γ_1 and

$$A_1(\boldsymbol{u}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_\Gamma(\mathbf{curl}, \mathbb{R}^3\backslash\bar{B}_1), \tag{6.9}$$

where the bilinear form A_1: $\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3\backslash\bar{B}_1) \times \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3\backslash\bar{B}_1) \to \mathbb{C}$ is defined by

$$A_1(\boldsymbol{u}, \boldsymbol{v}) = \int_{\mathbb{R}^3\backslash\bar{B}_1} \big(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} - k^2\mathbb{A}^{-1}\boldsymbol{u} \cdot \boldsymbol{v}\big).$$

It is easy to calculate the eigenvalues of $\mathbb{B}$ which are

$$\lambda_1 = \lambda_2 = \alpha, \qquad \lambda_3 = \alpha + \rho\alpha'. \tag{6.10}$$

Write $\boldsymbol{x} = \rho(\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)^\top$ with $\theta \in [0, 2\pi]$ and $\phi \in [-\pi/2, \pi/2]$. The associated eigenvectors are real and read as follows

$$\begin{aligned} \boldsymbol{\xi}_1 &= (-\sin\theta, \cos\theta, 0)^\top, \\ \boldsymbol{\xi}_2 &= (\cos\theta\sin\phi, \sin\theta\sin\phi, -\cos\phi)^\top, \\ \boldsymbol{\xi}_3 &= (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)^\top. \end{aligned} \tag{6.11}$$

Clearly $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ are also eigenvectors of $\mathbb{A}$ and belong respectively to the eigenvalues

$$\nu_1 = \nu_2 = (\alpha + \rho\alpha')^{-1}, \qquad \nu_3 = (\alpha + \rho\alpha')\alpha^{-2}. \tag{6.12}$$

For any $\boldsymbol{\xi} = \sum_{i=1}^3 t_i \boldsymbol{\xi}_i$ with $t_1, t_2, t_3 \in \mathbb{C}$, we find that

$$\mathbb{A}\boldsymbol{\xi} \cdot \bar{\boldsymbol{\xi}} = \sum_{i=1}^3 t_i \nu_i \boldsymbol{\xi}_i \cdot \sum_{j=1}^3 \bar{t}_j \boldsymbol{\xi}_j = \sum_{i=1}^3 \nu_i |t_i|^2, \qquad \mathbb{A}^{-1}\boldsymbol{\xi} \cdot \bar{\boldsymbol{\xi}} = \sum_{i=1}^3 \nu_i^{-1} |t_i|^2. \tag{6.13}$$

By (4.1)–(4.2) and (6.3), there is a constant $C > 0$ independent of σ_0 such that

$$\mathrm{Im}(\mathbb{A}\boldsymbol{\xi} \cdot \bar{\boldsymbol{\xi}}) \le -C\sigma_0^{-1} |\boldsymbol{\xi}|^2, \quad \mathrm{Im}(\mathbb{A}^{-1}\boldsymbol{\xi} \cdot \bar{\boldsymbol{\xi}}) \ge C|\boldsymbol{\xi}|^2 \quad \text{in } \mathbb{R}^3\backslash\bar{B}_1, \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^3.$$

Therefore, the bilinear form A_1 is coercive on $\boldsymbol{H}_{\Gamma_1}(\mathbf{curl}, \mathbb{R}^3 \backslash \bar{B}_1)$

$$|A_1(\boldsymbol{v}, \bar{\boldsymbol{v}})| \geq -\operatorname{Im} A_1(\boldsymbol{v}, \bar{\boldsymbol{v}}) \geq C\sigma_0^{-1} \min(1, k_1) \|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \backslash \bar{B}_1)}. \tag{6.14}$$

So problem (6.9) attains a unique solution.

