CONVERGENCE OF THE UNIAXIAL PERFECTLY MATCHED LAYER METHOD FOR TIME-HARMONIC SCATTERING PROBLEMS IN LAYERED MEDIA*

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Abstract. In this paper, we propose a uniaxial perfectly matched layer (PML) method for solving the time-harmonic scattering problems in layered media. The exterior region of the scatterer is divided into two half spaces by an infinite plane, on two sides of which the wave number takes different values. We surround the computational domain where the scattering field is interested by a PML layer with the uniaxial medium property. By imposing homogenous boundary condition on the outer boundary of the PML layer, we show that the solution of the PML problem converges exponentially to the solution of the original scattering problem in the computational domain as either the PML absorbing coefficient or the thickness of the PML layer tends to infinity.

1. Introduction. We propose and study the uniaxial perfectly matched layer (PML) method for solving Helmholtz scattering problems in layered media:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}, \tag{1.1}$$

$$u = g \quad \text{on } \Gamma_D, \tag{1.2}$$

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathbf{i}k\,u\right) \to 0 \quad \text{as } r = |x| \to \infty. \tag{1.3}$$

Here $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D and $g \in H^{1/2}(\Gamma_D)$. We assume the wave number k is positive and piecewise constant, defined by

$$k(x) = \begin{cases} k_1, & \text{if } x \in \mathbb{R}^2_+, \\ k_2, & \text{if } x \in \mathbb{R}^2_-, \end{cases} \tag{1.4}$$

where $\mathbb{R}^2_\pm = \{(x_1, x_2) \in \mathbb{R}^2 : \ \pm x_2 > 0\}$. Without loss of generality we asuume in this paper that $k_2 > k_1 > 0$. Additional continuity conditions are needed across the interface $\Sigma = \{(x_1, 0) \ : \ -\infty < x_1 < \infty\}$:

$$[u]_\Sigma = \left[\frac{\partial u}{\partial x_2}\right]_\Sigma = 0, \tag{1.5}$$

where $[u]_\Sigma := u_+ - u_-$ is the jump of u across Σ from above to below. We remark that the boundary condition (1.2) and the continuity conditions (1.5) are not essential for our results. In fact, (1.2) can be replaced by other boundary conditions such as Neumann or impedance boundary conditions on Γ_D, and (1.5) can be replaced by other continuity conditions (cf. e.g. [16]). We refer to Coyle and Monk [13] and Monk [20] for finite element methods solving scattering problems in layered media.

Since the work of Bérénger [3] which proposed a PML method for solving the time dependent Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Hagstrom [14], Turkel and Yefet [23], Teixeira and Chew [22] for the reviews). The basic idea of the PML method

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is to surround the computational domain by a layer of finite thickness with specially designed model medium that absorbs all the waves that propagate from inside the computational domain.

The convergence of the PML method for homogeneous background materials has drawn considerable attention in the literature. Lassas and Somersalo [17], [18], Hohage et al [15] studied the acoustic scattering problems for circular and smooth PML layers. It is proved in [15, 17, 18] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinite. In the practical application of PML methods, the adaptive PML method was proposed in Chen and Wu [5] for a scattering problem by periodic structures (the grating problem), in Chen and Liu [6] for the acoustic scattering problem, and in Chen and Chen [4] for Maxwell scattering problems. The main idea of the adaptive PML method is to use the a posteriori error estimate to determine the PML parameters and to use the adaptive finite element method to solve the PML equations. The adaptive PML method provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer.

The purpose of this paper is to study the convergence of the uniaxial PML (UPML) method for the scattering problem (1.1)–(1.5). The UPML method is widely used in the engineering literatures and has the advantage over the circular PML method in that it provides greater flexibility and efficiency to solve problems involving anisotropic scatterers. The convergence result established in this paper constitutes an important step in studying efficient numerical methods such as adaptive UPML method for solving scattering problems in layered media. In Chen and Wu [7] the adaptive UPML method is proposed for Helmholtz scattering problems with constant wave number.

Since the background materials in the upper and lower half spaces are different, the scattering waves will change their directions at the interface Σ and split into reflective and refractive waves on two sides of Σ. The Green function of the scattering problem in layered media becomes very complicated. Our convergence proof is based on the Cagniard de-Hoop transformation of the Green function and the idea of the complex coordinate stretching. By using the integral representation of the exterior Helmholtz equation and some elaborated estimation of the modified Green function, we show that the solution of the UPML problem converges exponentially to the solution of (1.1)–(1.5) as either the PML absorbing coefficient or the thickness of the PML layer tends to infinity.

The layout of the paper is as follows. In section 2 we study the Green function for the scattering problem in layered media. We recall the derivation of the Green function by the method of Fourier transform and derive an alternative form of the Green function which is crucial for the convergence analysis by using the Cagniard-de Hoop transformation. In section 3 we prove an integral representation of the exterior Helmholtz equation in layered media. In section 4 we introduce the UPML formulation for (1.1)–(1.5) by following the method of complex coordinate stretching in Chew and Weedon [7], Collino and Monk [11]. In section 5 we study the stability of the Dirichlet problem of UPML equation in the PML layer. In section 6 we study the exponential decay estimate for the modified Green function in the PML layer. In section 7 we prove the convergence of the UPML method.

2. Green function. In this section we study the Green function for the layered media

$$\Delta_x G(x,y) + k^2 G(x,y) = -\delta_y(x) \quad \text{ in } \mathbb{R}^2, \tag{2.1}$$

$$[G]_\Sigma = \left[\frac{\partial G}{\partial x_2}\right]_\Sigma = 0, \tag{2.2}$$

where $\delta_y(x)$ is the Dirac source at $y \in \mathbb{R}^2_+$ or $y \in \mathbb{R}^2_-$. We will first derive the formula for the Green function by using the method of Fourier transform and the Sommerfeld Integral Path. Next we will use the Cagniard-de Hoop transform to obtain a new formula for the Green function which will be crucial for us to prove the exponential decay of the PML extension in Section 6.

2.1. The method of Fourier transform. We first consider the case $y \in \mathbb{R}^2_+$, that is, $y_2 > 0$. Let

$$\hat{G}(\xi, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G_y(x_1, x_2) e^{-\mathbf{i}(x_1 - y_1)\xi} \mathrm{d}x_1$$

be the Fourier transform of $G_y(x) = G(x,y)$ for the first variable. By taking the Fourier transform of (2.1) in the first variable, we obtain

$$\frac{\partial^2 \hat{G}}{\partial x_2^2} + (k^2 - \xi^2)\hat{G} = -\frac{1}{\sqrt{2\pi}} \delta_{y_2}(x_2). \tag{2.3}$$

Throughout the paper we will always assume that for $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Re}\,(z^{1/2}) \geq 0$. This corresponds to the left half real axis as the branch cut in the complex plane.

Denote $v_j = e^{\mathbf{i}\mu_j |x_2 - y_2|}$, where $\mu_j = (k_j^2 - \xi^2)^{1/2}$, $j = 1, 2$. It is easy to see that

$$\frac{\partial^2 v_j}{\partial x_2^2} + (k_j^2 - \xi^2) v_j = 2\mathbf{i}\mu_j \delta_{y_2}(x_2). \tag{2.4}$$

For $y_2 > 0$, we write the solution of (2.3) as

$$\hat{G}(\xi, x_2) = \frac{1}{\sqrt{2\pi}} \begin{cases} \dfrac{\mathbf{i}}{2\mu_1} e^{\mathbf{i}\mu_1 |x_2 - y_2|} + \hat{G}_1(\xi, x_2) & \text{if } x_2 > 0, \\ \hat{G}_2(\xi, x_2) & \text{if } x_2 < 0. \end{cases}$$

Combining (2.3) and (2.4) we find that

$$\frac{\partial^2 \hat{G}_j}{\partial x_2^2} + (k_j^2 - \xi^2)\hat{G}_j = 0,$$

which has two fundamental solutions $e^{\mathbf{i}\mu_j x_2}$ and $e^{-\mathbf{i}\mu_j x_2}$, $j = 1, 2$. Since only outgoing waves are allowed for the Green function, we choose $\hat{G}_1 = A e^{\mathbf{i}\mu_1 x_2}$ and $\hat{G}_2 = B e^{-\mathbf{i}\mu_2 x_2}$. By using the matching conditions $[\hat{G}]_\Sigma = 0$, $\left[\dfrac{\partial \hat{G}}{\partial x_2}\right]_\Sigma = 0$ which follow from (2.2), we know that

$$A = \frac{\mathbf{i}}{2\mu_1} \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} e^{\mathbf{i}\mu_1 y_2}, \quad B = \frac{\mathbf{i}}{\mu_1 + \mu_2} e^{\mathbf{i}\mu_1 y_2}.$$

Therefore

$$\hat{G}(\xi, x_2) = \frac{1}{\sqrt{2\pi}} \begin{cases} \dfrac{\mathbf{i}}{2\mu_1} e^{\mathbf{i}\mu_1|x_2-y_2|} + \dfrac{\mathbf{i}}{2\mu_1} \dfrac{\mu_1-\mu_2}{\mu_1+\mu_2} e^{\mathbf{i}\mu_1(x_2+y_2)}, & \text{if } x_2 > 0, \\ \dfrac{\mathbf{i}}{\mu_1+\mu_2} e^{\mathbf{i}(\mu_1 y_2 - \mu_2 x_2)}, & \text{if } x_2 < 0. \end{cases}$$

The desired Green function should be obtained by taking the inverse Fourier transform of $\hat{G}(\xi, x_2)$. Unfortunately, one cannot simply take the inverse Fourier transform in the above formula because the branch cuts for μ_j are the half lines $(-\infty, -k_j]$ and $[k_j, \infty)$, $j = 1, 2$, in the complex ξ-plane. One way to solve the problem is to take the Sommerfeld Integral Path (SIP) as the integral path for the inverse Fourier transform (see Figure 2.1 for the SIP for the real wave number k_1 and k_2). We refer to [10, Chapter 2] for more discussion on the Sommerfeld Integral Paths.

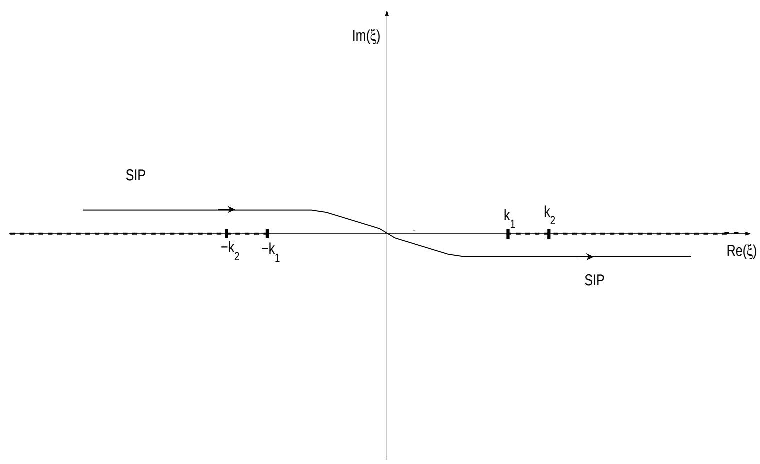

FIG. 2.1. *The Sommerfeld Integral Path.*

Recall that the Green function for the Helmholtz equation with constant wave number k_1 is $\Phi(k_1, x, y) = \frac{\mathbf{i}}{4} H_0^{(1)}(k_1|x-y|)$ which satisfies (cf. e.g. [10])

$$\Phi(k_1, x, y) = \frac{\mathbf{i}}{4\pi} \int_{\text{SIP}} \frac{1}{\mu_1} e^{\mathbf{i}(x_1-y_1)\xi + \mathbf{i}\mu_1|x_2-y_2|} \mathrm{d}\xi. \tag{2.5}$$

By taking the inverse Fourier transform of $\hat{G}(\xi, x_2)$ using the SIP, we obtain the Green function $G(x, y)$ for $x \in \mathbb{R}^2_+$, $y \in \mathbb{R}^2_+$,

$$G(x, y) = \Phi(k_1, x, y) - \Phi(k_1, x, y') + \frac{\mathbf{i}}{2\pi} \int_{\text{SIP}} \frac{1}{\mu_1+\mu_2} e^{\mathbf{i}\xi(x_1-y_1) + \mathbf{i}\mu_1(x_2+y_2)} \mathrm{d}\xi, \tag{2.6}$$

where $y' = (y_1, -y_2)$ is the image of $y = (y_1, y_2)$, and for $x \in \mathbb{R}^2_-$, $y \in \mathbb{R}^2_+$,

$$G(x, y) = \frac{\mathbf{i}}{2\pi} \int_{\text{SIP}} \frac{1}{\mu_1+\mu_2} e^{\mathbf{i}\xi(x_1-y_1) + \mathbf{i}(\mu_1 y_2 - \mu_2 x_2)} \mathrm{d}\xi. \tag{2.7}$$

Similarly we can deduce the Green function for $x \in \mathbb{R}^2_+$, $y \in \mathbb{R}^2_-$,

$$G(x, y) = \frac{\mathbf{i}}{2\pi} \int_{\text{SIP}} \frac{1}{\mu_1+\mu_2} e^{\mathbf{i}\xi(x_1-y_1) + \mathbf{i}(\mu_1 x_2 - \mu_2 y_2)} \mathrm{d}\xi, \tag{2.8}$$

and for $x \in \mathbb{R}^2_-$, $y \in \mathbb{R}^2_-$,

$$(2.9)\ G(x,y) = \Phi(k_2,x,y) - \Phi(k_2,x,y') + \frac{\mathbf{i}}{2\pi}\int_{\mathrm{SIP}} \frac{1}{\mu_1+\mu_2} e^{\mathbf{i}\xi(x_1-y_1)-\mathbf{i}\mu_2(x_2+y_2)}\mathrm{d}\xi.$$

In (2.6)–(2.7), the point source is located at $y \in \mathbb{R}^2_+$. The first term on the righthand side of (2.6) stands for the incident waves coming from the source, and the other terms are the reflected waves by the interface. The righthand side of (2.7) stands for the refractive waves below the interface. Similarly (2.9) represents the combination of incident and reflected waves and (2.8) represents refractive waves when the point source is located at $y \in \mathbb{R}^2_-$.

