AN EFFICIENT EDDY CURRENT MODEL FOR NONLINEAR MAXWELL EQUATIONS WITH LAMINATED CONDUCTORS*

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Abstract. In this paper, we propose a new eddy current model for the nonlinear Maxwell equations with laminated conductors. Direct simulation of three-dimensional (3D) eddy currents in grain-oriented (GO) silicon steel laminations is very challenging since the coating film over each lamination is only several microns thick and the magnetic reluctivity is nonlinear and anisotropic. The system of GO silicon steel laminations has multiple sizes and the ratio of the largest scale to the smallest scale can amount to 10^6. The new model omits coating films and thus reduces the scale ratio by 2–3 orders of magnitude. It avoids very fine or very anisotropic mesh in coating films and can save computations greatly in computing 3D eddy currents. We establish the wellposedness of the new model and prove the convergence of the solution of the original problem to the solution of the new model as the thickness of coating films tends to zero. The new model is validated by finite element computations of an engineering benchmark problem — Team Workshop Problem 21^c–M1.

Key words. Nonlinear eddy current problem, Maxwell's equations, finite element method, GO silicon steel lamination, Team Workshop Problem 21

AMS subject classifications. 35Q60, 65N30, 78A25

1. Introduction.

We propose to study the following eddy current problem in magnetic and anisotropic materials

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\, \boldsymbol{E} = 0 \quad \text{in } \mathbb{R}^3, \qquad \text{(Farady's law)} \tag{1.1a}$$

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \quad \text{in } \mathbb{R}^3, \qquad \text{(Ampere's law)} \tag{1.1b}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic flux, $\boldsymbol{H}$ is the magnetic field, and $\boldsymbol{J}$ is the current density defined by:

$$\boldsymbol{J} = \begin{cases} \sigma \boldsymbol{E} & \text{in } \Omega_c, & \text{(conducting region)} \\ \boldsymbol{J}_s & \text{in } \mathbb{R}^3 \backslash \bar{\Omega}_c. & \text{(nonconducting region)} \end{cases} \tag{1.2}$$

Here $\sigma \geq 0$ is the electric conductivity, $\boldsymbol{J}_s$ is the source current density carried by some coils, and Ω_c denotes the conducting region. For magnetic and anisotropic materials, $\boldsymbol{B} = (B_1, B_2, B_3)$ is a nonlinear vector function of $\boldsymbol{H} = (H_1, H_2, H_3)$ in the form of $B_i = B_i(H_i)$, $i = 1, 2, 3$.

The eddy current problem is a quasi-static approximation of Maxwell's equations at very low frequency by neglecting the displacement currents in Ampere's law (see [2]). For linear eddy current problems, there are many interesting works in the literature on numerical methods (cf. e.g. [5, 9, 13, 15, 22]) and on the regularity of the solution (cf. e.g. [10]). But the mathematical theory and numerical analysis for nonlinear eddy current problems are still rare in the literature. In [4], Bachinger et al studied the numerical analysis of nonlinear multi-harmonic eddy current problems

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in isotropic materials. In this paper, we shall study the eddy current problem with nonlinear and anisotropic reluctivity.

GO silicon steel laminations are widely used in iron cores and shielding structures of large power transformers [7]. The complex structure is made of many laminated steel sheets and each sheet is only $0.18-0.35mm$ thick. Moreover, each steel sheet is coated with a layer of insulating film whose thickness is only $2-5\mu m$ so that the electric current can not flow into its neighboring sheets (see Figure 1.1). Usually the lamination stack has multi-scale sizes and the ratio of the largest scale to the smallest scale can amount to 10^6. Full 3D finite element modeling for (1.1) is extremely difficult because of extensive unknowns from meshing the laminations and the coating films. There are very few works on the computation of 3D eddy currents inside the laminations in the literature.

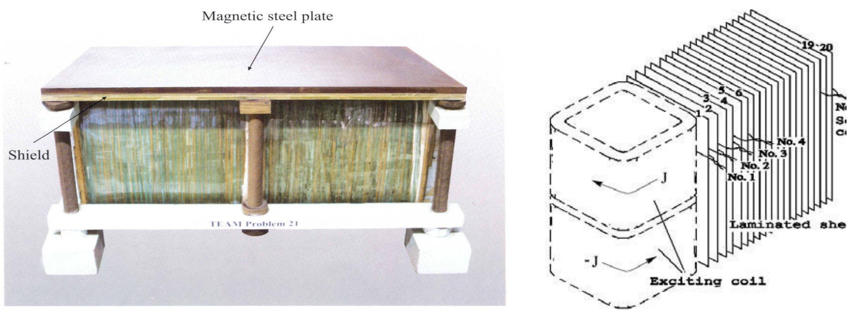

Fig. 1.1. *Left: the magnetic shield for protecting the magnetic plate. Right: the magnetic shield made of laminated steel sheets.*

In recent years, there are considerable papers devoted to developing efficient numerical methods for nonlinear eddy current problems in steel laminations in the engineering community. Among them, most works pay attention to effective reluctivities and conductivities of the lamination stack (cf. e.g. [6, 12, 14, 17, 19]). The main idea is to replace physical parameters with equivalent (or homogenized) parameters for Maxwell's equations. Since the effective conductivity is anisotropic and has zero value in the perpendicular direction to the lamination plane, the numerical eddy current is thus two-dimensional in the lamination stack. When the leakage magnetic flux is very strong and enters the lamination plane perpendicularly, for example, in the outer laminations of a large power transformer core, the eddy current loss induced there must be taken into account in electromagnetic design. It is preferable to accurately compute 3D eddy currents at least in a few laminations close to the source, that is, to use the zoned treatment for practical approaches (see Figure 1.2). In the 3D eddy current region, one usually has to mesh both the laminations and the coating films [8].

The main objective of this paper is to propose a new eddy current model for the Maxwell equations with laminated conductors. This model omits coating films from the lamination system and thus saves computations greatly in numerical solution. For the original model, the eddy current is confined in each steel sheet by the coating film. The treatment of this conservation property plays the key role in designing accurate numerical methods for computing 3D eddy currents. Without the coating film, the new model still conserves the eddy current inside each steel sheet, that is, the eddy current can not flow across the interface between neighboring steel sheets. Based on the magnetic potential $\boldsymbol{A}$, we established the following theories:

1. the existence and uniqueness of the solution of the new eddy current model,

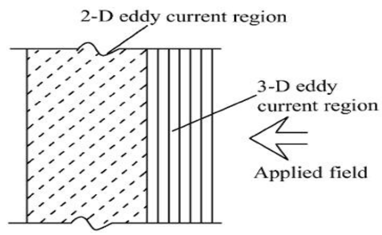

FIG. 1.2. *Zoned treatment of the lamination stack.*

2. the stability of the solution with respect to the source current,
3. and the convergence of the solution of the original problem to the solution of the new model as the thickness of coating films tends to zero.

The magnetic potential $\boldsymbol{A}$ is not unique and makes no physical sense in the nonconducting region. It is just $\sigma\boldsymbol{A}$ and $\mathbf{curl}\,\boldsymbol{A}$ that are important in electrical engineering and usually used in computing iron loss. We proved the convergence of $\sigma\boldsymbol{A}$ and $\mathbf{curl}\,\boldsymbol{A}$ as the thickness of the coating film tends to zero. To validate the new eddy current model numerically, we computed an engineering benchmark problem, Team Workshop Problem 21^c–M1 [7], by the hybrid of edge element method and nodal element method. The numerical results show good agreements with the experimental values and demonstrate our theory.

The layout of the paper is organized as follows. In section 2 we present some notation and Sobolev spaces used in this paper and study the $\boldsymbol{A}$-formulation of (1.1). In section 3 we propose a new eddy current model for laminated conductors by omitting coating films. In section 4 we prove the well-posedness of the new model. In section 5, we prove the convergence of the solution of the original problem to the solution of the new model. In section 6 we present a numerical experiment to validate the new eddy current model.

2. The $\boldsymbol{A}$-formulation of the eddy current problem. Let Ω be a truncated cube which encloses all inhomogeneities, such as coils and conductors. Let $L^2(\Omega)$ be the usual Hilbert space of square integrable functions equipped with the following inner product and norm:

$$(u,\, v) := \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \quad \text{and} \quad \|u\|_{L^2(\Omega)} := (u,\, u)^{1/2}.$$

Define $H^m(\Omega) := \{v \in L^2(\Omega) : D^\xi v \in L^2(\Omega), |\xi| \le m\}$ where ξ represents nonnegative triple index. Let $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ whose functions have zero traces on $\partial\Omega$. Throughout the paper we denote vector-valued quantities by boldface notation, such as $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^3$.