Let $\hat{\Omega}_1$ be defined in (6.4). Since γ_t: $\boldsymbol{H}(\mathbf{curl}, \hat{\Omega}_1) \to \boldsymbol{H}^{-1/2}(\mathrm{Div}, \partial\hat{\Omega}_1)$ is surjective, there is a $\boldsymbol{u}_0 \in \boldsymbol{H}(\mathbf{curl}, \hat{\Omega}_1)$ such that $\gamma_t \boldsymbol{u}_0 = \boldsymbol{\lambda}$ on Γ_1, $\gamma_t \boldsymbol{u}_0 = 0$ on $\partial\hat{B}_1$, and

$$\|\boldsymbol{u}_0\|_{\boldsymbol{H}(\mathbf{curl}, \hat{\Omega}_1)} \leq C \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)}. \tag{6.15}$$

Extend $\boldsymbol{u}_0$ by zero to the exterior of $\hat{B}_1$. From (6.5), we deduce that

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_0\|^2_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \backslash \bar{B}_1)} &\leq C\sigma_0 |A_1(\boldsymbol{u} - \boldsymbol{u}_0, \bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_0)| = C\sigma_0 |A_1(\boldsymbol{u}_0, \bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_0)| \\ &\leq C\sigma_0 \|\boldsymbol{u}_0\|_{\boldsymbol{H}(\mathbf{curl}, \hat{\Omega}_1)} \|\boldsymbol{u} - \boldsymbol{u}_0\|_{\boldsymbol{H}(\mathbf{curl}, \hat{\Omega}_1)}. \end{aligned}$$

Together with (6.15), this yields $\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \backslash \bar{B}_1)} \leq C\sigma_0 \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)}$.

It is left to show the integral representation of $\boldsymbol{u}$. For any $\boldsymbol{x} \in \mathbb{R}^3_\pm \backslash \bar{B}_1$, let $B(\rho)$ be a sufficiently large ball which contains $\boldsymbol{x}$. Write $\Omega_\rho = B(\rho) \backslash \bar{B}_1$ and $\Gamma_\rho = \partial B(\rho)$. Using (5.3), (6.7), and the formula of integration by part, we have

$$\begin{aligned} \mathbb{B}^{-1}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) &= \int_{\Omega_\rho \cap \mathbb{R}^3_\pm} \Big[\mathbf{curl}\Big(\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(\tilde{\boldsymbol{x}}, \cdot)\Big) - k_\pm^2 \mathbb{A}^{-1}\tilde{\mathbb{G}}(\tilde{\boldsymbol{x}}, \cdot)\Big]^\top \boldsymbol{u} \\ &= \int_{\Gamma_1} \Big[\tilde{\mathbb{G}}^\top(\tilde{\boldsymbol{x}}, \cdot)\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) + (\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}})^\top(\tilde{\boldsymbol{x}}, \cdot)\gamma_t\boldsymbol{u}\Big] - I(\rho), \end{aligned}$$

where the second term is defined by

$$I(\rho) = \int_{\partial B(\rho)} \Big[\tilde{\mathbb{G}}^\top(\tilde{\boldsymbol{x}}, \cdot)\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}) + (\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}})^\top(\tilde{\boldsymbol{x}}, \cdot)\gamma_t\boldsymbol{u}\Big].$$

By Lemma 4.7, there is a constant $C > 0$ independent of ρ such that

$$\begin{aligned} |I(\rho)| &\leq Ce^{-\frac{1}{2}k_+\sigma_0\rho}\Big[\|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \backslash \bar{B}_1)} + \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \backslash \bar{B}_1)}\Big] \\ &\leq Ce^{-\frac{1}{2}k_+\sigma_0\rho}\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \backslash \bar{B}_1)}, \qquad \text{as } \rho \to \infty. \end{aligned}$$

This shows $\lim_{\rho\to\infty} I(\rho) = 0$. The proof is complete. ☐

LEMMA 6.2. *There exists a constant $C > 0$ depending only on k, R_0 such that*

$$\|\mathcal{G}(\boldsymbol{\lambda})\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)} \leq C\sigma_0 \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)} \quad \forall \boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1),$$

Proof. Let $\boldsymbol{u}$ be the solution of (6.7). By Lemma 6.1, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \backslash \bar{B}_1)} \leq C\sigma_0 \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)}.$$