2.2. The Cagniard-de Hoop transform. Let h be a bounded analytic function in $\mathbb{C} \setminus ((-\infty,-k_1] \cup [k_1,\infty))$. For any $a \in \mathbb{R}, b > 0$, we denote

$$(2.10)\qquad I(h;a,b) = \frac{\mathbf{i}}{2\pi}\int_{\mathrm{SIP}} \frac{h(\xi)}{\mu_1+\mu_2} e^{\mathbf{i}\xi a+\mathbf{i}\mu_1 b}\mathrm{d}\xi.$$

It is easy to see that the Green function $G(x,y)$ can be represented as follows: for $y \in \mathbb{R}^2_+$,

$$(2.11)\ G(x,y) = \begin{cases} \Phi(k_1,x,y) - \Phi(k_1,x,y') + I\left(1; x_1-y_1, x_2+y_2\right), & \text{if } x_2 > 0, \\ I\left(e^{\mathbf{i}(\mu_1-\mu_2)x_2}; x_1-y_1, -x_2+y_2\right), & \text{if } x_2 < 0, \end{cases}$$

and for $y \in \mathbb{R}^2_-$,

$$(2.12)\quad G(x,y) = \begin{cases} I\left(e^{\mathbf{i}(\mu_1-\mu_2)y_2}, x_1-y_1, x_2-y_2\right), & \text{if } x_2 > 0, \\ \Phi(k_2,x,y) - \Phi(k_2,x,y') & \\ \qquad + I\left(e^{\mathbf{i}(\mu_1-\mu_2)(x_2+y_2)}; x_1-y_1, -x_2-y_2\right), & \text{if } x_2 < 0. \end{cases}$$

LEMMA 2.1. *Let $a \in \mathbb{R}$, $b > 0$, $\rho = \sqrt{a^2+b^2}$, and h be a bounded analytic function in $\mathbb{C} \setminus ((-\infty,-k_1] \cup [k_1,\infty))$ satisfying $h(\xi) = h(-\xi)$ and $\overline{h(\xi)} = h(\bar{\xi})$. Then*

$$I(h;a,b) = \frac{1}{\pi}\int_1^\infty \frac{1}{\sqrt{t^2-1}} \mathrm{Re}\left[\left(\frac{\mu_1}{\mu_1+\mu_2}h\right)(\xi)\right] e^{\mathbf{i}k_1\rho t}\mathrm{d}t,$$

where $\xi = \dfrac{k_1|a|t}{\rho} + \mathbf{i}\dfrac{k_1 b\sqrt{t^2-1}}{\rho}$ *and* $\mu_j = \left(k_j^2-\xi^2\right)^{1/2}$, $j = 1,2$.

Proof. Notice that the Sommerfeld Integral Path is symmetric with respect to the origin of the complex ξ-plane (see Figure 2.1). From (2.10) and $h(\xi) = h(-\xi)$ we find that

$$I(h;a,b) = I(h;-a,b) = \frac{\mathbf{i}}{2\pi}\int_{\mathrm{SIP}} \frac{h(\xi)}{\mu_1+\mu_2} e^{\mathbf{i}\xi|a|+\mathbf{i}\mu_1 b}\mathrm{d}\xi.$$

Without loss of generality we assume $a \geq 0$ in the rest of the proof.

We use the method of Cagniard-de Hoop transform to prove the lemma. Let $t' = \xi a + \mu_1 b$. Then it is easy to see that

$$(2.13)\qquad \xi = \frac{k_1 a t}{\rho} \pm \mathbf{i}\frac{k_1 b\sqrt{t^2-1}}{\rho}, \quad t = \frac{t'}{k_1\rho}.$$

Let $\xi = \xi_1 + \mathbf{i}\xi_2$, $\xi_1, \xi_2 \in \mathbb{R}$, then

$$b^2\xi_1^2 - a^2\xi_2^2 = k_1^2 a^2 b^2/\rho^2. \tag{2.14}$$

For $t \in [1, \infty)$, $\xi(t)$ is the right branch of hyperbola that intersects the real axis at $\xi_0 = k_1 a/\rho$. It is easy to see that

$$k_1^2 - \xi^2 = \left(\frac{k_1 b t}{\rho} \mp \mathbf{i}\frac{k_1 a}{\rho}\sqrt{t^2-1}\right)^2.$$

Since $\mathrm{Re}\,(\mu_1) \geq 0$, we have, for $t \in [1, \infty)$,

$$\mu_1 = \frac{k_1 b t}{\rho} \mp \mathbf{i}\frac{k_1 a}{\rho}\sqrt{t^2-1}, \tag{2.15}$$

and consequently $\dfrac{\mathrm{d}\xi}{\mathrm{d}t} = \pm\mathbf{i}\dfrac{\mu_1}{\sqrt{t^2-1}}$.

Let Γ_+, Γ_- be respectively the parts of the hyperbola in the upper-half complex plane and the lower-half complex plane. For any $r > 0$, denote C_r^+, C_r^- be respectively the part of the circle $\{\xi : |\xi| = r\}$ that are bounded by the SIP and Γ_+ or by the SIP and Γ_-. The geometry is depicted in Figure 2.2.

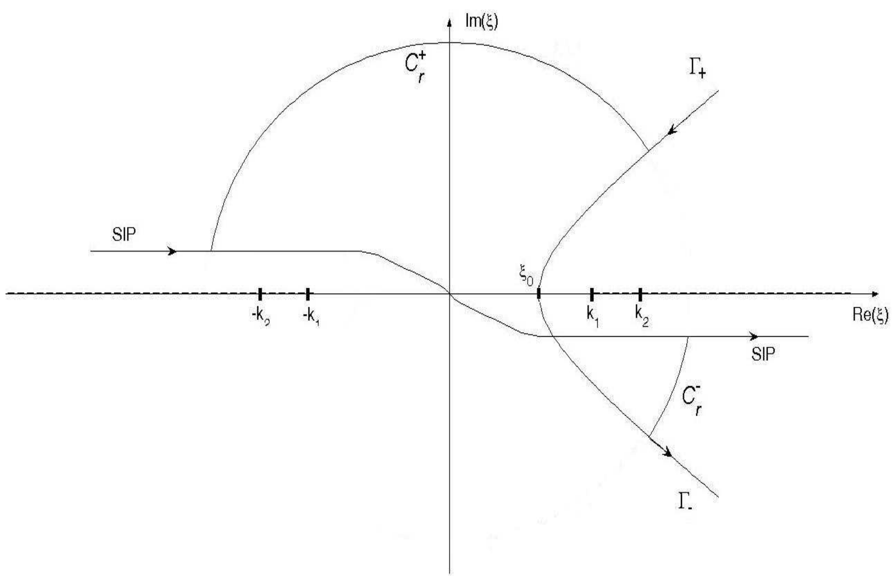

Fig. 2.2. *Cagniard-de Hoop transform from the SIP to $\Gamma_+ \cup \Gamma_-$.*

For the integrals on $C_r^\pm$ we have

$$\lim_{r\to\infty}\left|\int_{C_r^\pm}\frac{h}{\mu_1+\mu_2}e^{\mathbf{i}\xi a + \mathbf{i}\mu_1 b}\mathrm{d}\xi\right| = 0. \tag{2.16}$$

We will postpone the proof of (2.16) at the end of the proof of this lemma. Now notice that for $b > 0$, $\xi_0 < k_1$, by using Cauchy integral theorem and letting $r \to \infty$, we obtain

$$I(h; a, b) = \frac{\mathbf{i}}{2\pi}\int_{\mathrm{SIP}}\frac{h}{\mu_1+\mu_2}e^{\mathbf{i}\xi a + \mathbf{i}\mu_1 b}\mathrm{d}\xi = \frac{\mathbf{i}}{2\pi}\int_{\Gamma_+\cup\Gamma_-}\frac{h}{\mu_1+\mu_2}e^{\mathbf{i}\xi a + \mathbf{i}\mu_1 b}\mathrm{d}\xi,$$

which by the transform (2.13) yields

$$I(h;a,b)=\frac{1}{2\pi}\int_1^\infty\frac{1}{\sqrt{t^2-1}}\left[\left(\frac{\mu_1}{\mu_1+\mu_2}h\right)(\xi_+)+\left(\frac{\mu_1}{\mu_1+\mu_2}h\right)(\xi_-)\right]e^{\mathbf{i}k_1\rho t}\mathrm{d}t.$$

This implies the desired formula by using the fact that $\xi_+=\xi$, $\xi_-=\bar{\xi}$, and $h(\bar{\xi})=\overline{h(\xi)}$.

Now we prove (2.16). By the convention that $\mathrm{Re}\,(z^{1/2})\geq 0$ for any $z\in\mathbb{C}$, we have, for $z=z_1+\mathbf{i}z_2, z_1,z_2\in\mathbb{R}$,

$$z^{1/2}=\sqrt{\frac{|z|+z_1}{2}}+\mathbf{i}\mathrm{sgn}(z_2)\sqrt{\frac{|z|-z_1}{2}}. \tag{2.17}$$

Let $\xi=re^{\mathbf{i}\theta}$. It is easy to check that for $r>k_2>k_1$,

$$|\mu_1+\mu_2|\geq\frac{\sqrt{2}}{2}\left(\sqrt{r^2-k_1^2}+\sqrt{r^2-k_2^2}\right).$$

Since h is bounded in $\mathbb{C}\setminus((-\infty,-k_1]\cup[k_1,\infty))$ whose upper bound is denoted as $M(h)$, we obtain

$$\begin{aligned}\left|\int_{C_r^+}\frac{h}{\mu_1+\mu_2}e^{\mathbf{i}\xi a+\mathbf{i}\mu_1 b}\mathrm{d}\xi\right| &\leq\left|\int_{\theta_1}^{\theta_2}\frac{|h\,(\xi)\,|}{|\mu_1(\xi)+\mu_2(\xi)|}e^{-\mathrm{Im}\,(\xi a+\mu_1(\xi)b)}r\mathrm{d}\theta\right|\\ &\leq\sqrt{2}M(h)\int_{\theta_1}^{\theta_2}\frac{r}{\sqrt{r^2-k_1^2}+\sqrt{r^2-k_2^2}}e^{-\mathrm{Im}\,(\xi a+\mu_1 b)}\mathrm{d}\theta,\end{aligned}$$

where $\xi_2=re^{\mathbf{i}\theta_2}$ and $\xi_1=re^{\mathbf{i}\theta_1}$ are respectively the intersection point of C_r^+ with SIP and Γ_+. We have $0<\theta_1<\theta_2<\pi$.

If $\theta\in(\pi/2,\theta_2)$, we have $\mathrm{Im}\,(\mu_1^2)=\mathrm{Im}\,(-r^2\sin 2\theta)>0$ and thus

$$\mathrm{Im}\,(\xi a+\mu_1 b)\geq ar\sin\theta>ar\sin\theta_2.$$

If $\theta\in(\theta_1,\pi/2)$, we have $\mathrm{Im}\,(\mu_1^2)=\mathrm{Im}\,(-r^2\sin 2\theta)<0$ and consequently

$$\mathrm{Im}\,(\xi a+\mu_1 b)=ar\sin\theta-\left[\frac{|k_1^2-r^2e^{2\mathbf{i}\theta}|-(k_1^2-r^2\cos 2\theta)}{2}\right]^{1/2}b,$$

which is an increasing function in $[\theta_1,\pi/2]$ for $r^2\geq 2k_1^2$. Thus, for $\xi=re^{\mathbf{i}\theta},\theta_1\leq\theta\leq\pi/2$,

$$\mathrm{Im}\,(\xi a+\mu_1 b)\geq\mathrm{Im}\,(\xi_1 a+\mu_1(\xi_1)b)=k_1\rho t_1,$$

where $t_1\in(1,\infty)$ satisfies $\xi_1=re^{\mathbf{i}\theta_1}=\dfrac{k_1at_1}{\rho}+\mathbf{i}\dfrac{k_1b\sqrt{t_1^2-1}}{\rho}$. It is clear that $t_1\to\infty$ as $r\to\infty$. Therefore

$$\begin{aligned}\left|\int_{C_r^+}\frac{h}{\mu_1+\mu_2}e^{\mathbf{i}\xi a+\mathbf{i}\mu_1 b}\mathrm{d}\xi\right| &\leq\sqrt{2}M(h)\int_{\pi/2}^{\theta_2}\frac{r}{\sqrt{r^2-k_1^2}+\sqrt{r^2-k_2^2}}e^{-ar\sin\theta_2}\mathrm{d}\theta\\ &\quad+\sqrt{2}M(h)\int_{\theta_1}^{\pi/2}\frac{r}{\sqrt{r^2-k_1^2}+\sqrt{r^2-k_2^2}}e^{-k_1\rho t_1}\mathrm{d}\theta\\ &\to 0,\quad \text{as } r\to\infty.\end{aligned}$$

This shows (2.16) for the integral on C_r^+. The proof for the integral on C_r^- is similar and we omit the details. This completes the proof. $\square$

The following theorem indicates that the Green function $G(x,y)$ satisfies radiation condition (1.3) at infinity.

THEOREM 2.2. *For any fixed $y \in \mathbb{R}^2_\pm$, we have*

$$\sqrt{r}\left(\frac{\partial G(x,y)}{\partial r} - \mathbf{i}kG(x,y)\right) \to 0, \quad \text{as } r = |x| \to \infty,$$

where $k = k_1$ for $x \in \mathbb{R}^2_+$ and $k = k_2$ for $x \in \mathbb{R}^2_-$.

