A DELTA-REGULARIZATION FINITE ELEMENT METHOD FOR A DOUBLE CURL PROBLEM WITH DIVERGENCE-FREE CONSTRAINT

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Abstract. To deal with the divergence-free constraint in a double curl problem: $\operatorname{curl} \mu^{-1} \operatorname{curl} u = f$ and $\operatorname{div} \varepsilon u = 0$ in Ω, where μ and ε represent the physical properties of the materials occupying Ω, we develop a δ-regularization method: $\operatorname{curl} \mu^{-1} \operatorname{curl} u_\delta + \delta \varepsilon u_\delta = f$ to completely ignore the divergence-free constraint $\operatorname{div} \varepsilon u = 0$. It is shown that u_δ converges to u in $H(\operatorname{curl}; \Omega)$ norm as $\delta \to 0$. The edge finite element method is then analyzed for solving u_δ. With the finite element solution $u_{\delta,h}$, quasi-optimal error bound in $H(\operatorname{curl}; \Omega)$ norm is obtained between u and $u_{\delta,h}$, including a uniform (with respect to δ) stability of $u_{\delta,h}$ in $H(\operatorname{curl}; \Omega)$ norm. All the theoretical analysis is done in a general setting, μ and ε may be discontinuous, anisotropic and inhomogeneous, and the solution may have a very low piecewise regularity on each material subdomain Ω_j with $u, \operatorname{curl} u \in (H^r(\Omega_j))^3$ for some $0 < r < 1$, where r may be not greater than $1/2$. To establish the uniform stability and the error bound for $r \leq 1/2$, we have respectively developed a new theory for the $\mathscr{K}_h$ ellipticity (related to mixed methods) and a new theory for the Fortin interpolation operator. A series of numerical experiments are performed to illustrate the proposed δ-regularization method.

Key words. double curl problem, divergence-free constraint, regularization, edge element, uniform stability, $\mathscr{K}_h$ ellipticity, error bound, Fortin operator

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1. Introduction. Given a simply-connected Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$, with a connected boundary $\partial\Omega$. Let $\mu, \varepsilon : \Omega \mapsto \mathbb{R}^{3 \times 3}$ be given matrix functions, representing the physical properties (such as permeability and permittivity) of the material occupying Ω. We assume that μ and ε are piecewise with respect to a finite partition $\mathscr{P}$ of Ω, $\mathscr{P} = \{\Omega_j, j = 1, 2, \cdots, J\}$, where every Ω_j is a simply-connected Lipschitz polyhedron with connected boundary. Let $\mathscr{S}_{\text{int}}$ and $\mathscr{S}_{\text{ext}}$ denote the collect of the faces of $\mathscr{P}$ contained in Ω and the collect of the faces of $\mathscr{P}$ contained in $\partial\Omega$, respectively. Let $[q]|_S$ denote the jump of q across $S \in \mathscr{S}_{\text{int}}$. Given $f : \Omega \mapsto \mathbb{R}^3$, satisfying $\operatorname{div} f = 0$. Consider the double curl problem as follows:

$$\operatorname{curl} \mu^{-1} \operatorname{curl} u = f \quad \text{in } \Omega_j, 1 \leq j \leq J, \tag{1.1}$$

$$\operatorname{div} \varepsilon u = 0 \quad \text{in } \Omega_j, 1 \leq j \leq J, \tag{1.2}$$

$$[u \times n]|_S = 0, \quad [\mu^{-1} \operatorname{curl} u \times n]|_S = 0, \quad [\varepsilon u \cdot n]|_S = 0 \quad \forall S \in \mathscr{S}_{\text{int}}, \tag{1.3}$$

$$u \times n|_S = 0 \quad \forall S \in \mathscr{S}_{\text{ext}}. \tag{1.4}$$

Problem (1.1)-(1.4) arises from computational electromagnetism [9][44][40][41]. An example is the vector potential method [29] for some divergence-free unknown which may be expressed as the curl of u(the vector potential), where the divergence-free constraint (1.2) is set up to ensure the uniqueness of the solution. Otherwise, problem (1.1), (1.3) and (1.4) would have infinitely many solutions, due to the infinite dimensional kernel of the curl operator consisting of the form ∇p where $p \in H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_S = 0, \forall S \in \mathscr{S}_{\text{ext}}\}$. Another example [44] is from the stabilization of the time-harmonic Maxwell's equation $\operatorname{curl} \mu^{-1} \operatorname{curl} u - \kappa^2 u = f$ with a very low frequency number κ, where the divergence-free constraint (1.2) may be introduced to play the stabilization role so that even $\kappa = 0$ a unique solution can exist. By introducing Hilbert spaces

$$H(\operatorname{div}; \Omega) = \{v \in (L^2(\Omega))^3 : \operatorname{div} v \in L^2(\Omega)\},$$

$$H(\operatorname{curl}; \Omega) = \{v \in (L^2(\Omega))^3 : \operatorname{curl} v \in (L^2(\Omega))^3\},$$

$$H_0(\operatorname{curl}; \Omega) = \{v \in H(\operatorname{curl}; \Omega) : u \times n|_S = 0 \quad \forall S \in \mathscr{S}_{\text{ext}}\},$$

$$H(\mathrm{div}\,;\varepsilon;\Omega) = \{v \in (L^2(\Omega))^3 : \mathrm{div}\,\varepsilon v \in L^2(\Omega)\},$$

$$H(\mathrm{div}^{\,0};\varepsilon;\Omega) = \{v \in H(\mathrm{div}\,;\varepsilon;\Omega) : \mathrm{div}\,\varepsilon v = 0\}.$$

Corresponding to (1.1)-(1.4), we may state a variational problem as follows: Find $u \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^{\,0};\varepsilon;\Omega)$ such that

$$(\mu^{-1}\mathrm{curl}\,u, \mathrm{curl}\,v) = (f,v) \quad \forall v \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^{\,0};\varepsilon;\Omega), \tag{1.5}$$

where $(\cdot,\cdot)$ stands for the L^2-inner product, i.e., $(u,v) = \int_\Omega uv$.

When discretized by a finite element method for solving problem (1.5), we would naturally seek the finite element solution in a finite element subspace of $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^{\,0};\varepsilon;\Omega)$. However, as is well-known, it is quite difficult to construct a finite element space consisting of lower-order piecewise polynomials to satisfy the divergence-free constraint. With rather restrictive finite element triangulations (e.g., multiply-refined composite elements) for lower-order (e.g., quadratic) elements or with higher-order (at least sextic) elements together with some relatively less restrictive but still quite structured finite element triangulations, one could construct divergence-free elements in the case where ε itself is piecewise polynomial [49].

In practice, the rule for dealing with the divergence-free constraint is to let it be satisfied weakly. This could be done by including the divergence-free constraint directly into the variational formulation and by seeking the solution in some bigger Hilbert space U without the divergence-free constraint other than the restricted $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^{\,0};\varepsilon;\Omega)$ with the divergence-free constraint. There are ways: the divergence-regularization method and the mixed method. The former is to find $u \in U$ such that

$$(\mu^{-1}\mathrm{curl}\,u, \mathrm{curl}\,v) + \langle \mathrm{div}\,\varepsilon u, \mathrm{div}\,\varepsilon v\rangle = (f,v) \quad \forall v \in U.$$

where $\langle\cdot,\cdot\rangle$ may stand for the L^2 inner product or the weighted L^2 inner product or the L^2-projected L^2 inner product or the H^{-1} inner product of the dual Hilbert space H^{-1} (the dual of $H_0^1(\Omega)$), and U may be correspondingly taken as $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}\,;\varepsilon;\Omega)$ or some weighted Hilbert space or $H_0(\mathrm{curl};\Omega)$ for the latter two, see [33][23][27][10]. In the case where $\langle\cdot,\cdot\rangle$ is simply taken as the L^2 inner product, the above is a second-order $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}\,;\varepsilon;\Omega)$-elliptic problem and the method is referred to as the plain regularization (PR) method [33][22]. For smooth ε, one may consider the classical continuous finite element method for the PR method. But, when the solution is not smooth and is only in H^r for some $r < 1$, the continuous finite element method cannot give a correct solution [34][44][9][23][22]. On the other hand, no finite element methods are immediately available for discontinuous ε when only based on the PR formulation. To deal with the discontinuous ε, by the introduction of amounts of jumps into the above formulation, one may consider the discontinuous Galerkin method with the use of discontinuous elements. But, likewise, the discontinuous Galerkin method cannot accommodate the nonsmooth solution with a low-regularity in H^r with r less than one [47]. Some of the combination of the nonconforming element method and the discontinuous Galerkin method may lead to a correct approximation of the nonsmooth solution [13]. One may also still consider to use the continuous element if adopting the recently developed L^2 projected continuous finite element method [27][26] to solve the problem where ε may be discontinuous and the solution may be nonsmooth. Both the H^{-1} method with $\langle\cdot,\cdot\rangle$ being the H^{-1} inner product and the weighted method with $\langle\cdot,\cdot\rangle$ being the weighted L^2 inner product can also allow the use of continuous elements for approximating the nonsmooth solution [23][15][43]. All these methods involve sophisticated modifications.

The mixed method [44] for dealing with the divergence-free constraint is to find $u \in H_0(\mathrm{curl};\Omega)$ and $p \in H_0^1(\Omega)$ such that

$$\begin{aligned}(\mu^{-1}\mathrm{curl}\,u, \mathrm{curl}\,v) + \quad &(\nabla p, \varepsilon v) = (f,v) \quad \forall v \in H_0(\mathrm{curl};\Omega),\\ &(\varepsilon u, \nabla q) = 0 \quad \forall q \in H_0^1(\Omega).\end{aligned} \tag{1.6}$$

With the introduction of the Lagrange multiplier p, the solution is only required to belong to the Hilbert space $H_0(\mathrm{curl};\Omega)$. The Lagrange multiplier $p \in H_0^1(\Omega)$ satisfies the following weak problem

$$(\varepsilon\nabla p, \nabla q) = (f, \nabla q) \quad \forall q \in H_0^1(\Omega). \tag{1.7}$$

Note that p is actually equal to zero with a compatible f which satisfies $\mathrm{div}\,f = 0$. For the mixed problem (1.6) one may consider to use the edge element for u and the continuous element for p, or the discontinuous Galerkin method [38]. The difficulty would be the verification of $\mathscr{K}$-ellipticity in the classical theory for saddle-point problems [14]. In the literature [3], the verification relies on the continuous embedding: there exists some real number s which is greater than $1/2$ such that the following continuous embedding holds:

$$H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^{\,0};\Omega) \hookrightarrow (H^s(\Omega))^3 \quad \text{where } s > 1/2. \tag{1.8}$$

But, in some cases, such $s > 1/2$ does not exist for the above embedding to hold true, e.g., when Ω is only Lipschitz, we can only find $s = 1/2$, see [21]. We will come back to this point again later. The most difficult would be of course the saddle-point structure of the mixed problem, since the indefiniteness of the saddle-point system would thwart many classical iterative algorithms, such as conjugate gradient algorithm. The preconditioning is necessary to have a good iterative algorithm for solving the saddle-point system [5].

In our paper, we shall develop a new and much simpler method to deal with the divergence-free constraint. We just completely neglect the divergence-free constraint and instead we consider a δ-perturbed problem: with $\delta > 0$ decreasing to zero, we are to find a family of $u_\delta \in H_0(\mathrm{curl}\,;\Omega)$ such that

$$(\mu^{-1}\mathrm{curl}\, u_\delta, \mathrm{curl}\, v) + \delta(\varepsilon u_\delta, v) = (f, v) \quad \forall v \in H_0(\mathrm{curl}\,;\Omega). \tag{1.9}$$

The δ-perturbed method will be called the δ-regularization method, since problem (1.9) is free of the divergence-free constraint and since problem (1.9) is $H_0(\mathrm{curl}\,;\Omega)$-elliptic. In comparisons with previous existing methods, there are several obvious features of the present method: a) it is no longer subject to the divergence-free constraint; b) it is more suitable for discontinuous ε, since no $\mathrm{div}\,\varepsilon v$ appears; c) it only involves a space $H_0(\mathrm{curl}\,;\Omega)$ which can be discretized by edge elements composing of lower-order piecewise polynomials, since no Lagrange multiplier is introduced; d) it is always well-posed and results in a symmetric, positive definite system in the finite element discretization, so the resultant algbraic system may be implemented more readily. In fact, since it results in a symmetric, positive definite system, the δ-regularization problem may be conveniently readily solved by any direct or iterative methods [31]. Moreover, nowadays there are highly efficient multigrid methods and preconditioning techniques available for solving (1.9) where multigrid convergence and preconditioned conditioning are uniform with respect to the parameter δ [4][35].

With the compatible source f satisfying $\mathrm{div}\, f = 0$, for all $\delta > 0$, we can verify that u_δ satisfies the divergence-free condition:

$$\mathrm{div}\,\varepsilon u_\delta = 0. \tag{1.10}$$

We show that u_δ converges to the original u in both the $H(\mathrm{curl}\,;\Omega)$-norm and the $H(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,;\varepsilon;\Omega)$-norm, namely,

$$||u - u_\delta||_{0,\mathrm{curl},\mathrm{div},\varepsilon} = ||u - u_\delta||_{0,\mathrm{curl}} \leq C\delta||u||_0, \tag{1.11}$$

where $||v||_{0,\mathrm{curl}}^2 := ||v||_0^2 + ||\mathrm{curl}\, v||_0^2$, $||v||_{0,\mathrm{curl},\mathrm{div},\varepsilon}^2 := ||v||_0^2 + ||\mathrm{curl}\, v||_0^2 + ||\mathrm{div}\,\varepsilon v||_0^2$, and $||\cdot||_0$ represents the L^2-norm. We then analyze the edge finite element method for the δ-regularization problem in the finite element space $U_h \subset H_0(\mathrm{curl}\,;\Omega)$, under the general setting where μ, ε may be discontinuous, anisotropic and inhomogeneous and $u, \mathrm{curl}\, u$ are nonsmooth, which may also have very low regularity only in $\prod_{j=1}^{J}(H^r(\Omega_j))^3$ for some $0 < r < 1$. Assume that U_h allows the usual both L^2 and $H(\mathrm{curl})$-orthogonal decomposition [44]. For the lowest-order edge/Nédélec element of first-family [45][44][34], we establish the following error estimates, which is optimal with respect to the regularity of the solution,

$$||u - u_{\delta,h}||_{0,\mathrm{curl}} \leq C(\delta + h^r)||f||_0, \tag{1.12}$$

where $u_{\delta,h}$ is the finite element solution of the δ-regularization problem, $f \in H(\mathrm{div}\,^0;\Omega) := H(\mathrm{div}\,^0;1;\Omega)$, and C does not depend on δ. From (1.12) we may choose $\delta \leq h^r$ to have the optimal error bound in the usual sense. A ready choice is $\delta = h$. Note that if the solution and its curl are more regular, higher-order edge elements can be employed to result in higher-order error bounds. At the same time, a uniform stability for any given compatible $f \in H(\mathrm{div}\,^0;\Omega)$ is obtained where C does not depend on δ as follows:

$$||u_{\delta,h}||_{0,\mathrm{curl}} \leq C||f||_0. \tag{1.13}$$

Very interesting, the theory for the uniform stability (1.13) is closely related to the well-posedness of the mixed problem (1.6). In fact, (1.13) is essentially the consequence of the $\mathscr{K}_h$-ellipticity (which, together with the Inf-Sup condition, ensures the well-posedness of the mixed problem (1.6)). In the literature [44][3], the $\mathscr{K}_h$-ellipticity was only shown under the assumption (1.8) with $s > 1/2$. In this paper, we shall establish the $\mathscr{K}_h$-ellipticity using Assumption A3) (regular-singular decomposition) in section 4, instead of (1.8). The Assumption A3) is much weaker than (1.8), because the former can generally hold but the latter may not hold.