We define the spaces of functions having square integrable curl by

$$\begin{aligned}\boldsymbol{H}(\mathbf{curl},\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,:\, \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\},\\ \boldsymbol{H}_0(\mathbf{curl},\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\Omega) \,:\, \boldsymbol{n} \times \boldsymbol{v} = 0 \ \text{ on } \partial\Omega\}.\end{aligned}$$

which are equipped with the following inner product and norm

$$(\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{H}(\mathbf{curl},\Omega)} := (\boldsymbol{v},\boldsymbol{w}) + (\mathbf{curl}\,\boldsymbol{v},\mathbf{curl}\,\boldsymbol{w}), \quad \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} := \sqrt{(\boldsymbol{v},\boldsymbol{v})_{\boldsymbol{H}(\mathbf{curl},\Omega)}}\ .$$

Here $\boldsymbol{n}$ denotes the unit outer normal to $\partial\Omega$. We shall also use the spaces of functions having square integrable divergence

$$\boldsymbol{H}(\mathrm{div},\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \, : \, \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega)\},$$
$$\boldsymbol{H}_0(\mathrm{div},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \, : \, \boldsymbol{n}\cdot\boldsymbol{v} = 0 \ \text{ on } \partial\Omega\}.$$

which are equipped with the following inner product and norm

$$(\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{H}(\mathrm{div},\Omega)} := (\boldsymbol{v},\boldsymbol{w}) + (\mathrm{div}\,\boldsymbol{v}, \mathrm{div}\,\boldsymbol{w}), \quad \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} := \sqrt{(\boldsymbol{v},\boldsymbol{v})_{\boldsymbol{H}(\mathrm{div},\Omega)}}\,.$$

Throughout the paper, we make the following assumptions on the material parameters and the source current which are usually satisfied in electrical engineering:

(H1) The electric conductivity σ is piecewise constant and there exist two constants $\sigma_{\min}, \sigma_{\max}$ such that

$$0 < \sigma_{\min} \le \sigma \le \sigma_{\max} \quad \text{in } \ \Omega_c \qquad \text{and} \qquad \sigma \equiv 0 \quad \text{in } \ \Omega_{nc} := \Omega \setminus \overline{\Omega}_c,$$

(H2) Let $\boldsymbol{H} = (H_1, H_2, H_3)$ and $\boldsymbol{B} = (B_1, B_2, B_3)$ be the magnetic field and the magnetic flux respectively. Each H_i is a Lipschitz continuous function of B_i satisfying $H_i(0) = 0$ and $H_i(B_i) = \nu_0 B_i$ in Ω_{nc}. Moreover, there exist two constants $\nu_{\min}, \nu_{\max}$ such that

$$0 < \nu_{\min} \le H_i'(B_i) \le \nu_{\max} \qquad \text{a.e. in } \ \Omega, \quad i = 1,2,3.$$

(H3) The source current density satisfies

$$\boldsymbol{J}_s \in \boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega)) \qquad \text{and} \qquad \mathrm{div}\,\boldsymbol{J}_s = 0 \quad \text{in } \ \Omega.$$

In (H2), ν_0 is the magnetic reluctivity in the empty space and the nonlinear functions $H_i = H_i(B_i)$ are usually obtained by spline interpolations using experimental data. Figure 2.1 shows the BH-curves in two different directions of the GO silicon steel laminations in large power transformers [7].

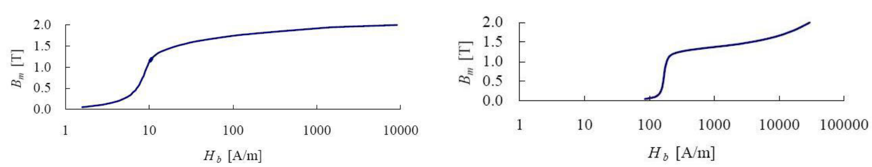

FIG. 2.1. *BH-curves in rolling (left) and transverse (right) directions of silicon steel laminations.*

Denote the boundary of Ω by $\Gamma = \partial\Omega$. We impose the initial and boundary conditions for (1.1) as follows

$$\boldsymbol{B}(\cdot,0) = 0 \quad \text{in } \ \Omega \qquad \text{and} \qquad \boldsymbol{E}\times\boldsymbol{n} = 0 \quad \text{on } \ \Gamma. \tag{2.1}$$

Then (1.1a) indicates that $\mathrm{div}\,\boldsymbol{B} = 0$ in Ω. There exists a magnetic potential $\boldsymbol{a}$ such that $\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{a}$ in Ω. Thus (1.1a) turns into

$$\mathbf{curl}\left(\frac{\partial \boldsymbol{a}}{\partial t} + \boldsymbol{E}\right) = 0 \quad \text{in } \ \Omega.$$

Thus there is a scalar electric potential p such that

$$\boldsymbol{E} + \frac{\partial \boldsymbol{a}}{\partial t} = -\nabla p \quad \text{in } \Omega.$$

Write $\psi(\cdot, t) = \int_0^t p(\cdot, s)\mathrm{d}s$ and set $\boldsymbol{A} = \boldsymbol{a} + \nabla\psi$. It follows that

$$\boldsymbol{E} = -\frac{\partial}{\partial t}(\boldsymbol{a} + \nabla\psi) = -\frac{\partial \boldsymbol{A}}{\partial t} \qquad \text{and} \qquad \boldsymbol{B} = \mathbf{curl}\,\boldsymbol{a} = \mathbf{curl}\,\boldsymbol{A}.$$

Substituting the identities into (1.1b) and using (2.1), we obtain the following initial boundary value problem

$$\sigma\frac{\partial \boldsymbol{A}}{\partial t} + \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl}\,\boldsymbol{A}) = \boldsymbol{J}_s \qquad \text{in } \Omega\times[0,T], \tag{2.2a}$$

$$\boldsymbol{A}\times\boldsymbol{n} = 0 \qquad \text{on } \Gamma\times[0,T], \tag{2.2b}$$

$$\boldsymbol{A}(\cdot,0) = 0 \qquad \text{in } \Omega_c, \tag{2.2c}$$

where $T > 0$ is the final time and $\boldsymbol{H} = \boldsymbol{H}(\mathbf{curl}\,\boldsymbol{A})$ is nonlinear with respect to $\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A}$ and usually defined by the BH-curves [7] (see Figure 2.1). We remark that (2.2) is understood in a distributional sense.

A weak formulation equivalent to (2.2) reads: Find $\boldsymbol{A} \in \boldsymbol{L}^2(0,T;\boldsymbol{H}_0(\mathbf{curl},\Omega))$ such that $\boldsymbol{A}(\cdot,0) = 0$ in Ω_c and

$$\int_\Omega \sigma\frac{\partial \boldsymbol{A}}{\partial t}\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{H}(\mathbf{curl}\,\boldsymbol{A})\cdot\mathbf{curl}\,\boldsymbol{v} = \int_\Omega \boldsymbol{J}_s\cdot\boldsymbol{v} \qquad \forall\,\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega). \tag{2.3}$$

We remark that (2.3) is meant in the sense of distributions in time. It is obvious that the solution of (2.3) is not unique in the insulating region Ω_{nc}. In fact, if $\boldsymbol{A}$ solves (2.3), then $\boldsymbol{A} + \xi\nabla\phi$ also solves (2.3) for any $\xi \in C^1([0,T])$ and

$$\phi \in H^1_{\mathrm{c}}(\Omega) := \left\{p \in H^1_0(\Omega),\ p = \text{Const. in } \overline{\Omega}_c\right\}.$$

To study the wellposedness of the weak solution, we define

$$\boldsymbol{X} = \left\{\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega):\ (\boldsymbol{v},\nabla p) = 0 \quad \forall\,p\in H^1_{\mathrm{c}}(\Omega)\right\}. \tag{2.4}$$

Then $\boldsymbol{H}_0(\mathbf{curl},\Omega)$ admits the orthogonal decomposition

$$\boldsymbol{H}_0(\mathbf{curl},\Omega) = \boldsymbol{X}\oplus\nabla H^1_{\mathrm{c}}(\Omega). \tag{2.5}$$

The following lemma is well-known (cf. e.g. [4, 9]) and will play an important role in our analysis.

Lemma 2.1. *Let $\boldsymbol{X}$ be endowed with the inner product*

$$(\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{X}} = \int_{\Omega_c}\boldsymbol{v}\cdot\boldsymbol{w} + \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w} \qquad \forall\,\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{X}. \tag{2.6}$$

Then $\|\cdot\|_{\boldsymbol{X}} = \sqrt{(\cdot,\cdot)_{\boldsymbol{X}}}$ is an equivalent norm to $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$ on $\boldsymbol{X}$.

A weak formulation on the subspace $\boldsymbol{X}$ reads: Find $\boldsymbol{u}\in\boldsymbol{L}^2(0,T;\boldsymbol{X})$ such that $\boldsymbol{u}(\cdot,0) = 0$ in Ω_c and

$$\int_\Omega \sigma\frac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{H}(\mathbf{curl}\,\boldsymbol{u})\cdot\mathbf{curl}\,\boldsymbol{v} = \int_\Omega \boldsymbol{J}_s\cdot\boldsymbol{v} \qquad \forall\,\boldsymbol{v}\in\boldsymbol{X}. \tag{2.7}$$

From (2.5), it is easy to see that $\boldsymbol{u}$ also satisfies

$$(2.8)\qquad \int_\Omega \sigma \frac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{v} + \int_\Omega \boldsymbol{H}(\mathbf{curl}\, \boldsymbol{u}) \cdot \mathbf{curl}\, \boldsymbol{v} = \int_\Omega \boldsymbol{J}_s \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega).$$

This means that $\boldsymbol{u}$ is one solution of (2.3). Here (2.7) and (2.8) are also meant in the sense of distributions in time. Although the solution $\boldsymbol{A}$ of (2.3) is not unique, the current density and the magnetic flux density are unique, namely,

$$\frac{\partial}{\partial t}(\sigma \boldsymbol{A}) = \frac{\partial}{\partial t}(\sigma \boldsymbol{u}), \qquad \mathbf{curl}\, \boldsymbol{A} = \mathbf{curl}\, \boldsymbol{u} \qquad \text{in } \Omega.$$

Therefore, we are only interested in $\sigma \boldsymbol{u}$ and $\mathbf{curl}\, \boldsymbol{u}$ throughout this paper.

THEOREM 2.2. *Let (H1)–(H3) be satisfied. Then* (2.7) *has a unique solution and there exists a constant* $C > 0$ *only depending on* T, Ω, $\sigma_{\min}$, $\nu_{\min}$ *such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{X})} \le C \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}.$$

In the next section, we shall propose an approximate model of (2.7) for laminated conductors. The proof of Theorem 2.2 uses similar arguments as the proof of Theorem 4.7 for the approximate problem, but is much easier. We omit the proof of Theorem 2.2 for simplicity.