The proof of this theorem depends on the asymptotic behavior of the oscillating integrals. It is an application of the stationary phase approximation theorem (see e.g. [24, Example 5.14, Page 295]). In this paper we are interested in the convergence analysis of the PML method and thus will not elaborate on the details here.

3. The scattering problem. We start with the well-posedness of the scattering problem (1.1)-(1.5).

THEOREM 3.1. *For any $g \in H^{1/2}(\Gamma_D)$, the scattering problem* (1.1) – (1.5) *has a unique solution $u \in H^1_{\rm loc}(\mathbb{R}^2\backslash\bar{D})$.*

Proof. The uniqueness of the solution can be proved by the similar argument as that in [16]. In view of (2.11)–(2.12), $G(x,y)$ has the same singularities as $\Phi(k_1,x,y)$ for $x_2, y_2 > 0$ and $\Phi(k_2,x,y)$ for $x_2,y_2 < 0$. The existence of the solution can then be proved upon using the integral method in Colton and Kress [12, Chapter 3]. Here we omit the details. $\square$

Now we show the integral representation of the solution of the exterior Dirichlet problem which plays an important role in our subsequent analysis for the PML problem.

LEMMA 3.2. *Any solution u of the exterior Dirichlet problem* (1.1)-(1.5) *satisfies*

$$u = -\Psi_{\rm SL}(\lambda) + \Psi_{\rm DL}(g) \quad \text{ in } \mathbb{R}^2\backslash\bar{D}, \tag{3.1}$$

where $\lambda = \partial u/\partial \mathbf{n}_D \in H^{-1/2}(\Gamma_D)$ is the Neumann trace of u on Γ_D, and $\Psi_{\rm SL}, \Psi_{\rm DL}$ are respectively the single and double layer potentials

$$\Psi_{\rm SL}(\lambda)(x) = \int_{\Gamma_D} G(x,y)\lambda(y)\,{\rm d}s(y), \qquad \forall\, \lambda \in H^{-1/2}(\Gamma_D), \tag{3.2}$$

$$\Psi_{\rm DL}(g)(x) = \int_{\Gamma_D} \frac{\partial G(x,y)}{\partial \mathbf{n}_D(y)} g(y)\,{\rm d}s(y), \qquad \forall\, g \in H^{1/2}(\Gamma_D). \tag{3.3}$$

Here $\mathbf{n}_D$ is the unit outer normal to Γ_D.

Proof. For any $R > 0$ sufficiently large such that $B_R = \{x : |x| \le R\}$ includes D, by the third Green formula, we know that, for any $x \in B_R\backslash\bar{D}$,

$$u(x) = \int_{\Gamma_D \cup \Gamma_R} \left[\frac{\partial u(y)}{\partial \mathbf{n}(y)} G(x,y) - \frac{\partial G(x,y)}{\partial \mathbf{n}(y)} u(y)\right] {\rm d}s(y),$$

where $\mathbf{n}$ is the unit outer normal to $\partial(B_R\backslash\bar{D})$. Since $\mathbf{n} = -\mathbf{n}_D$ on Γ_D, it is clear that we need only to show that

$$\int_{\Gamma_R} \left[\frac{\partial u(y)}{\partial \mathbf{n}(y)} G(x,y) - \frac{\partial G(x,y)}{\partial \mathbf{n}(y)} u(y)\right] {\rm d}s(y) \to 0, \quad \text{as } R \to \infty.$$

To do that, by the radiation condition (1.3), we know that as $R \to \infty$,

$$\int_{\Gamma_R} \left[\left| \frac{\partial u}{\partial r} \right|^2 + k^2 |u|^2 + 2k \mathrm{Im} \left(u \frac{\partial \bar{u}}{\partial r} \right) \right] \mathrm{d}s = \int_{\Gamma_R} \left| \frac{\partial u}{\partial r} - \mathbf{i} k u \right|^2 \mathrm{d}s \to 0. \tag{3.4}$$

On the other hand, for any fixed $R_0 < R$ such that B_{R_0} includes $\bar{D}$, again by the third Green formula we know that

$$\int_{\Gamma_R} \mathrm{Im} \left(u \frac{\partial \bar{u}}{\partial r} \right) \mathrm{d}s = \int_{\Gamma_{R_0}} \mathrm{Im} \left(u \frac{\partial \bar{u}}{\partial r} \right) \mathrm{d}s < \infty.$$

Thus we deduce from (3.4) that $\|u\|_{L^2(\Gamma_R)}$ is bounded as $R \to \infty$. Similarly, by Theorem 2.2, we know that $\|G(\cdot, y)\|_{L^2(\Gamma_R)}$ is also bounded as $R \to \infty$. Now the lemma follows from (1.3), Theorem 2.2, and the fact that

$$\begin{aligned} &\int_{\Gamma_R} \left[\frac{\partial u(y)}{\partial \mathbf{n}(y)} G(x, y) - \frac{\partial G(x, y)}{\partial \mathbf{n}(y)} u(y) \right] \mathrm{d}s(y) \\ = &\int_{\Gamma_R} \left[\left(\frac{\partial u(y)}{\partial \mathbf{n}(y)} - \mathbf{i} k u(y) \right) G(x, y) - \left(\frac{\partial G(x, y)}{\partial \mathbf{n}(y)} - \mathbf{i} k G(x, y) \right) u(y) \right] \mathrm{d}s(y). \end{aligned}$$

This completes the proof. □

We conclude this section by introducing the equivalent weak formulation of the original scattering problem (1.1)-(1.3). Let D be contained in the interior of the rectangle $B_1 = \{x \in \mathbb{R}^2 : |x_1| < L_1/2, |x_2| < L_2/2\}$. Let $\Gamma_1 = \partial B_1$ and $\mathbf{n}_1$ the unit outer normal to Γ_1. We start by introducing the Dirichlet-to-Neumann operator $T : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$. Given $f \in H^{1/2}(\Gamma_1)$, we define $Tf = \dfrac{\partial \chi}{\partial \mathbf{n}_1}$ on Γ_1, where χ is the solution of the following exterior Dirichlet problem of the Helmholtz equation

$$\Delta \chi + k^2 \chi = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{B}_1, \tag{3.5}$$

$$[\chi]_\Sigma = \left[\frac{\partial \chi}{\partial x_2} \right]_\Sigma = 0, \tag{3.6}$$

$$\chi = f \quad \text{on } \Gamma_1, \qquad \sqrt{r} \left(\frac{\partial \chi}{\partial r} - \mathbf{i} k \chi \right) \to 0 \quad \text{as } r = |x| \to \infty. \tag{3.7}$$

By Theorem 3.1, $T : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$ is well-defined and is a continuous linear operator.

Let $a : H^1(\Omega_1) \times H^1(\Omega_1) \to \mathbb{C}$, where $\Omega_1 = B_1 \backslash \bar{D}$, be the sesquilinear form

$$a(\varphi, \psi) = \int_{\Omega_1} \left(\nabla \varphi \cdot \nabla \bar{\psi} - k^2 \varphi \bar{\psi} \right) \mathrm{d}x - \langle T\varphi, \psi \rangle_{\Gamma_1}, \tag{3.8}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_1}$ stands for the inner product on $L^2(\Gamma_1)$ or the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$. The scattering problem (1.1)-(1.5) is equivalent to the following weak formulation: Given $g \in H^{1/2}(\Gamma_D)$, find $u \in H^1(\Omega_1)$ such that $u = g$ on Γ_D and

$$a(u, \psi) = 0, \qquad \forall\, \psi \in H^1_{\Gamma_D}(\Omega_1), \tag{3.9}$$

where $H^1_{\Gamma_D}(\Omega_1) = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \Gamma_D\}$. By Theorem 3.1, the general theory in Babuška and Aziz [2, Chap. 5] implies that there exists a constant $C > 0$ such that the following inf-sup condition is satisfied

$$\sup_{0 \neq \psi \in H^1_{\Gamma_D}(\Omega_1)} \frac{|a(\varphi, \psi)|}{\|\, \psi \,\|_{H^1(\Omega_1)}} \geq C \|\, \varphi \,\|_{H^1(\Omega_1)}, \qquad \forall\, \varphi \in H^1_{\Gamma_D}(\Omega_1). \tag{3.10}$$

4. The uniaxial PML equation. Now we introduce the absorbing PML layer. Let $B_2 = \{x \in \mathbb{R}^2 : |x_1| < L_1/2 + d_1, |x_2| < L_2/2 + d_2\}$ be the rectangle which contains B_1. Let $\alpha_1(t) = 1 + \mathbf{i}\sigma_1(t), \alpha_2(t) = 1 + \mathbf{i}\sigma_2(t)$ be the model medium property which satisfy

$$\sigma_j \geq 0, \quad \sigma_j(t) = \sigma_j(-t), \quad \text{and} \quad \sigma_j = 0 \quad \text{for} \quad |t| \leq L_j/2, \quad j = 1, 2.$$

Denote by $\tilde{x}_j$ the complex coordinate defined by

$$\tilde{x}_j = \int_0^{x_j} \alpha_j(t)\mathrm{d}t, \qquad j = 1, 2. \tag{4.1}$$

Notice that $\tilde{x}_j$ depends only on x_j and for this reason the method is called the uniaxial PML method. The complex distance is defined by

$$\rho(\tilde{x}, y) = \left[(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2\right]^{1/2}.$$

We follow the method of complex coordinate stretching [9, 5] to introduce the PML equation. We define

$$\tilde{G}(x, y) := G(\tilde{x}, y), \quad \forall x, y \in \mathbb{R}^2.$$

From (2.11)–(2.12) and (4.1), it is easy to see that $\tilde{G}$ is smooth for $x \in \mathbb{R}^2 \backslash \bar{B}_1$ and $y \in \bar{B}_1$. Now we can define the modified single and double layer potentials

$$\tilde{\Psi}_{\mathrm{SL}}(\lambda)(x) = \int_{\Gamma_1} \tilde{G}(x, y)\lambda(y)\,\mathrm{d}s(y), \qquad \forall\, \lambda \in H^{-1/2}(\Gamma_1), \tag{4.2}$$

$$\tilde{\Psi}_{\mathrm{DL}}(f)(x) = \int_{\Gamma_1} \frac{\partial \tilde{G}(x, y)}{\partial \mathbf{n}_1(y)} f(y)\,\mathrm{d}s(y), \qquad \forall\, f \in H^{1/2}(\Gamma_1). \tag{4.3}$$

It is clear that $\tilde{\Psi}_{\mathrm{SL}}(\lambda), \tilde{\Psi}_{\mathrm{DL}}(f)$ are smooth in $\mathbb{R}^2 \backslash \bar{B}_1$, and

$$\begin{aligned} \gamma_D^+ \tilde{\Psi}_{\mathrm{SL}}(\lambda) &= \gamma_D^+ \Psi_{\mathrm{SL}}(\lambda), \qquad \forall\, \lambda \in H^{-1/2}(\Gamma_1), \\ \gamma_D^+ \tilde{\Psi}_{\mathrm{DL}}(f) &= \gamma_D^+ \Psi_{\mathrm{DL}}(f), \qquad \forall\, f \in H^{1/2}(\Gamma_1), \end{aligned} \tag{4.4}$$

where $\gamma_D^+ : H^1_{loc}(\mathbb{R}^2 \backslash \bar{B}_1) \to H^{1/2}(\Gamma_1)$ is the trace operator.

For any $f \in H^{1/2}(\Gamma_1)$, let $\mathbb{E}(f)(x)$ be the PML extension given by

$$\mathbb{E}(f)(x) = -\tilde{\Psi}_{\mathrm{SL}}(Tf) + \tilde{\Psi}_{\mathrm{DL}}(f) \qquad \text{for } x \in \mathbb{R}^2 \backslash \bar{B}_1. \tag{4.5}$$

By (4.4) and (3.7) we know that, for any $f \in H^{1/2}(\Gamma_1)$,

$$\gamma_D^+ \mathbb{E}(f) = -\gamma_D^+ \Psi_{\mathrm{SL}}(Tf) + \gamma_D^+ \Psi_{\mathrm{DL}}(f) = \gamma_D^+ \chi = f \quad \text{on } \Gamma_1.$$

For the solution u of the scattering problem (3.9), let $\tilde{u} = \mathbb{E}(u|_{\Gamma_1})$ be the PML extension of $u|_{\Gamma_1}$ which satisfies $\gamma_D^+ \tilde{u} = u|_{\Gamma_1}$ on Γ_1. It is obvious that $\tilde{u}$ satisfies

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{x}_1^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{x}_2^2} + k^2 \tilde{u} = 0 \ \text{ in } \mathbb{R}^2 \backslash \bar{B}_1, \qquad [\tilde{u}]_\Sigma = \left[\frac{\partial \tilde{u}}{\partial \tilde{x}_2}\right]_\Sigma = 0,$$

which yields the desired UPML equation by the chain rule

$$\nabla \cdot (A \nabla \tilde{u}) + \alpha_1 \alpha_2 k^2 \tilde{u} = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{B}_1,$$

where $A = \mathrm{diag}(\alpha_2(x_2)/\alpha_1(x_1), \alpha_1(x_1)/\alpha_2(x_2))$ is a diagonal matrix. The continuity condition across Σ reads

$$\left[\frac{\partial \tilde{u}}{\partial \tilde{x}_2}\right]_\Sigma = 0 \quad \Leftrightarrow \quad [(A\nabla\tilde{u})\cdot\mathbf{n}]_\Sigma = 0 \quad \Leftrightarrow \quad \left[\frac{\partial \tilde{u}}{\partial x_2}\right]_\Sigma = 0.$$

The UPML solution $\hat{u}$ in $\Omega_2 = B_2\backslash\bar{D}$ is defined as the solution of the following system

$$\nabla\cdot(A\nabla\hat{u}) + \alpha_1\alpha_2 k^2\hat{u} = 0 \quad \text{in } \Omega_2, \tag{4.6}$$

$$[\hat{u}]_\Sigma = \left[\frac{\partial \hat{u}}{\partial x_2}\right]_\Sigma = 0 \quad \text{on } \Sigma\cap\Omega_2, \tag{4.7}$$

$$\hat{u} = g \quad \text{on } \Gamma_D, \qquad \hat{u} = 0 \quad \text{on } \Gamma_2. \tag{4.8}$$

The well-posedness of the UPML problem (4.6)-(4.8) and the convergence of the solution $\hat{u}$ to the solution of the original scattering problem will be studied in next section.