In addition, the error bound in $H(\mathrm{curl})$-norm in (1.12) is obtained with the help of the Fortin operator [8]. Likewise, in the literature, the well-posedness and the error estimate of the Fortin operator rely on the assumption (1.8) with $s > 1/2$. Under the much weaker Assumption A3) again, in this paper we shall provide

a theory for the well-posedness and the error estimate for the Fortin operator, so that we can establish the error estimate (1.12) in $H(\text{curl};\Omega)$ norm for very low regular solution, i.e., r in (1.12) can be not greater than $1/2$. Note that for interface problem, not only the global regularity of the solution is very low (this is a well-known fact), but also its piecewise regularity over each material subdomain Ω_j may be still possibly very low, see [24]. Since the method and the theory of this paper are valid to the solution with a very low piecewise regularity, i.e., $u, \text{curl}\, u \in \prod_{j=1}^{J}(H^r(\Omega_j))^3$, where r may be less than $1/2$ and even close to zero, the theory is also applicable to the edge finite element methods and the related discontinuous methods in computational electromagnetism. This is in sharp contrast to the numerous existing literature, where r and s are usually assumed to be greater than $1/2$, e.g., see [2] [8] [17] [36] [44] [37], [38], just to name a few.

Before closing this section, we also remark that the method and theory developed here also cover the following more general model which is widely employed in computational electromagnetism: given f, g, α and a third material matrix ε_1, to find the solution pair u and p such that

$$\text{curl}\,\mu^{-1}\text{curl}\, u + \alpha\varepsilon u + \varepsilon_1\nabla p = f \quad \text{in } \Omega, \tag{1.14}$$

$$\text{div}\,\varepsilon u = g \quad \text{in } \Omega, \tag{1.15}$$

$$u \times n|_S = 0, \quad p|_S = 0 \quad \forall S \in \mathscr{S}_{\text{ext}}, \tag{1.16}$$

In fact, we can first solve in parallel, simultaneously the two second-order elliptic interface problems in $H_0^1(\Omega)$ space for p and some p^* which solves $\text{div}\,\varepsilon\nabla p^* = g$, and then we are left with a problem, similar to (1.1)-(1.4), which the δ-regularization method can be applied to.

The rest of this paper is arranged as follows: In section 2, we obtain the convergence of the solution of problem (1.9) to the solution of problem (1.5). In section 3, the edge finite element method is defined and the uniform stability (1.13) is obtained under the assumption (1.8). In section 4, a general $\mathscr{K}_h$ ellipticity is established without the assumption (1.8), instead under Assumption A3) the regular-singular decomposition. As a result of the $\mathscr{K}_h$ ellipticity, the uniform stability (1.13) holds. In section 5, for a concrete choice of the edge element in [45], the error estimates of the finite element solution of problem (1.9) is established, especially the error bound in $H(\text{curl})$-norm is obtained with the help of the Fortin operator. In section 6, under Assumption A3) we present the general theory for the Fortin operator without the assumption (1.8). In section 7, some numerical experiments are performed to illustrate the proposed method. In the last section, a conclusion remark is given and how to extend the proposed method to solve (1.14)-(1.16) is briefly discussed.

2. Convergence for continuous problem. Throughout the paper, we shall use the standard Hilbert and Sobolev spaces $H^s(\Omega)$ for any $s \in \mathbb{R}$ and $L^p(\Omega)$ for any $p \geq 2$, see [1][32].

Assume that μ and ε are symmetric, positive definite, satisfying

$$\xi'\mu(x)\xi, \qquad \xi'\varepsilon(x)\xi \geq C|\xi|^2 \quad \forall \xi \in \mathbb{R}^3 \quad \text{almost everywhere over } \bar{\Omega},$$

$$\mu, \varepsilon \in (L^\infty(\Omega))^{3\times 3}, \qquad \mu|_{\Omega_j}, \varepsilon|_{\Omega_j} \in (W^{1,\infty}(\Omega_j))^{3\times 3}, 1 \leq j \leq J,$$

where μ, ε are required to be piecewise smooth so that we could obtain the regularity of the solution of problem (1.1)-(1.4). Let $||v||_{0,\mu^{-1}}^2 := (\mu^{-1}v, v)$ and $||v||_{0,\varepsilon}^2 := (\varepsilon v, v)$ be the μ^{-1}-weighted and ε-weighted L^2 norm respectively.

We first give a lemma about the divergence of u_δ.

Lemma 2.1 *Let $u_\delta \in H_0(\text{curl};\Omega)$ denote the solution of problem (1.9). For the compatible $f \in H(\text{div}^0;\Omega)$, then $u_\delta \in H_0(\text{curl};\Omega) \cap H(\text{div};\varepsilon;\Omega)$ satisfies*

$$\text{div}\,\varepsilon u_\delta = 0. \tag{2.1}$$

Proof. Since $(C_0^\infty(\Omega))^3$ is dense in $H_0(\text{curl};\Omega)$, and taking $v \in (C_0^\infty(\Omega_j))^3$, $1 \leq j \leq J$, from (1.9) we find that

$$\text{curl}\,\mu^{-1}\text{curl}\, u_\delta + \delta\varepsilon u_\delta = f \quad \text{in } \Omega_j, 1 \leq j \leq J, \tag{2.2}$$

holds in the distributional sense. But, $f, u_\delta \in (L^2(\Omega_j))^3$, (2.2) also holds in $(L^2(\Omega_j))^3$. We then take $v \in (C_0^\infty(\Omega))^3$ in (1.9) to have $[\mu^{-1}\text{curl}\, u \times n]|_S = 0$ for all $S \in \mathscr{S}_{\text{int}}$. Hence, (2.2) is in fact valid globally in Ω. Since $f \in H(\text{div}^0;\Omega)$, by applying the divergence operator to both sides of (2.2) we immediately conclude the proof of the lemma. $\square$

We next recall a Poincaré-Friedrichs' inequality over $H_0(\mathrm{curl}\,;\Omega)\cap H(\mathrm{div}^{\,0};\varepsilon;\Omega)$ by stating it in a proposition as follows.

Proposition 2.1 [28] *For any Lipschitz domain Ω and for any ε as assumed earlier, we have*

$$||v||_0 \leq C||\mathrm{curl}\,v||_0 \quad \forall v \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}^{\,0};\varepsilon;\Omega). \tag{2.3}$$

We are now in a position to investigate the existence and uniqueness and the convergence of u_δ.

Theorem 2.1 *For any given $\delta > 0$ and for any $f \in (L^2(\Omega))^3$ problem (1.9) has a unique solution u_δ. And, for the compatible $f \in H(\mathrm{div}^{\,0};\Omega)$, there exists a constant $C > 0$, independent of δ, such that*

$$||u_\delta||_{0,\mathrm{curl}} \leq C||f||_0. \tag{2.4}$$

Proof. Let

$$\mathscr{A}_\delta(u,v) := (\mu^{-1}\mathrm{curl}\,u, \mathrm{curl}\,v) + \delta(\varepsilon u, v). \tag{2.5}$$

We find that $\mathscr{A}_\delta(u,v)$ is a bilinear form from $H_0(\mathrm{curl}\,;\Omega) \times H_0(\mathrm{curl}\,;\Omega) \to \mathbb{R}$, satisfying

$$|\mathscr{A}_\delta(u,v)| \leq C \max(1,\delta)||u||_{0,\mathrm{curl}}||v||_{0,\mathrm{curl}} \quad \forall u,v \in H_0(\mathrm{curl}\,;\Omega) \tag{2.6}$$

and

$$\mathscr{A}_\delta(v,v) = ||\mathrm{curl}\,v||_{0,\mu^{-1}}^2 + \delta||v||_{0,\varepsilon}^2 \geq C \min(1,\delta)||v||_{0,\mathrm{curl}}^2 \quad \forall v \in H_0(\mathrm{curl}\,;\Omega). \tag{2.7}$$

We also see that

$$|(f,v)| \leq ||f||_0||v||_0 \leq ||f||_0||v||_{0,\mathrm{curl}} \quad \forall v \in H_0(\mathrm{curl}\,;\Omega). \tag{2.8}$$

Thus, from the classical Lax-Milgram lemma [12], we conclude that problem (1.9) admits a unique solution $u_\delta \in H_0(\mathrm{curl}\,;\Omega)$ for any given $\delta > 0$ and for any given $f \in (L^2(\Omega))^3$. From Lemma 2.1 u_δ is in fact in $H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}^{\,0};\varepsilon;\Omega)$, and from (2.3) in Proposition 2.1 we have

$$||u_\delta||_0 \leq C||\mathrm{curl}\,u_\delta||_0 \leq C||\mathrm{curl}\,u_\delta||_{0,\mu^{-1}}, \tag{2.9}$$

and we take $v = u_\delta$ in (1.9) to have

$$(\mu^{-1}\mathrm{curl}\,u_\delta, \mathrm{curl}\,u_\delta) + \delta(\varepsilon u_\delta, u_\delta) = (f, u_\delta) \leq C||f||_0||\mathrm{curl}\,u_\delta||_{0,\mu^{-1}}. \tag{2.10}$$

We thus have

$$||\mathrm{curl}\,u_\delta||_{0,\mu^{-1}} \leq C||f||_0. \tag{2.11}$$

It follows from (2.9) and (2.11) that (2.4) holds. $\square$

Remark 2.1 With the compatible $f \in H(\mathrm{div}^{\,0};\Omega)$, from Proposition 2.1 we have a stability for the original u, the solution to problem (1.5), as follows:

$$||u||_{0,\mathrm{curl}} \leq C||f||_0. \tag{2.12}$$

Theorem 2.2 *Let u and u_δ denote the solutions of problem (1.5) and problem (1.9), respectively. Assuming a compatible $f \in H(\mathrm{div}^{\,0};\Omega)$, we have the convergence*

$$||u - u_\delta||_{0,\mathrm{curl}} = ||u - u_\delta||_{0,\mathrm{curl}\,,\mathrm{div}\,,\varepsilon} \leq C\delta||u||_0, \tag{2.13}$$

where C does not depend on δ.

Proof. From Lemma 2.1 and (1.2) we observe that $||u - u_\delta||_{0,\mathrm{curl}\,,\mathrm{div}\,,\varepsilon} = ||u - u_\delta||_{0,\mathrm{curl}}$. Taking $v := u - u_\delta \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}^{\,0};\varepsilon;\Omega)$, from (1.5) and (1.9) we have

$$||\mathrm{curl}\,(u - u_\delta)||_{0,\mu^{-1}}^2 + \delta||u - u_\delta||_{0,\varepsilon}^2 = \delta(\varepsilon u, u - u_\delta). \tag{2.14}$$

Hence, from (2.3) in Proposition 2.1 with $u - u_\delta \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}^{\,0};\varepsilon;\Omega)$, we have

$$||u - u_\delta||_{0,\mathrm{curl}}^2 \leq C||\mathrm{curl}\,(u - u_\delta)||_{0,\mu^{-1}}^2 \leq C|\delta(\varepsilon u, u - u_\delta)| \leq C\delta||u||_0||\mathrm{curl}\,(u - u_\delta)||_{0,\mu^{-1}}, \tag{2.15}$$

where we used $||u - u_\delta||_0 \leq C||\mathrm{curl}\,(u - u_\delta)||_0 \leq C||\mathrm{curl}\,(u - u_\delta)||_{0,\mu^{-1}}$, and we have (2.13). $\square$

Remark 2.2 From (2.4) and (2.13) we have the convergence

$$||u - u_\delta||_{0,\mathrm{curl}} = ||u - u_\delta||_{0,\mathrm{curl}\,,\mathrm{div}\,,\varepsilon} \leq C\delta||f||_0. \tag{2.16}$$

3. Edge finite element method. For any given $h > 0$, let $\mathcal{T}_h$ denote the shape-regular conforming triangulation of Ω into tetrahedra [18][12], where $h := \max_{T \in \mathcal{T}_h} h_T$, and h_T denotes the diameter of T. We assume that $\mathcal{T}_h$ is also conforming along every interface $S \in \mathscr{S}_{\text{int}}$ and every boundary face $S \in \mathscr{S}_{\text{ext}}$. Let

$$U_h \subset H_0(\text{curl}; \Omega) \tag{3.1}$$

denote the finite element subspace, which is usually composed of piecewise polynomials with respect to $\mathcal{T}_h$. We then state the finite element problem corresponding to problem (1.9): To find $u_{\delta,h} \in U_h$ such that

$$(\mu^{-1}\text{curl}\, u_{\delta,h}, \text{curl}\, v) + \delta(\varepsilon u_{\delta,h}, v) = (f, v) \quad \forall v \in U_h. \tag{3.2}$$

Theorem 3.1 *For any $f \in (L^2(\Omega))^3$, for any $\delta > 0$ and for any h, problem (3.2) is well-posed.*

Proof. Thanks to the conformity of U_h in $H_0(\text{curl}; \Omega)$, from the coercivity (2.7) we easily infer that problem (3.2) has a unique solution $u_{\delta,h} \in U_h$ and $||u_{\delta,h}||_{0,\text{curl}} \leq C\delta^{-1}||f||_0$. $\quad\square$

As seen earlier, from the coercivity (2.7), we cannot obtain a uniform stability on $u_{\delta,h}$ with respect to the parameter δ. In order to have a uniform stability like (2.4) for the finite element solution $u_{\delta,h}$, we are to make the following very useful decomposition assumption about U_h.