Let $\hat{\Omega}_1$ be given in (6.4). Using (6.7a) and (6.5), we deduce that

$$\begin{aligned} \|\mathcal{G}(\boldsymbol{\lambda})\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)} &= \|\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u})\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)} \leq C \|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \hat{\Omega}_1)} \\ &\leq C\Big\{\|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\hat{\Omega}_1)} + \big\|k^2\mathbb{A}^{-1}\boldsymbol{u}\big\|_{\boldsymbol{L}^2(\hat{\Omega}_1)}\Big\} \\ &\leq C\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \hat{\Omega}_1)} \leq C\sigma_0 \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)}. \end{aligned}$$

The proof is complete. ☐

6.2. A weak formulation of (6.2)**.** Based on the DtN operator, we define a bilinear form on $\boldsymbol{H}(\mathbf{curl}, \Omega_1)$ as follows

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega_1} \left(\mathbb{A}\, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} - k^2 \mathbb{A}^{-1} \boldsymbol{u} \cdot \boldsymbol{v} \right) + \langle \mathcal{G}(\gamma_t \boldsymbol{u}), \gamma_T \boldsymbol{v} \rangle_{\Gamma_1} . \tag{6.16}$$

An equivalent weak formulation of (6.2) reads: Find $\tilde{\boldsymbol{E}} \in \boldsymbol{H}(\mathbf{curl}, \Omega_1)$ such that $\gamma_t \tilde{\boldsymbol{E}} = \boldsymbol{g}$ on Γ_D and

$$a(\tilde{\boldsymbol{E}}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega_1). \tag{6.17}$$

We are going to prove the inf-sup condition of a and the well-posedness of (6.17).

Following [7], we define a bilinear form on the unbounded domain D_c by

$$A(\boldsymbol{u}, \boldsymbol{v}) = \int_{D_c} \left(\mathbb{A}\, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} - k^2 \mathbb{A}^{-1} \boldsymbol{u} \cdot \boldsymbol{v} \right) \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, D_c).$$

LEMMA 6.3. *There exists a constant $C > 0$ depending only on k, R_0 such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, D_c)} \le C \sigma_0^7 \sup_{\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)} \frac{|A(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, D_c)}} \qquad \forall\, \boldsymbol{u} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c).$$

Proof. For given $\boldsymbol{u}$, we define the linear functional $l \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)'$ by

$$l(\boldsymbol{v}) = \int_{D_c} (\mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} + \boldsymbol{u} \cdot \bar{\boldsymbol{v}}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c).$$

It is clear that

$$l(\boldsymbol{u}) = \|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathbf{curl}, D_c)}, \qquad |l(\boldsymbol{v})| \le \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, D_c)} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, D_c)} .$$

We consider the weak formulation: Find $\boldsymbol{u}_+ \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ such that

$$A_+(\boldsymbol{u}_+, \boldsymbol{v}) = l(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c), \tag{6.18}$$

where $A_+ \colon \boldsymbol{H}(\mathbf{curl}, D_c) \times \boldsymbol{H}(\mathbf{curl}, D_c) \to \mathbb{C}$ is the bilinear form defined by

$$A_+(\boldsymbol{u}, \boldsymbol{v}) = \int_{D_c} \left(\mathbb{A}\, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + k^2 \mathbb{A}^{-1} \boldsymbol{u} \cdot \boldsymbol{v} \right).$$

By virtue of (4.1)–(4.2) and (6.12)–(6.13), direct calculations show that

$$|A_+(\boldsymbol{v}, \bar{\boldsymbol{v}})| \ge \operatorname{Re} A_+(\boldsymbol{v}, \bar{\boldsymbol{v}}) \ge \beta \sigma_0^{-1} \|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl}, D_c)},$$

for some constant $\beta > 0$ depending only on k. This means that A_+ is coercive on $\boldsymbol{H}(\mathbf{curl}, D_c)$. So problem (6.18) has a unique solution which satisfies

$$\|\boldsymbol{u}_+\|_{\boldsymbol{H}(\mathbf{curl}, D_c)} \le C \sigma_0 \|l\|_{\boldsymbol{H}(\mathbf{curl}, D_c)'} \le C \sigma_0 \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, D_c)} . \tag{6.19}$$