3. A new eddy current model for laminated conductors. In this section, we shall propose an approximate model which omits coating films. To simplify the setting, we assume that the conducting domain consists of cuboid steel sheets which are laminated along the x_1-direction, that is, $\Omega_c = \bigcup_{i=1}^{I} \Omega_i$ where

$$(3.1)\qquad \begin{aligned} &\Omega_i := (X_{i-1},\ X_i) \times (Y_1,\ Y_2) \times (Z_1,\ Z_2), && \text{for odd } i, \\ &\Omega_i := (X_{i-1} + d,\ X_i - d) \times (Y_1,\ Y_2) \times (Z_1,\ Z_2), && \text{for even } i, \end{aligned} \qquad 1 \le i \le I,$$

and $\Omega_I = (X_{I-1} + d,\ X_I) \times (Y_1,\ Y_2) \times (Z_1,\ Z_2)$ if I is even. Here $d > 0$ stands for the thickness of the coating film between neighboring steel sheets. For example, Figure 3.1 shows the geometric sizes of silicon steel laminations in Team Workshop Problem 21^c–M1. We also assume that $\sigma = \sigma_i > 0$ is constant in Ω_i for all $1 \le i \le I$. We remark that the assumptions on $\Omega_1, \cdots, \Omega_I$ and σ are not essential for our theory. In fact, the results can be easily extended to polyhedral conductors and to the case that σ is not piecewise constant.

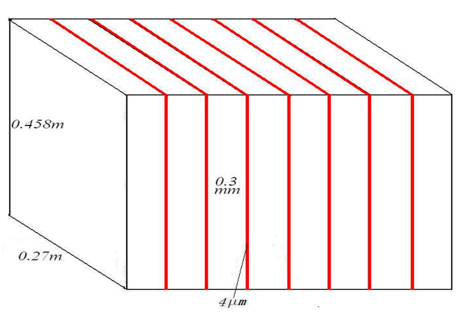

FIG. 3.1. *Geometric size of silicon steel laminations in Team Workshop Problem* 21^c*–M1.*

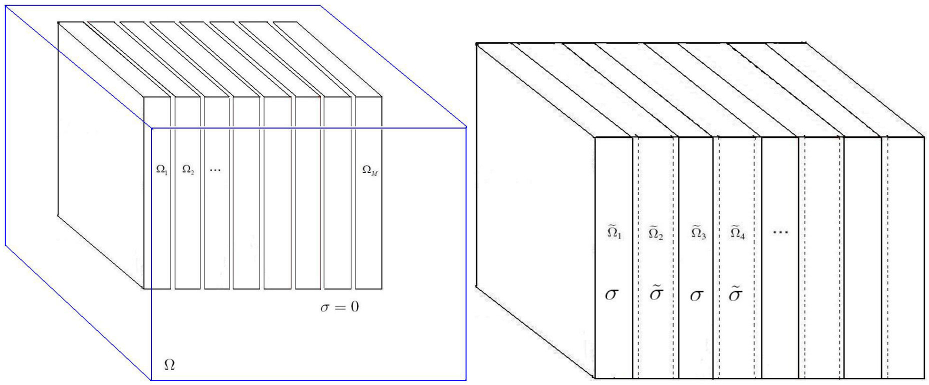

FIG. 3.2. *Left: isolated conductors with the coating film. Right: extended conductors by merging the coating film into even conductors.*

We define the extended conductors by (see Figure 3.2 (right))

$$\tilde\Omega_c := (X_0, X_I) \times (Y_1, Y_2) \times (Z_1, Z_2),$$
$$\tilde\Omega_i := (X_{i-1}, X_i) \times (Y_1, Y_2) \times (Z_1, Z_2), \quad i = 1, 2, \cdots, I.$$

Clearly $\tilde\Omega_i$ and $\tilde\Omega_{i+1}$ are adjacent conductors and have an interface $\Gamma_i = \partial\tilde\Omega_i \cap \partial\tilde\Omega_{i+1}$ for $1 \le i \le I-1$. Let $\tilde\Omega_{nc}$ be the complement of $\tilde\Omega_c$ in Ω. We define the modified material parameters as follows:

$$(3.2) \quad \begin{cases} \tilde\sigma := \sigma_i \qquad \text{in } \tilde\Omega_i, \quad 1 \le i \le I, \\ \tilde{\boldsymbol{H}}(\boldsymbol{B}) = \boldsymbol{H}(\boldsymbol{B}) \qquad \text{in } \tilde\Omega_{nc} \bigcup \Omega_c, \\ \tilde{\boldsymbol{H}}(\boldsymbol{B}) \text{ is defined by the nonlinear BH-curves satisfying (H2) in } \tilde\Omega_c \backslash \overline{\Omega}_c \,. \end{cases}$$

We shall propose an eddy current model which does not allow the eddy current flowing across each Γ_i, or which insures $\boldsymbol{J} \cdot \boldsymbol{n} = 0$ on $\partial\tilde\Omega_i$ for all $1 \le i \le I$.

First we consider the isolated conductors $\Omega_1, \cdots, \Omega_I$ (see Figure 3.2 (left)). By $\operatorname{div} \boldsymbol{J}_s = 0$ and taking $\boldsymbol{v} = \nabla\varphi$, (2.3) shows that

$$(3.3) \qquad \int_{\Omega_c} \sigma \frac{\partial \boldsymbol{A}}{\partial t} \cdot \nabla\varphi = 0 \qquad \forall\, \varphi \in H_0^1(\Omega),$$

which is equivalent to

$$(3.4) \qquad \int_{\Omega_i} \sigma \frac{\partial \boldsymbol{A}}{\partial t} \cdot \nabla\varphi_i = 0 \qquad \forall\, \varphi_i \in H^1(\Omega_i), \quad 1 \le i \le I.$$

This leads to the conservation of the current density $\boldsymbol{J} = -\sigma \dfrac{\partial \boldsymbol{A}}{\partial t}$ in each conductor:

$$(3.5) \qquad \operatorname{div} \boldsymbol{J} = 0 \quad \text{in } \Omega_i \qquad \text{and} \qquad \boldsymbol{J} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega_i.$$

Now we consider the extended conducting domain $\tilde\Omega_c$. Let $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{J}} = -\tilde\sigma \dfrac{\partial \tilde{\boldsymbol{A}}}{\partial t}$ be the modified magnetic vector potential and current density respectively. Similarly the conservation property should be satisfied

$$\operatorname{div} \tilde{\boldsymbol{J}} = 0 \quad \text{in } \tilde\Omega_i \qquad \text{and} \qquad \tilde{\boldsymbol{J}} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\tilde\Omega_i, \quad 1 \le i \le I.$$

This comes directly from

$$\int_{\tilde{\Omega}_i} \tilde{\sigma}\frac{\partial \tilde{\boldsymbol{A}}}{\partial t}\cdot\nabla\varphi_i = 0 \qquad \forall\,\varphi_i\in H^1(\tilde{\Omega}_i), \quad 1\le i\le I,$$

which for adjacent conductors is equivalent to

$$\int_{\tilde{\Omega}_c} \tilde{\sigma}\frac{\partial \tilde{\boldsymbol{A}}}{\partial t}\cdot\nabla\varphi = 0 \qquad \forall\,\varphi\in H_0^1(\Omega), \tag{3.6}$$

$$\int_{\tilde{\Omega}_i} \tilde{\sigma}\frac{\partial \tilde{\boldsymbol{A}}}{\partial t}\cdot\nabla\varphi_i = 0 \qquad \forall\,\varphi_i\in H^1(\tilde{\Omega}_i)\ \text{ and even }\ i. \tag{3.7}$$

We shall also use the notation that

$$\tilde{\Omega}_{\text{odd}} = \Omega_{\text{odd}} := \bigcup_{\substack{\text{odd } i\\ 1\le i\le I}} \Omega_i, \qquad \Omega_{\text{even}} := \bigcup_{\substack{\text{even } i\\ 1< i\le I}} \Omega_i, \qquad \tilde{\Omega}_{\text{even}} := \bigcup_{\substack{\text{even } i\\ 1< i\le I}} \tilde{\Omega}_i.$$

A comparison of (3.6)–(3.7) with (3.3) inspires us to enlarge the test function space in (2.3) from $\boldsymbol{H}_0(\mathbf{curl},\Omega)$ to $\boldsymbol{H}_0(\mathbf{curl},\Omega)+\chi\nabla H_0^1(\Omega)$, where χ is the characteristic function satisfying

$$\chi = \begin{cases} 1 & \text{in } \tilde{\Omega}_{\text{even}}\,, \\ 0 & \text{elsewhere.} \end{cases}$$

Accordingly, we define the modified curl operator by

$$\widetilde{\mathbf{curl}}\,(\boldsymbol{v}+\chi\nabla v) := \mathbf{curl}\,\boldsymbol{v} \qquad \forall\,\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl},\Omega),\ v\in H^1(\Omega). \tag{3.8}$$

It is clear that $\widetilde{\mathbf{curl}}$ is just the normal curl operator on $\boldsymbol{H}(\mathbf{curl},\Omega)$:

$$\widetilde{\mathbf{curl}}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{v} \qquad \forall\,\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl},\Omega). \tag{3.9}$$

Clearly $\boldsymbol{H}_0(\mathbf{curl},\Omega)+\chi\nabla H_0^1(\Omega)$ is not a direct sum since $\chi\nabla\phi\in\boldsymbol{H}_0(\mathbf{curl},\Omega)$ for any $\phi\in H_0^1(\Omega)$ satisfying $\text{supp}(\phi)\subset\tilde{\Omega}_{\text{even}}$. To find a direct sum, we define

$$\boldsymbol{X}_{\text{odd}} = \big\{\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega):\ (\boldsymbol{v},\nabla\varphi)=0 \quad \forall\,\varphi\in H^1_{\text{odd}}(\Omega)\big\}, \tag{3.10}$$

where

$$H^1_{\text{odd}}(\Omega) := \big\{\varphi\in H_0^1(\Omega),\ \varphi=\text{Const. in }\ \tilde{\Omega}_{\text{odd}}\big\}.$$