5. The PML equation in the PML layer. In this section we consider the Dirichlet problem of the PML equation in the layer:

$$\nabla\cdot(A\nabla w) + \alpha_1\alpha_2 k^2 w = 0 \quad \text{in } \ \Omega_{\mathrm{PML}} = B_2\setminus\bar{B}_1, \tag{5.1}$$

$$[w]_\Sigma = \left[\frac{\partial w}{\partial x_2}\right]_\Sigma = 0 \quad \text{on } \ \Sigma\cap\Omega_{\mathrm{PML}}, \tag{5.2}$$

$$w = 0 \quad \text{on } \ \Gamma_1, \qquad w = q \quad \text{on } \ \Gamma_2, \tag{5.3}$$

where $q \in H^{1/2}(\Gamma_2)$. Introduce the sesquilinear form c: $H^1(\Omega_{\mathrm{PML}})\times H^1(\Omega_{\mathrm{PML}}) \to \mathbb{C}$ as

$$c(\varphi,\psi) = \int_{\Omega_{\mathrm{PML}}} \left(A\nabla\varphi\cdot\nabla\bar{\psi} - \alpha_1\alpha_2 k^2\varphi\bar{\psi}\right)\mathrm{d}x, \quad \forall\,\varphi,\psi\in H_0^1(\Omega_{\mathrm{PML}}).$$

Then the weak formulation of (5.1) – (5.3) is: Find $w\in H^1(\Omega_{\mathrm{PML}})$ such that $w=0$ on Γ_1, $w=q$ on Γ_2, and

$$c(w,v) = 0, \quad \forall\, v\in H_0^1(\Omega_{\mathrm{PML}}). \tag{5.4}$$

Notice that, for any $\varphi\in H^1(\Omega_{\mathrm{PML}})$,

$$\mathrm{Re}\,[c(\varphi,\varphi)] = \int_{\Omega_{\mathrm{PML}}}\left[\frac{1+\sigma_1\sigma_2}{1+\sigma_1^2}\left|\frac{\partial\varphi}{\partial x_1}\right|^2 + \frac{1+\sigma_1\sigma_2}{1+\sigma_2^2}\left|\frac{\partial\varphi}{\partial x_2}\right|^2 + k^2(\sigma_1\sigma_2-1)|\varphi|^2\right]\mathrm{d}x.$$

Since

$$\frac{1+\sigma_1\sigma_2}{1+\sigma_1^2} \geq \frac{1}{1+\sigma_m^2}, \quad \frac{1+\sigma_1\sigma_2}{1+\sigma_2^2} \geq \frac{1}{1+\sigma_m^2},$$

where $\sigma_m = \max\limits_{x\in\Omega_{\mathrm{PML}}}(\sigma_1(x_1),\,\sigma_2(x_2)) > 0$, we know by using the spectral theory of compact operators that for any given k_2, (5.4) has a unique solution for every real k_1 except possibly for a discrete set of values of k_1 (see Collino and Monk [11, Theorem 2] for a similar discussion on the PML equation in the polar coordinates). In this section we will make the following assumption on the medium property

(H1) $\sigma_j(t) \equiv \sigma > 0, \ \ \forall |t| \geq L_j/2, \quad j = 1, 2$, where σ is a positive constant.

This assumption which allows us to prove the coercivity of the sesquilinear form c is not very restrictive in practical applications. In particular, our numerical experiences with the adaptive uniaxial PML for Maxwell scattering problems [8] indicate that the constant medium property leads to better preconditioning techniques for the discrete PML problems as opposed to the continuous medium properties.

Throughout the paper we will use the weighted H^1-norm

$$|||\varphi|||_{H^1(\Omega)} = \left(\|\nabla\varphi\|^2_{L^2(\Omega)} + \|k\varphi\|^2_{L^2(\Omega)}\right)^{1/2},$$

for any bounded domain $\Omega \subset \mathbb{R}^2$. For any $\varphi \in H^1(\Omega_{\rm PML})$, we define an equivalent norm on $H^1(\Omega_{\rm PML})$ by

$$\|\varphi\|_{*,\Omega_{\rm PML}} = \left(\|A\nabla\varphi\|^2_{L^2(\Omega_{\rm PML})} + \|k\alpha_1\alpha_2\varphi\|^2_{L^2(\Omega_{\rm PML})}\right)^{1/2}.$$

Lemma 5.1. *Let (H1) be satisfied. Then* (5.4) *has a unique solution and it holds that*

$$\sup_{0\neq\psi\in H^1_0(\Omega_{\rm PML})} \frac{|c(\varphi,\psi)|}{|||\psi|||_{H^1(\Omega_{\rm PML})}} \geq \hat{C}\|\varphi\|_{*,\Omega_{\rm PML}}, \quad \forall\, \varphi \in H^1_0(\Omega_{\rm PML}), \tag{5.5}$$

where

$$\hat{C} = \frac{\min(1,\sigma^3)}{2(1+\sigma^2)^2\max(1,k_2^2d^2)}, \qquad d = \max(d_1,d_2).$$

Proof. It is clear that $\bar{\Omega}_{\rm PML} = \bar{\Omega}_c \cup \bar{\Omega}_1 \cup \bar{\Omega}_2$, where

$$\begin{aligned}
\Omega_c &= \{x \in \Omega_{\rm PML}:\ |x_1| > L_1/2,\ |x_2| > L_2/2\},\\
\Omega_1 &= \{x \in \Omega_{\rm PML}:\ |x_1| > L_1/2,\ |x_2| < L_2/2\},\\
\Omega_2 &= \{x \in \Omega_{\rm PML}:\ |x_2| > L_2/2,\ |x_1| < L_1/2\}.
\end{aligned}$$

Since $\sigma_1 = \sigma_2 = \sigma$ in Ω_c, $\sigma_2 = 0$ in Ω_1, and $\sigma_1 = 0$ in Ω_2, it is east to check that

$$\begin{aligned}
\frac{1}{\sigma}{\rm Im}\ [c(\varphi,\varphi)] = {} & \left\|\frac{\partial\varphi}{\partial x_2}\right\|^2_{L^2(\Omega_1)} + \left\|\frac{\partial\varphi}{\partial x_1}\right\|^2_{L^2(\Omega_2)} - \frac{1}{1+\sigma^2}\sum_{j=1}^2\left\|\frac{\partial\varphi}{\partial x_j}\right\|^2_{L^2(\Omega_j)}\\
& -2\,\|k\varphi\|^2_{L^2(\Omega_c)} - \|k\varphi\|^2_{L^2(\Omega_1\cup\Omega_2)}.
\end{aligned}$$

The key observation is that

$$\begin{aligned}
& {\rm Re}\,[c(\varphi,\varphi)] - \frac{1}{\sigma}{\rm Im}\,[c(\varphi,\varphi)]\\
& = \|\nabla\varphi\|^2_{L^2(\Omega_c)} + (1+\sigma^2)\,\|k\varphi\|^2_{L^2(\Omega_c)} + \frac{2}{1+\sigma^2}\sum_{j=1}^2\left\|\frac{\partial\varphi}{\partial x_j}\right\|^2_{L^2(\Omega_j)}.
\end{aligned}$$

Therefore, for any $\gamma > 0$,

$$\begin{aligned}
& {\rm Re}\,[c(\varphi,\varphi)] + \frac{\gamma-1}{\sigma}{\rm Im}\,[c(\varphi,\varphi)]\\
& = \|\nabla\varphi\|^2_{L^2(\Omega_c)} + \gamma\left\|\frac{\partial\varphi}{\partial x_2}\right\|^2_{L^2(\Omega_1)} + \gamma\left\|\frac{\partial\varphi}{\partial x_1}\right\|^2_{L^2(\Omega_2)} + \frac{2-\gamma}{1+\sigma^2}\sum_{j=1}^2\left\|\frac{\partial\varphi}{\partial x_j}\right\|^2_{L^2(\Omega_j)}\\
& \quad +(1+\sigma^2-2\gamma)\|\varphi\|^2_{L^2(\Omega_c)} - \gamma\|k\varphi\|^2_{L^2(\Omega_1\cup\Omega_2)}.
\end{aligned} \tag{5.6}$$

Since $\varphi = 0$ on Γ_2, we deduce easily that

$$\|\varphi\|^2_{L^2(\Omega_1)} \le d_1^2 \left\| \frac{\partial \varphi}{\partial x_1} \right\|^2_{L^2(\Omega_1)}, \qquad \|\varphi\|^2_{L^2(\Omega_2)} \le d_2^2 \left\| \frac{\partial \varphi}{\partial x_2} \right\|^2_{L^2(\Omega_2)},$$

which implies

$$\begin{aligned} -\gamma \|k\varphi\|^2_{L^2(\Omega_1\cup\Omega_2)} &= \gamma \|k\varphi\|^2_{L^2(\Omega_1\cup\Omega_2)} - 2\gamma \|k\varphi\|^2_{L^2(\Omega_1\cup\Omega_2)} \\ &\ge \gamma \|k\varphi\|^2_{L^2(\Omega_1\cup\Omega_2)} - 2\gamma k_2^2 d^2 \sum_{j=1}^{2} \left\| \frac{\partial \varphi}{\partial x_j} \right\|^2_{L^2(\Omega_j)}, \end{aligned}$$

where $d = \max(d_1, d_2)$. Substitute the above estimate into (5.6) we obtain

$$\begin{aligned} &\mathrm{Re}\,[c(\varphi,\varphi)] + (\gamma-1)\sigma^{-1}\mathrm{Im}\,[c(\varphi,\varphi)] \\ &\ge \|\nabla\varphi\|^2_{L^2(\Omega_c)} + \gamma \left\| \frac{\partial \varphi}{\partial x_2} \right\|^2_{L^2(\Omega_1)} + \gamma \left\| \frac{\partial \varphi}{\partial x_1} \right\|^2_{L^2(\Omega_2)} \\ &\quad + \left(\frac{2-\gamma}{1+\sigma^2} - 2\gamma k_2^2 d^2 \right) \sum_{j=1}^{2} \left\| \frac{\partial \varphi}{\partial x_j} \right\|^2_{L^2(\Omega_j)} \\ &\quad + (1+\sigma^2 - 2\gamma)\|\varphi\|^2_{L^2(\Omega_c)} + \gamma \|k\varphi\|^2_{L^2(\Omega_1\cup\Omega_2)}. \end{aligned}$$

Now we take $\gamma = 1/(2\max(1, k_2^2 d^2)(1+\sigma^2))$, then

$$\frac{2-\gamma}{1+\sigma^2} - 2\gamma k_2^2 d^2 \ge \frac{1}{2(1+\sigma^2)} \ge \gamma, \qquad 1+\sigma^2 - 2\gamma \ge \sigma^2.$$

Thus

$$\begin{aligned} \max(1,\sigma^{-1})|c(\varphi,\varphi)| &\ge \mathrm{Re}\,[c(\varphi,\varphi)] + (\gamma-1)\sigma^{-1}\mathrm{Im}\,[c(\varphi,\varphi)] \\ &\ge \min(\gamma,\sigma^2)|||\varphi|||^2_{H^1(\Omega_{\mathrm{PML}})}. \end{aligned}$$

Since $\min(\gamma,\sigma^2)/\max(1,\sigma^{-1}) \ge \gamma\min(1,\sigma^3)$, we have

$$\begin{aligned} \sup_{0\ne\psi\in H_0^1(\Omega_{\mathrm{PML}})} \frac{|c(\varphi,\psi)|}{|||\psi|||_{H^1(\Omega_{\mathrm{PML}})}} &\ge \frac{|c(\varphi,\varphi)|}{|||\varphi|||_{H^1(\Omega_{\mathrm{PML}})}} \ge \gamma\min(1,\sigma^3)|||\varphi|||_{H^1(\Omega_{\mathrm{PML}})} \\ &\ge \gamma \frac{\min(1,\sigma^3)}{1+\sigma^2}\|\varphi\|_{*,\Omega_{\mathrm{PML}}}. \end{aligned}$$

This completes the proof. ∎

LEMMA 5.2. *Let (H1) be satisfied and w be the solution of the PML equation in the layer* (5.1)–(5.3). *Then, for any $\zeta \in H^1(\Omega_{\mathrm{PML}})$ such that $\zeta = 0$ on Γ_1 and $\zeta = q$ on Γ_2,*

$$\left\| \frac{\partial w}{\partial \mathbf{n}} \right\|_{H^{-1/2}(\Gamma_1)} \le (1+\hat{C}^{-1})\|\zeta\|_{*,\Omega_{\mathrm{PML}}}. \tag{5.7}$$

Proof. Since $w - \zeta \in H_0^1(\Omega_{\mathrm{PML}})$, by (5.5) we have

$$\begin{aligned} \hat{C}\|w-\zeta\|_{*,\Omega_{\mathrm{PML}}} &\le \sup_{\varphi\in H_0^1(\Omega_{\mathrm{PML}})} \frac{|c(w-\zeta,\varphi)|}{|||\varphi|||_{H^1(\Omega_{\mathrm{PML}})}} \\ &= \sup_{\varphi\in H_0^1(\Omega_{\mathrm{PML}})} \frac{|c(\zeta,\varphi)|}{|||\varphi|||_{H^1(\Omega_{\mathrm{PML}})}} \le \|\zeta\|_{*,\Omega_{\mathrm{PML}}}, \end{aligned}$$

where we have used (5.4) and the fact that $|c(\zeta,\varphi)| \le \|\zeta\|_{*,\Omega_{\rm PML}} |||\varphi|||_{H^1(\Omega_{\rm PML})}$. This proves $\|w\|_{*,\Omega_{\rm PML}} \le (1+\hat{C}^{-1})\|\zeta\|_{*,\Omega_{\rm PML}}$.