Let $\boldsymbol{u}_1 \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ be the solution of (5.18) where the righthand side is given by $\boldsymbol{f} = 2k^2 \mathbb{A}^{-1} \boldsymbol{u}_+$. Multiplying both sides of (5.18a) with $\boldsymbol{v} \in \boldsymbol{C}_0^\infty(D_c)$ and using integration by part, we find that

$$A(\boldsymbol{u}_1, \boldsymbol{v}) = \int_{D_c} \boldsymbol{f} \cdot \boldsymbol{v} = l(\boldsymbol{v}) - A(\boldsymbol{u}_+, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{C}_0^\infty(D_c).$$

The density of $\boldsymbol{C}_0^\infty(D_c)$ in $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ indicates

$$A(\boldsymbol{u}_1, \boldsymbol{v}) = \int_{D_c} \boldsymbol{f}\cdot\boldsymbol{v} = l(\boldsymbol{v}) - A(\boldsymbol{u}_+, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v}\in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c). \tag{6.20}$$

Using Lemma 5.4 and (6.19), we have

$$\|\boldsymbol{u}_1\|_{\boldsymbol{H}(\mathbf{curl},D_c)} \le C\sigma_0^5 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(D_c)} \le C\sigma_0^7 \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}. \tag{6.21}$$

Clearly $\boldsymbol{w} = \boldsymbol{u}_1 + \boldsymbol{u}_+ \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ and satisfies

$$A(\boldsymbol{w}, \boldsymbol{v}) = l(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v}\in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c).$$

Since $A(\cdot,\cdot)$ is symmetric, combining (6.19)-(6.21) yields the desired inf-sup condition

$$\sup_{\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},D_c)} \frac{|A(\boldsymbol{u},\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}} \ge \frac{|A(\boldsymbol{u},\boldsymbol{w})|}{\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}} = \frac{|l(\boldsymbol{u})|}{\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}} \ge \frac{\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}}{C\sigma_0^7}.$$

This completes the proof. ∎

Lemma 6.4. *For any $\boldsymbol{u}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)$, there is a constant $C_{\rm inf}$ depending only on k, R_0, D such that*

$$\sup_{\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)} \frac{|a(\boldsymbol{u},\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}} \ge C_{\rm inf}\sigma_0^{-7}\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}. \tag{6.22}$$

Proof. Remember that $\mathcal{G}(\gamma_t\boldsymbol{u}) = \boldsymbol{n}\times\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}$ where $\boldsymbol{\xi}\in\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3\backslash\bar{B}_1)$ satisfies

$$\begin{aligned} \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}) - k^2\mathbb{A}^{-1}\boldsymbol{\xi} = 0 &\quad \text{in } \mathbb{R}^3_\pm\setminus\bar{B}_1,\\ \big[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n}\big] = \big[\boldsymbol{\xi}\times\boldsymbol{n}\big] = 0 &\quad \text{on } \Sigma,\\ \gamma_t\boldsymbol{\xi} = \gamma_t\boldsymbol{u} &\quad \text{on } \Gamma_1, \end{aligned}$$

where $\boldsymbol{n}$ is the outer normal to B_1. Using integration by part, we have

$$\begin{aligned} \langle\mathcal{G}(\gamma_t\boldsymbol{u}), \gamma_T\boldsymbol{v}\rangle_{\Gamma_1} &= \int_{\Gamma_1}(\boldsymbol{n}\times\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\cdot\gamma_T\boldsymbol{v} = \int_{\Gamma_1}(\boldsymbol{n}\times\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\cdot\boldsymbol{v}\\ &= \int_{\mathbb{R}^3\backslash\bar{B}_1}\left[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{v} - \mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\cdot\boldsymbol{v}\right]\\ &= \int_{\mathbb{R}^3\backslash\bar{B}_1}\left[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{v} - k^2\mathbb{A}^{-1}\boldsymbol{\xi}\cdot\boldsymbol{v}\right] \quad \forall\,\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},D_c). \end{aligned}$$

Let $\tilde{\boldsymbol{u}}$ be the extension of $\boldsymbol{u}$ defined by

$$\tilde{\boldsymbol{u}} = \begin{cases} \boldsymbol{u} & \text{in } \Omega_1,\\ \boldsymbol{\xi} & \text{in } \mathbb{R}^3\backslash\bar{B}_1. \end{cases}$$