Assumption A1) Let $(\cdot, \cdot)_{0,\varepsilon} = (\varepsilon \cdot, \cdot)$ denote the ε-weighted L^2 inner product. We assume that U_h admits an L^2-orthogonal decomposition with respect to the ε-weighted L^2 inner product $(\cdot, \cdot)_{0,\varepsilon}$:

$$U_h = Z_h(\varepsilon) + \nabla Q_h, \tag{3.3}$$

where $Z_h(\varepsilon), \nabla Q_h$ are closed subspaces of U_h, defined by

$$Z_h(\varepsilon) = \{v \in U_h : (\varepsilon v, \nabla q) = 0, \forall q \in Q_h\}, \tag{3.4}$$

$$Q_h \subset H_0^1(\Omega). \tag{3.5}$$

Clearly, the above decomposition is also an ε-and μ^{-1}-weighted $H_0(\text{curl}; \Omega)$ orthogonal, i.e., for all $z \in Z_h(\varepsilon)$ and for all $q \in Q_h$,

$$(z, \nabla q)_{H_0(\text{curl};\Omega),\varepsilon,\mu^{-1}} := (\varepsilon z, \nabla q) + (\mu^{-1}\text{curl}\, z, \text{curl}\, \nabla q) = 0.$$

Remark 3.1 In fact, for any given $v \in U_h$, we first let $p_h \in Q_h$ uniquely solve $(\varepsilon \nabla p_h, \nabla q) = (\varepsilon v, \nabla q)$ for all $q \in Q_h$. We have $(\varepsilon(v - \nabla p_h), \nabla q) = 0$ for all $q \in Q_h$. Putting $z_h := v - \nabla p_h \in U_h$, we see that $v = z_h + \nabla p_h$ is the desired, with $z_h \in Z_h(\varepsilon)$. $\quad\square$

We next make an assumption about the regularity of $H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega)$.

Assumption A2) We assume that there exists a $s > 0$ such that

$$H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega) \hookrightarrow (H^s(\Omega))^3 \tag{3.6}$$

is a continuously embedding, satisfying for all $v \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega)$

$$||v||_s \leq C||\text{curl}\, v||_0. \tag{3.7}$$

Remark 3.2 When $s > 1/2$, Assumption A2) is (1.8). For Ω being Lipschitz polyhedron, we have $s > 1/2$, see [3]. For general Lipschitz domains, $s = 1/2$, see [21]. Possibly, $s < 1/2$, e.g., for non-Lipschitz, non-simply-connected domains with screening parts [22].

Proposition 3.1 *Assume that $s > 1/2$ in Assumption A2). We have the $\mathscr{K}_h$-ellipticity as follows:*

$$||v||_0 \leq C||\text{curl}\, v||_0 \quad \forall v \in \mathscr{K}_h := Z_h(\varepsilon). \tag{3.8}$$

Proof. For $\varepsilon = 1$, (3.8) is proven in [3]. For a general ε, (3.8) is essentially proven in [44]. $\quad\square$

Remark 3.3 Note that $s > 1/2$ of Assumption A2) is used in the literature [3][44]. As we mentioned earlier, s may not be greater than $1/2$, i.e., $s \leq 1/2$ possibly. We have not been aware of any work in the literature in which the $\mathscr{K}_h$ ellipticity (3.8) was shown without this requirement, so we will show (3.8) in a different way but not using Assumption A2) in the next section.

Remark 3.4 We refer (3.8) to as the $\mathscr{K}_h$-ellipticity using the terminology in the classical theory for the mixed problem (1.6) where $\mathscr{K}_h = Z_h(\varepsilon)$, since there is some relationship between the δ-regularization problem and the mixed problem (1.6). Nevertheless, problem (3.2) does not need the $\mathscr{K}_h$-ellipticity to ensure the well-posedness, but problem (1.6) does.

With (3.8) in Proposition 3.1 at hand, we can establish the uniform stability of $u_{\delta,h}$ below.

Theorem 3.2 *Let $u_{\delta,h} \in U_h$ be the solution to problem (3.2). Assume Assumption A1) and the $\mathscr{K}_h$-ellipticity (3.8), we have the following uniform stability for the compatible $f \in H(\mathrm{div}^0; \Omega)$*

$$||u_{\delta,h}||_{0,\mathrm{curl}} \leq C||f||_0, \tag{3.9}$$

where C does not depend on δ.

Before proving (3.9), we give a lemma for what the true space is for the finite element solution $u_{\delta,h}$ of problem (3.2).

Lemma 3.1 *Let $u_{\delta,h} \in U_h$ be the solution of problem (3.2). Then, under Assumption A1), for a compatible $f \in H(\mathrm{div}^0; \Omega)$, we have $u_{\delta,h} \in Z_h(\varepsilon)$.*

Proof. From Assumption A1), we have the following ε-weighted L^2-orthogonal decomposition

$$u_{\delta,h} = z_{\delta,h} + \nabla p_{\delta,h}, \tag{3.10}$$

where $z_{\delta,h} \in Z_h(\varepsilon)$ and $p_{\delta,h} \in Q_h$. From Assumption A1) again, problem (3.2) may re-cast into a mixed problem (in fact, two decoupled subproblems): Find $z_{\delta,h} \in Z_h(\varepsilon)$ and $p_{\delta,h} \in Q_h$ such that

$$(\mu^{-1}\mathrm{curl}\, z_{\delta,h}, \mathrm{curl}\, z) + \delta(\varepsilon z_{\delta,h}, z) = (f, z) \quad \forall z \in Z_h(\varepsilon), \tag{3.11}$$

$$\delta(\varepsilon \nabla p_{\delta,h}, \nabla q) = (f, \nabla q) \quad \forall q \in Q_h. \tag{3.12}$$

But, $f \in H(\mathrm{div}^0; \Omega)$, we find that $p_{\delta,h}$ is equally zero, and we conclude that $u_{\delta,h} = z_{\delta,h} \in Z_h(\varepsilon)$, solving (3.11). □

Proof of Theorem 3.2 From Lemma 3.1, (3.8) and (3.11) it immediately follows that (3.9) holds. □

4. A general verification of $\mathscr{K}_h$-ellipticity. In order to establish the $\mathscr{K}_h$-ellipticity (3.8) without using Assumption A2), we make the following assumption instead of Assumption A2).

Assumption A3) We assume that for any $v \in H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}^0; \Omega)$ it admits a regular-singular decomposition in the following:

$$v = z_0 + \nabla p_0, \tag{4.1}$$

where $z_0 \in H_0(\mathrm{curl}; \Omega) \cap (H^t(\Omega))^3$ for some $t > 1/2$ and $p_0 \in H_0^1(\Omega)$, satisfying

$$||z_0||_t + ||p_0||_1 \leq C||\mathrm{curl}\, v||_0. \tag{4.2}$$

Remark 4.1 For Lipschitz domains, in [7] it is shown that $t = 1$. In this paper, we need only $t > 1/2$.

Lemma 4.1 *Assuming Assumption A3) we have the $\mathscr{K}_h$ ellipticity (3.8), i.e.*

$$||v||_0 \leq C||\mathrm{curl}\, v||_0 \quad \forall v \in Z_h(\varepsilon). \tag{4.3}$$

Before proving Lemma 4.1, we recall the L^2 orthogonal decomposition for $(L^2(\Omega))^3$ and recall the finite element interpolation theory of U_h.

Proposition 4.1 [28] *For any $v \in (L^2(\Omega))^3$ and for any Lipschitz domain Ω, it can be written as the following L^2-orthogonal decomposition:*

$$v = v_1 + \nabla p_1, \tag{4.4}$$

where $v_1 \in H(\mathrm{div}^0; \Omega)$, $p_1 \in H_0^1(\Omega)$.

For proving Lemma 4.1 we need to specify U_h. We use the first family of edge/Nédélec elements in [45], see also [44] [30] for a unified description. Over every $T \in \mathcal{T}_h$, let $\mathcal{P}_l(T)$ and $\widetilde{\mathcal{P}}_l(T)$ denote the space of polynomials of total degree not greater than the integer $l \geq 0$ and the subspace of homogeneous polynomials of total degree l, respectively. On every $T \in \mathcal{T}_h$, for $l \geq 1$ putting the Nédélec element of order l as follows:

$$\mathcal{N}_l(T) := \mathrm{span}\{a + b, a \in (\mathcal{P}_{l-1}(T))^3, b \in (\widetilde{\mathcal{P}}_l(T))^3, b \cdot x = 0\}, \tag{4.5}$$

we define U_h by

$$U_h = \{v \in H_0(\mathrm{curl}; \Omega) : v|_T \in \mathcal{N}_l(T), \forall T \in \mathcal{T}_h\}. \tag{4.6}$$

The degrees of freedom for any function $v \in \mathcal{N}_l(T)$ on each $T \in \mathcal{T}_h$, with the tangential vector τ along the edge e and the normal vector n to the face F, are as follows:

$$\int_e v \cdot \tau q \quad \forall q \in \mathcal{P}_{l-1}(e), \quad \int_F (v \times n) \cdot q \quad \forall q \in (\mathcal{P}_{l-2}(F))^2, \quad \int_T v \cdot q \quad \forall q \in (\mathcal{P}_{l-3}(T))^3.$$

For any given u with suitable regularity (see Remark 4.3 below) we can define a unique $\Pi_T u \in \mathcal{N}_l(T)$ using the above degrees of freedom, i.e.,

$$\int_e \Pi_T u \cdot \tau q = \int_e u \cdot \tau q \quad \forall q \in \mathcal{P}_{l-1}(e), \forall e \in \partial F, \forall F \in \partial T,$$

$$\int_F (\Pi_T u \times n) \cdot q = \int_F (u \times n) \cdot q \quad \forall q \in (\mathcal{P}_{l-2}(F))^2, \forall F \in \partial T,$$

$$\int_T \Pi_T u \cdot q = \int_T u \cdot q \quad \forall q \in (\mathcal{P}_{l-3}(T))^3.$$

Remark 4.2 For U_h defined as above, Assumption A1) holds, where

$$Q_h = \{q \in H_0^1(\Omega) : q|_T \in \mathcal{P}_l(T), \forall T \in \mathcal{T}_h\}.$$

Throughout this paper, we shall focus on the lowest-order edge element method, namely, $l = 1$, since we are interested in the low regular solution of H^r function for some $r \leq 1$. It is of course straightforward to consider higher-order elements if the solution is more regular.

We also need the auxiliary Raviart-Thomas finite element space of $H_0(\mathrm{div}\,; \Omega)$, denoted by X_h, which is closely related to U_h, see [45][44][14][30]. On each $T \in \mathcal{T}_h$, for $l \geq 1$ putting the Raviart-Thomas element of order l as follows:

$$\mathcal{RT}_l(T) = \mathrm{span}\{a + bx, a \in (\mathcal{P}_{l-1}(T))^3, b \in \widetilde{\mathcal{P}}_{l-1}(T)\}, \tag{4.7}$$

$$X_h = \{v \in H_0(\mathrm{div}\,; \Omega) : v|_T \in \mathcal{RT}_l(T), \forall T \in \mathcal{T}_h\}. \tag{4.8}$$

The degrees of freedom for any function $v \in \mathcal{RT}_l(T)$ on each $T \in \mathcal{T}$ are as follows:

$$\int_F v \cdot n q \quad \forall q \in \mathcal{P}_{l-1}(F), \quad \int_T v \cdot q \quad \forall q \in (\mathcal{P}_{l-2}(T))^3.$$

For any given u with suitable regularity (see Remark 4.3 below) we can define a unique $\Upsilon_T u \in \mathcal{RT}_l(T)$ using the above degrees of freedom, i.e.,

$$\int_F \Upsilon_T u \cdot n q = \int_F u \cdot n q \quad \forall q \in \mathcal{P}_{l-1}(F), \forall F \in \partial T, \qquad \int_T \Upsilon_T u \cdot q = \int_T u \cdot q \quad \forall q \in (\mathcal{P}_{l-2}(T))^3.$$

Let Π_h and Υ_h respectively denote the finite element interpolant onto U_h and X_h, with $\Pi_h u|_T = \Pi_T u$ and $\Upsilon_h u|_T = \Upsilon_T u$ for all $T \in \mathcal{T}_h$. Let u be a given function making $\Pi_h u \in U_h$ and $\Upsilon_h \mathrm{curl}\, u \in X_h$ well-defined. Then $\mathrm{curl}\, U_h \subset X_h$ and $\mathrm{curl}\, \Pi_h u = \Upsilon_h \mathrm{curl}\, u$.

Remark 4.3 If $u \in H(\mathrm{div}\,; T) \cap (L^p(T))^3$ for some $p > 2$, then $\Upsilon_T u$ is well-defined , see [14]. If u satisfies the following regularity

$$\{u \in (L^p(T))^3 : \mathrm{curl}\, u \in (L^p(T))^3, v \times n \in (L^p(\partial T))^3\},$$

where $p > 2$, then $\Pi_T u$ is well-defined, see [3]. There are several examples of u for which $\Pi_T u$ is well-defined. 1) $u \in (H^s(T))^3$ for some $s > 1/2$ and $\mathrm{curl}\, u \in (L^p(T))^3$ for some $p > 2$, $\Pi_T u$ is well-defined, since by Sobolev embedding theorem $H^s(T)$ for some $s > 1/2$ is continuously embedded into $L^p(\partial T)$ with $p = \dfrac{6}{3 - 2(s - 1/2)} > 2$; 2) When $u, \mathrm{curl}\, u \in (H^s(T))^3$ for some $s > 1/2$, $\Pi_T u$ is also well-defined, since by Sobolev embedding theorem $H^s(T)$ is continuously embedded into $L^p(T)$ with $p = \dfrac{6}{3 - 2s} > 2$; 3) When $u \in (H^{s_1}(T))^3$ for some $s_1 > 1/2$ and $\mathrm{curl}\, u \in (H^{s_2}(T))^3$ for some $s_2 > 0$, $\Pi_T u$ is well-defined, since by

Sobolev embedding theorem we know that $H^{s_1}(T)$ for $s_1 > 1/2$ is continuously embedded into $L^q(\partial T)$ with $q = \dfrac{6}{3 - 2(s_1 - 1/2)} > 2$, and $H^{s_i}(T)$ is continuously embedded into $L^{p_i}(T)$ with $p_i = \dfrac{6}{3 - 2s_i} > 2$. Take $p := \min(q, p_1, p_2) > 2$; 4) For $u \in (H^s(T))^3$ for some $s > 1/2$ and $\operatorname{curl} u \in X_h$, $\Pi_T u$ is well-defined.

Below we recall the finite element interpolation property we shall use.