To show (5.7), for any $\varphi \in H^1(\Omega_{\rm PML})$ such that $\varphi = 0$ on Γ_2, by testing (5.1) with φ and using (5.2) we know that

$$\left|\left\langle \frac{\partial w}{\partial \mathbf{n}}, \varphi \right\rangle_{\Gamma_1}\right| = |c(w,\varphi)| \le \|w\|_{*,\Omega_{\rm PML}} |||\varphi|||_{H^1(\Omega_{\rm PML})}.$$

This completes the proof. ∎

6. Estimation of the modified Green function. The convergence analysis for the UPML problem depends crucially on the exponential decay estimate for the modified Green's function $\tilde{G}(x,y)$ which is the goal of this section. We start with the following assumption on the fictitious medium property, which is rather mild in the practical application of the UPML method.

(H2) $\displaystyle\int_0^{\frac{L_1}{2}+d_1} \sigma_1(t){\rm d}t = \int_0^{\frac{L_2}{2}+d_2} \sigma_2(t){\rm d}t =: \bar{\sigma}, \quad \bar{\sigma} > 0$ is a constant.

We also remark that the result of this section does not depend on the assumption of constant medium property (H1). The following elementary lemma is from [7, Lemma 3.2].

Lemma 6.1. *For any $z_1 = a_1 + \mathbf{i}b_1$ and $z_2 = a_2 + \mathbf{i}b_2$ with $a_1, b_1, a_2, b_2 \in \mathbb{R}$ such that $a_1b_1 + a_2b_2 > 0$ and $a_1^2 + a_2^2 > 0$, we have*

$$\operatorname{Im}(z_1^2 + z_2^2)^{1/2} \ge \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2}}.$$

Proof. Because the proof is short, we rewrite it here for the completeness. Since $\operatorname{Im}(z_1^2 + z_2^2) = 2(a_1b_1 + a_2b_2) > 0$, by (2.17) we know that

$$\operatorname{Re}(a+\mathbf{i}b)^{1/2} = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}, \quad \operatorname{Im}(a+\mathbf{i}b)^{1/2} = \sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}. \tag{6.1}$$

It is easy to check that $\operatorname{Im}(a+\mathbf{i}b)^{1/2}$ is a decreasing function in $a \in \mathbb{R}$. Let $z_1^2 + z_2^2 = a + \mathbf{i}b$, then

$$a + \mathbf{i}b = \left(\sqrt{a_1^2+a_2^2} + \mathbf{i}\,\frac{a_1b_1+a_2b_2}{\sqrt{a_1^2+a_2^2}}\right)^2 - \frac{(a_2b_1 - a_1b_2)^2}{a_1^2+a_2^2}.$$

Let $a' = a + \dfrac{(a_2b_1 - a_1b_2)^2}{a_1^2+a_2^2}$. Since $a_1b_1 + a_2b_2 \ge 0$, we have

$$\operatorname{Im}(a' + \mathbf{i}b)^{1/2} = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2+a_2^2}}.$$

On the other hand, since $a' > a$, we know that $\operatorname{Im}(a+\mathbf{i}b)^{1/2} \ge \operatorname{Im}(a'+\mathbf{i}b)^{1/2}$. This completes the proof. ∎

The following lemma is the complex counterpart of (2.13) and (2.15).

LEMMA 6.2. *For any $z_1 = a_1 + \mathbf{i}b_1$ and $z_2 = a_2 + \mathbf{i}b_2$ with $a_1, b_1, a_2, b_2 \in \mathbb{R}$ such that $a_1, a_2, b_1, b_2 \geq 0$, define*

$$\xi = \frac{k_1}{\rho}\left(z_1 t + \mathbf{i} z_2 \sqrt{t^2-1}\right), \quad \rho = (z_1^2 + z_2^2)^{1/2}, \quad \forall\, t \in [1, \infty). \tag{6.2}$$

Then $\mu_1(\xi) = \left(k_1^2 - \xi^2\right)^{1/2}$ satisfies

$$\mu_1 = \frac{k_1}{\rho}\left(z_2 t - \mathbf{i} z_1 \sqrt{t^2-1}\right), \qquad \forall\, t \in [1, \infty). \tag{6.3}$$

Proof. For any $t \in [1, \infty)$, let $\mu = k_1\left(z_2 t - \mathbf{i} z_1 \sqrt{t^2-1}\right)/\rho$. Clearly $\mu^2 = k_1^2 - \xi^2 = \mu_1^2$. By the convention in (2.17), the lemma follows from $\mathrm{Re}\,(\mu) \geq 0$ which can be proved by direct calculations. Here we omit the details. $\square$

For ξ given in (6.2). Let $\xi = \xi_1 + \mathbf{i}\xi_2$ with $\xi_1, \xi_2 \in \mathbb{R}$. It is easy to check that

$$\xi_1\xi_2 = k_1^2 \frac{(a_1a_2 + b_1b_2)(|z_1|^2 + |z_2|^2)}{|\rho|^4} f_1(t), \quad f_1(t) := tt' - \beta(t^2 + t'^2), \tag{6.4}$$

$$\xi_1^2 - \xi_2^2 = \frac{1}{2}k_1^2 + \frac{1}{2}k_1^2 \frac{|z_1|^4 - |z_2|^4}{|\rho|^4} f_2(t), \quad f_2(t) := t^2 + t'^2 - 4\beta tt', \tag{6.5}$$

where $t' = \sqrt{t^2 - 1}$ and

$$\beta := \frac{a_1b_2 - a_2b_1}{|z_1|^2 + |z_2|^2}. \tag{6.6}$$

Consequently, for $\rho = (z_1^2 + z_2^2)^{1/2}$,

$$|\rho|^4 = |z_1|^4 + |z_2|^4 + 2(a_1a_2 + b_1b_2)^2 - 2(a_2b_1 - a_1b_2)^2. \tag{6.7}$$

It is easy to check that $\mu_2(\xi) = \left(k_2^2 - \xi^2\right)^{1/2}$ satisfies

$$|\mu_2|^4 = 4(k_2^4 - k_2^2k_1^2)\frac{(a_1a_2 + b_1b_2)^2}{|\rho|^4} + \frac{k_1^4}{4}\frac{\left(|z_1|^2 + |z_2|^2\right)^2}{|\rho|^4}\left(f_2(t) - M\right)^2, \tag{6.8}$$

where

$$M := \frac{2k_2^2 - k_1^2}{k_1^2}\frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2}. \tag{6.9}$$

LEMMA 6.3. *For any $z_1 = a_1 + \mathbf{i}b_1$ and $z_2 = a_2 + \mathbf{i}b_2$ with $a_1, b_1, a_2, b_2 \in \mathbb{R}$ such that $a_1, a_2, b_1, b_2 \geq 0$, define $\rho = (z_1^2 + z_2^2)^{1/2}$. Assume that*

$$b_1^2 + b_2^2 \geq a_1^2 + a_2^2. \tag{6.10}$$

Then for any $\mu_j(\xi) = \left(k_j^2 - \xi^2\right)^{1/2}$ with ξ given in (6.2), $j = 1, 2$, we have

$$\mathrm{Im}\,[(\mu_1 - \mu_2)z_2] \leq 0.$$

Proof. Denote $\mu_j = p_j + \mathbf{i}q_j$ with $p_j, q_j \in \mathbb{R}$, $j = 1, 2$. Since $\mu_2^2 - \mu_1^2 = k_2^2 - k_1^2$, we have

$$p_2^2 - q_2^2 = k_2^2 - k_1^2 + p_1^2 - p_2^2, \quad p_1 q_1 = p_2 q_2.$$

We recall (2.17) and find

$$p_2 = \left[\frac{\sqrt{\left(k_2^2 - k_1^2 + p_1^2 - q_1^2\right)^2 + 4p_1^2 q_1^2} + \left(k_2^2 - k_1^2 + p_1^2 - q_1^2\right)}{2} \right]^{1/2}, \tag{6.11}$$

$$q_2 = \operatorname{sgn}(q_1) \left[\frac{\sqrt{\left(k_2^2 - k_1^2 + p_1^2 - q_1^2\right)^2 + 4p_1^2 q_1^2} - \left(k_2^2 - k_1^2 + p_1^2 - q_1^2\right)}{2} \right]^{1/2}. \tag{6.12}$$

Direct calculations show that

$$p_2 \geq p_1 \geq 0, \quad |q_2| \leq |q_1|, \tag{6.13}$$

where we have used $p_1 \geq 0$ from Lemma 6.2. Since

$$\operatorname{Im}\left[(\mu_1 - \mu_2) z_2\right] = a_2(q_1 - q_2) + b_2(p_1 - p_2), \tag{6.14}$$

the lemma now follows obviously for $q_1 \leq 0$. The rest of the proof is devoted to the case of $q_1 > 0$.

By the assumption (6.10) we know that $\operatorname{Re}(z_1^2 + z_2^2) \leq 0$. Thus (2.17) implies $\rho_1 \leq \rho_2$, where $\rho_1 = \operatorname{Re}\rho$, $\rho_2 = \operatorname{Im}\rho$. By Lemma 6.2 we know that

$$p_1 = \frac{k_1}{|\rho|^2}\left[t(\rho_1 a_2 + \rho_2 b_2) + t'(\rho_1 b_1 - \rho_2 a_1)\right], \tag{6.15}$$

$$q_1 = \frac{k_1}{|\rho|^2}\left[t(\rho_1 b_2 - \rho_2 a_2) - t'(\rho_1 a_1 + \rho_2 b_1)\right]. \tag{6.16}$$

This yields

$$\begin{aligned} p_1 - q_1 &= \frac{k_1}{|\rho|^2}\Big\{ t\left[a_2(\rho_2 + \rho_1) + b_2(\rho_2 - \rho_1)\right] + t'\left[b_1(\rho_2 + \rho_1) + a_1(\rho_1 - \rho_2)\right] \Big\} \\ &\geq \frac{k_1 t'(\rho_2 - \rho_1)}{|\rho|^2}(a_2 + b_2 + b_1 - a_1) \geq 0. \end{aligned}$$

Since $q_1 > 0$, (6.16) implies that $\rho_1 b_2 > \rho_2 a_2$, which together with $\rho_1 \leq \rho_2$ implies $b_2 \geq a_2$. Therefore, by (6.14), $q_1 > 0$, and $p_1 q_1 = p_2 q_2$,

$$\begin{aligned} \operatorname{Im}\left[(\mu_1 - \mu_2) z_2\right] = \frac{1}{q_1}(q_2 - q_1)(b_2 p_2 - a_2 q_1) &\leq \frac{a_2}{q_1}(q_2 - q_1)(p_2 - q_1) \\ &\leq \frac{a_2}{q_1}(q_2 - q_1)(p_1 - q_1) \leq 0, \end{aligned}$$

where we have used (6.13). This completes the proof. ∎

Lemma 6.4. *For $z_1, z_2 \in \mathbb{C}$ and $\rho = (z_1^2 + z_2^2)^{1/2}$. Suppose $k_1 \operatorname{Im}\rho \geq 1$ and let $h(t)$ be a bounded function in $[1, \infty)$. Then the function*

$$\Phi_1(h; z_1, z_2) = \frac{1}{2\pi}\int_1^\infty \frac{h(t)}{\sqrt{t^2 - 1}} e^{\mathbf{i}k_1\rho t} \mathrm{d}t$$

satisfies the estimate

$$|\Phi_1(h;z_1,z_2)| \le CM(h)e^{-k_1 \operatorname{Im}\rho},$$

where $M(h)$ is the upper bound of $|h|$ and C is independent of k_1, z_1, z_2.

Proof. Since $k_1 \operatorname{Im}\rho \ge 1$, we have

$$\begin{aligned}|\Phi_1(h;z_1,z_2)| &\le CM(h)\int_1^\infty \frac{1}{\sqrt{t^2-1}}e^{-k_1 \operatorname{Im}\rho\, t}\mathrm{d}t \\ &\le CM(h)e^{-(k_1 \operatorname{Im}\rho-1)}\int_1^\infty \frac{e^{-t}}{\sqrt{t^2-1}}\mathrm{d}t \\ &= CM(h)e^{-k_1 \operatorname{Im}\rho}.\end{aligned}$$

This completes the proof. $\square$

In the following we will always denote

$$z_1 = \left[(\tilde{x}_1-y_1)^2\right]^{1/2}, \quad z_2 = \left[(\tilde{x}_2)^2\right]^{1/2} + |y_2|, \qquad \forall x \in \Gamma_2,\ y \in \Gamma_1. \tag{6.17}$$

Let $z_j = a_j + \mathbf{i}b_j, a_j, b_j \in \mathbb{R}, j = 1,2$. Then $a_j, b_j \ge 0$. By Lemma 6.1, $\rho(\tilde{x},y) = (z_1^2+z_2^2)^{1/2}$ satisfies

$$\begin{aligned}\operatorname{Im}\rho(\tilde{x},y) \ge &\frac{|x_1-y_1|}{\sqrt{|x_1-y_1|^2+(|x_2|+|y_2|)^2}}\left|\int_0^{x_1}\sigma_1(t)\mathrm{d}t\right| \\ &+\frac{|x_2|+|y_2|}{\sqrt{|x_1-y_1|^2+(|x_2|+|y_2|)^2}}\left|\int_0^{x_2}\sigma_2(t)\mathrm{d}t\right|.\end{aligned}$$

Now by (H2) we have, for any $x \in \Gamma_2$ and $y \in \Gamma_1$,

$$\operatorname{Im}\rho(\tilde{x},y) \ge \gamma_0\bar{\sigma}, \qquad \gamma_0 = \frac{\min(d_1, d_2+L_2/2)}{\sqrt{(L_1+d_1)^2+(L_2+d_2)^2}}. \tag{6.18}$$

The following lemma on the estimate of the modified Green function $\tilde{G}$ will play an important role in the following analysis.