It induces a subspace of $\boldsymbol{H}_0(\mathbf{curl},\Omega)+\chi\cdot\nabla H_0^1(\Omega)$ as follows

$$\tilde{\boldsymbol{X}} := \boldsymbol{X}_{\text{odd}}+\chi\cdot\nabla H_0^1(\Omega). \tag{3.11}$$

Lemma 3.1. *The righthand side of* (3.11) *is a direct sum in the sense that, for any* $\hat{\boldsymbol{v}}\in\boldsymbol{X}_{\text{odd}}$ *and* $v\in H_0^1(\Omega)$,

$$\hat{\boldsymbol{v}}+\chi\nabla v = 0 \qquad \textit{if and only if} \qquad \hat{\boldsymbol{v}}=0,\ \ \chi\nabla v=0. \tag{3.12}$$

Moreover, $\tilde{\boldsymbol{X}}$ *is a Hilbert space under the inner product and norm*

$$(\boldsymbol{v},\boldsymbol{w})_{\tilde{\boldsymbol{X}}} := \int_{\tilde{\Omega}_c}\boldsymbol{v}\cdot\boldsymbol{w}+\int_{\Omega}\widetilde{\mathbf{curl}}\,\boldsymbol{v}\cdot\widetilde{\mathbf{curl}}\,\boldsymbol{w},\ \ \|\boldsymbol{v}\|_{\tilde{\boldsymbol{X}}} := \sqrt{(\boldsymbol{v},\boldsymbol{v})_{\tilde{\boldsymbol{X}}}} \qquad \forall\,\boldsymbol{v},\boldsymbol{w}\in\tilde{\boldsymbol{X}}. \tag{3.13}$$

Proof. Suppose that $\hat{\boldsymbol{v}} + \chi\nabla v = 0$ in Ω for $\hat{\boldsymbol{v}} \in \boldsymbol{X}_{\rm odd}$ and $v \in H_0^1(\Omega)$. Then

$$\hat{\boldsymbol{v}} = -\nabla v \quad \text{in } \tilde{\Omega}_{\rm even} \qquad \text{and} \qquad \hat{\boldsymbol{v}} = 0 \quad \text{elsewhere}.$$

From (3.10), we know that $v_i := v|_{\tilde{\Omega}_i}$ solves the elliptic problem

$$-\Delta v_i = \operatorname{div} \hat{\boldsymbol{v}} = 0 \quad \text{in } \tilde{\Omega}_i, \qquad \nabla v_i \times \boldsymbol{n} = \hat{\boldsymbol{v}} \times \boldsymbol{n} = 0 \quad \text{on } \partial\tilde{\Omega}_i, \tag{3.14}$$

for any $1 < i \le I$ and even i. Clearly (3.14) only has constant solutions so that $\nabla v_i = 0$ in $\tilde{\Omega}_i$. We have $\chi\nabla v = 0$ and $\hat{\boldsymbol{v}} = 0$ in Ω. Thus (3.12) is a direct sum.

Next we prove that $\tilde{\boldsymbol{X}}$ is complete. Since $\chi\cdot\nabla H_0^1(\Omega)$ is isomorphic to $\nabla H^1(\tilde{\Omega}_{\rm even})$, it suffices to prove the completeness of $\boldsymbol{X}_{\rm odd}$. By Lemma 2.1, one equivalent norm on $\boldsymbol{X}_{\rm odd}$ is defined by

$$\|\boldsymbol{v}\|_{\boldsymbol{X}_{\rm odd}} := \Big(\int_{\tilde{\Omega}_{\rm odd}} |\boldsymbol{v}|^2 + \int_\Omega |\mathbf{curl}\, \boldsymbol{v}|^2 \Big)^{1/2} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{X}_{\rm odd}.$$

Let $\{\boldsymbol{v}_n\}_{n=1}^\infty \subset \boldsymbol{X}_{\rm odd}$ be a Cauchy sequence under the norm $\|\cdot\|_{\boldsymbol{X}_{\rm odd}}$. Then it is also a Cauchy sequence under $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$. There exists a $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl},\Omega)$ such that

$$\lim_{n\to\infty} \|\boldsymbol{v}_n - \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} = 0, \qquad (\boldsymbol{v}, \nabla\varphi) = \lim_{n\to\infty} (\boldsymbol{v}_n, \nabla\varphi) = 0 \quad \forall\, \varphi \in H^1_{\rm odd}(\Omega).$$

Thus $\boldsymbol{v} \in \boldsymbol{X}_{\rm odd}$ and $\lim_{n\to\infty} \|\boldsymbol{v}_n - \boldsymbol{v}\|_{\boldsymbol{X}_{\rm odd}} = 0$. Then $\boldsymbol{X}_{\rm odd}$ is complete, and so is $\tilde{\boldsymbol{X}}$.

Now let $\boldsymbol{v} = \hat{\boldsymbol{v}} + \chi\nabla v$ satisfy $\|\boldsymbol{v}\|_{\tilde{\boldsymbol{X}}} = 0$ where $\hat{\boldsymbol{v}} \in \boldsymbol{X}_{\rm odd}$ and $v \in H_0^1(\Omega)$. Then (3.8) and (3.13) show that

$$\hat{\boldsymbol{v}} = 0 \quad \text{in } \tilde{\Omega}_{\rm odd}, \qquad \hat{\boldsymbol{v}} + \nabla v = 0 \quad \text{in } \tilde{\Omega}_{\rm even}, \qquad \mathbf{curl}\, \hat{\boldsymbol{v}} = 0 \quad \text{in } \Omega.$$

This indicates $\|\hat{\boldsymbol{v}}\|_{\boldsymbol{X}_{\rm odd}} = 0$. We conclude that $\hat{\boldsymbol{v}} = 0$ in Ω and thus $\boldsymbol{v} = 0$ in Ω. Therefore, $\|\cdot\|_{\tilde{\boldsymbol{X}}}$ is a norm on $\tilde{\boldsymbol{X}}$ so that $\tilde{\boldsymbol{X}}$ is a Hilbert space. □

We end this section by the approximate problem to (2.7): Find $\tilde{\boldsymbol{u}} \in \boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})$ such that $\tilde{\boldsymbol{u}}(\cdot,0) = 0$ in $\tilde{\Omega}_c$ and

$$\int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \cdot \boldsymbol{v} + \int_\Omega \tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}}) \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v} = \int_\Omega \boldsymbol{J}_s \cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \tilde{\boldsymbol{X}}. \tag{3.15}$$

The formulation is meant in the sense of distributions in time.

4. Well-posedness of the approximate problem. First we prove the uniqueness of the solution of (3.15).

THEOREM 4.1. *Let (H1)–(H3) be satisfied. Then* (3.15) *has at most one solution.*

Proof. Suppose $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ are two solutions of (3.15). Then

$$\int_\Omega \tilde{\sigma} \frac{\partial}{\partial t}\big(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2\big) \cdot \boldsymbol{v} + \int_\Omega \big\{\tilde{\boldsymbol{H}}\big(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}}_1\big) - \tilde{\boldsymbol{H}}\big(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}}_2\big)\big\} \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v} = 0 \quad \forall\, \boldsymbol{v} \in \tilde{\boldsymbol{X}}.$$

It means that, for almost every $t \in (0,T]$ and all $\boldsymbol{v} \in \boldsymbol{L}^2(0,t;\tilde{\boldsymbol{X}})$,

$$\int_0^t \int_\Omega \tilde{\sigma} \frac{\partial}{\partial t}\big(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2\big) \cdot \boldsymbol{v} + \int_0^t \int_\Omega \big\{\tilde{\boldsymbol{H}}\big(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}}_1\big) - \tilde{\boldsymbol{H}}\big(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}}_2\big)\big\} \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v} = 0.$$

Taking $\boldsymbol{v} = \tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2$, the above equality shows that

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{\tilde{\sigma}}{2} \left|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2\right|^2 + \int_0^t \int_\Omega \left\{ \tilde{\boldsymbol{H}}\big(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_1\big) - \tilde{\boldsymbol{H}}\big(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_2\big) \right\} \cdot \widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2) = 0.$$

From initial conditions $\tilde{\boldsymbol{u}}_1(\cdot, 0) = \tilde{\boldsymbol{u}}_2(\cdot, 0) = 0$ in $\tilde{\Omega}_c$, we find that

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{\tilde{\sigma}}{2} \left|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2\right|^2 = \frac{1}{2} \int_\Omega \tilde{\sigma} \left|\tilde{\boldsymbol{u}}_1(t) - \tilde{\boldsymbol{u}}_2(t)\right|^2.$$

And the strict monotonicity of $\tilde{\boldsymbol{H}}$ shows that (see (3.2) and (H2))

$$\int_\Omega \left\{ \tilde{\boldsymbol{H}}\big(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_1\big) - \tilde{\boldsymbol{H}}\big(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_2\big) \right\} \cdot \widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2) \geq \nu_{\min} \left\| \widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2) \right\|^2_{\boldsymbol{L}^2(\Omega)}.$$

It follows that, for almost every $t \in (0, T]$,

$$\frac{1}{2} \int_\Omega \tilde{\sigma} \left|\tilde{\boldsymbol{u}}_1(t) - \tilde{\boldsymbol{u}}_2(t)\right|^2 + \nu_{\min} \int_0^t \left\| \widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2) \right\|^2_{L^2(\Omega)} \leq 0.$$

This shows $\|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2\|_{\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})} = 0$. From Lemma 3.1 we have $\tilde{\boldsymbol{u}}_1 = \tilde{\boldsymbol{u}}_2$. ◻

We shall use Rothe's method (cf. e.g. [18]) to prove the existence of the solution of (3.15). Let N be a positive integer and let

$$t_n = n\tau, \quad n = 0, 1, \cdots, N, \qquad \tau = T/N,$$

be a uniform partition of $[0, T]$. We consider the semi-discrete approximation to (3.15): Given $\tilde{\boldsymbol{u}}_0 = 0$, find $\tilde{\boldsymbol{u}}_n \in \tilde{\boldsymbol{X}}$, $1 \leq n \leq N$ such that

$$\left(\tilde{\sigma}\frac{\tilde{\boldsymbol{u}}_n - \tilde{\boldsymbol{u}}_{n-1}}{\tau}, \boldsymbol{v}\right) + \left(\tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_n), \widetilde{\mathbf{curl}}\,\boldsymbol{v}\right) = (\boldsymbol{J}_n, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \tilde{\boldsymbol{X}}, \tag{4.1}$$

where $\boldsymbol{J}_n = \tau^{-1} \int_{t_{n-1}}^{t_n} \boldsymbol{J}_s(\cdot, t)\mathrm{d}t$ is the temporal average of $\boldsymbol{J}_s$ over (t_{n-1}, t_n).