Proposition 4.2 [44] *For $u \in (H^s(\Omega))^3$ for some $s > 1/2$ and $\operatorname{curl} u \in X_h$, $\Pi_T u$ is well-defined and we have*

$$||\Pi_T u - u||_{0,T} \leq C h_T^s (||u||_{s,T} + ||\operatorname{curl} u||_{0,T}) \quad \forall T \in \mathcal{T}_h. \tag{4.9}$$

For $u \in (H^{s_1}(\Omega))^3$ for some $s_1 > 1/2$ and $\operatorname{curl} u \in (H^{s_2}(\Omega))^3$ for some $s_2 > 0$, $\Pi_T u$ is well-defined and we have

$$||\Pi_T u - u||_{0,T} \leq C h_T^{s_1} (||u||_{s_1,T} + ||\operatorname{curl} u||_{s_2,T}) \quad \forall T \in \mathcal{T}_h, \tag{4.10}$$

$$||\operatorname{curl}(\Pi_T u - u)||_{0,T} \leq C h_T^{s_2} ||\operatorname{curl} u||_{s_2,T} \quad \forall T \in \mathcal{T}_h. \tag{4.11}$$

Proof of Lemma 4.1. For $v \in Z_h(\varepsilon) \subset U_h$, it admits a both L^2 and $H_0(\operatorname{curl}; \Omega)$ orthogonal decomposition (see Assumption A1) with $\varepsilon = 1$ as follows:

$$v = \widetilde{v} + \nabla p_h, \tag{4.12}$$

where $\widetilde{v} \in Z_h(1)$ and $p_h \in Q_h$. Since $v \in Z_h(\varepsilon)$, we have

$$(\varepsilon v, v) = (\varepsilon v, v - \nabla p_h) = (\varepsilon v, \widetilde{v}). \tag{4.13}$$

Hence

$$||v||_0^2 \leq C||v||_{0,\varepsilon}^2 = C(\varepsilon v, v) = C(\varepsilon v, \widetilde{v}) \leq C||v||_{0,\varepsilon}||\widetilde{v}||_0, \tag{4.14}$$

that is

$$||v||_0 \leq C||\widetilde{v}||_0. \tag{4.15}$$

Noticing that

$$\operatorname{curl} v = \operatorname{curl} \widetilde{v}, \tag{4.16}$$

if we can show

$$||\widetilde{v}||_0 \leq C||\operatorname{curl} \widetilde{v}||_0 \tag{4.17}$$

then we have the conclusion (4.3). In what follows, we are to show (4.17).

According to Proposition 4.1, with the given $\widetilde{v} \in U_h \subset H_0(\operatorname{curl}; \Omega)$ we first have the L^2 orthogonal decomposition

$$\widetilde{v} = v_1 + \nabla p_1, \quad v_1 \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div}^0; \Omega), \quad p_1 \in H_0^1(\Omega), \tag{4.18}$$

then from Assumption A3) we have the regular-singular decomposition for $v_1 \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div}^0; \Omega)$:

$$v_1 = v_0 + \nabla p_0, \quad v_0 \in (H^t(\Omega))^3, \quad p_0 \in H_0^1(\Omega), \tag{4.19}$$

where, $t > 1/2$, and

$$||v_0||_t \leq C||\operatorname{curl} v_1||_0 = C||\operatorname{curl} \widetilde{v}||_0. \tag{4.20}$$

We thus have

$$\widetilde{v} = v_0 + \nabla p, \quad p := p_0 + p_1 \in H_0^1(\Omega), \tag{4.21}$$

$$\operatorname{curl} v_0 = \operatorname{curl} \widetilde{v} \in X_h. \tag{4.22}$$

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From Proposition 4.2 we know that $\Pi_h v_0$ is well-defined, satisfying

$$||\Pi_h v_0 - v_0||_0 \leq Ch^t(||v_0||_t + ||\operatorname{curl} v_0||_0) \leq Ch^t||\operatorname{curl} \widetilde{v}||_0, \tag{4.23}$$

from which and (4.20) we have

$$||\Pi_h v_0||_0 \leq C||\operatorname{curl} \widetilde{v}||_0. \tag{4.24}$$

Consequently, $\Pi_h \nabla p$ is also well-defined, and it is a well-known result [30] that $\Pi_h \nabla p = \nabla q_h$ for some $q_h \in Q_h$. Therefore,

$$||\widetilde{v}||_0^2 = (\widetilde{v}, \widetilde{v}) = (\widetilde{v}, v_0 + \nabla p) = (\widetilde{v}, v_0 + \nabla p - \Pi_h(v_0 + \nabla p)) + (\widetilde{v}, \Pi_h v_0 + \nabla q_h), \tag{4.25}$$

where, since $\Pi_h v_h = v_h$ for all $v_h \in U_h$, we have

$$(\widetilde{v}, v_0 + \nabla p - \Pi_h(v_0 + \nabla p)) = (\widetilde{v}, \widetilde{v} - \Pi_h \widetilde{v}) = 0, \tag{4.26}$$

and since $\widetilde{v} \in Z_h(1)$, we have

$$(\widetilde{v}, \Pi_h v_0 + \nabla q_h) = (\widetilde{v}, \Pi_h v_0) \leq ||\widetilde{v}||_0 ||\Pi_h v_0||_0 \leq C||\widetilde{v}||_0 ||\operatorname{curl} \widetilde{v}||_0. \tag{4.27}$$

Thus, (4.17) follows from (4.25)-(4.27). $\quad \square$

Remark 4.4 The difference between Proposition 3.1 and Lemma 4.1 is that the former requires the continuously embedding (3.6) for some $s > 1/2$ in Assumption A2); while the latter instead requires a weaker Assumption A3). For Lipschitz domains, only $s = 1/2$ in Assumption A2), while Assumption A3) holds with $t = 1$.

Remark 4.5 Theorem 3.2 holds, since the $\mathscr{K}_h$ ellipticity (4.3) and Assumption A1) with U_h defined by (4.6) hold.

5. Error estimates. In this section, we will establish the convergence of $u_{\delta,h}$.

From Proposition 4.2 we first recall the finite element interpolation theory of Π_h in U_h for a u with a piecewise regularity with respect to the material subdomains.

Proposition 5.1 Let $u, \operatorname{curl} u \in \prod_{j=1}^{J}(H^r(\Omega_j))^3$ for some $r > 1/2$. Then, $\Pi_h u$ is well-defined and satisfies

$$||u - \Pi_h u||_{0,\operatorname{curl}} \leq Ch^r(\sum_{j=1}^{J} ||u||_{r,\Omega_j} + ||\operatorname{curl} u||_{r,\Omega_j}). \tag{5.1}$$

Since $\mathscr{A}_\delta(u,v)$ given by (2.5) is coercive over $H_0(\operatorname{curl}; \Omega)$ and the consistency property between problem (1.9) and problem (3.2) holds because of the conformity of $U_h \subset H_0(\operatorname{curl}; \Omega)$, it is not difficult to have the following quasi-optimal error estimates following the classical Céa's argument in [18].

Proposition 5.2 Let $u_\delta \in H_0(\operatorname{curl}; \Omega)$ be the solution of problem (1.9) and $u_{\delta,h} \in U_h$ the solution of problem (3.2). Then

$$||u_\delta - u_{\delta,h}||_{\mathscr{A}_\delta} = \inf_{v \in U_h} ||u_\delta - v||_{\mathscr{A}_\delta}, \tag{5.2}$$

where

$$||v||_{\mathscr{A}_\delta}^2 := \mathscr{A}_\delta(v,v) = (\mu^{-1}\operatorname{curl} v, \operatorname{curl} v) + \delta(\varepsilon v, v). \tag{5.3}$$

From Proposition 5.1 and Proposition 5.2 we can obtain the error estimates following the classical finite element theory in [18].

Proposition 5.3 Let $u_\delta \in H_0(\operatorname{curl}; \Omega)$ be the solution of problem (1.9) and $u_{\delta,h} \in U_h$ the solution of problem (3.2). Assume that $u_\delta, \operatorname{curl} u_\delta \in \prod_{j=1}^{J}(H^r(\Omega_j))^3$ for some $r > 1/2$. Then

$$||u_\delta - u_{\delta,h}||_{\mathscr{A}_\delta} \leq Ch^r(\sum_{j=1}^{J} ||u_\delta||_{r,\Omega_j} + ||\operatorname{curl} u_\delta||_{r,\Omega_j}). \tag{5.4}$$

To relate the right-hand side of (5.4) to the source function f, we need to make the following assumption.

Assumption A4)   We assume that there exists a $r > 0$ such that

$$H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div}{}^0;\varepsilon;\Omega) \hookrightarrow \prod_{j=1}^{J}(H^r(\Omega_j))^3 \tag{5.5}$$

is a continuously embedding, satisfying for all $v \in H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div}{}^0;\varepsilon;\Omega)$

$$\sum_{j=1}^{J}||v||_{r,\Omega_j} \le C||\operatorname{curl} v||_0. \tag{5.6}$$

Note that this Assumption A4) reduces to Assumption A2) for $\varepsilon = 1$.

Assumption A5)   Introduce $H_0(\operatorname{div}{}^0;\mu;\Omega) = \{v \in (L^2(\Omega))^3 : \operatorname{div}\mu v = 0, \mu v \cdot n|_S = 0, \forall S \in \mathscr{S}_{\text{ext}}\}$. We assume that there exists a $r > 0$, the same as in Assumption A4), such that

$$H(\operatorname{curl};\Omega) \cap H_0(\operatorname{div}{}^0;\mu;\Omega) \hookrightarrow \prod_{j=1}^{J}(H^r(\Omega_j))^3 \tag{5.7}$$

is a continuously embedding, satisfying for all $v \in H(\operatorname{curl};\Omega) \cap H_0(\operatorname{div}{}^0;\mu;\Omega)$

$$\sum_{j=1}^{J}||v||_{r,\Omega_j} \le C||\operatorname{curl} v||_0. \tag{5.8}$$

Remark 5.1   The regularity in Assumption A4) and Assumption A5) may be possibly different, but here we assume the same r.

Lemma 5.1   *Given a compatible $f \in H(\operatorname{div}{}^0;\Omega)$. Under Assumption A4) and Assumption A5), the solution u_δ of problem (1.9) and the solution u of problem (1.5), together with $\operatorname{curl} u_\delta$ and $\operatorname{curl} u$, are in $\prod_{j=1}^{J}(H^r(\Omega_j))^3$, and the following hold:*

$$\sum_{j=1}^{J}||u_\delta||_{r,\Omega_j} + ||\operatorname{curl} u_\delta||_{r,\Omega_j} \le C||f||_0, \tag{5.9}$$

$$\sum_{j=1}^{J}||u||_{r,\Omega_j} + ||\operatorname{curl} u||_{r,\Omega_j} \le C||f||_0. \tag{5.10}$$

Proof.   Observe that u_δ and u belong to $H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div}{}^0;\varepsilon;\Omega)$ and satisfy $||u_\delta||_{0,\operatorname{curl}}, ||u||_{0,\operatorname{curl}} \le C||f||_0$ (See Lemma 2.1, Theorem 2.1 and Remark 2.1). At the same time, both $\mu^{-1}\operatorname{curl} u_\delta$ and $\mu^{-1}\operatorname{curl} u$ belong to $H(\operatorname{curl};\Omega) \cap H_0(\operatorname{div}{}^0;\mu;\Omega)$, satisfying

$$||\operatorname{curl}\mu^{-1}\operatorname{curl} u||_0 = ||f||_0, \tag{5.11}$$

$$||\operatorname{curl}\mu^{-1}\operatorname{curl} u_\delta||_0 = ||f - \delta\varepsilon u_\delta||_0 \le ||f||_0 + C||u_\delta||_0 \le C||f||_0. \tag{5.12}$$

Hence, from Assumption A4) and Assumption A5) we conclude that Lemma 5.1 holds, noting that $\mu|_{\Omega_j}$ is in $(W^{1,\infty}(\Omega_j))^{3\times 3}$. $\square$

Theorem 5.1   *Assume that $r > 1/2$ holds in Assumption A4) and Assumption A5). Then, the solution u of problem (1.5) and the solution $u_{\delta,h}$ of problem (3.2) satisfy the following error estimation:*

$$||u - u_{\delta,h}||_{\mathscr{A}_\delta} \le C(\delta + h^r)||f||_0. \tag{5.13}$$

Proof.   From Proposition 5.3 and Lemma 5.1 we obtain

$$||u_\delta - u_{\delta,h}||_{\mathscr{A}_\delta} \le Ch^r||f||_0. \tag{5.14}$$

Secondly, from Theorem 2.2 and Remark 2.2 we have

$$||u - u_\delta||_{\mathscr{A}_\delta} \le C||u - u_\delta||_{0,\operatorname{curl}} \le C\delta||u||_0 \le C\delta||f||_0. \tag{5.15}$$

By the triangle inequality we immediately obtain (5.13) from (5.14) and (5.15). $\square$

Remark 5.2 The inadequacy of the error estimates of (5.13) in Theorem 5.1 is the δ-dependent norm $||\cdot||_{\mathscr{A}_\delta}$. However, we can have uniform error estimates, i.e., (5.13) can hold for the error $u - u_{\delta,h}$ in the standard $H(\mathrm{curl}\,;\Omega)$ norm $||v||_{0,\mathrm{curl}}^2 = ||v||_0^2 + ||\mathrm{curl}\, v||_0^2$. We will achieve this by using the Fortin-type finite element interpolation [8].

In what follows, we only consider the case where $\mu = \varepsilon = 1$ and use the Fortin-operator to obtain the uniform error estimates under the same assumptions in Theorem 5.1. The study for the case with general μ, ε and with very low regular solution will be deferred to the next section.

Let π_h be the Fortin operator defined by seeking $\pi_h u \in Z_h(1) \subset U_h$ for a given $u \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$ such that

$$(\mathrm{curl}\,\pi_h u, \mathrm{curl}\, v) = (\mathrm{curl}\, u, \mathrm{curl}\, v) \quad \forall v \in U_h, \tag{5.16}$$

$$(\pi_h u, \nabla q) = 0 \quad \forall q \in Q_h. \tag{5.17}$$

Proposition 5.4 [4][8][44][34] Let $u \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$ and let $\pi_h u$ be defined by (5.16) and (5.17). Assuming the same Assumptions as in Theorem 5.1, we have

$$||\pi_h u - u||_{0,\mathrm{curl}} \leq C h^r (||u||_r + ||\mathrm{curl}\, u||_r). \tag{5.18}$$

Remark 5.3 Under an additional assumption, i.e., the following continuous embedding holds:

$$\{v \in H_0(\mathrm{div}\,^0;\Omega) : -\Delta v \in (L^2(\Omega))^3, \mathrm{curl}\, v \times n|_{\partial\Omega} = 0\} \hookrightarrow (H^r(\Omega))^3 \quad \text{for some } r > 1/2,$$

a little stronger result in [8] than (5.18) is obtained as follows:

$$||\pi_h u - u||_0 \leq C h^r ||u||_r, \tag{5.19}$$

where $\mathrm{curl}\, u$ does not appear in the right-hand side of (5.19).