Then inserting the last equality into (6.16), we find that

$$a(\boldsymbol{u},\boldsymbol{v}) = A(\tilde{\boldsymbol{u}},\boldsymbol{v}) \qquad \forall\,\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},D_c).$$

We conclude the inf-sup condition from Lemma 6.3 as follows

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \le \|\tilde{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl},D_c)} &\le C\sigma_0^7 \sup_{\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},D_c)} \frac{|A(\tilde{\boldsymbol{u}},\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}}\\ &\le C\sigma_0^7 \sup_{\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)} \frac{|a(\boldsymbol{u},\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}}. \end{aligned}$$

The proof is completed. □

THEOREM 6.5. *For any* $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$*, let* $\boldsymbol{E}$ *be the solution of* (1.1) *and let* $\tilde{\boldsymbol{E}}$ *be defined in* (6.1). *Then* $\tilde{\boldsymbol{E}}$ *is the unique solution of* (6.17) *and satisfies*

$$\left\| \tilde{\boldsymbol{E}} \right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \le C \left\| \boldsymbol{g} \right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} .$$

Proof. By Lemma 6.2 and Lemma 6.4, we know that $a(\cdot,\cdot)$ provides a continuous and coercive bilinear form on $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega_1)$. So $\tilde{\boldsymbol{E}}$ is the unique solution of (6.17). The stability is a direct consequence of Theorem 2.2 and (6.3). □

7. The PML problem. The purpose of this section is to study the PML approximation to the scattering problem (1.1) or to the exterior problem (6.2). Let $R_2 \ge 2R_0$ and define $B_2 = B(R_2)$, $\Omega_2 = B_2 \backslash \bar{D}$, and $\Gamma_2 = \partial B_2$. For convenience, let the wave-absorbing layer and its thickness be denoted respectively by

$$\Omega_{\mathrm{PML}} := B_2 \backslash \bar{B}_1, \qquad d = \mathrm{distance}(\Gamma_1, \Gamma_2) = R_2 - R_1.$$

We consider the PML problem with homogeneous boundary condition on the truncation boundary:

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{E}}) - k^2 \mathbb{A}^{-1} \hat{\boldsymbol{E}} = 0 \quad \text{in } \Omega_2, \tag{7.1a}$$

$$\left[\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{E}} \times \boldsymbol{n}\right] = \left[\hat{\boldsymbol{E}} \times \boldsymbol{n}\right] = 0 \quad \text{on } \Sigma, \tag{7.1b}$$

$$\gamma_t \hat{\boldsymbol{E}} = \boldsymbol{g} \quad \text{on } \Gamma_D, \qquad \gamma_t \hat{\boldsymbol{E}} = 0 \quad \text{on } \Gamma_2. \tag{7.1c}$$

We first introduce the PML Dirichlet-to-Neumann operator $\hat{\mathcal{G}}$: $\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1) \to \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$ as follows: for any $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$, let $\hat{\mathcal{G}}(\boldsymbol{\lambda}) = \boldsymbol{n} \times \mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{u}}$ on Γ_1 where $\hat{\boldsymbol{u}}$ solves the Dirichlet boundary value problem

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{u}}) - k_\pm^2 \mathbb{A}^{-1} \hat{\boldsymbol{u}} = 0 \quad \text{in } \Omega_{\mathrm{PML}} \cap \mathbb{R}^3_\pm, \tag{7.2a}$$

$$\left[\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{u}} \times \boldsymbol{n}\right] = \left[\hat{\boldsymbol{u}} \times \boldsymbol{n}\right] = 0 \quad \text{on } \Sigma, \tag{7.2b}$$

$$\gamma_t \hat{\boldsymbol{u}} = \boldsymbol{\lambda} \quad \text{on } \Gamma_1, \qquad \gamma_t \hat{\boldsymbol{u}} = 0 \quad \text{on } \Gamma_2. \tag{7.2c}$$

By arguments similar the proofs of Lemma 6.1–6.2, we have the lemma on the stability of $\hat{\mathcal{G}}$. The proof is omitted here for simplicity.