Lemma 6.5. *Let (H2) and $\gamma_0\bar{\sigma} \ge \max(k_1^{-1}, \min(d_1, d_2+L_2/2))$ be satisfied. There exists a constant C independent of k_j, L_j, and d_j, $j=1,2$, such that*

$$|\tilde{G}(x,y)| \le C\gamma_1 e^{-k_1\gamma_0\bar{\sigma}}, \quad \forall\, x \in \Gamma_2,\ \ y \in \Gamma_1,$$

where $\gamma_1 := e^{L_2\sqrt{k_2^2-k_1^2}/2}$.

Before we prove the lemma we remark that the condition

$$\gamma_0\bar{\sigma} \ge \max(k_1^{-1}, \min(d_1, d_2+L_2/2))$$

in the lemma is rather mild in practical applications because one achieves the exponential convergence of the PML method by enlarging $\bar{\sigma}$ which can be realized by either enlarging the thickness of the PML layer d_j or enlarging the medium property $\tilde{\sigma}_j$. Moreover, the condition $\gamma_0\bar{\sigma} \ge \min(d_1, d_2+L_2/2)$ implies $b_1^2+b_2^2 \ge a_1^2+a_2^2$ which is the condition (6.10) in Lemma 6.3.

Proof. We only prove the case for $y \in \mathbb{R}^2_+$, the proof of the other case $y \in \mathbb{R}^2_-$ is similar. By (2.5) and the method of Cagniard-de Hoop transform (cf. [10]) we know the Green function $\Phi(k_1, x, y) = \frac{\mathbf{i}}{4} H_0^{(1)}(k_1|x-y|)$ satisfies

$$\Phi(k_1, x, y) = \frac{1}{2\pi} \int_1^{\infty} \frac{1}{\sqrt{t^2-1}} e^{\mathbf{i}k_1|x-y|t} dt.$$

Thus by (2.11) and Lemma 2.1, Lemma 6.4, we know that, for $x_2 > 0$,

$$\tilde{G}(x, y) = \Phi_1(1; z_1, z_2') - \Phi_1(1; z_1, z_2) + \Phi_1(h_1; z_1, z_2), \tag{6.19}$$

where z_1, z_2 are given in (6.17), $z_2' = \tilde{x}_2 - y_2$, and

$$h_1(t) = \mathrm{Re}\left(\frac{2\mu_1}{\mu_1 + \mu_2}\right). \tag{6.20}$$

Since $\mathrm{Im}\,(\mu_1^2) = \mathrm{Im}\,(\mu_2^2)$, by (2.17), $\mathrm{sgn}(\mathrm{Im}\,\mu_1) = \mathrm{sgn}(\mathrm{Im}\,\mu_2)$. Recalling the convention of choosing the analytic branch of $\sqrt{z}$, we know $\mathrm{Re}\,\mu_1 \geq 0, \mathrm{Re}\,\mu_2 \geq 0$. It follows that

$$\left|\frac{\mu_1}{\mu_1 + \mu_2}\right| \leq 1, \tag{6.21}$$

which yields $|h_1(t)| \leq 2$. Then using (6.19), Lemma 6.4, and (6.18), we have

$$|\tilde{G}(x, y)| \leq Ce^{-k_1\gamma_0\bar{\sigma}}, \quad \text{for } x_2 > 0.$$

Now we consider the case $x \in \mathbb{R}^2_-$. By (2.11), Lemma 2.1, and Lemma 6.4, we know that

$$\tilde{G}(x, y) = \Phi_1(h_2; z_1, z_2), \tag{6.22}$$

where

$$h_2 := \mathrm{Re}\left(\frac{2\mu_1}{\mu_1 + \mu_2} e^{\mathbf{i}(\mu_1 - \mu_2)(\tilde{x}_2 - y_2)} e^{\mathbf{i}y_2(\mu_1 - \mu_2)}\right).$$

Let $\mu_j = p_j + \mathbf{i}q_j$ with $p_j, q_j \in \mathbb{R}$, $j = 1, 2$. By the remark after this lemma we can use Lemma 6.3 and (6.21) to obtain

$$|h_2(t)| \leq 2e^{|q_1 - q_2||y_2|}.$$

Since q_1 and q_2 have the same sign, we deduce that

$$|q_1 - q_2| \leq |\mu_1 - \mu_2| = \frac{k_2^2 - k_1^2}{|\mu_1 + \mu_2|} \leq \frac{k_2^2 - k_1^2}{|q_1 + q_2|} \leq \frac{k_2^2 - k_1^2}{|q_1 - q_2|}.$$

Then

$$|h_2(t)| \leq e^{\sqrt{k_2^2 - k_1^2}|y_2|} \leq e^{L_2\sqrt{k_2^2 - k_1^2}/2}, \quad \forall y \in \Gamma_2.$$

This completes the proof by (6.22) and Lemma 6.4. □

To estimate the derivatives of the modified Green function, we need to estimate the lower bound of $\mu_2(\xi)$ for all ξ in (6.2) with z_1, z_2 in (6.17). We distinguish several cases.

1°) If $b_1 = 0$, we have $|x_1| \le L_1/2$. Then (H2) and the fact $x \in \Gamma_2$ indicate that $b_2 = \bar{\sigma}$. It follows from $\gamma_0 \bar{\sigma} \ge \min(d_1, d_2 + L_2/2)$ that $\bar{\sigma} \ge \sqrt{(L_1 + d_1)^2 + (L_2 + d_2)^2}$. Thus

$$|z_2| \ge \bar{\sigma} \ge d_1 + L_1 \ge |x_1 - y_1| = |z_1|, \quad \forall x \in \Gamma_2, y \in \Gamma_1.$$

Since $|\beta| \le 1/2$ by Cauchy-Schwarz inequality, we have $f_2(t) \ge 0$ for any $t \in [1, \infty)$ and consequently we deduce from (6.5) that

$$|\mu_2| \ge \sqrt{\mathrm{Re}\,(k_2^2 - \xi^2)} \ge \sqrt{k_2^2 - \frac{1}{2} k_1^2} \ge \frac{1}{\sqrt{2}} k_2. \tag{6.23}$$

2°) If $b_1 > 0$ and $b_2 > 0$, then

$$\text{either } |x_1| = L_1/2 + d_1, |x_2| \ge L_2/2 \ \text{ or } |x_2| = L_2/2 + d_2, |x_1| \ge L_1/2.$$

Thus $a_2 = |x_2| + |y_2| \ge L_2/2$. If $|x_1| = L_1/2 + d_1$, $a_1 = |x_1 - y_1| \ge d_1$ for any $y \in \Gamma_1$. In the other case, $|x_2| = L_2/2 + d_2$ and hence $b_2 = \bar{\sigma}$. If $|z_2| \le |z_1|$, then $a_2^2 + b_2^2 \le a_1^2 + b_1^2$ and consequently $a_1^2 \ge a_2^2 + b_2^2 - b_1^2 = a_2^2 + \bar{\sigma}^2 - b_1^2 \ge a_2^2$. That is $a_1 \ge a_2 \ge L_2/2$. From (6.8) we have

$$|\mu_2| \ge (k_2^4 - k_2^2 k_1^2)^{1/4} \frac{\sqrt{\min(d_1, L_2/2) L_2}}{|\rho|}.$$

On the other hand, if $|z_2| \ge |z_1|$, we obtain as in (6.23) that $|\mu_2| \ge \frac{1}{\sqrt{2}} k_2$. In summary we have

$$|\mu_2| \ge \frac{1}{\sqrt{2}} k_2 \quad \text{or} \quad |\mu_2| \ge (k_2^4 - k_2^2 k_1^2)^{1/4} \frac{\sqrt{\min(d_1, L_2/2) L_2}}{|\rho|}. \tag{6.24}$$

The trickiest case of $b_2 = 0$ is the objective of next lemma.

Lemma 6.6. *Suppose $b_2 = 0$ and $b_1^2 + b_2^2 \ge a_1^2 + a_2^2$. Define*

$$R = \sqrt{\frac{(t - t_0)^2}{2(t_0^2 - 1)^2} + \frac{a_2^2}{b_1^2}},$$

where $t_0 = k_2/k_1$. Then

$$|\mu_2| \ge \sqrt{k_2^2 - k_1^2} \frac{\sqrt{a_1 b_1}}{|\rho|} \min\left(1, \sqrt{R}\right), \quad \forall\, t \in [1, \infty).$$

Proof. For $b_2 = 0$, we note that (6.8) becomes

$$k_1^{-4} |\rho|^4 |\mu_2(t)|^4 = 4(t_0^4 - t_0^2) a_1^2 a_2^2 + \frac{1}{4} \left(|z_1|^2 + |z_2|^2\right)^2 \left(f_2(t) - M\right)^2. \tag{6.25}$$

By (6.5) we have

$$f_2(t) - M = 2(t^2 - t_0^2 - 2\beta t t') + 2(2t_0^2 - 1) \frac{|z_2|^2}{|z_1|^2 + |z_2|^2}.$$

Note the elementary inequality

$$(X + B)^2 \ge (1 - \varepsilon) X^2 + (1 - \varepsilon^{-1}) B^2, \quad \forall\, \varepsilon > 0. \tag{6.26}$$

By taking $X = t^2 - t_0^2 - 2\beta t t'$, $B = \dfrac{(2t_0^2-1)|z_2|^2}{|z_1|^2+|z_2|^2}$, we know from (6.26) that

$$\frac{1}{4}\left(f_2(t)-M\right)^2 \geq (1-\varepsilon)(t^2-t_0^2-2\beta t t')^2 + (1-\varepsilon^{-1})(2t_0^2-1)^2\frac{|z_2|^4}{\left(|z_1|^2+|z_2|^2\right)^2}.$$

Again using (6.26) with $\varepsilon = 1/2$, we have, for $1 \leq t \leq t_0$, that

$$(t^2-t_0^2-2\beta t t')^2 \geq \frac{1}{2}(t^2-t_0^2)^2 - 4\beta^2 t^2 t'^2 \geq (t-t_0)^2 - \frac{4a_2^2 b_1^2}{(|z_1|^2+|z_2|^2)^2}(t_0^4-t_0^2).$$

Now since $b_2 = 0$, $\beta \leq 0$ by (6.6). The above inequality is also valid for $t \geq t_0 \geq 1$ because $t^2 - t_0^2 - 2\beta t t' \geq t^2 - t_0^2 \geq t - t_0 \geq 0$ for $t \geq t_0$.

Using (6.25) we have

$$\begin{aligned} k_1^{-4}|\rho|^4|\mu_2|^4 \geq{}& (1-\varepsilon)(|z_1|^2+|z_2|^2)^2(t-t_0)^2 + (1-\varepsilon^{-1})(2t_0^2-1)^2 a_2^4 \\ &+4a_2^2(t_0^4-t_0^2)\left[a_1^2-(1-\varepsilon)b_1^2\right]. \end{aligned}$$

By the assumption $b_1^2+b_2^2 \geq a_1^2+a_2^2$ and $b_2 = 0$, we have $b_1^2 \geq a_1^2$. Therefore we can set $\varepsilon = 1 - a_1^2/(2b_1^2) > 0$ in above inequality to obtain

$$k_1^{-4}|\rho|^4|\mu_2|^4 \geq \frac{a_1^2}{2b_1^2}(|z_1|^2+|z_2|^2)^2(t-t_0)^2 + a_1^2 a_2^2\left[2(t_0^4-t_0^2) - \frac{a_2^2}{b_1^2}(2t_0^2-1)^2\right],$$

where we have used the fact that $1-\varepsilon^{-1} = -a_1^2/(2b_1^2-a_1^2) \geq -a_1^2/b_1^2$.

If $t_0^4 - t_0^2 \geq \dfrac{a_2^2}{b_1^2}(2t_0^2-1)^2$, since $|z_1|^2+|z_2|^2 \geq b_1^2$, we have

$$k_1^{-4}|\rho|^4|\mu_2|^4 \geq \frac{1}{2}a_1^2 b_1^2(t-t_0)^2 + a_1^2 a_2^2(t_0^4-t_0^2) = a_1^2 b_1^2(t_0^2-1)^2 R^2.$$

If $t_0^4 - t_0^2 \leq \dfrac{a_2^2}{b_1^2}(2t_0^2-1)^2$ which is equivalent to $a_2^2 > \dfrac{t_0^4-t_0^2}{(2t_0^2-1)^2}\, b_1^2$, we deduce from (6.25) that

$$k_1^{-4}|\rho|^4|\mu_2(t)|^4 \geq 4(t_0^4-t_0^2)a_1^2\,\frac{(t_0^4-t_0^2)^2}{(2t_0^2-1)^2}\,b_1^2 \geq (t_0^2-1)^2 a_1^2 b_1^2. \tag{6.27}$$

This completes the proof. ☐

Lemma 6.7. *Let (H2) and $\gamma_0\bar\sigma \geq \max(k_1^{-1}, \min(d_1, d_2+L_2/2))$ be satisfied. There exists a constant C depending only on γ_0, k_2/k_1, L_2/L_1 but independent of k_j, L_j, and d_j, $j = 1, 2$, such that, for any $x \in \Gamma_2$ and $y \in \Gamma_1$,*

$$\left|\frac{\partial \tilde G(x,y)}{\partial y_j}\right| \leq C\gamma_1 k_1\left(1+\frac{1}{k_1 L_1}\right)e^{-k_1\gamma_0\bar\sigma}, \tag{6.28}$$

$$\left|\frac{\partial \tilde G(x,y)}{\partial x_j}\right| \leq C\gamma_1 k_1 \alpha_m\left(1+\frac{1}{k_1 L_1}\right)e^{-k_1\gamma_0\bar\sigma}. \tag{6.29}$$

Here $\alpha_m = \max_{x\in\Omega_{\rm PML}}(|\alpha_1(x_1)|, |\alpha_2(x_2)|)$, γ_1 is defined in Lemma 6.5.