Theorem 5.2 Under the same assumptions in Theorem 5.1, we have the following uniform error estimation:

$$||u - u_{\delta,h}||_{0,\mathrm{curl}} \leq C(\delta + h^r)||f||_0. \tag{5.20}$$

Proof. Let $v := u_{\delta,h} - \pi_h u_\delta \in U_h$, where u_δ is the solution of problem (1.9). Note that $\mu = \varepsilon = 1$. From the consistency property between problem (1.9) and problem (3.2) and the definition of the Fortin operator π_h by (5.16) and (5.17) we have

$$
\begin{aligned}
||v||_{\mathscr{A}_\delta}^2 &= (\mathrm{curl}\, v, \mathrm{curl}\, v) + \delta(v, v) \\
&= (\mathrm{curl}\,(u_{\delta,h} - \pi_h u_\delta), \mathrm{curl}\, v) + \delta(u_{\delta,h} - \pi_h u_\delta, v) \\
&= (\mathrm{curl}\,(u_{\delta,h} - u_\delta), \mathrm{curl}\, v) + (\mathrm{curl}\,(u_\delta - \pi_h u_\delta), \mathrm{curl}\, v) + \delta(u_{\delta,h} - u_\delta, v) + \delta(u_\delta - \pi_h u_\delta, v) \\
&= \delta(u_\delta - \pi_h u_\delta, v) \\
&\leq \delta||u_\delta - \pi_h u_\delta||_0 ||v||_0 \\
&\leq \delta^{\frac{1}{2}} ||u_\delta - \pi_h u_\delta||_0 ||v||_{\mathscr{A}_\delta},
\end{aligned}
\tag{5.21}
$$

that is

$$||\mathrm{curl}\,(u_{\delta,h} - \pi_h u_\delta)||_0 + \delta^{\frac{1}{2}}||u_{\delta,h} - \pi_h u_\delta||_0 \leq C\delta^{\frac{1}{2}}||u_\delta - \pi_h u_\delta||_0, \tag{5.22}$$

$$||u_{\delta,h} - \pi_h u_\delta||_0 \leq C||u_\delta - \pi_h u_\delta||_0, \tag{5.23}$$

$$||\mathrm{curl}\,(u_{\delta,h} - \pi_h u_\delta)||_0 \leq C\delta^{\frac{1}{2}}||u_\delta - \pi_h u_\delta||_0. \tag{5.24}$$

From Proposition 5.4 with $u_\delta \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$, by the triangle inequality we obtain from (5.23) and (5.24)

$$
\begin{aligned}
||u_\delta - u_{\delta,h}||_{0,\mathrm{curl}} &\leq ||u_\delta - \pi_h u_\delta||_{0,\mathrm{curl}} + ||u_{\delta,h} - \pi_h u_\delta||_{0,\mathrm{curl}} \\
&\leq C||u_\delta - \pi_h u_\delta||_{0,\mathrm{curl}} \\
&\leq C h^r (||u_\delta||_r + ||\mathrm{curl}\, u_\delta||_r),
\end{aligned}
\tag{5.25}
$$

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and from Lemma 5.1 we further have

$$||u_\delta - u_{\delta,h}||_{0,\text{curl}} \le Ch^r ||f||_0. \tag{5.26}$$

Finally, from Theorem 2.2 and Remark 2.2 we obtain the desired (5.20). $\quad\square$

Remark 5.4 Clearly, a more refined error estimation for $u_\delta - u_{\delta,h}$ can be concluded in the following

$$||u_\delta - u_{\delta,h}||_0 \le C||u_\delta - \pi_h u_\delta||_0, \quad ||\text{curl}\,(u_\delta - u_{\delta,h})||_0 \le C||\text{curl}\,(u_\delta - \pi_h u_\delta)||_0 + C\delta^{\frac{1}{2}}||u_\delta - \pi_h u_\delta||_0.$$

Remark 5.5 The assumption $r > 1/2$ of the regularity of u and $\text{curl}\,u$ over Ω_j is not that restrictive in practice. In fact, such regularity assumption is commonly used in the literature [17][19][36][2][44]. Meanwhile, it has been shown $r > 1/2$ or even $r = 1$ for practical interface problems [16][39].

Remark 5.6 On the other hand, the interface problem from electromagnetism would have a possible very low regularity solution, i.e., $r \le 1/2$, see [24]. In addition, even if $\mu = \varepsilon = 1$, s in Assumption A2) or r in both Assumption A4) and Assumption A5) is still possibly less than or equal to $1/2$. For example, for a general Lipschitz domain, $s = 1/2$ in Assumption A2).

Without the requirements $r > 1/2$ and $s > 1/2$, we shall obtain the uniform error estimates next section. We have seen that the uniform error estimates rely on the Fortin operator, so it suffices that the finite element interpolation property (5.18) for the Fortin operator hold even if $0 < r, s \le 1/2$. In addition, we shall deal with general μ and ε as assumed in section 2.

6. A general Fortin operator. The advantage of the Fortin operator over the finite element interpolation operator Π_h is the former is a projection in the sense of equation (5.16). We have seen that it is based on the projection property of π_h that we have established the uniform error bound in the previous section. Although, from the definition (5.16) we may infer that the Fortin operator π_h would be well-defined for any $u \in H_0(\text{curl}\,;\Omega) \cap H(\text{div}^{\,0};\Omega)$ even if the interpolated function u does not have the regularity with $r > 1/2$. Unfortunately, in the literature, the well-definedness and the interpolation error property of the Fortin operator indeed depend on Assumption A2) with $s > 1/2$ (or (1.8) mentioned in Introduction section) and on the regularity $r > 1/2$ of the interpolated function u.

In this section, we consider the following general Fortin operator: Given $u \in H_0(\text{curl}\,;\Omega)$, to find $\pi_h u \in U_h$ such that

$$(\mu^{-1}\text{curl}\,\pi_h u, \text{curl}\,v) = (\mu^{-1}\text{curl}\,u, \text{curl}\,v) \quad \forall v \in U_h, \tag{6.1}$$

$$(\varepsilon\pi_h u, \nabla q) = (\varepsilon u, \nabla q) \quad \forall q \in Q_h. \tag{6.2}$$

It will be shown that $\pi_h u$ is well-defined and satisfy (5.18) without assuming the regularity index r of the interpolated function u to be greater than $1/2$ and without requiring $s > 1/2$ in Assumption A2). In fact, we only assume Assumption A3) which is already stated in section 4. The difference between Assumption A3) and Assumption A2) is addressed in Remark 4.4 and will be further addressed in Remark 6.4.

Lemma 6.1 *Assume that Assumption A3) holds. For any given* $u \in H_0(\text{curl}\,;\Omega)$, $\pi_h u$ *is well-defined.*
Proof. Observe that $\pi_h u$ is the first component of the solution pair $(u_h, p_h) \in U_h \times Q_h$ such that

$$(\mu^{-1}\text{curl}\,u_h, \text{curl}\,v) + (\varepsilon v, \nabla p_h) = (\mu^{-1}\text{curl}\,u, \text{curl}\,v) \quad \forall v \in U_h, \tag{6.3}$$

$$(\varepsilon u_h, \nabla q) = (\varepsilon u, \nabla q) \quad \forall q \in Q_h. \tag{6.4}$$

In fact, since $U_h = Z_h(\varepsilon) + \nabla Q_h$ as stated in Assumption A1) which is verified with U_h defined by (4.6), p_h is equal to zero in the above mixed problem. If we have shown $\mathcal{K}_h^t \subset \mathcal{K}^t$, $\mathcal{K}_h$-ellipticity and the Inf-Sup condition, then the classical theory in [14] for saddle-point problems yields the well-posedness of the above mixed problem. The inclusion $\mathcal{K}_h^t \subset \mathcal{K}^t$ is verified by noticing that $\mathcal{K}^t = \{q \in H_0^1(\Omega) : (\varepsilon v, \nabla q) = 0, \forall v \in H_0(\text{curl}\,;\Omega)\} = \{0\}$ and $\mathcal{K}_h^t = \{q_h \in Q_h : (\varepsilon v_h, \nabla q_h) = 0, \forall v_h \in U_h\} = \{0\}$, and the verification of the Inf-Sup condition follows from the decomposition $U_h = Z_h(\varepsilon) + \nabla Q_h$

$$\sup_{v \in U_h} \frac{(\varepsilon v, \nabla q)}{||v||_{0,\varepsilon} + ||\text{curl}\,v||_{0,\mu^{-1}}} \ge \frac{(\varepsilon\nabla q, \nabla q)}{||\nabla q||_{0,\varepsilon}} \ge C||q||_1 \quad \forall q \in Q_h. \tag{6.5}$$

We are left to verify the $\mathcal{K}_h$ ellipticity, where $\mathcal{K}_h = Z_h(\varepsilon) = \{v \in U_h : (\varepsilon v, \nabla q) = 0, \forall q \in Q_h\}$. But, this is just the conclusion of Lemma 4.1 since Assumption A3) holds. $\quad\square$

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Remark 6.1   Lemma 6.1 implies the following invariance property over U_h

$$\pi_h u = u \quad \forall u \in U_h \subset H_0(\mathrm{curl}\,;\Omega) \tag{6.6}$$

and the following boundedness property

$$\|\pi_h u\|_{0,\mathrm{curl}} \leq C\|u\|_{0,\mathrm{curl}} \quad \forall u \in H_0(\mathrm{curl}\,;\Omega). \tag{6.7}$$

Lemma 6.2   *For any $p \in H_0^1(\Omega)$ with $\nabla p \in H_0(\mathrm{curl}\,;\Omega)$, we have $\pi_h \nabla p = \nabla p_h$, where $p_h \in Q_h$ satisfies*

$$(\varepsilon \nabla p_h, \nabla q) = (\varepsilon \nabla p, \nabla q) \quad \forall q \in Q_h. \tag{6.8}$$

Proof.   With $u := \nabla p \in H_0(\mathrm{curl}\,;\Omega)$ and $\pi_h u \in U_h \subset H_0(\mathrm{curl}\,;\Omega)$ at hand, we find from equation (6.1) that $\mathrm{curl}\,\pi_h \nabla p = 0$. Thus, from [30][44] we have some $p_h \in Q_h$ satisfies $\pi_h \nabla p = \nabla p_h$, and we obtain (6.8) from equation (6.2).   □

Lemma 6.3   *For $p \in H_0^1(\Omega) \cap \prod\limits_{j=1}^{J} H^{1+r}(\Omega_j)$ for some $r > 0$, there exists $I_h p \in Q_h$ such that*

$$\|p - I_h p\|_1 \leq C h^r \sum_{j=1}^{J} \|p\|_{1+r,\Omega_j}. \tag{6.9}$$

Proof.   If $p \in H_0^1(\Omega) \cap H^{1+r}(\Omega)$, then (6.7) is a classical result from the finite element interpolation theory [12][18][20][48]. Now, p is piecewise $H^{1+r}(\Omega_j)$, no immediate literature could be located. Here, we give an approach to obtain $I_h p \in Q_h$ that can satisfies (6.9). Firstly, taking any Ω_j, putting

$$Q_{h,\Omega_j} := Q_h|_{\Omega_j}, \tag{6.10}$$

we using the Clément interpolation[20][6] or Scott-Zhang interpolation[48] to have $\mathscr{C}_{h,j} p \in Q_{h,\Omega_j}$ satisfying

$$\|\mathscr{C}_{h,j} p - p\|_{1,\Omega_j} + \left(\sum_{F \in \mathscr{S}_{\Omega_j}} h_F^{-1} \|\mathscr{C}_{h,j} p - p\|_{0,F}^2 \right)^{\frac{1}{2}} \leq C h^r \|p\|_{1+r,\Omega_j}, \tag{6.11}$$

where $\mathscr{S}_{\Omega_j}$ denotes the collect of all element faces in $\mathcal{T}_h|_{\Omega_j}$. Introducing a function p_h defined by

$$p_h|_{\Omega_j} := \mathscr{C}_{h,j} p \quad 1 \leq j \leq J. \tag{6.12}$$

In general, such p_h is discontinuous when crossing any interface S in $\mathscr{S}_{\mathrm{int}}$. However, by an averaging procedure (see Remark 6.1 below) we can find a new finite element function $I_h p \in Q_h$ to satisfy $I_h p(a) = p_h(a)$ for all interior nodes inside Ω_j, $1 \leq j \leq J$, for all boundary nodes on $\mathscr{S}_{\mathrm{ext}}$, and

$$\sum_{j=1}^{J} \|I_h p - p_h\|_{1,\Omega_j}^2 \leq C \sum_{F \in \mathscr{S}_{\mathrm{int}}} h_F^{-1} \int_F |[p_h]|^2. \tag{6.13}$$

But, $p \in H_0^1(\Omega)$, we have

$$\begin{aligned}
\left(\sum_{F \in \mathscr{S}_{\mathrm{int}}} h_F^{-1} \int_F |[p_h]|^2 \right)^{\frac{1}{2}} &= \left(\sum_{F \in \mathscr{S}_{\mathrm{int}}} h_F^{-1} \int_F |[p_h - p]|^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{j=1}^{J} \sum_{F \in \mathscr{S}_{\Omega_j}} h_F^{-1} \|\mathscr{C}_{h,j} p - p\|_{0,F}^2 \right)^{\frac{1}{2}} \\
&\leq C h^r \sum_{j=1}^{J} \|p\|_{1+r,\Omega_j}.
\end{aligned} \tag{6.14}$$

Hence, using the triangle inequality, we obtain (6.9) from (6.11),(6.13) and (6.14).   □

Remark 6.2   For linear element, the finite element function $I_h p$ which is constructed from an averaging approach may be referred to [11]. For higher-order elements, readers may refer to [42] for a general approach to construct $I_h p$.

Now we state the main result in Theorem 6.1 below for the Fortin operator π_h with the special $u := \nabla p$.

Theorem 6.1 *Let $p_h = \pi_h \nabla p \in Q_h$ be given by (6.8). For $p \in H_0^1(\Omega) \cap \prod\limits_{j=1}^{J} H^{1+r}(\Omega_j)$ we have*

$$||\pi_h \nabla p - \nabla p||_0 \leq C h^r \sum_{j=1}^{J} ||p||_{1+r,\Omega_j}. \tag{6.15}$$

Proof. Let $I_h p$ be constructed in Lemma 6.3. Since p_h is the finite element solution to (6.8), we have

$$||\nabla(p_h - p)||_{0,\varepsilon}^2 = (\varepsilon \nabla(p_h - p), \nabla(p_h - p)) = (\varepsilon \nabla(p_h - p), \nabla(I_h p - p)) \leq C||\nabla(p_h - p)||_{0,\varepsilon}||I_h p - p||_1. \tag{6.16}$$

We then have

$$||\pi_h \nabla p - \nabla p||_0 = ||\nabla(p_h - p)||_0 \leq C||\nabla(p_h - p)||_{0,\varepsilon} \leq C||I_h p - p||_1 \leq C h^r \sum_{j=1}^{J} ||p||_{1+r,\Omega_j}. \tag{6.17}$$

$\square$

Lemma 6.4 *Assume that Assumption A3) holds. For any $z \in H_0(\mathrm{curl}\,; \Omega)$, we have the following regular-singular decomposition:*

$$z = z_0 + \nabla p, \tag{6.18}$$

where $z_0 \in H_0(\mathrm{curl}\,; \Omega) \cap (H^t(\Omega))^3$ with the same t in Assumption A3) and $p \in H_0^1(\Omega)$, satisfying

$$||z_0||_t \leq C||\mathrm{curl}\, z||_0, \tag{6.19}$$

$$||p||_1 \leq C||z||_{0,\mathrm{curl}}. \tag{6.20}$$

Proof. From Proposition 4.1 we first have the following L^2 orthogonal decomposition

$$z = z_1 + \nabla p_1, \tag{6.21}$$

where $z_1 \in H_0(\mathrm{curl}\,; \Omega) \cap H(\mathrm{div}^{\,0}; \Omega)$ and $p_1 \in H_0^1(\Omega)$, satisfying

$$||z||_0^2 = ||z_1||_0^2 + ||\nabla p_1||_0^2, \tag{6.22}$$

$$\mathrm{curl}\, z_1 = \mathrm{curl}\, z. \tag{6.23}$$

From Assumption A3) we may write

$$z_1 = z_0 + \nabla p_0, \tag{6.24}$$

where $z_0 \in (H^t(\Omega))^3 \cap H_0(\mathrm{curl}\,; \Omega)$ for some $t > 1/2$ and $p_0 \in H_0^1(\Omega)$, satisfying

$$||z_0||_t + ||p_0||_1 \leq C||\mathrm{curl}\, z_1||_0 = C||\mathrm{curl}\, z||_0. \tag{6.25}$$

Putting

$$p := p_1 + p_0, \tag{6.26}$$

We thus have

$$z = z_0 + \nabla p. \tag{6.27}$$

$\square$

Lemma 6.5 *Assume that Assumption A3) holds. For $z \in H_0(\mathrm{curl}\,; \Omega) \cap \prod\limits_{j=1}^{J} (H^{r_1}(\Omega_j))^3$ with some $r_1 > 1/2$ and $\mathrm{curl}\, z \in \prod\limits_{j=1}^{J} (H^{r_2}(\Omega_j))^3$ with some $r_2 > 0$, we have*

$$||z - \pi_h z||_0 \leq C h^{r_2} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\mathrm{curl}\, z||_{r_2,\Omega_j}), \tag{6.28}$$

$$||\mathrm{curl}\,(z - \pi_h z)||_0 \leq C h^{r_2} \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.29}$$

Proof. Since Assumption A3) holds, from Lemma 6.1 it follows that $\pi_h z$ is well-defined for $z \in H_0(\mathrm{curl};\Omega)$. We are now ready to estimate the difference between z and $\pi_h z$. For the case where $\mu = \varepsilon = 1$ and $r_1 = r_2 > 1/2$, see Proposition 5.4. For general μ and ε as assumed in section 2 and for general $r_1 > 1/2$ and $r_2 > 0$ as assumed respectively for the regularity of z and $\mathrm{curl}\, z$, we prove (6.28) and (6.29) in the following.