LEMMA 7.1. *There exists a constant* $C > 0$ *depending only on* k, R_0 *such that*

$$\left\| \hat{\mathcal{G}}(\boldsymbol{\lambda}) \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \le C\sigma_0 \left\| \boldsymbol{\lambda} \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} .$$

LEMMA 7.2. *Assume* $R_2 \ge 2R_0$ *and* $\sigma_0 \ge 4$. *There is a constant* C *depending only on* k, R_0 *such that, for any* $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$,

$$\left\| \mathcal{G}(\boldsymbol{\lambda}) - \hat{\mathcal{G}}(\boldsymbol{\lambda}) \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \le C\sigma_0^5 d e^{-\frac{1}{2}k_+ \sigma_0 d} \left\| \boldsymbol{\lambda} \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} .$$

Proof. Given $\boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_1)$, let $\boldsymbol{u}$ be the solution of problem (6.7). By Lemma 6.1, $\boldsymbol{u}$ admits the integral representation

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbb{B}(\boldsymbol{x}) \int_{\Gamma_1} \left[\gamma_T \tilde{\mathbb{G}}^\top(\tilde{\boldsymbol{x}}, \cdot) \mathcal{G}(\boldsymbol{\lambda}) + \gamma_T \big(\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}\big)^\top (\tilde{\boldsymbol{x}}, \cdot) \boldsymbol{\lambda} \right] \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3 \backslash \bar{B}_1.$$

By Lemma 6.2, we have

$$|\boldsymbol{u}(\boldsymbol{x})| \le C\sigma_0^2 \left[\left\| \tilde{\mathbb{G}}(\tilde{\boldsymbol{x}},\cdot) \right\|_{\boldsymbol{H}(\mathbf{curl},B_1)} + \left\| \mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}(\tilde{\boldsymbol{x}},\cdot) \right\|_{\boldsymbol{H}(\mathbf{curl},B_1)} \right] \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}$$
$$\le C\sigma_0^2 e^{-\frac{1}{2}k_+\sigma_0 d} \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3 \backslash B_2.$$

Moreover, it is easy to see that

$$\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x}) = J(\boldsymbol{x})\mathbb{B}^{-1}(\boldsymbol{x})\tilde{\nabla} \times \int_{\Gamma_1} \left[\tilde{\mathbb{G}}^\top(k;\tilde{\boldsymbol{x}},\cdot)\,\mathcal{G}(\boldsymbol{\lambda}) + \left(\mathbb{A}\,\mathbf{curl}\,\tilde{\mathbb{G}}\right)^\top(k;\tilde{\boldsymbol{x}},\cdot)\,\boldsymbol{\lambda} \right].$$

Similarly, from Lemma 4.7 we have

$$|\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x})| \le C\sigma_0^3 e^{-\frac{1}{2}k_+\sigma_0 d} \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3 \backslash B_2.$$

By (6.6) and (7.2), $\mathcal{G}(\boldsymbol{\lambda}) - \hat{\mathcal{G}}(\boldsymbol{\lambda}) = \boldsymbol{n} \times \mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}$ where $\boldsymbol{\xi}$ solves the Dirichlet boundary value problem in the layer

$$\begin{aligned} &\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}) - k_\pm^2 \mathbb{A}^{-1}\boldsymbol{\xi} = 0 \quad \text{in } \Omega_{\mathrm{PML}} \cap \mathbb{R}^3_\pm, \\ &\left[\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi} \times \boldsymbol{n}\right] = \left[\boldsymbol{\xi} \times \boldsymbol{n}\right] = 0 \quad \text{on } \Sigma, \\ &\gamma_t \boldsymbol{\xi} = 0 \quad \text{on } \Gamma_1, \quad \gamma_t\boldsymbol{\xi} = \gamma_t \boldsymbol{u} \quad \text{on } \Gamma_2. \end{aligned} \tag{7.3}$$