Lemma 4.2. *Let (H1)–(H3) be satisfied and assume* $\mathrm{supp}(\boldsymbol{J}_s) \cap \tilde{\Omega}_c = \emptyset$. *For any* $1 \leq n \leq N$, *(4.1) has a unique solution* $\tilde{\boldsymbol{u}}_n \in \tilde{\boldsymbol{X}}$. *Then there exists a constant C only depending on Ω such that*

$$\max_{1\leq n\leq N} \left\| \tilde{\sigma}^{1/2} \tilde{\boldsymbol{u}}_n \right\|^2_{\boldsymbol{L}^2(\Omega)} + \nu_{\min} \sum_{n=1}^N \tau \left\| \widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_n \right\|^2_{L^2(\Omega)} \leq C\nu_{\min}^{-1} \left\| \boldsymbol{J}_s \right\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}. \tag{4.2}$$

Proof. The proof of this lemma is provided in Appendix A. ◻

Lemma 4.3. *There exists a constant $C > 0$ only depending on Ω such that, for any $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ satisfying* $\mathrm{div}\, \boldsymbol{f} = 0$,

$$|(\boldsymbol{f}, \boldsymbol{v})| \leq C \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega).$$

Proof. For any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$, we have the orthogonal decomposition $\boldsymbol{v} = \nabla\phi + \boldsymbol{w}$, where $\phi \in H^1_0(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ satisfying $\mathrm{div}\, \boldsymbol{w} = 0$. By the embedding theorem in [3], there exists a constant C only depending on Ω such that

$$\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C \left(\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{w}\|_{L^2(\Omega)} \right) = C \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}.$$

Then by Hölder's inequality and $\operatorname{div} \boldsymbol{f} = 0$, we have

$$|(\boldsymbol{f}, \boldsymbol{v})| = |(\boldsymbol{f}, \boldsymbol{w})| \le \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)} \le C \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}.$$

The proof is completed. □

We define the piecewise constant and piecewise linear interpolations in time by

$$\bar{\boldsymbol{u}}_\tau(\cdot, t) = \tilde{\boldsymbol{u}}_n, \quad \tilde{\boldsymbol{u}}_\tau(\cdot, t) = l_n(t)\tilde{\boldsymbol{u}}_n + (1 - l_n(t))\tilde{\boldsymbol{u}}_{n-1} \qquad \forall\, t \in (t_{n-1},\ t_n], \tag{4.3}$$

where $1 \le n \le N$ and $l_n(t) := (t - t_{n-1})/\tau$. Clearly we have $\bar{\boldsymbol{u}}_\tau \in \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}})$ and $\tilde{\boldsymbol{u}}_\tau \in \boldsymbol{C}(0, T; \tilde{\boldsymbol{X}})$. Then the following lemma is a direct consequence of Lemma 4.2.

Lemma 4.4. *Let (H1)–(H3) be satisfied and assume* $\operatorname{supp}(\boldsymbol{J}_s) \cap \tilde{\Omega}_c = \emptyset$. *There exists a constant C only depending on Ω, T, $\nu_{\min}$, $\sigma_{\min}$ such that*

$$\|\bar{\boldsymbol{u}}_\tau\|_{\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})} + \|\tilde{\boldsymbol{u}}_\tau\|_{\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})} \le C \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}.$$

Lemma 4.5. *Let (H1)–(H3) be satisfied and assume* $\operatorname{supp}(\boldsymbol{J}_s) \cap \tilde{\Omega}_c = \emptyset$. *There exists a $\tilde{\boldsymbol{u}} \in \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}})$ such that*

$$\lim_{\tau\to 0} \tilde{\boldsymbol{u}}_\tau = \lim_{\tau\to 0} \bar{\boldsymbol{u}}_\tau = \tilde{\boldsymbol{u}} \qquad \textit{weakly in } \ \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}}), \tag{4.4}$$

$$\lim_{\tau\to 0} \tilde{\sigma}\frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} = \tilde{\sigma}\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \qquad \textit{weakly* in } \ \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}}'). \tag{4.5}$$

Proof. Clearly (4.1) and (4.3) indicate that

$$\int_0^T \int_\Omega \tilde{\sigma}\frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \cdot \boldsymbol{v} = \int_0^T \int_\Omega \left\{ \boldsymbol{J}_s \cdot \boldsymbol{v} - \tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\, \bar{\boldsymbol{u}}_\tau) \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v} \right\} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}}).$$

Write $\boldsymbol{v} = \hat{\boldsymbol{v}} + \chi\nabla v$ with $\hat{\boldsymbol{v}} \in \boldsymbol{X}_{\rm odd}$ and $v \in H_0^1(\Omega)$. Then Lemma 4.3 shows that

$$\int_\Omega \boldsymbol{J}_s \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{J}_s \cdot \hat{\boldsymbol{v}} \le C \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)} \|\mathbf{curl}\, \hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2(\Omega)} = C \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)} \|\widetilde{\mathbf{curl}}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}.$$

Together with Lemma 4.4, we deduce that

$$\int_0^T \int_\Omega \tilde{\sigma}\frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \cdot \boldsymbol{v} \le C \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}}).$$

This implies

$$\tilde{\sigma}\frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \in \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}}') \quad \text{and} \quad \left\| \tilde{\sigma}\frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \right\|_{\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}}')} \le C \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}. \tag{4.6}$$

Since $\boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}})$ is self-reflective, from (4.2), there are two subsequences still denoted by $\{\tilde{\boldsymbol{u}}_\tau\}_{\tau>0}$ and $\{\bar{\boldsymbol{u}}_\tau\}_{\tau>0}$ such that

$$\lim_{\tau\to 0} \tilde{\boldsymbol{u}}_\tau = \tilde{\boldsymbol{u}}, \quad \lim_{\tau\to 0} \bar{\boldsymbol{u}}_\tau = \bar{\boldsymbol{u}} \qquad \text{weakly in } \ \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}}).$$

Similarly from (4.6) we have a subsequence such that

$$\lim_{\tau\to 0} \tilde{\sigma}\frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} = \tilde{\boldsymbol{u}}' \qquad \text{weakly* in } \ \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}}').$$

Then (4.4) requires to show $\tilde{\boldsymbol{u}} = \bar{\boldsymbol{u}}$ and (4.5) requires to show $\tilde{\boldsymbol{u}}' = \tilde{\sigma}\dfrac{\partial \tilde{\boldsymbol{u}}}{\partial t}$.

Next we fix the integer $N > 0$ and define $\delta = T/N$ and $t_n = n\delta$ for any $0 \le n \le N$. Let $m > 0$ be an integer and define $\tau = \delta/m$. From (4.2), there exists a constant $C > 0$ independent of n, τ such that

$$\sqrt{\tau}\left\|\tilde{\boldsymbol{u}}_{mn} - \tilde{\boldsymbol{u}}_{m(n-1)}\right\|_{\tilde{\boldsymbol{X}}} \le C \left\|\boldsymbol{J}_s\right\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}.$$

Let ζ_n be the characteristic function satisfying

$$\zeta_n(t) = \begin{cases} 1 & \text{if } t \in (t_{n-1}, t_n), \\ 0 & \text{elsewhere}, \end{cases} \qquad 1 \le n \le N.$$

We introduce the space of piecewise constant functions for the time variable

$$\tilde{\boldsymbol{X}}_N = \left\{\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{w}(\boldsymbol{x})\zeta(t):\ \boldsymbol{w} \in \tilde{\boldsymbol{X}},\ \zeta \in \mathrm{Span}\left\{\zeta_n,\ 1 \le n \le N\right\}\right\}.$$

For any $\zeta_n \boldsymbol{w} \in \tilde{\boldsymbol{X}}_N$, the weak convergence of $\tilde{\boldsymbol{u}}_\tau$ and $\bar{\boldsymbol{u}}_\tau$ shows that

$$\begin{aligned}\int_0^T (\tilde{\boldsymbol{u}} - \bar{\boldsymbol{u}}, \zeta_n \boldsymbol{w})_{\tilde{\boldsymbol{X}}} &= \lim_{\tau\to 0}\int_0^T (\tilde{\boldsymbol{u}}_\tau - \bar{\boldsymbol{u}}_\tau, \zeta_n \boldsymbol{w})_{\tilde{\boldsymbol{X}}} = \lim_{\tau\to 0}\int_{t_{n-1}}^{t_n} (\tilde{\boldsymbol{u}}_\tau - \bar{\boldsymbol{u}}_\tau, \boldsymbol{w})_{\tilde{\boldsymbol{X}}} \\ &= \lim_{\tau\to 0}\sum_{k=m(n-1)}^{mn-1}\int_{k\tau}^{(k+1)\tau}\frac{t-k\tau}{\tau}(\tilde{\boldsymbol{u}}_{k+1} - \tilde{\boldsymbol{u}}_k, \boldsymbol{w})_{\tilde{\boldsymbol{X}}}\mathrm{d}t \\ &= \lim_{\tau\to 0}\frac{\tau}{2}\left(\tilde{\boldsymbol{u}}_{mn} - \tilde{\boldsymbol{u}}_{m(n-1)}, \boldsymbol{w}\right)_{\tilde{\boldsymbol{X}}} = 0.\end{aligned}$$

The density of $\tilde{\boldsymbol{X}}_N$ in $\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})$ as $N \to \infty$ implies that $\int_0^T (\tilde{\boldsymbol{u}} - \bar{\boldsymbol{u}}, \boldsymbol{v})_{\tilde{\boldsymbol{X}}} = 0$ for any $\boldsymbol{v} \in \boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})$. This proves $\tilde{\boldsymbol{u}} = \bar{\boldsymbol{u}}$.