Noticing that $\Pi_h z \in U_h$ is well-defined, since $z \in \prod_{j=1}^{J}(H^{r_1}(\Omega_j))^3$ for some $r_1 > 1/2$ and $\mathrm{curl}\, z = \mathrm{curl}\, z \in \prod_{j=1}^{J}(H^{r_2}(\Omega_j))^3$ for some $r_2 > 0$ (see Remark 4.2), from Proposition 4.2 we have

$$||z - \Pi_T z||_{0,T} \le Ch_T^{r_1}(||z||_{r_1,T} + ||\mathrm{curl}\, z||_{r_2,T}) \quad \forall T \in \mathcal{T}_h, \tag{6.30}$$

$$||\mathrm{curl}\,(z - \Pi_T z)||_{0,T} \le Ch_T^{r_2}||\mathrm{curl}\, z||_{r_2,T} \quad \forall T \in \mathcal{T}_h, \tag{6.31}$$

and we have

$$||z - \Pi_h z||_0 \le Ch^{r_1} \sum_{j=1}^{J}(||z||_{r_1,\Omega_j} + ||\mathrm{curl}\, z||_{r_2,\Omega_j}), \tag{6.32}$$

$$||\mathrm{curl}\,(z - \Pi_h z)||_0 \le Ch^{r_2} \sum_{j=1}^{J}||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.33}$$

We first estimate the difference between $\Pi_h z$ and $\pi_h z$.

From Lemma 6.4 we decompose $\Pi_h z - \pi_h z$ as follows:

$$\Pi_h z - \pi_h z = z_0 + \nabla p, \tag{6.34}$$

where $z_0 \in H_0(\mathrm{curl};\Omega) \cap (H^t(\Omega))^3$ for some $t > 1/2$ and $p \in H_0^1(\Omega)$, satisfying

$$||z_0||_t \le C||\mathrm{curl}\,(\Pi_h z - \pi_h z)||_0, \tag{6.35}$$

$$||p||_1 \le C||\Pi_h z - \pi_h z||_{0,\mathrm{curl}}, \tag{6.36}$$

$$\mathrm{curl}\, z_0 = \mathrm{curl}\,(\Pi_h z - \pi_h z) \in X_h. \tag{6.37}$$

But, from Remark 6.1 and (6.33) we have

$$||\mathrm{curl}\,(\Pi_h z - \pi_h z)||_0 = ||\mathrm{curl}\, \pi_h(\Pi_h z - z)||_0 \le C||\mathrm{curl}\,(\Pi_h z - z)||_0 \le Ch^{r_2} \sum_{j=1}^{J}||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.38}$$

Thus, it follows from (6.35), (6.37) and (6.38) that

$$||z_0||_{0,\mathrm{curl}} \le Ch^{r_2} \sum_{j=1}^{J}||\mathrm{curl}\, z||_{r_2,\Omega_j}, \tag{6.39}$$

$$||z_0||_t \le Ch^{r_2} \sum_{j=1}^{J}||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.40}$$

Note that $\Pi_h z_0$ is well-defined, because $z_0 \in (H^t(\Omega))^3$ for some $t > 1/2$ and $\mathrm{curl}\, z_0 \in X_h$. From Proposition 4.2 and (6.39), (6.40) we have

$$||z_0 - \Pi_h z_0||_0 \le Ch^t(||z_0||_t + ||\mathrm{curl}\, z_0||_0) \le Ch^{t+r_2} \sum_{j=1}^{J}||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.41}$$

In addition, $\Pi_h \nabla p$ is also well-defined since $\nabla p = \Pi_h z - \pi_h z - z_0$, and we have some $q_h \in Q_h$ such that $\Pi_h \nabla p = \nabla q_h$, see [30]. And, we have

$$\Pi_h z - \pi_h z = \Pi_h(\Pi_h z - \pi_h z) = \Pi_h z_0 + \Pi_h \nabla p = \Pi_h z_0 + \nabla q_h. \tag{6.42}$$

We next estimate $z - \pi_h z$.

We have

$$||z - \pi_h z||^2_{0,\varepsilon} = (\varepsilon(z - \pi_h z), z - \pi_h z) = (\varepsilon(z - \pi_h z), z - \Pi_h z) + (\varepsilon(z - \pi_h z), \Pi_h z - \pi_h z), \qquad (6.43)$$

where, from (6.32), we have

$$(\varepsilon(z - \pi_h z), z - \Pi_h z) \leq C||z - \pi_h z||_{0,\varepsilon}||z - \Pi_h z||_0 \leq C||z - \pi_h z||_{0,\varepsilon} h^{r_1} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\text{curl}\, z||_{r_2,\Omega_j}), \quad (6.44)$$

and from (6.2) in the definition of π_h with $u := z$ here, i.e., $(\varepsilon\pi_h z, \nabla q) = (\varepsilon z, \nabla q)$ holds for all $q \in Q_h$, and from (6.42), (6.39) and (6.41), we have

$$
\begin{aligned}
(\varepsilon(z - \pi_h z), \Pi_h z - \pi_h z) &= (\varepsilon(z - \pi_h z), \Pi_h z_0 + \nabla q_h) \\
&= (\varepsilon(z - \pi_h z), \Pi_h z_0) \\
&= (\varepsilon(z - \pi_h z), \Pi_h z_0 - z_0) + (\varepsilon(z - \pi_h z), z_0) \\
&\leq C||z - \pi_h z||_{0,\varepsilon}||z_0 - \Pi_h z_0||_0 + C||z - \pi_h z||_{0,\varepsilon}||z_0||_0 \\
&\leq C||z - \pi_h z||_{0,\varepsilon} h^{r_2} \sum_{j=1}^{J} ||\text{curl}\, z||_{r_2,\Omega_j}.
\end{aligned}
\qquad (6.45)
$$

Hence, from (6.43), (6.44) and (6.45) we obtain

$$||z - \pi_h z||_0 \leq C||z - \pi_h z||_{0,\varepsilon} \leq C h^{r_2} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\text{curl}\, z||_{r_2,\Omega_j}). \qquad (6.46)$$

This competes the proof of (6.28). Regarding (6.29), we find from (6.1) in the definition of π_h that for all $v \in U_h$,

$$
\begin{aligned}
||\text{curl}\,(z - \pi_h z)||^2_{0,\mu^{-1}} &= (\mu^{-1}\text{curl}\,(z - \pi_h z), \text{curl}\,(z - \pi_h z)) \\
&= (\mu^{-1}\text{curl}\,(z - \pi_h z), \text{curl}\,(z - v)) \\
&\leq ||\text{curl}\,(z - \pi_h z)||_{0,\mu^{-1}}||\text{curl}\,(z - v)||_{0,\mu^{-1}}
\end{aligned}
\qquad (6.47)
$$

i.e.

$$||\text{curl}\,(z - \pi_h z)||_{0,\mu^{-1}} \leq \inf_{v \in U_h} ||\text{curl}\,(z - v)||_{0,\mu^{-1}}, \qquad (6.48)$$

and taking $v = \Pi_h z \in U_h$, we have

$$||\text{curl}\,(z - \pi_h z)||_0 \leq C||\text{curl}\,(z - \pi_h z)||_{0,\mu^{-1}} \leq C||\text{curl}\, z - \text{curl}\,\Pi_h z||_0 \leq C h^{r_2} \sum_{j=1}^{J} ||\text{curl}\, z||_{r_2,\Omega_j}. \qquad (6.49)$$

$\square$

Remark 6.3 Compared with the error bound (6.32) of $\Pi_h z$, the error bound (6.28) of $\pi_h z$ may be improved, since we would expect r_1 in (6.28), i.e., the following

$$||z - \pi_h z||_0 \leq C h^{r_1} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\text{curl}\, z||_{r_2,\Omega_j}). \qquad (6.50)$$

We are not aware of any work in the literature that dealt with this issue where z and $\text{curl}\, z$ have different regularity. At the same time, we did not find the way to obtain (6.50). However, if Assumption A2) holds for some $s > 1/2$ and Assumption A4) and Assumption A5) hold for some $r > 0$, using a different argument from the one in proving Lemma 6.5, we can obtain

$$||z - \pi_h z||_0 \leq C h^{\min(r_1, s+r_2, r+r_2)} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\text{curl}\, z||_{r_2,\Omega_j}),$$

which is not the same as (6.50) but is better than (6.28). Here, we will not deal with this issue any further, since (6.28) and (6.29) are sufficient for the main result in the following.

We now state the main result for the Fortin operator π_h for the general $u := z$ with suitable regularity.

Theorem 6.2 *Assume that Assumption A3) holds for some $t > 1/2$ and that $z \in H_0(\mathrm{curl}\,;\Omega)$ and $z, \mathrm{curl}\, z \in \prod_{j=1}^{J}(H^r(\Omega_j))^3$, where $0 < r \le t$. Then*

$$||z - \pi_h z||_0 \le Ch^r(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||z||_{r,\Omega_j} + \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r,\Omega_j}), \qquad (6.51)$$

$$||\mathrm{curl}\,(z - \pi_h z)||_0 \le Ch^r \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r,\Omega_j}. \qquad (6.52)$$

Proof. From Lemma 6.4 we write $z \in H_0(\mathrm{curl}\,;\Omega)$ into the following regular-singular decomposition:

$$z = z_0 + \nabla p, \qquad (6.53)$$

where $z_0 \in H_0(\mathrm{curl}\,;\Omega) \cap (H^t(\Omega))^3$ for some $t > 1/2$ and $p \in H_0^1(\Omega)$, satisfying

$$||z_0||_t \le C||\mathrm{curl}\, z||_0, \qquad (6.54)$$

$$||p||_1 \le C||z||_{0,\mathrm{curl}}. \qquad (6.55)$$

Note that $\Pi_h z_0 \in U_h$ is well-defined, since $z_0 \in (H^t(\Omega))^3$ for some $t > 1/2$ and $\mathrm{curl}\, z_0 = \mathrm{curl}\, z \in \prod_{j=1}^{J}(H^r(\Omega_j))^3$ for some $r > 0$, see Remark 4.2. From Proposition 4.2 we have

$$||z_0 - \Pi_T z_0||_{0,T} \le Ch_T^t(||z_0||_{t,T} + ||\mathrm{curl}\, z_0||_{r,T}) \quad \forall T \in \mathcal{T}_h, \qquad (6.56)$$

$$||\mathrm{curl}\,(z_0 - \Pi_T z_0)||_{0,T} \le Ch_T^r ||\mathrm{curl}\, z_0||_{r,T} \quad \forall T \in \mathcal{T}_h, \qquad (6.57)$$

and from (6.54) and $\mathrm{curl}\, z_0 = \mathrm{curl}\, z$ we have

$$||z_0 - \Pi_h z_0||_0 \le Ch^t(||z_0||_t + \sum_{j=1}^{J} ||\mathrm{curl}\, z_0||_{r,\Omega_j}) \le Ch^t(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r,\Omega_j}), \qquad (6.58)$$

$$||\mathrm{curl}\,(z_0 - \Pi_h z_0)||_0 \le Ch^r \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r,\Omega_j}. \qquad (6.59)$$

On the other hand, since $z \in \prod_{j=1}^{J}(H^r(\Omega_j))^3$ for some $r > 0$ and $z_0 \in (H^t(\Omega))^3$ with $t \ge r$, from (6.53) we know that $p \in \prod_{j=1}^{J} H^{1+r}(\Omega_j)$, satisfying

$$\sum_{j=1}^{J} ||p||_{1+r,\Omega_j} \le C(\sum_{j=1}^{J} ||z||_{r,\Omega_j} + ||z_0||_t) \le C(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||z||_{r,\Omega_j}). \qquad (6.60)$$

Thus, from Theorem 6.1 we have

$$||\pi_h \nabla p - \nabla p||_0 \le Ch^r \sum_{j=1}^{J} ||p||_{1+r,\Omega_j} \le Ch^r(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||z||_{r,\Omega_j}). \qquad (6.61)$$

We are now in a position to estimate the difference $z - \pi_h z$ in the following.
Observe that

$$||z - \pi_h z||_{0,\varepsilon}^2 = (\varepsilon(z - \pi_h z), z - \pi_h z) = (\varepsilon(z - \pi_h z), z_0 - \pi_h z_0) + (\varepsilon(z - \pi_h z), \nabla p - \pi_h \nabla p), \qquad (6.62)$$

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where, from Lemma 6.5 for this z_0 with $r_1 := t$ and $r_2 := r$ we have

$$
\begin{aligned}
(\varepsilon(z - \pi_h z), z_0 - \pi_h z_0) &\leq C||z - \pi_h z||_{0,\varepsilon}||z_0 - \pi_h z_0||_0 \\
&\leq C||z - \pi_h z||_{0,\varepsilon} h^r(||z_0||_t + \sum_{j=1}^{J}||\text{curl}\, z_0||_{r,\Omega_j}) \\
&\leq C||z - \pi_h z||_{0,\varepsilon} h^r(||\text{curl}\, z||_0 + \sum_{j=1}^{J}||\text{curl}\, z||_{r,\Omega_j}),
\end{aligned}
\tag{6.63}
$$

and from (6.61), we have

$$
(\varepsilon(z - \pi_h z), \nabla p - \pi_h \nabla p) \leq C||z - \pi_h z||_{0,\varepsilon}||\nabla p - \pi_h \nabla p||_0 \leq C||z - \pi_h z||_{0,\varepsilon} h^r(||\text{curl}\, z||_0 + \sum_{j=1}^{J}||z||_{r,\Omega_j}), \tag{6.64}
$$

It then follows from (6.62)-(6.64) that

$$
||z - \pi_h z||_0 \leq C||z - \pi_h z||_{0,\varepsilon} \leq h^r(||\text{curl}\, z||_0 + \sum_{j=1}^{J}||z||_{r,\Omega_j} + \sum_{j=1}^{J}||\text{curl}\, z||_{r,\Omega_j}). \tag{6.65}
$$

Regarding (6.52), by the definition of π_h we have

$$
||\text{curl}\,(z - \pi_h z)||_{0,\mu^{-1}} \leq ||\text{curl}\, z - \text{curl}\, v||_{0,\mu^{-1}} \quad \forall v \in U_h. \tag{6.66}
$$

But, $\text{curl}\, z = \text{curl}\, z_0$ and $\Pi_h z_0$ is well-defined, from (6.59) we have

$$
\inf_{v \in U_h} ||\text{curl}\, z - \text{curl}\, v||_{0,\mu^{-1}} \leq C||\text{curl}\, z_0 - \text{curl}\, \Pi_h z_0||_0 \leq Ch^r \sum_{j=1}^{J}||\text{curl}\, z||_{r,\Omega_j}. \tag{6.67}
$$

We therefore have

$$
||\text{curl}\,(z - \pi_h z)||_0 \leq C||\text{curl}\,(z - \pi_h z)||_{0,\mu^{-1}} \leq Ch^r \sum_{j=1}^{J}||\text{curl}\, z||_{r,\Omega_j}. \tag{6.68}
$$

$\square$

Following the argument in proving Theorem 5.2 we can obtain the following $H(\text{curl})$-error bound for very low regular solution with r not greater than $1/2$.