Proof. By the symmetry of the Green function $G(x,y)$ we know that $\dfrac{\partial G(\tilde{x},y)}{\partial x_j} = -\alpha(x_j)\dfrac{\partial G(\tilde{x},y)}{\partial y_j}$. Thus we only need to prove (6.28) which will be given only for $x, y \in \mathbb{R}^2_+$. The proof for other cases is similar.

In view of (6.19), we have, for any $x, y \in \mathbb{R}^2_+$,

$$\left|\frac{\partial \tilde{G}(x,y)}{\partial y_j}\right| \le \left|\frac{\partial \Phi_1}{\partial z_j}(1; z_1, z_2')\right| + \left|\frac{\partial \Phi_1}{\partial z_j}(1; z_1, z_2)\right| + \left|\frac{\partial \Phi_1}{\partial z_j}(h_1; z_1, z_2)\right|.$$

Since the estimation for $\dfrac{\partial \Phi_1}{\partial z_j}(1; z_1, z_2')$ and $\dfrac{\partial \Phi_1}{\partial z_j}(1; z_1, z_2)$ is simpler, we consider only $\dfrac{\partial \Phi_1}{\partial z_j}(h_1; z_1, z_2)$ in the sequel.

From the definition of $\Phi_1(h_1; z_1, z_2)$ in Lemma 6.4 and $\partial\rho/\partial z_j = z_j/\rho$ we know that

$$\frac{\partial \Phi_1}{\partial z_j}(h_1; z_1, z_2) = \Phi_1\left(\frac{\partial h_1}{\partial z_j} + \mathbf{i}k_1 h_1 \frac{z_j}{\rho} t; z_1, z_2\right), j = 1, 2. \tag{6.30}$$

Since the remark below Lemma 6.5 we know that the assumption of the lemma implies $b_1^2 + b_2^2 \ge a_1^2 + a_2^2$, by (6.18) and the fact that $|b_j| \le \bar{\sigma}$ for $x \in \Gamma_2, y \in \Gamma_1$, we know that

$$\left|\frac{z_j}{\rho}\right| \le \frac{\sqrt{a_j^2 + b_j^2}}{\gamma_0 \bar{\sigma}} \le \frac{\sqrt{3}}{\gamma_0}. \tag{6.31}$$

Recall Lemma 6.2 for the expressions of ξ and μ_1. There exists a constant depending only on γ_0 but independent of k_j, d_j, L_j, $j = 1, 2$, such that

$$\left|\frac{\partial \xi}{\partial z_j}\right| + \left|\frac{\partial \mu_1}{\partial z_j}\right| \le C\frac{k_1 t}{|\rho|^j}, \quad j = 0, 1, 2. \tag{6.32}$$

By the chain rule $\dfrac{\partial \mu_2}{\partial z_j} = \dfrac{\mu_1}{\mu_2}\dfrac{\partial \mu_1}{\partial z_j}$, we deduce by direct calculation and using (6.32) that

$$\left|\frac{\partial h_1}{\partial z_j}\right| \le C\left|\frac{\mu_1 - \mu_2}{\mu_2(\mu_1 + \mu_2)}\right|\left|\frac{\partial \mu_1}{\partial z_j}\right| \le C\,\frac{k_1 t}{|\rho||\mu_2|}, \tag{6.33}$$

where we have used $\left|\dfrac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\right| \le 1$. Then by using (6.30)-(6.31) and Lemma 6.4 we have that

$$\left|\frac{\partial \Phi_1}{\partial z_j}(h_1; z_1, z_2)\right| \le C\, k_1 e^{-k_1\gamma_0\bar{\sigma}}\left(1 + K_1\right), \qquad K_1 = \int_1^\infty \frac{te^{-t}}{|\rho||\mu_2|\sqrt{t^2-1}}\mathrm{d}t.$$

To estimate K_1, notice that under the assumption $\gamma_0\bar{\sigma} \ge \min(d_1, d_2 + L_2/2)$, $\bar{\sigma} \ge \sqrt{(L_1+d_1)^2 + (L_2+d_2)^2}$. Thus $|\rho| \ge \gamma_0\bar{\sigma} \ge CL_1$. Now by (6.23)-(6.24) and Lemma 6.6 we have

$$|\rho||\mu_2| \ge Ck_1 L_1 \min(1, \sqrt{|t - t_0|}),$$

which implies easily $|K_1| \le C/(k_1 L_1)$. Thus

$$\left|\frac{\partial \Phi_1}{\partial z_j}(h_1; z_1, z_2)\right| \le C\, k_1 \left(1 + \frac{1}{k_1 L_1}\right) e^{-k_1 \gamma_0 \bar{\sigma}}.$$

This completes the proof. ☐

LEMMA 6.8. *Let (H2) and $\gamma_0 \bar{\sigma} \ge \max(k_1^{-1}, \min(d_1, d_2 + L_2/2))$ be satisfied. There exists a constant C depending only on γ_0, k_2/k_1, L_2/L_1 but independent of k_j, L_j, and d_j, $j = 1, 2$, such that, for any $x \in \Gamma_2$ and $1 \le p < 4/3$,*

$$\left\|\frac{\partial^2 \tilde{G}(x, \cdot)}{\partial x_i \partial y_j}\right\|_{L^p(\Gamma_1)} \le C \gamma_1 k_1^2 L_1^{1/p} \left(1 + \frac{1}{k_1 L_1}\right)^2 \left(1 + \frac{\bar{\sigma}}{L_1}\right)^2 \alpha_m e^{-k_1 \gamma_0 \bar{\sigma}}.$$

Proof. We start by estimating $\dfrac{\partial^2 \tilde{G}}{\partial x_i \partial y_j}(x, y)$ for any $x \in \Gamma_2$ and $y \in \Gamma_1$. We first consider the case when $x, y \in \mathbb{R}^2_+$. In view of (6.19), we have, for any $x, y \in \mathbb{R}^2_+$,

$$\left|\frac{\partial^2 \tilde{G}(x, y)}{\partial x_i \partial y_j}\right| \le |\alpha_i| \left(\left|\frac{\partial^2 \Phi_1}{\partial z_i \partial z_j}(1; z_1, z_2')\right| + \left|\frac{\partial^2 \Phi_1}{\partial z_i \partial z_j}(1; z_1, z_2)\right| + \left|\frac{\partial^2 \Phi_1}{\partial z_i \partial z_j}(h_1; z_1, z_2)\right|\right).$$

Since the estimation for $\dfrac{\partial^2 \Phi_1}{\partial z_i \partial z_j}(1; z_1, z_2')$ and $\dfrac{\partial^2 \Phi_1}{\partial z_i \partial z_j}(1; z_1, z_2)$ is simpler, we only consider $\dfrac{\partial^2 \Phi_1}{\partial z_i \partial z_j}(h_1; z_1, z_2)$ in the sequel.

It is easy to see that

$$\begin{aligned}\frac{\partial^2 \Phi_1}{\partial z_i \partial z_j}(h_1; z_1, z_2) = {} & \Phi_1\left(\frac{\partial^2 h_1}{\partial z_i \partial z_j} + \mathbf{i} k_1 t \frac{\partial}{\partial z_i}\left(h_1 \frac{z_j}{\rho}\right); z_1, z_2\right) \\ & + \Phi_1\left(\left(\frac{\partial h_1}{\partial z_j} + \mathbf{i} k_1 h_1 \frac{z_j}{\rho} t\right) \mathbf{i} k_1 t \frac{z_i}{\rho}; z_1, z_2\right).\end{aligned} \tag{6.34}$$

By the chain rule $\dfrac{\partial \mu_2}{\partial z_j} = \dfrac{\mu_1}{\mu_2} \dfrac{\partial \mu_1}{\partial z_j}$ and (6.32), there exists a constant C depending only on γ_0, k_2/k_1 but independent of k_j and d_j, $j = 1, 2$, such that

$$\begin{aligned}\left|\frac{\partial^2 h_1}{\partial z_i \partial z_j}\right| &\le \left|\frac{\mu_1 - \mu_2}{\mu_2(\mu_1 + \mu_2)}\right| \left|\frac{\partial^2 \mu_1}{\partial z_i \partial z_j}\right| + \left|\frac{2\mu_2 + \mu_1}{\mu_2^3}\right| \left|\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\right| \left|\frac{\partial \mu_1}{\partial z_i}\right| \left|\frac{\partial \mu_1}{\partial z_j}\right| \\ &\le C \frac{k_1 t}{|\rho|^2 |\mu_2|} + C \frac{k_1^3 t^3}{|\rho|^2 |\mu_2|^3},\end{aligned}$$

where we have used $\left|\dfrac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\right| \le 1$ and $|\mu_2| \le k_2 + |\mu_1| \le C k_1 t$ for $t \ge 1$. By the assumption $\gamma_0 \bar{\sigma} \ge \min(d_1, d_2 + L_2/2)$, $k_1 |\rho| \ge k_1 \mathrm{Im}\, \rho \ge k_1 \gamma_0 \bar{\sigma} \ge 1$, we then have

$$\left|\frac{\partial^2 h_1}{\partial z_i \partial z_j}\right| \le C \frac{k_1^2 t}{|\rho| |\mu_2|} + C \frac{k_1^3 t^3 |\rho|}{|\rho|^3 |\mu_2|^3}.$$

Similarly, by using (6.33) we can obtain

$$\begin{aligned}\left|\mathbf{i} k_1 t \frac{\partial}{\partial z_i}\left(h_1 \frac{z_j}{\rho}\right)\right| &\le C k_1^2 t^2 \frac{1}{|\rho| |\mu_2|}, \\ \left|\left(\frac{\partial h_1}{\partial z_j} + \mathbf{i} k_1 h_1 \frac{z_j}{\rho} t\right) \mathbf{i} k_1 t \frac{z_i}{\rho}\right| &\le C k_1^2 t^2 \left(1 + \frac{1}{|\rho| |\mu_2|}\right).\end{aligned}$$

By Lemma 6.4,

$$\left|\frac{\partial^2\Phi_1}{\partial z_i\partial z_j}(h_1;z_1,z_2)\right| \le Ck_1^2e^{-k_1\gamma_0\bar\sigma}(1+K_1+K_2+K_3), \tag{6.35}$$

where K_1 is defined in the proof of last lemma, and

$$K_2 := \int_1^\infty \frac{t^2e^{-t}}{|\rho||\mu_2|\sqrt{t^2-1}}\mathrm{d}t, \qquad K_3 := \int_1^\infty \frac{k_1t^3|\rho|e^{-t}}{|\rho|^3|\mu_2|^3\sqrt{t^2-1}}\mathrm{d}t. \tag{6.36}$$

The estimate (6.35) is also valid for the other cases when x and y are not both in $\mathbb{R}^2_+$ if the constant C is replaced by $C\gamma_1$.