Moreover, using the formula of integration by parts, we find that $\tilde{\boldsymbol{u}}'$ satisfies

$$\int_0^T \langle \tilde{\boldsymbol{u}}', \boldsymbol{v}\rangle_{\tilde{\boldsymbol{X}}' \times \tilde{\boldsymbol{X}}} = \lim_{\tau\to 0}\int_0^T \left(\tilde{\sigma}\frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t}, \boldsymbol{v}\right) = -\lim_{\tau\to 0}\int_0^T \left(\tilde{\sigma}\tilde{\boldsymbol{u}}_\tau, \frac{\partial \boldsymbol{v}}{\partial t}\right) = -\int_0^T \left(\tilde{\sigma}\tilde{\boldsymbol{u}}, \frac{\partial \boldsymbol{v}}{\partial t}\right),$$

for any $\boldsymbol{v} \in C_0^\infty(0,T;\tilde{\boldsymbol{X}})$. This implies $\tilde{\boldsymbol{u}}' = \tilde{\sigma}\dfrac{\partial \tilde{\boldsymbol{u}}}{\partial t}$ in a distributional sense. □

Lemma 4.6. *Let (H1)–(H3) be satisfied and assume* $\mathrm{supp}(\boldsymbol{J}_s) \cap \tilde{\Omega}_c = \emptyset$. *Let* $\tilde{\boldsymbol{u}}$ *be the weak limit of* $\tilde{\boldsymbol{u}}_\tau$ *in* $\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})$. *Then* $\tilde{\boldsymbol{u}}(\cdot,0) = 0$ *in* $\tilde{\Omega}_c$ *and*

$$\lim_{\tau\to 0}\tilde{\boldsymbol{u}}_\tau(\cdot,T) = \tilde{\boldsymbol{u}}(\cdot,T) \quad \textit{weakly in } \boldsymbol{L}^2(\tilde{\Omega}_c).$$

Proof. From (4.2), it is easy to see

$$\left\|\tilde{\boldsymbol{u}}_\tau(\cdot,T)\right\|_{\boldsymbol{L}^2(\tilde{\Omega}_c)} \le C\left\|\boldsymbol{J}_s\right\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \qquad \forall\,\tau > 0. \tag{4.7}$$

So $\left\{\tilde{\boldsymbol{u}}_\tau(\cdot,T)\right\}_{\tau>0}$ has a subsequence such that $\lim\limits_{\tau\to 0}\tilde{\boldsymbol{u}}_\tau(\cdot,T) = \boldsymbol{q}$ weakly in $\boldsymbol{L}^2(\tilde{\Omega}_c)$. It is left to show $\boldsymbol{q} = \tilde{\boldsymbol{u}}(\cdot,T)$ in $\tilde{\Omega}_c$.

Take any $\phi \in C^\infty([0,T])$ satisfying $\phi(0) = 0$ and $\phi(T) = 1$. Then Lemma 4.5 and the formula of integration by parts show that, for any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl},\Omega)$,

$$\begin{aligned}\int_{\tilde{\Omega}_c}\tilde{\sigma}\boldsymbol{q}\cdot\boldsymbol{v} &= \lim_{\tau\to 0}\int_\Omega \tilde{\sigma}\tilde{\boldsymbol{u}}_\tau(\cdot,T)\cdot\boldsymbol{v}\phi(T) = \lim_{\tau\to 0}\int_0^T\int_\Omega \tilde{\sigma}\left[\tilde{\boldsymbol{u}}_\tau\cdot\frac{\partial(\phi\boldsymbol{v})}{\partial t} + \frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t}\cdot(\phi\boldsymbol{v})\right] \\ &= \int_0^T\int_\Omega \tilde{\sigma}\left[\tilde{\boldsymbol{u}}\cdot\frac{\partial(\phi\boldsymbol{v})}{\partial t} + \frac{\partial \tilde{\boldsymbol{u}}}{\partial t}\cdot(\phi\boldsymbol{v})\right] = \int_{\tilde{\Omega}_c}\tilde{\sigma}\tilde{\boldsymbol{u}}(\cdot,T)\cdot\boldsymbol{v}.\end{aligned}$$

We conclude that $\boldsymbol{q} = \tilde{\boldsymbol{u}}(\cdot, T)$ in $\tilde{\Omega}_c$ by the density of $\boldsymbol{H}(\mathbf{curl}, \tilde{\Omega}_c)$ in $\boldsymbol{L}^2(\tilde{\Omega}_c)$.

Notice that $\tilde{\boldsymbol{u}}_\tau(\cdot, 0) = 0$ for all $\tau > 0$. The proof for $\tilde{\boldsymbol{u}}(\cdot, 0) = 0$ in $\tilde{\Omega}_c$ is similar and we do not elaborate on the details here. ▯

THEOREM 4.7. *Let (H1)–(H3) be satisfied and assume* $\operatorname{supp}(\boldsymbol{J}_s) \cap \tilde{\Omega}_c = \emptyset$. *Then* (3.15) *has a unique solution* $\tilde{\boldsymbol{u}}$ *and there exists a constant* $C > 0$ *only depending on* Ω, T, $\nu_{\min}$, $\sigma_{\min}$ *such that*

$$\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})} \leq C \, \|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \, .$$

Proof. Let $\tilde{\boldsymbol{u}}_\tau$ and $\bar{\boldsymbol{u}}_\tau$ be defined in (4.3). For any $\boldsymbol{v} \in \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}})$, (4.1) yields

$$\int_0^T \int_\Omega \tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\, \bar{\boldsymbol{u}}_\tau) \cdot \widetilde{\mathbf{curl}}\, (\boldsymbol{v} - \bar{\boldsymbol{u}}_\tau) = I_\tau^{(1)} - I_\tau^{(2)}, \tag{4.8}$$

where

$$I_\tau^{(1)} := \int_0^T \int_\Omega \boldsymbol{J}_s \cdot (\boldsymbol{v} - \bar{\boldsymbol{u}}_\tau), \qquad I_\tau^{(2)} := \int_0^T \int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \cdot (\boldsymbol{v} - \bar{\boldsymbol{u}}_\tau).$$

An application of (4.4) shows that

$$\lim_{\tau \to 0} I_\tau^{(1)} = \int_0^T \int_\Omega \boldsymbol{J}_s \cdot (\boldsymbol{v} - \tilde{\boldsymbol{u}}). \tag{4.9}$$

Now we are going to study the limit of $I_\tau^{(2)}$. From (4.5) we have

$$\lim_{\tau \to 0} \int_0^T \int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \cdot \boldsymbol{v} = \int_0^T \int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{L}^2(0, T; \tilde{\boldsymbol{X}}). \tag{4.10}$$

Using (A.5) and the initial condition $\tilde{\boldsymbol{u}}_0 = 0$, we deduce that

$$\int_0^T \int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \cdot \bar{\boldsymbol{u}}_\tau = \sum_{n=0}^N \big(\tilde{\sigma}(\tilde{\boldsymbol{u}}_n - \tilde{\boldsymbol{u}}_{n-1}), \tilde{\boldsymbol{u}}_n\big) \geq \frac{1}{2} \left\| \tilde{\sigma}^{1/2} \tilde{\boldsymbol{u}}_N \right\|^2_{L^2(\Omega)} .$$

From Lemma 4.6, the righthand side has the weak limit

$$\lim_{\tau \to 0} \tilde{\sigma}^{1/2} \tilde{\boldsymbol{u}}_N = \lim_{\tau \to 0} \tilde{\sigma}^{1/2} \tilde{\boldsymbol{u}}_\tau(\cdot, T) = \tilde{\sigma}^{1/2} \tilde{\boldsymbol{u}}(\cdot, T) \qquad \text{weakly in } \boldsymbol{L}^2(\Omega).$$

It follows that $\left\| \tilde{\sigma}^{1/2} \tilde{\boldsymbol{u}}(\cdot, T) \right\|_{L^2(\Omega)} \leq \varliminf_{\tau \to 0} \left\| \tilde{\sigma}^{1/2} \tilde{\boldsymbol{u}}_N \right\|_{L^2(\Omega)}$. This implies

$$\varliminf_{\tau \to 0} \int_0^T \int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \cdot \bar{\boldsymbol{u}}_\tau \geq \frac{1}{2} \left\| \tilde{\sigma}^{1/2} \tilde{\boldsymbol{u}}(\cdot, T) \right\|^2_{L^2(\Omega)} . \tag{4.11}$$

Combining (4.10) and (4.11) leads to

$$\begin{aligned} \varlimsup_{\tau \to 0} I_\tau^{(2)} &= \varlimsup_{\tau \to 0} \left\{ \int_0^T \int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \cdot \boldsymbol{v} - \int_0^T \int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}_\tau}{\partial t} \cdot \bar{\boldsymbol{u}}_\tau \right\} \\ &\leq \int_0^T \int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \cdot \boldsymbol{v} - \frac{1}{2} \left\| \tilde{\sigma}^{1/2} \tilde{\boldsymbol{u}}(\cdot, T) \right\|^2_{L^2(\Omega)} = \int_0^T \int_\Omega \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \cdot (\boldsymbol{v} - \tilde{\boldsymbol{u}}). \end{aligned} \tag{4.12}$$