A weak formulation reads: Find $\boldsymbol{\xi} \in \boldsymbol{H}_{\Gamma_1}(\mathbf{curl}, \Omega_{\mathrm{PML}})$ such that $\gamma_t\boldsymbol{\xi} = \gamma_t\boldsymbol{u}$ on Γ_2 and

$$A_{\mathrm{PML}}(\boldsymbol{\xi}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\mathrm{PML}}),$$

where $A_{\mathrm{PML}}\colon \boldsymbol{H}(\mathbf{curl}, \Omega_{\mathrm{PML}}) \times \boldsymbol{H}(\mathbf{curl}, \Omega_{\mathrm{PML}}) \to \mathbb{C}$ is defined by

$$A_{\mathrm{PML}}(\boldsymbol{\xi}, \boldsymbol{v}) = \int_{\Omega_1} \left(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\boldsymbol{v} - k^2 \mathbb{A}^{-1}\boldsymbol{\xi} \cdot \boldsymbol{v} \right).$$

Similar to (6.14), A_{PML} is coercive on $\boldsymbol{H}(\mathbf{curl}, \Omega_{\mathrm{PML}})$, namely,

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega_{\mathrm{PML}})} \le C\sigma_0 \left| A_{\mathrm{PML}}(\boldsymbol{v}, \bar{\boldsymbol{v}}) \right| \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega_{\mathrm{PML}}). \tag{7.4}$$

Therefore, the weak problem has a unique solution.

Multiplying both sides of (7.3) with $\bar{\boldsymbol{\xi}}$ and using integration by part, we have

$$A_{\mathrm{PML}}(\boldsymbol{\xi}, \bar{\boldsymbol{\xi}}) = -\int_{\Gamma_2} \gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}) \cdot \gamma_T \bar{\boldsymbol{\xi}} = -\int_{\Gamma_2} \gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}) \cdot \gamma_T \bar{\boldsymbol{u}}. \tag{7.5}$$

Define $O_- = B_2 \backslash \overline{B(R_2 - R_0/8)}$ and $O_+ = B(R_2 + R_0/8) \backslash \overline{B_2}$. Clearly O_+ and O_- share the boundary Γ_2. Combining (7.4) and (7.5), we deduce that

$$\begin{aligned} \|\boldsymbol{\xi}\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega_{\mathrm{PML}})} &\le C\sigma_0 \left\|\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi})\right\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{Div},\Gamma_2)} \left\|\gamma_T \boldsymbol{u}\right\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{Curl},\Gamma_2)} \\ &\le C\sigma_0 \left\|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\right\|_{\boldsymbol{H}(\mathbf{curl},O_-)} \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},O_+)} \\ &\le C\sigma_0^2 \|\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathbf{curl},O_-)} \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},O_+)}. \end{aligned}$$

This shows that

$$\|\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\mathrm{PML}})} \le C\sigma_0^2 \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},O_+)} \le C\sigma_0^5 d e^{-\frac{1}{2}k_+\sigma_0 d} \|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}.$$

Let $\hat{\Omega}_1$ be defined in (6.4). Using (7.3) and (6.5), we conclude that

$$\begin{aligned}\left\|\mathcal{G}(\boldsymbol{\lambda}) - \hat{\mathcal{G}}(\boldsymbol{\lambda})\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} &\le \|\mathbb{A}\,\mathbf{curl}\,\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathbf{curl},\hat{\Omega}_1)} \le C\,\|\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathbf{curl},\hat{\Omega}_1)} \\ &\le C\sigma_0^5 d e^{-\frac{1}{2}k_+\sigma_0 d}\,\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)}.\end{aligned}$$

The proof is complete. ◻

Based on the PML DtN operator, we define the bilinear form $\hat{a}$: $\boldsymbol{H}(\mathbf{curl},\Omega_1) \times \boldsymbol{H}(\mathbf{curl},\Omega_1) \to \mathbb{C}$ by

$$\hat{a}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega_1} \left(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} - k^2\mathbb{A}^{-1}\boldsymbol{u}\cdot\boldsymbol{v}\right) + \langle\hat{\mathcal{G}}(\gamma_t\boldsymbol{u}),\gamma_T\boldsymbol{v}\rangle_{\Gamma_1}.$$

Then an equivalent weak formulation of (7.1) reads as follows: Find $\hat{\boldsymbol{E}} \in \boldsymbol{H}(\mathbf{curl},\Omega_1)$ such that $\gamma_t\hat{\boldsymbol{E}} = \boldsymbol{g}$ on Γ_D and

$$\hat{a}(\hat{\boldsymbol{E}},\boldsymbol{v}) = 0 \qquad \forall\,\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1). \tag{7.6}$$