From the proof of last lemma we know that $|K_1| \le C/(k_1L_1)$. Similarly we can prove $|K_2| \le C/(k_1L_1)$. It remains to estimate K_3. For $|x_2| > L_2/2$ we have $b_2 > 0$ and by (6.23)-(6.24) we know that $|\rho||\mu_2| \ge Ck_1L_1$ and thus $K_3 \le Ck_1|\rho|/(k_1L_1)^3$ which yields

$$\left|\int_{\Gamma_1}|K_3|^p ds(y)\right|^{1/p} \le C\frac{1}{k_1^2L_1^2}\left(\frac{\bar\sigma}{L_1}\right)L_1^{1/p}.$$

For $|x_2| \le L_2/2$ we know that $b_2 = 0$, $b_1 = \bar\sigma$, and $a_1 = |x_1-y_1| \ge d_1 \ge \gamma_0^{-1}L_1$, $a_2 = |x_2|^2+y_2^2 \ge y_2^2$. Thus by Lemma 6.6,

$$|\rho||\mu_2| \ge Ck_1\sqrt{L_1\bar\sigma}\min(1,\sqrt{r}), \quad r=\sqrt{(t-t_0)^2+y_2^2/\bar\sigma^2}.$$

Thus, since $|\rho| \le C\bar\sigma$, $\bar\sigma \ge \sqrt{(L_1+d_1)^2+(L_2+d_2)^2}$,

$$\left|\int_{\Gamma_1}|K_3|^p ds(y)\right|^{1/p} \le C\frac{1}{k_1^2L_1^2}\left(\int_{\Gamma_1}\left|\int_1^\infty \frac{t^3e^{-t}}{\min(1,r^{3/2})\sqrt{t^2-1}}\mathrm{d}t\right|^p \mathrm{d}s(y)\right)^{1/p}.$$

Notice that

$$r^{-3/2} \le \left(\frac{2}{t_0-1}\right)^{3/2} \le C, \quad \forall\, t\in\left[1,\frac{t_0+1}{2}\right]\cup\left[\frac{3t_0-1}{2},\infty\right).$$

An application of Holder's inequality yields that

$$\begin{aligned}\int_{\Gamma_1}\left|\int_1^\infty \frac{t^3e^{-t}}{r^{3/2}\sqrt{t^2-1}}\mathrm{d}t\right|^p \mathrm{d}s(y) &\le CL_1 + C\int_{\Gamma_1}\left|\int_{\frac{t_0+1}{2}}^{\frac{3t_0-1}{2}} \frac{t^3e^{-t}}{r^{3/2}\sqrt{t^2-1}}\mathrm{d}t\right|^p \mathrm{d}s(y)\\ &\le CL_1 + C\int_{\Gamma_1}\int_{\frac{t_0+1}{2}}^{\frac{3t_0-1}{2}} r^{-\frac{3p}{2}}\mathrm{d}t\mathrm{d}s(y).\end{aligned}$$

On the part of the boundary of Γ_1 where $|y_2| = L_2/2$, we have $r^{-1} \le \bar\sigma/L_2$, thus

$$\begin{aligned}\int_{\Gamma_1}\left|\int_1^\infty \frac{t^3e^{-t}}{r^{3/2}\sqrt{t^2-1}}\mathrm{d}t\right|^p \mathrm{d}s(y) &\le CL_1 + CL_1\left(\frac{\bar\sigma}{L_2}\right)^{\frac{3p}{2}} + C\int_0^{\frac{L_2}{2}}\int_{\frac{t_0+1}{2}}^{\frac{3t_0-1}{2}} r^{-\frac{3p}{2}}\mathrm{d}t\mathrm{d}y_2\\ &\le CL_1 + CL_1\left(\frac{\bar\sigma}{L_2}\right)^2 + C\bar\sigma, \quad \text{if } p<4/3.\end{aligned}$$

This completes the proof. $\square$

7. The convergence. In this section, we are going to show the exponential convergence of the solution of (4.6)–(4.8) to the solution of the original scattering problem (1.1)–(1.5). We first introduce the approximate Dirichlet-to-Neumann operator $\hat{T}$: $H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$ associated with the UPML problem. Given $f \in H^{1/2}(\Gamma_1)$, let $\hat{T}f = \left.\dfrac{\partial \phi}{\partial \mathbf{n}}\right|_{\Gamma_1}$, where ϕ is the solution of the following Dirichlet problem in the PML layer:

$$\nabla \cdot (A\nabla\phi) + \alpha_1\alpha_2 k^2\phi = 0 \qquad \text{in } \Omega_{\mathrm{PML}}, \tag{7.1}$$

$$[\phi]_\Sigma = \left[\frac{\partial \phi}{\partial x_2}\right]_\Sigma = 0 \qquad \text{on } \Sigma \cap \Omega_{\mathrm{PML}}, \tag{7.2}$$

$$\phi = f \quad \text{on } \Gamma_1, \qquad \phi = 0 \qquad \text{on } \Gamma_2. \tag{7.3}$$

From (5.5) we know that (7.1)–(7.3) has a unique solution and thus $\hat{T}$ is well-defined. Then (4.6)–(4.8) is reduced to the following weak formulation: Find $\hat{u} \in H^1(\Omega_2)$ such that $\hat{u} = g$ on Γ_D and

$$\hat{a}(\hat{u}, v) = 0, \qquad \forall\, v \in H^1_{\Gamma_D}(\Omega_1), \tag{7.4}$$

where the sesquilinear form $\hat{a} : H^1(\Omega_1) \times H^1(\Omega_1) \to \mathbb{C}$ is defined by

$$\hat{a}(\varphi, \psi) = \int_{\Omega_1} (\nabla\varphi \cdot \nabla\bar{\psi} - k^2\varphi\bar{\psi})\mathrm{d}x - \langle \hat{T}\varphi, \psi\rangle_{\Gamma_1}, \qquad \forall\, \varphi, \psi \in H^1(\Omega_1).$$

In view of (3.9) and (7.4), we are in the position to estimate the error $Tf - \hat{T}f$ for any $f \in H^{1/2}(\Gamma_1)$. It is obvious that

$$Tf - \hat{T}f = \left.\frac{\partial w}{\partial \mathbf{n}}\right|_{\Gamma_1}, \tag{7.5}$$

where $w \in H^1(\Omega_{\mathrm{PML}})$ is the solution of the Dirichlet problem (5.1)–(5.3) with $q = \mathbb{E}(f)$.

For any $f \in H^{1/2}(\Gamma_1)$, denote

$$X(f) := \left\{\zeta \in H^1(\Omega_{\mathrm{PML}}) : \ \zeta = 0 \ \text{ on } \ \Gamma_1, \ \zeta = \mathbb{E}(f) \ \text{ on } \ \Gamma_2\right\}.$$

By Lemma 5.2 we know that

$$\|Tf - \hat{T}f\|_{H^{-1/2}(\Gamma_1)} \le (1 + \hat{C}^{-1}) \inf_{\zeta \in X(f)} \|\zeta\|_{*,\Omega_{\mathrm{PML}}}. \tag{7.6}$$

We introduce the weighted $H^{1/2}(\Gamma_j)$–norm, $j = 1, 2$,

$$\|v\|_{H^{1/2}(\Gamma_j)} := \left(|\Gamma_j|^{-1}\|v\|^2_{L^2(\Gamma_j)} + |v|^2_{\frac{1}{2},\Gamma_j}\right)^{1/2}, \qquad \forall\, v \in H^{1/2}(\Gamma_j), \tag{7.7}$$

where

$$|v|^2_{\frac{1}{2},\Gamma_j} := \int_{\Gamma_j}\int_{\Gamma_j} \frac{|v(x) - v(x')|^2}{|x - x'|^2}\mathrm{d}s(x)\,\mathrm{d}s(x'). \tag{7.8}$$

LEMMA 7.1. *Let (H2) and $\gamma_0\bar{\sigma} \ge \max(k_1^{-1}, \min(d_1, d_2 + L_2/2))$ be satisfied. There exists a constant C depending only on γ_0, k_2/k_1, L_2/L_1 but independent of k_j, L_j, and d_j, $j = 1, 2$, such that, for any $f \in H^{1/2}(\Gamma_1)$,*

$$\inf_{\zeta \in X(f)} \|\zeta\|_{*,\Omega_{\mathrm{PML}}} \le C\gamma_1 (1 + k_1L_1)^3 \alpha_m^3 \left(1 + \frac{\bar{\sigma}}{L_1}\right)^2 \|f\|_{H^{1/2}(\Gamma_1)}.$$

Proof. By definition

$$\begin{aligned}\|\zeta\|_{*,\Omega_{\rm PML}}^2 &= \|\, A\nabla\zeta\,\|_{L^2(\Omega_{\rm PML})}^2 + \|\, k\alpha_1\alpha_2\zeta\,\|_{L^2(\Omega_{\rm PML})}^2 \\ &\le C\,(1+k_1L_1)^2\,\alpha_m^4\left(\|\,\nabla\zeta\,\|_{L^2(\Omega_{\rm PML})}^2 + |\Gamma_1|^{-2}\|\,\zeta\,\|_{L^2(\Omega_{\rm PML})}^2\right).\end{aligned}$$

which by the trace inequality implies that

$$\inf_{\zeta\in X}\|\zeta\|_{*,\Omega_{\rm PML}} \le C\,(1+k_1L_1)\,\alpha_m^2\|\mathbb{E}(f)\|_{H^{1/2}(\Gamma_2)}.$$

The definition of the $H^{1/2}$–norm shows that

$$\inf_{\zeta\in X}\|\zeta\|_{*,\Omega_{\rm PML}} \le C\,(1+kL_1)\,\alpha_m^2\left(\|\mathbb{E}(f)\|_{L^\infty(\Gamma_2)} + |\Gamma_1|\,|\mathbb{E}(f)|_{W^{1,\infty}(\Gamma_2)}\right).$$

On the other hand,

$$\begin{aligned}|\mathbb{E}(f)| &= |-\tilde\Psi_{\rm SL}(\lambda)+\tilde\Psi_{\rm DL}(f)| \\ &= \Big|-\int_{\Gamma_1}\tilde G(x,y)\lambda(y){\rm d}s(y) + \int_{\Gamma_1}\frac{\partial\tilde G(x,y)}{\partial\mathbf{n}_1(y)}f(y){\rm d}s(y)\Big| \\ &\le C\|\tilde G(x,\cdot)\|_{H^{1/2}(\Gamma_1)}\|\,\lambda\,\|_{H^{-1/2}(\Gamma_1)} + \|\,\partial\tilde G(x,\cdot)/\partial\mathbf{n}_1(y)\,\|_{L^\infty(\Gamma_1)}\|\,f\,\|_{L^1(\Gamma_1)}.\end{aligned}$$

Since $\|\,\lambda\,\|_{H^{-1/2}(\Gamma_1)} \le C\|\,f\,\|_{H^{1/2}(\Gamma_1)}$, we then obtain

$$|\mathbb{E}(f)| \le C(1+k_1L_1)\max_{y\in\Gamma_1}\left(|\tilde G(x,\cdot)| + L_1|\nabla_y\tilde G(x,\cdot)|\right)\|\,f\,\|_{H^{1/2}(\Gamma_1)},$$

which implies by Lemmas 6.5 and Lemma 6.7 that

$$|\mathbb{E}(f)| \le C\gamma_1(1+k_1L_1)^2e^{-k_1\gamma_0\bar\sigma}.$$

Similarly, for any $1<p<4/3$, we know from the embedding theorem that $W^{1,p}(\Gamma_1)$ is embedded to $H^{1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$ is embedded to $L^{p'}(\Gamma_1)$, where $1/p+1/p'=1$. Then

$$\begin{aligned}|\nabla_x\mathbb{E}(f)| &\le \|\nabla_x\tilde G(x,\cdot)\|_{H^{1/2}(\Gamma_1)}\|\lambda\|_{H^{-1/2}(\Gamma_1)} + \|\nabla_x\nabla_y\tilde G(x,\cdot)\|_{L^p(\Gamma_1)}\|f\|_{L^{1/p'}(\Gamma_1)} \\ &\le CL_1^{-1/p}(\|\nabla_x\tilde G(x,\cdot)\|_{L^p(\Gamma_1)} + L_1\|\nabla_x\nabla_y\tilde G(x,\cdot)\|_{L^p(\Gamma_1)})\|f\|_{H^{1/2}(\Gamma_1)}.\end{aligned}$$

This implies by using Lemmas 6.7-6.8 that

$$L_1|\nabla_x\mathbb{E}(f)| \le C\alpha_m(1+k_1L_1)^2\left(1+\frac{\bar\sigma}{L_1}\right)^2e^{-k_1\gamma_0\bar\sigma}\|f\|_{H^{1/2}(\Gamma_1)}.$$

This competes the proof. ∎

Now we are ready to present the main result of this paper.

THEOREM 7.2. *Let (H1)-(H2) and $\gamma_0\bar\sigma\ge\max(k_1^{-1},\min(d_1,d_2+L_2/2))$ be satisfied. Let u be the solution of* (1.1) – (1.5). *Then for sufficiently large $\bar\sigma$, the UPML problem* (4.6) – (4.8) *has a unique solution $\hat u$. Moreover, there exists a constant C depending only on γ_0, k_2/k_1, L_2/L_1 but independent of k_j, L_j, and d_j, $j=1,2$, such that*

$$\|u-\hat u\|_{H^1(\Omega_1)} \le C(1+\hat C^{-1})\gamma_1(1+k_1L_1)^3\alpha_m^3\left(1+\frac{\bar\sigma}{L_1}\right)^2e^{-k_1\gamma_0\bar\sigma}\|\hat u\|_{H^{1/2}(\Gamma_1)}. \quad (7.9)$$

Proof. We prove the estimate (7.9) first. Suppose the solution $\hat{u}$ of (4.6) – (4.8) exists. By (3.9) and (7.4), simple integration by parts implies

$$a(u-\hat{u},\varphi)=\hat{a}(\hat{u},\varphi)-a(\hat{u},\varphi)=\langle T\hat{u}-\hat{T}\hat{u},\varphi\rangle_{\Gamma_1},\qquad \forall\,\varphi\in H^1(\Omega_{\rm PML}).$$

Using (3.10) and Lemma 7.1, we obtain (7.9).

Now we turn to the well-posedness of the UPML problem. By the Fredholm alternative theorem we only need to show the uniqueness of the UPML problem (4.6) – (4.8). For that purpose we assume (4.6) – (4.8) has a solution $\hat{u}$ for $g=0$ in (4.8). By the uniqueness of the scattering problem we know that the corresponding scattering solution $u=0$ in Ω_1. Thus (7.9) implies

$$\|\hat{u}\|_{H^1(\Omega_1)}\le C\left(1+\hat{C}^{-1}\right)\gamma_1(1+k_1L_1)^3\alpha_m^3\left(1+\frac{\bar{\sigma}}{L_1}\right)^2e^{-k_1\gamma_0\bar{\sigma}}\|\hat{u}\|_{H^{1/2}(\Gamma_1)}.$$

Thus for sufficiently large $\bar{\sigma}$ we conclude that $\hat{u}=0$ on Ω_1. That $\hat{u}$ also vanishes in Ω_2 is a direct consequence of the unique continuation theorem (cf. e.g. Monk [20, Page 92]). Here we omit the details. ◻

8. Concluding remarks. In this paper we proved that the solution of the UPML problem converges exponentially to the solution of the Helmholtz scattering problem in layered media. The convergence can be realized by either enlarging the thickness of the PML layer or enlarging the PML absorbing coefficients. The proof is based the method of complex coordinate stretching and a new representation of the Green function which is essential for the estimate for the modified Green function. We will extend the results of this paper to design an adaptive UPML method and report the numerical examples in a forthcoming paper.

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