Inserting (4.9) and (4.12) into (4.8) leads to

$$\underline{\lim}_{\tau\to 0}\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\bar{\boldsymbol{u}}_\tau)\cdot\tilde{\mathbf{curl}}\,(\boldsymbol{v}-\bar{\boldsymbol{u}}_\tau)\geq\int_0^T\Big(\boldsymbol{J}_s-\tilde{\sigma}\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}\Big)\cdot(\boldsymbol{v}-\tilde{\boldsymbol{u}}). \tag{4.13}$$

On one hand, taking $\boldsymbol{v}=\tilde{\boldsymbol{u}}$, (4.13) shows that

$$\overline{\lim}_{\tau\to 0}\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\bar{\boldsymbol{u}}_\tau)\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\tilde{\boldsymbol{u}})\leq 0.$$

On the other hand, the strict monotonicity of $\tilde{\boldsymbol{H}}$ shows that, as $\tau\to 0$,

$$\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\bar{\boldsymbol{u}}_\tau)\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\tilde{\boldsymbol{u}})\geq\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\tilde{\boldsymbol{u}})\to 0.$$

Thus we conclude that

$$\lim_{\tau\to 0}\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\bar{\boldsymbol{u}}_\tau)\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\tilde{\boldsymbol{u}})=0. \tag{4.14}$$

Let $\boldsymbol{w}=\tilde{\boldsymbol{u}}+s(\boldsymbol{v}-\tilde{\boldsymbol{u}})$ with $s>0$. The strict monotonicity of $\tilde{\boldsymbol{H}}$ yields

$$\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\bar{\boldsymbol{u}}_\tau)\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\boldsymbol{w})\geq\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\boldsymbol{w})\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\boldsymbol{w}),$$

which is equivalent to

$$\begin{aligned} s\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\bar{\boldsymbol{u}}_\tau)\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\boldsymbol{v})&\geq s\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\boldsymbol{w})\cdot\tilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}-\boldsymbol{v})\\ &+\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\boldsymbol{w})\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\tilde{\boldsymbol{u}})+(s-1)\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\bar{\boldsymbol{u}}_\tau)\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\tilde{\boldsymbol{u}}). \end{aligned}$$

Taking the lower limit of the above inequality as $\tau\to 0$ and using (4.4) and (4.14), we find that

$$\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\boldsymbol{w})\cdot\tilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}-\boldsymbol{v})\leq\underline{\lim}_{\tau\to 0}\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\bar{\boldsymbol{u}}_\tau)\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\boldsymbol{v}).$$

Since $\tilde{\boldsymbol{H}}$ is Lipschitz continuous, letting $s\to 0$ in the above inequality yields

$$\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})\cdot\tilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}-\boldsymbol{v})\leq\underline{\lim}_{\tau\to 0}\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\bar{\boldsymbol{u}}_\tau)\cdot\tilde{\mathbf{curl}}\,(\bar{\boldsymbol{u}}_\tau-\boldsymbol{v}). \tag{4.15}$$

Combining (4.13) and (4.15), we have

$$\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})\cdot\tilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}-\boldsymbol{v})\leq\int_0^T\int_\Omega\Big(\boldsymbol{J}_s-\tilde{\sigma}\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}\Big)\cdot(\tilde{\boldsymbol{u}}-\boldsymbol{v}).$$

Since $\boldsymbol{v}$ is arbitrary, we conclude that

$$\int_0^T\int_\Omega \tilde{\boldsymbol{H}}(\tilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}})\cdot\tilde{\mathbf{curl}}\,\boldsymbol{v}=\int_0^T\int_\Omega\Big(\boldsymbol{J}_s-\tilde{\sigma}\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}\Big)\cdot\boldsymbol{v}\qquad\forall\,\boldsymbol{v}\in\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}}). \tag{4.16}$$

Thus $\tilde{\boldsymbol{u}}$ solves (3.15) in a distributional sense.

The initial condition $\tilde{\boldsymbol{u}}(\cdot,0)=0$ in $\tilde{\Omega}_c$ has been proved in Lemma 4.6. The stability comes from the weak convergence of $\tilde{\boldsymbol{u}}_\tau$ and Lemma 4.4:

$$\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})}\leq\underline{\lim}_{\tau\to 0}\|\tilde{\boldsymbol{u}}_\tau\|_{\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})}\leq C\,\|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\,.$$

The proof is completed. □

5. Convergence of the approximate problem. The purpose of this section is to study the convergence of the solution of (2.7) to the solution of (3.15) as $d \to 0$, where $d = \mathrm{dist}(\Omega_i; \Omega_{i+1})$ is the thickness of coating films. For convenience, we append the solution of (2.7) with a subscript d, namely, $\boldsymbol{u}_d \in \boldsymbol{X}$ denotes the solution. In electrical engineering, it is just $\sigma \boldsymbol{u}_d$ and $\mathbf{curl}\, \boldsymbol{u}_d$ that are important and used in computing iron loss. Therefore, we shall study the convergence of $\sigma \boldsymbol{u}_d$ and $\mathbf{curl}\, \boldsymbol{u}_d$ as $d \to 0$.

5.1. Nonlinear and time-dependent problems. First we consider the nonlinear and time-dependent eddy current problems (2.7) and (3.15).

THEOREM 5.1. *Let* $\mathrm{supp}(\boldsymbol{J}_s) \cap \tilde{\Omega}_c = \emptyset$ *and (H1)–(H3) be satisfied. Let* $\boldsymbol{u}_d$, $\tilde{\boldsymbol{u}}$ *be the solutions of* (2.7), (3.15) *respectively and assume* $\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \in \boldsymbol{L}^2(0,T; \boldsymbol{L}^2(\tilde{\Omega}_c))$. *Then*

$$\lim_{d\to 0} \left\{ \|\sigma(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d)\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d)\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \right\} = 0.$$

Proof. From (2.8) we know that

$$\int_\Omega \left[\sigma \frac{\partial \boldsymbol{u}_d}{\partial t} \cdot \boldsymbol{v} + \boldsymbol{H}(\mathbf{curl}\, \boldsymbol{u}_d) \cdot \mathbf{curl}\, \boldsymbol{v}\right] = \int_\Omega \boldsymbol{J}_s \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega).$$

Since $\mathrm{supp}(\boldsymbol{J}_s) \cap \Omega_c = \emptyset$ and $\mathrm{dist}(\Omega_i; \Omega_j) \geq d > 0$ for any $i \neq j$, we have

$$\int_\Omega \sigma \frac{\partial \boldsymbol{u}_d}{\partial t} \cdot (\chi \nabla \varphi) = \int_{\tilde{\Omega}_{\mathrm{even}}} \sigma \frac{\partial \boldsymbol{u}_d}{\partial t} \cdot \nabla \varphi = \int_{\Omega_{\mathrm{even}}} \sigma \frac{\partial \boldsymbol{u}_d}{\partial t} \cdot \nabla \varphi = 0 \qquad \forall \varphi \in H_0^1(\Omega).$$

Adding up the two equalities leads to

$$\int_\Omega \left[\sigma \frac{\partial \boldsymbol{u}_d}{\partial t} \cdot \boldsymbol{v} + \boldsymbol{H}(\widetilde{\mathbf{curl}}\, \boldsymbol{u}_d) \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v}\right] = \int_\Omega \boldsymbol{J}_s \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \tilde{\boldsymbol{X}}, \tag{5.1}$$

where we have used $\mathbf{curl}\, \boldsymbol{u}_d = \widetilde{\mathbf{curl}}\, \boldsymbol{u}_d$. And (3.15) reads

$$\int_\Omega \left[\tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \cdot \boldsymbol{v} + \tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}}) \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v}\right] = \int_\Omega \boldsymbol{J}_s \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \tilde{\boldsymbol{X}}. \tag{5.2}$$

Subtracting (5.1) from (5.2), we find that, for any $\boldsymbol{v} \in \tilde{\boldsymbol{X}}$,

$$\begin{aligned} \int_\Omega \sigma \frac{\partial}{\partial t}(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d) \cdot \boldsymbol{v} + &\int_\Omega \left[\boldsymbol{H}(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}}) - \boldsymbol{H}(\widetilde{\mathbf{curl}}\, \boldsymbol{u}_d)\right] \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v} \\ &= \int_\Omega (\sigma - \tilde{\sigma}) \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \cdot \boldsymbol{v} + \int_\Omega \left[\boldsymbol{H}(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}}) - \tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}})\right] \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v} \\ &= -\int_{\Omega_d} \tilde{\sigma} \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \cdot \boldsymbol{v} + \int_{\Omega_d} \left[\nu_0\, \widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}})\right] \cdot \widetilde{\mathbf{curl}}\, \boldsymbol{v}, \end{aligned} \tag{5.3}$$

where $\Omega_d := \tilde{\Omega}_c \backslash \bar{\Omega}_c$ denotes the region of coating films. Taking $\boldsymbol{v} = \tilde{\boldsymbol{u}} - \boldsymbol{u}_d$, (5.3) shows that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sigma \left|\tilde{\boldsymbol{u}} - \boldsymbol{u}_d\right|^2 + \nu_{\min} \int_\Omega \left|\widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}} - \boldsymbol{u}_d)\right|^2 \leq (\nu_{\max} + \sigma_{\max}) \mathcal{E}_d \left\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_d\right\|_{\tilde{\boldsymbol{X}}}, \tag{5.4}$$

where $\mathcal{E}_d$ is a function of t and defined by

$$\mathcal{E}_d := \left\{ \left\|\frac{\partial \tilde{\boldsymbol{u}}}{\partial t}\right\|^2_{\boldsymbol{L}^2(\Omega_d)} + \|\widetilde{\mathbf{curl}}\, \tilde{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega_d)} \right\}^{1/2}.$$