Corollary 6.1 *Assume that Assumption A3) holds for some $t > 1/2$ and that Assumption A4) and Assumption A5) hold for some $0 < r \leq t$. Given any compatible $f \in H(\text{div}^0; \Omega)$. Let u be the solution of problem (1.5) and $u_{\delta,h} \in U_h$ the finite element solution of problem (3.2), where U_h is taken as (4.6) which satisfies Assumption A1). We have*

$$
||u - u_{\delta,h}||_{0,\text{curl}} \leq C(\delta + h^r)||f||_0. \tag{6.69}
$$

Remark 6.4 We have used Assumption A3) to ensure that π_h is well-defined. We also used this assumption to establish the error estimates for the Fortin operator. Assumption A3) is a regular-singular decomposition where $t > 1/2$. This t is different from the s in Assumption A2) and the r in both Assumption A4) and Assumption A5) where s and r may be not greater than $1/2$. For example, for Lipschitz domains we can have $t = 1$, but $s = 1/2$ only. For interface problem, we may still have $t = 1$, but r may be close to zero [24]. In fact, the regular-singular decomposition in Assumption A3) depends little on the domain boundary and on the material occupying Ω, since it has been established mainly from the H^1 existence of the Poisson equation of Laplace operator and the extension of $H_0(\text{curl}; \Omega)$ to the $H(\text{curl}; \mathbb{R}^3)$ [25][7]. On the contrary, the continuous embedding in Assumption A2), Assumption A4) and Assumption A5), and the regularity of the solution and its curl counterpart of problem (1.1)-(1.4) are determined by the domain boundary singularities (due to reentrant corners and edges, etc), the property of the materials occupying Ω, and the topology of Ω (i.e., simply-connected or multi-connected, etc), see [22][24]. In general, these are profoundly related to the singularities of the solution of the second-order elliptic problem of Laplace operator in nonsmooth domains [32].

Remark 6.5 As highlighted in Remark 6.4, $r \leq t$ is generally true, since $t = 1$ usually. If r is larger than t and $r > 1/2$, then the theory has already been developed in section 5. For more regular solution, say $r > 1$, we may use higher-order elements, and the theory in section 5 can be easily applied to obtain higher-order error bounds.

7. Numerical test. We report some numerical results to support the method and the theory developed in the previous sections. We present four examples for numerical experiments associated with problem (1.1)-(1.4) in three-dimensions. All the domains in these examples are partitioned into uniform tetrahedra, with the mesh reduction of factor two, i.e., $h = 1/4, 1/8, 1/16$,, etc. In the δ-regularization problem, we use the lowest-order Nédélec element of first-family and choose $\delta = h$.

Example 1. Given the thick L-domain in $\mathbb{R}^3$: $\Omega = [-1, 1]^2 \setminus ([0, 1] \times [-1, 0]) \times [0, 1]$ in the $O - xyz$-coordinates system. There is a reentrant edge of the opening angle $3\pi/2$ along the positive z-axis of Ω. We choose $\mu = \varepsilon = 1$ and f so that the exact solution is

$$u = (\partial_y p, -\partial_x p, 0)$$

where $p(x, y) = (1 - x^2)^2(1 - y^2)^2 r^{\frac{2}{3}} \cos(\frac{2\theta}{3})$ in the polar coordinates system of $\mathbb{R}^2$, $x = r\cos(\theta), y = r\sin(\theta)$, r is the distance to the origin $O(0, 0)$ and θ is the angular degree between 0 and 2π. The regularity of u and $\mathrm{curl}\, u = (0, 0, -\Delta p)$ is $(H^{2/3-\epsilon}(\Omega))^3$ for any $\epsilon > 0$, i.e., r is approximately 2/3. From the theoretical results we should expect that the ratio of the error reduction is approximately $2^{2/3} \approx 1.5874$ for the mesh reduction of factor two and that the finite element solution is uniform stable independent of the regularization parameter δ deceasing to zero. The computed results in L^2 semi-norm and curl semi-norm which are listed in Table 1 and Table 2, as we have expected, are consistent with the theoretical results.

TABLE 1   Errors in L^2 semi-norm and curl semi-norm

h	1/4	1/8	1/16	1/32	1/64
$\|u - u_{\delta,h}\|_0$	0.0809	0.0503	0.0314	0.0196	0.0124
Ratio		1.6083	1.6019	1.6020	1.5806
$\|\mathrm{curl}\,(u - u_{\delta,h})\|_0$	1.9963	0.9973	0.5125	0.2795	0.1721
Ratio		2.0020	1.9460	1.8336	1.6241

TABLE 2   Stability in L^2 semi-norm and curl semi-norm
for a fixed mesh size $h = 1/16$ but δ decreases

δ	1/10	1/20	1/30	1/40	1/50
$\|u - u_{\delta,h}\|_0$	0.0821	0.0826	0.0827	0.0828	0.0829
Ratio		0.9939	0.9988	0.9988	0.9988
$\|\mathrm{curl}\,(u - u_{\delta,h})\|_0$	1.9960	1.9960	1.9961	1.9961	1.9961
Ratio		1.0000	1.0000	1.0000	1.0000

Example 2. Given the thick cracked domain in $\mathbb{R}^3$: $\Omega = [-1, 1]^2 \setminus \{(x, 0) \in \mathbb{R}^2 : 0 < x < 1\} \times [0, 1]$ in the $O - xyz$-coordinates system. There is a "screen" of the opening angle 2π along the positive x-axis of Ω. We choose $\mu = \varepsilon = 1$ and f so that the exact solution is

$$u = (\partial_y p, -\partial_x p, 0)$$

where $p(x, y) = (1 - x^2)^2(1 - y^2)^2 r^{\frac{1}{2}} \cos(\frac{\theta}{2})$ in the polar coordinates system of $\mathbb{R}^2$, $x = r\cos(\theta), y = r\sin(\theta)$, r is the distance to the origin $O(0, 0)$ and θ is the angular degree between 0 and 2π. The regularity of u and $\mathrm{curl}\, u = (0, 0, -\Delta p)$ is $(H^{1/2-\epsilon}(\Omega))^3$ for any $\epsilon > 0$, i.e., r is approximately 1/2. From the theoretical results we should expect that the ratio of the error reduction is approximately $2^{1/2} \approx 1.4142$ for the mesh reduction of factor two and that the finite element solution is uniform stable independent of the regularization parameter δ deceasing to zero. The computed results in L^2 semi-norm and curl semi-norm which are listed in Table 3 and Table 4, as we have expected, are consistent with the theoretical results.

TABLE 3 Errors in L^2 semi-norm and curl semi-norm

h	1/4	1/8	1/16	1/32	1/64
$\|u - u_{\delta,h}\|_0$	0.1361	0.0913	0.0637	0.0451	0.0319
Ratio		1.4907	1.4333	1.4124	1.4138
$\|\mathrm{curl}\,(u - u_{\delta,h})\|_0$	2.9047	1.8437	1.2459	0.8872	0.6412
Ratio		1.5755	1.4798	1.4043	1.3837

TABLE 4 Stability in L^2 semi-norm and curl semi-norm
for a fixed mesh-size $h = 1/16$ but δ decreases

δ	1/10	1/20	1/30	1/40	1/50
$\|u - u_{\delta,h}\|_0$	0.0637	0.0637	0.0638	0.0638	0.0638
Ratio		1.0000	0.9984	1.0000	1.0000
$\|\mathrm{curl}\,(u - u_{\delta,h})\|_0$	1.2459	1.2458	1.2458	1.2458	1.2457
Ratio		1.0000	1.0000	1.0000	1.0000

Example 3. We consider the same domain as in Example 1 and still take $\mu = 1$. But, we assume there are three material subdomains in Ω, $\Omega_1 = [0,1]^3$, $\Omega_2 = [-1,0] \times [0,1]^2$, $\Omega_3 = [-1,0]^2 \times [0,1]$, which are introduced by the interfaces $\mathscr{S}_{\mathrm{int}} = \{(x,y,z) \in \mathbb{R}^3 : x = 0, 0 \le y \le 1, 0 \le z \le 1\} \cup \{(x,y,z) \in \mathbb{R}^3 : -1 \le x \le 0, y = 0, 0 \le z \le 1\}$, and ε is discontinuous and for three material subdomains ε are respectively given by

$$\varepsilon|_{\Omega_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \varepsilon|_{\Omega_2} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \qquad \varepsilon|_{\Omega_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this example, we are given a smooth source, i.e., $f = (1,1,1)$. Since the exact solution is not known, we only compute the $H(\mathrm{curl})$ norm of the numerical solution to illustrate the uniform stability independent of δ and h which decrease to zero. The computed results in $H(\mathrm{curl})$ norm which are given in Table 5 and Table 6, as we have expected, are consistent with the theoretical results.

TABLE 5 Stability in $H(\mathrm{curl})$ norm
for $\delta = h$ and h decreases

h	1/4	1/8	1/16	1/32	1/64
$\|u_{\delta,h}\|_{0,\mathrm{curl}}$	0.8978	0.9352	0.9446	0.9463	0.9468
Ratio		0.9600	0.9900	0.9982	0.9995

TABLE 6 Stability in $H(\mathrm{curl})$ norm
for a fixed mesh-size $h = 1/16$ but δ decreases

δ	1/10	1/20	1/30	1/40	1/50
$\|u_{\delta,h}\|_{0,\mathrm{curl}}$	0.9457	0.9480	0.9488	0.9491	0.9494
Ratio		0.9976	0.9992	0.9997	0.9997

Example 4. Given the domain in $\mathbb{R}^3$: $\Omega = ([-1,3] \times [-1,1] \setminus ([0,3] \times [-1,0] \cup \{(x,y) \in \mathbb{R}^2 : 2 < x < 3, y = \frac{1}{2}\})) \times [0,1]$ in the $O - xyz$-coordinates system. There is a reentrant edge originating from the origin $O(0,0,0)$ along the positive z axis with an opening angle $3\pi/2$ and a screen originating from the point $(2,1/2,0)$ along the positive x axis with an opening angle 2π in Ω. We take $\mu = 1$. But, we assume there are two material subdomains in Ω, $\Omega_1 = [1,3] \times [0,1]^2$, $\Omega_2 = [-1,1]^2 \setminus ([0,1] \times [-1,0]) \times [0,1]$, which are introduced by the interface $\mathscr{S}_{\mathrm{int}} = \{(x,y,z) \in \mathbb{R}^3 : x = 1, 0 \le y \le 1, 0 \le z \le 1\}$, and ε is discontinuous and

for two material subdomains ε are respectively given as follows:

$$\varepsilon|_{\Omega_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \varepsilon|_{\Omega_2} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

We choose f so that the exact solution

$$u|_{\Omega_j} = (\partial_y p|_{\Omega_j}, -\partial_x p|_{\Omega_j}, 0), \quad 1 \le j \le 2,$$

where $p|_{\Omega_1^-} = (1-x)^2(1-y)^2 q_1(x,y)$, with $q_1(x,y) = r^{\frac{2}{3}}\cos(\frac{2\theta}{3})$, and $p|_{\Omega_2^-} = (1-x)^2(3-x)^2(\frac{1}{4} - (y - \frac{1}{2})^2)^2 q_2(x-2, y-\frac{1}{2})$, with $q_2(x,y) = r^{\frac{1}{2}}\cos(\frac{\theta}{2})$, where $q_1(x,y) = r^{\frac{2}{3}}\cos(\frac{2\theta}{3})$ and $q_2(x,y) = r^{\frac{1}{2}}\cos(\frac{\theta}{2})$ in the polar coordinates system of $\mathbb{R}^2$, $x = r\cos(\theta), y = r\sin(\theta)$, r is the distance to the origin O and θ is the angular degree between 0 and 2π. The regularity of the exact solution u and its $\operatorname{curl} u = (0,0,-\Delta p)$ is the same on each material subdomain Ω_j and $u, \operatorname{curl} u \in (H^{r_j}(\Omega_j))^3$, r_j is respectively $2/3 - \epsilon, 1/2 - \epsilon$ for any $\epsilon > 0$. The solution u and its $\operatorname{curl} u$ belong to $\prod_{j=1}^2 (H^r(\Omega_j))^3$ for r being approximately $1/2$. From the theoretical results we should expect that the ratio of the error reduction is approximately $2^{1/2} \approx 1.4142$ for the mesh reduction of factor two and that the finite element solution is uniform stable independent of the regularization parameter δ deceasing to zero. The computed results in L^2 semi-norm and curl semi-norm which are listed in Table 7 and Table 8, as we expected, are consistent with the theoretical results. Also, we find that although the solution and its curl have higher regularity in Ω_1, the whole convergence rate is governed by the lower regularity in Ω_2.