THEOREM 7.3. *Assume $\sigma_0 \ge 4$ and R_2 is large enough. The bilinear form $\hat{a}$ satisfies the inf-sup condition*

$$\sup_{\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)} \frac{|\hat{a}(\boldsymbol{w},\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}} \ge \frac{C_{\inf}}{2\sigma_0^7}\,\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \qquad \forall\,\boldsymbol{w} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1),$$

where $C_{\inf}$ is the constant for the inf-sup condition in (6.22)*. Moreover, the PML problem* (7.1) *has a unique solution $\hat{\boldsymbol{E}} \in \boldsymbol{H}(\mathbf{curl},\Omega_1)$.*

Proof. For any $\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)$, Lemma 7.2 shows that

$$\begin{aligned}|\hat{a}(\boldsymbol{w},\boldsymbol{v})| &\ge |a(\boldsymbol{w},\boldsymbol{v})| - \left\|\mathcal{G}(\gamma_t\boldsymbol{w}) - \hat{\mathcal{G}}(\gamma_t\boldsymbol{w})\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \|\gamma_T\boldsymbol{v}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_1)} \\ &\ge |a(\boldsymbol{w},\boldsymbol{v})| - C\sigma_0^5 d e^{-\frac{1}{2}k_+\sigma_0 d}\,\|\gamma_t\boldsymbol{w}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \|\gamma_T\boldsymbol{v}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_1)} \\ &\ge |a(\boldsymbol{w},\boldsymbol{v})| - C\sigma_0^5 d e^{-\frac{1}{2}k_+\sigma_0 d}\,\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}.\end{aligned}$$

Let d be so large that $2C\sigma_0^{12} d e^{-\frac{1}{2}k_+\sigma_0 d} \le C_{\inf}$. Then the inf-sup condition comes from Lemma 6.4. So problem (7.1) has a unique solution. ◻

Finally, we arrive at the main theorem of this paper.

THEOREM 7.4. *Assume $\sigma_0 \ge 4$ and R_2 is large enough. Let $\boldsymbol{E}$, $\hat{\boldsymbol{E}}$ be the solutions of* (1.1) *and* (7.1) *respectively. There exists a constant $C > 0$ depending only on k, R_0, and D such that*

$$\left\|\boldsymbol{E} - \hat{\boldsymbol{E}}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_0)} \le C d\sigma_0^{12} e^{-\frac{1}{2}k_+\sigma_0 d}\,\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.$$

Proof. Write $\tilde{\boldsymbol{E}} = \mathbb{B}\boldsymbol{E}\circ\boldsymbol{F}$. By Theorem 6.5, $\tilde{\boldsymbol{E}}$ is the unique solution of problem (6.17). From (6.17) and (7.6), we have

$$\hat{a}(\hat{\boldsymbol{E}} - \tilde{\boldsymbol{E}},\boldsymbol{v}) = \langle\mathcal{G}(\gamma_t\tilde{\boldsymbol{E}}) - \hat{\mathcal{G}}(\gamma_t\tilde{\boldsymbol{E}}),\gamma_T\boldsymbol{v}\rangle_{\Gamma_1} \qquad \forall\,\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1).$$

By Lemma 7.2 and Lemma 7.3, we have

$$\begin{aligned}\left\|\tilde{\boldsymbol{E}} - \hat{\boldsymbol{E}}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)} &\le C\sigma_0^7 \sup_{\boldsymbol{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega_1)} \frac{\left|\hat{a}(\tilde{\boldsymbol{E}} - \hat{\boldsymbol{E}},\boldsymbol{v})\right|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}} \\ &\le C\sigma_0^7 \left\|\hat{\mathcal{G}}(\gamma_t\tilde{\boldsymbol{E}}) - \mathcal{G}(\gamma_t\tilde{\boldsymbol{E}})\right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_1)} \\ &\le C d\sigma_0^{12} e^{-\frac{1}{2}k_+\sigma_0 d}\left\|\tilde{\boldsymbol{E}}\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega_1)}.\end{aligned}$$

The proof is completed by using Theorem 6.5. ∎

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