TABLE 7   Errors in L^2 semi-norm and curl semi-norm

h	1/4	1/8	1/16	1/32	1/64
$\|u - u_{\delta,h}\|_0$	0.3955	0.2604	0.1566	0.0981	0.0700
Ratio		1.5188	1.6628	1.5963	1.4014
$\|\operatorname{curl}(u - u_{\delta,h})\|_0$	2.9515	1.5506	0.8720	0.5282	0.3517
Ratio		1.9035	1.7782	1.6509	1.5018

TABLE 8   Stability in L^2 semi-norm and curl semi-norm
for a fixed mesh-size $h = 1/16$ but δ decreases

δ	1/10	1/20	1/30	1/40	1/50
$\|u - u_{\delta,h}\|_0$	0.1510	0.1434	0.1419	0.1414	0.1412
Ratio		1.0530	1.0106	1.0035	1.0014
$\|\operatorname{curl}(u - u_{\delta,h})\|_0$	0.8541	0.8294	0.8247	0.8230	0.8223
Ratio		1.0300	1.0057	1.0021	1.0009

We change the value of $\varepsilon|_{\Omega_2}$ so that there is a discontinuity of ε with high ratio/contrast across the interface. We take two choices for $\varepsilon|_{\Omega_2}$:

$$\varepsilon|_{\Omega_2} = \begin{pmatrix} 1/100 & 0 & 0 \\ 0 & 1/100 & 0 \\ 0 & 0 & 1/100 \end{pmatrix}, \quad \text{or} \quad \varepsilon|_{\Omega_2} = \begin{pmatrix} 1/1000 & 0 & 0 \\ 0 & 1/1000 & 0 \\ 0 & 0 & 1/1000 \end{pmatrix}.$$

We find that the computed results are accurate to four decimal places as shown in Tables 7 and 8. This may be interpreted as follows. For δ decreasing to zero, the theoretical results show that the finite element solution is uniformly stable independent of δ and there holds optimal convergence with respect to $\delta + h$, thus, when the combination of $\delta\varepsilon$ with $\delta = h$ and $h\varepsilon_{\max}$ decrease to zero, all the theoretical results are expected to still be valid, where $\varepsilon_{\max}$ represents the upper bound of ε over Ω, and here $\varepsilon_{\max} = 1$. Therefore, the present method appears to cover the case where the discontinuous materials have high ratio/contrast across material subdomains, although we did not develop the related theory for this situation.

8. Conclusion and extension. In this paper, we have proposed a general approach, δ-regularization method, for dealing with the divergence-free constraint in a double curl problem which typically arises from computational electromagnetism. With this δ-regularization method, we can completely disregard the divergence-free constraint and instead we introduce a δ perturbation zero-term which couples the curlcurl operator to constitute a well-posed coercive problem for any given δ. Such δ-regularization method is shown to have a uniform stable finite element solution independent of the regularization parameter δ which decreases to zero. For nonsmooth solution, together with its curl, being H^r regularity for some $0 < r < 1$, we have established the optimal error bound $\mathcal{O}(h^r)$ in the natural $H(\text{curl})$ norm (which is independent of δ) for $\delta \leq Ch$ when using the lowest-order Nédélec element of first-family. Higher-order Nédélec elements can be used to yield higher-order accuracy if the exact solution is more regular.

Furthermore, we have developed the new theory for the $\mathscr{K}_h$ ellipticity (a Poincaré-Friedrichs' type inequality) and the new theory for the Fortin-interpolation operator. The $\mathscr{K}_h$ ellipticity is one of the two critical conditions (the other is the Inf-Sup condition) for the well-posedness and the optimal convergence for the mixed finite element method, while the Fortin operator is fundamental in the edge finite element method, as is well-known. These two theories generalize the existing ones to cover those problems whose solutions may have very low regularity. In fact, they are established only under the regular-singular decomposition assumption. Such assumption is true for general domains and does not depend on the material properties occupying the domain and the topology of the domain.

A series of numerical examples have been performed for three-dimensional problems to illustrate the method and the theoretical results. Moreover, the proposed δ-regularization method appears to cover the interface problem with high contrast/ratio material coefficients across material subdomains, although we did not have the theory for the latter situation. These have justified the capability of the δ-regularization method in dealing with divergence-free constraint. Meanwhile, these have exhibited the potential to deal with the discontinuous materials of high contrast/ratio among different material subdomains.

We should point out that although the proposed δ-regularization method is developed, analyzed and performed for the model problem in (1.1)-(1.4), but, in actual fact, it can cover a number of models of computational electromagnetism. To illustrate this point, we shall discuss the extension of the δ-regularization method we have developed in this paper. As we know, there are other widely used models arising from computational electromagnetism, for example, we often need to solve the following problem: to find u and p such that

$$\text{curl}\,\mu^{-1}\text{curl}\,u + \alpha\varepsilon u + \varepsilon_1\nabla p = f \quad \text{in } \Omega,$$

$$\text{div}\,\varepsilon u = g \quad \text{in } \Omega,$$

$$u \times n = 0, \quad p = 0 \quad \text{on } \partial\Omega,$$

where α is a given real number which may arise from either the time-discretization problems of the time-dependent Maxwell's equations with α inversely proportional to the time-step or the time-harmonic Maxwell's equations with $-\alpha$ amounting to the angular frequency, and ε_1 is a third material coefficient matrix. Below we simply show how to apply the proposed δ-regularization method to the above problem. This consists of two stages. We first parallel solve the two second-order elliptic problems: to find $p^* \in H_0^1(\Omega)$ such that

$$\text{div}\,\varepsilon\nabla p^* = g \quad \text{in } \Omega, \quad p^* = 0 \quad \text{on } \partial\Omega,$$

and to find $p \in H_0^1(\Omega)$ such that

$$\text{div}\,\varepsilon_1\nabla p = \text{div}\,f - \alpha g \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega,$$

and then we solve the problem: to find $w \in H_0(\text{curl};\Omega) \cap H(\text{div}^{\,0};\varepsilon;\Omega)$ such that

$$\text{curl}\,\mu^{-1}\text{curl}\,w + \alpha\varepsilon w = F := f - \varepsilon_1\nabla p - \alpha\varepsilon\nabla p^* \quad \text{in } \Omega,$$

$$\text{div}\,\varepsilon w = 0 \quad \text{in } \Omega,$$

$$w \times n = 0 \quad \text{on } \partial\Omega.$$

Clearly, we have

$$u = w + \nabla p^*.$$

However, we do not directly solve w. Instead, we solve the following δ regularization problem: to find w_δ such that

$$\operatorname{curl} \mu^{-1} \operatorname{curl} w_\delta + \alpha \varepsilon w_\delta + \delta \varepsilon w_\delta = F \quad \text{in } \Omega,$$

$$w_\delta \times n = 0 \quad \text{in } \Omega.$$

Noticing that α is known, we choose δ so that $\delta + \alpha \neq 0$, and we can analyze this δ regularization problem following the routine in previous sections. Hence, firstly simultaneously solving two symmetric, positive definite problems (second-order elliptic interface problems) in parallel, and then solving a δ-regularization problem, we can obtain the desired solution.

REFERENCES

[1] R. ADAMS, *Sobolev Spaces*, Academic Press, New York(1975).

[2] A. ALONSO AND A. VALLI, *An optimal domain decomposition preconditioner for lowfrequency time-harmonic Maxwell equations*, Math. Comp., 68 (1999), pp. 607–631.

[3] C. AMROUHE, C. BERNARDI, M. DAUGE, AND V. GIRAULT, *Vector potentials in three dimensional nonsmooth domains*, Math. Methods Appl. Sci., 21 (1998), pp. 823–864.

[4] D. N. ARNOLD, R. S. FALK, AND R. WINTHER, *Multigrid in H(div) and H(curl)*, Numer. Math., 85 (2000), pp. 197–218.

[5] M. BENZI, G. H. GOLUB, AND J. LIESEN, *Numerical Solution of Saddle Point Problems*, Acta Numerica, 14 (2005), pp. 1–137.

[6] C. BERNARDI AND V. GIRAULT, *A local regularization operator for triangular and quadrilateral finite elements*, SIAM J. Numer. Anal., 35(1998), pp. 1893-1916.

[7] M. SH. BIRMAN AND M. Z. SOLOMYAK, *L2 theory of the Maxwell operator in arbitrary domains*, Russian Math. Surveys, 42 (1987), pp. 75–96.

[8] D. BOFFI, *Fortin operators and discrete compactness for edge elements*, Numer. Math., 87 (2000), pp. 229–246.

[9] A. BOSSAVIT, *Computational Electromagnetism: Variational Formulations, Complementarity, Edge Elements*, Academic Press, New-York (1998).

[10] J. BRAMBLE AND J. PASCIAK, *A new approximation technique for div-curl systems*, Math. Comp., 73 (2004), pp. 1739–1762.

[11] S. C. BRENNER, *Korns inequalities for piecewise H1 vector fields*, Math. Comp., 73 (2004), pp. 106–1087.

[12] S.C. BRENNER AND L.R. SCOTT, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, Berlin(1996).

[13] S.C. BRENNER, J.CUI, F. LI, AND L.-Y. SUNG, *A nonconforming finite element method for a two-dimensional curl-curl and grad-div problem*, Numer. Math., 109 (2008), pp. 509–533.

[14] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New-York (1991).

[15] A. BUFFA, P. CIARLET, JR., AND E. JAMELOT, *Solving electromagnetic eigenvalue problems in polyhedral Domains*. Numer. Math., 113 (2009), pp. 497–518.

[16] G. CALOZ, M. DAUGE, AND V. PÉRON, *Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism*, J. Math. Anal. Appl., 370 (2010), pp. 555–572.

[17] Z. CHEN, Q. DU, AND J. ZOU, *Finite element methods with matching and nonmatching meshes for Maxwell equations with discontinuous coefficients*, SIAM J. Numer. Anal., 37 (2000), pp. 1542–1570.

[18] P.G. CIARLET, *Basic Error Estimates for Elliptic Problems*, in: Handbook of Numerical Analysis, Vol. II, Finite Element Methods (part 1), P. G. Ciarlet and J.-L. Lions eds, North-Holland, Amsterdam (1991).

[19] JR P. CIARLET AND J. ZOU, *Fully discrete finite element approaches for time dependent Maxwells equations*, Numer. Math., 82 (1999), pp. 193–219.

[20] P. CLÉMENT, *Approximation by finite element functions using local regularization*, RAIRO Numer. Anal., 9 (1975), pp. 77–84.

[21] M. COSTABEL, *A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains*, M3AS Math. Methods Appl. Sci., 12 (1990), pp. 365–368.

[22] M. COSTABEL AND M. DAUGE, *Singularities of electromagnetic fields in polyhedral domains*, Arch. Rational Mech. Anal., 151 (2000), pp. 221–276.

[23] M. COSTABEL AND M. DAUGE, *Weighted regularization of Maxwell equations in polyhedral domains*, Numer. Math., 93(2002), pp. 239–277.

[24] M. COSTABLE, M. DAUGE, AND S. NICAISE, *Singularities of Maxwell inteface problems*, M2AN Math. Modelling and Numer. Anal., 33 (1999), pp. 627–649.

[25] A. BONNET-BEN DHIA, C. HAZARD, AND S. LOHRENGEL, *A singular field method for the solution of Maxwells equations in polyhedral domains*, SIAM J. Appl. Math., 59 (1990), no. 6, pp. 2028–2044.

[26] H.Y. DUAN, P. LIN, AND ROGER C. E. TAN, *Analysis of a continuous finite element method for $H(\operatorname{curl}, \operatorname{div})$-elliptic interface problem*, Submitted.

[27] H.Y. DUAN, F. JIA, P. LIN, AND R.C.E. TAN, *The local L2 projected C0 finite element method for Maxwell problem*, SIAM J. Numer. Anal., 47(2009), pp. 1274–1303.

[28] P. FERNANDES AND G. GILARDI, *Magnetostatic and Electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions*, Math. Models and Methods Appl. Sci., 7(1997), pp. 957–991.

[29] P. Fernandes and I. Perugia, *Vector potential formulation for magnetostatics and modelling of permanent magnets*, IMA J. Appl. Math., 66(2001), pp. 293-318.

[30] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, New York, 1986.

[31] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 2ed., The Johns Hopkins University Press, Baltimore, MD (1989).

[32] P. Grisvard, *Elliptitc Problems in Nonsmooth Domains*, Pitman Advanced Publishing Program, London (1985).

[33] C. Hazard and M. Lenoir, *On the solution of time-harmonic scattering problems for Maxwell's equations*, SIAM J. Math. Anal., 27 (1996), pp. 1597–1630.

[34] R. Hiptmair, *Finite elements in computational electromagnetism*, Acta Numerica, 2002, pp. 237–339.

[35] R. Hiptmair and J. Xu, *Nodal auxiliary space preconditioning in H(curl) and H(div) spaces*, SIAM J. Numer. Anal., 45(2007), pp. 2483–2509.

[36] R. Hiptmair, J. Li, and J. Zou, *Convergence analysis of finite element methods for H(div;)-elliptic interface problems*, J. Numer. Math., 18 (2010), pp. 187–218.

[37] P. Houston, I. Perugia, A. Schneebeli & D. Schötzau, *Interior penalty method for indefinite time-harmonic Maxwell equations*, Numer. Math., 100(2005), pp. 485-518.

[38] P. Houston, I. Perugia, A. Schneebeli, and D. Schötzau,*Mixed discontinuous Galerkin approximation of the Maxwell operator*, SIAM J. NUMER. ANAL., 42(2004), pp. 434–459.

[39] J.-G. Huang and J. Zou, *Uniform a pripori estimates for elliptic and static Maxwell interface problems*, Dis. Con. Dyn. Sys., Ser. B(DCDS-B), 7 (2007), pp. 145–170.

[40] B.-N. Jiang, *The Least-Squares Finite Element Method*, Springer, New-York (1998).

[41] J. M. Jin, *The Finite Element Method in Electromagnetics*, 2nd Edition. New York: John Wiley & Sons (2002)

[42] O. A. Karakashian and F. Pascal, *A posteriori error estimation for a discontinuous Galerkin approximation of second order elliptic problems*, SIAM J. Numer. Anal. 41, (2003), pp. 2374–2399.

[43] E. J. Lee and T. A. Manteuffel, *FOSLL* method for the eddy current problem with three-dimensional edge singularities*, SIAM J. Numer. Anal., 45 (2007), pp. 787–809.

[44] P. Monk, *Finite Element Methods for Maxwell Equations*, Clarendon Press, Oxford, 2003.

[45] J. C. Nedelec, *Mixed finite elements in R^3*, Numer. Math., 35 (1980), pp. 315–341.

[46] S. Nicaise, *Edge elements on anisotropic meshes and approximation of the Maxwell equations*, SAIM J. Numer. Anal., 39(2001), pp. 784–816.

[47] I.Perugia and D. Schotzau, *The hp-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations*, Math.Comp., 72 (2002), pp. 1179-1214.

[48] L. R. Scott & S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54(1990), pp. 483–493.

[49] S.Y. Zhang, *Quadratic divergence-free finite elements on Powell-Sabin tetrahedral grids*, Calcolo, 48(2011), pp. 211-244.