Using Theorem 2.2 and 4.7, we have

$$\|\tilde{\boldsymbol{u}}-\boldsymbol{u}_d\|_{\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})} \le C\,\|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\,.$$

Integrating (5.4) from 0 to $t\in(0,T]$ and using $\tilde{\boldsymbol{u}}(\cdot,0)=\boldsymbol{u}_d(\cdot,0)=0$ in Ω_c, we have

$$\|\sigma(\tilde{\boldsymbol{u}}-\boldsymbol{u}_d)\|^2_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}-\boldsymbol{u}_d)\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\le C\int_0^T\mathcal{E}_d(t)\mathrm{d}t,$$

where C only depends on T, $\sigma_{\min}$, $\sigma_{\max}$, $\nu_{\min}$, $\nu_{\max}$, and $\|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}$. Since

$$\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}\in\boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\tilde{\Omega}_c)\big),\quad \widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}\in\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega)),\quad |\Omega_d|<Cd,$$

where $|\Omega_d|$ denotes the measure of Ω_d, we easily get

$$\lim_{d\to0}\int_0^T\mathcal{E}_d(t)\mathrm{d}t\le T^{1/2}\lim_{d\to0}\Big(\int_0^T|\mathcal{E}_d(t)|^2\,\mathrm{d}t\Big)^{1/2}=0.$$

This completes the proof. ▯

5.2. Linear time-harmonic problems. The purpose of this subsection is to derive an explicit error estimate with respect to d for linear eddy current problems. For simplicity, we assume that $\boldsymbol{H}(\boldsymbol{B})=\nu_0\boldsymbol{B}$ and $\boldsymbol{J}_s$ is periodic in time so that (2.7) and (3.15) can be written into the time-harmonic forms at a single frequency $\omega>0$:

$$\text{(5.5)}\quad \text{Find } \boldsymbol{u}\in\boldsymbol{X}:\qquad \mathbf{i}\omega(\sigma\boldsymbol{u},\boldsymbol{v})+\nu_0(\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})=(\boldsymbol{J}_s,\boldsymbol{v})\qquad\forall\,\boldsymbol{v}\in\boldsymbol{X},$$

$$\text{(5.6)}\quad \text{Find } \tilde{\boldsymbol{u}}\in\tilde{\boldsymbol{X}}:\qquad \mathbf{i}\omega(\tilde{\sigma}\tilde{\boldsymbol{u}},\boldsymbol{v})+\nu_0(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}},\widetilde{\mathbf{curl}}\,\boldsymbol{v})=(\boldsymbol{J}_s,\boldsymbol{v})\qquad\forall\,\boldsymbol{v}\in\tilde{\boldsymbol{X}}.$$

THEOREM 5.2. *Let (H1) be satisfied. Assume* $\boldsymbol{J}_s\in\boldsymbol{L}^2(\Omega)$, $\operatorname{div}\boldsymbol{J}_s=0$, *and* $\operatorname{supp}(\boldsymbol{J}_s)\cap\tilde{\Omega}_c=\emptyset$*. Then problem* (5.5) *has a unique solution* $\boldsymbol{u}\in\boldsymbol{X}$*, problem* (5.6) *has a unique solution* $\tilde{\boldsymbol{u}}\in\tilde{\boldsymbol{X}}$*, and there exists a constant* C *only depending on* $\sigma_{\min}$ *and* $\sigma_{\max}$ *such that*

$$\text{(5.7)}\qquad \|\boldsymbol{u}\|_{\boldsymbol{X}}\le C\,\|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)}\,,\qquad \|\tilde{\boldsymbol{u}}\|_{\tilde{\boldsymbol{X}}}\le C\,\|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)}\,.$$

Proof. The theorem is a direct consequence of Lemma 2.1 and Lemma 3.1. ▯

THEOREM 5.3. *Assume* $\sigma|_{\Omega_i}=\sigma_i>0$ *for each* $1\le i\le I$ *and that* $\boldsymbol{J}_s\in\boldsymbol{L}^2(\Omega)$ *satisfies* $\operatorname{div}\boldsymbol{J}_s=0$ *and* $\operatorname{supp}(\boldsymbol{J}_s)\cap\tilde{\Omega}_c=\emptyset$*. Let* $\boldsymbol{u}\in\boldsymbol{X}$ *and* $\tilde{\boldsymbol{u}}\in\tilde{\boldsymbol{X}}$ *be the solutions of* (5.5) *and* (5.6) *respectively. Then there exists a constant* $C>0$ *only depending on* $\tilde{\Omega}_{\text{even}}$ *and* $\|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)}$ *such that*

$$\left\|\sigma^{1/2}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\right\|^2_{\boldsymbol{L}^2(\Omega)}+\nu_0\omega^{-1}\,\|\widetilde{\mathbf{curl}}\,(\boldsymbol{u}-\tilde{\boldsymbol{u}})\|^2_{\boldsymbol{L}^2(\Omega)}\le C\rho(d)\,d^{1/3},\qquad \lim_{d\to0}\rho(d)=0.$$

Proof. By similar arguments as for (5.1) and (5.2), we know that $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ satisfy

$$\text{(5.8)}\qquad \mathbf{i}\omega(\sigma\boldsymbol{u},\boldsymbol{v})+\nu_0(\widetilde{\mathbf{curl}}\,\boldsymbol{u},\widetilde{\mathbf{curl}}\,\boldsymbol{v})=(\boldsymbol{J}_s,\boldsymbol{v})\qquad\forall\,\boldsymbol{v}\in\tilde{\boldsymbol{X}},$$

$$\text{(5.9)}\qquad \mathbf{i}\omega(\tilde{\sigma}\tilde{\boldsymbol{u}},\boldsymbol{v})+\nu_0(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}},\widetilde{\mathbf{curl}}\,\boldsymbol{v})=(\boldsymbol{J}_s,\boldsymbol{v})\qquad\forall\,\boldsymbol{v}\in\tilde{\boldsymbol{X}}.$$

Subtracting (5.8) from (5.9) shows that

$$\int_\Omega \left[\mathbf{i}\omega\sigma(\tilde{\boldsymbol{u}}-\boldsymbol{u})\cdot\boldsymbol{v}+\nu_0\widetilde{\mathbf{curl}}\,(\tilde{\boldsymbol{u}}-\boldsymbol{u})\cdot\widetilde{\mathbf{curl}}\,\boldsymbol{v}\right]=-\mathbf{i}\omega\int_{\Omega_d}\tilde{\sigma}\tilde{\boldsymbol{u}}\cdot\boldsymbol{v}.$$

Taking $\boldsymbol{v}=\tilde{\boldsymbol{u}}-\boldsymbol{u}$, we find that

$$\left\|\sigma^{1/2}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\right\|^2_{\boldsymbol{L}^2(\Omega)}+\nu_0\omega^{-1}\left\|\widetilde{\mathbf{curl}}\,(\boldsymbol{u}-\tilde{\boldsymbol{u}})\right\|^2_{\boldsymbol{L}^2(\Omega)}\le\rho(d)\left\|\tilde{\boldsymbol{u}}\right\|_{\boldsymbol{L}^2(\Omega_d)}. \tag{5.10}$$

where $\rho(d):=\|\tilde{\sigma}(\tilde{\boldsymbol{u}}-\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega_d)}$. By Theorem 5.2 and Lemma 2.1, we know that $\|\tilde{\boldsymbol{u}}-\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}$ is uniformly bounded with respect to d. Then we have $\lim_{d\to 0}\rho(d)=0$.

Since $\tilde{\sigma}=\sigma_i$ in each $\tilde{\Omega}_i$, taking $\boldsymbol{v}=\chi\nabla v$, then (5.9) shows that

$$\int_{\tilde{\Omega}_i}\tilde{\sigma}\tilde{\boldsymbol{u}}\cdot\nabla v=\sigma_i\int_{\tilde{\Omega}_i}\tilde{\boldsymbol{u}}\cdot\nabla v=0\qquad\forall\, v\in H^1(\tilde{\Omega}_i)\ \text{ and even }\ i.$$

This indicates that $\operatorname{div}\tilde{\boldsymbol{u}}=0$ in $\tilde{\Omega}_i$ and $\tilde{\boldsymbol{u}}\cdot\boldsymbol{n}=0$ on $\partial\tilde{\Omega}_i$, that is, $\tilde{\boldsymbol{u}}\in\boldsymbol{H}(\mathbf{curl},\tilde{\Omega}_i)\cap\boldsymbol{H}_0(\mathrm{div},\tilde{\Omega}_i)$ for even i. By [11, Theorem 3.9], we have $\tilde{\boldsymbol{u}}\in\boldsymbol{H}^1(\tilde{\Omega}_i)$. Then an application of Hölder's inequality shows that

$$\begin{aligned}\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_d)}&\le|\Omega_d|^{1/3}\,\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^6(\Omega_d)}\le Cd^{1/3}\,\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^6(\tilde{\Omega}_{\text{even}})}\le Cd^{1/3}\,\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\tilde{\Omega}_{\text{even}})}\\&\le Cd^{1/3}\,\|\mathbf{curl}\,\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\tilde{\Omega}_{\text{even}})}\le Cd^{1/3},\end{aligned}$$

where the generic constant C only depends on $\tilde{\Omega}_{\text{even}}$ and $\|\boldsymbol{J}_s\|_{\boldsymbol{L}^2(\Omega)}$. This completes the proof. □

6. Numerical experiments. The purpose of this section is to validate the approximation of the new eddy current model (3.15) to the original eddy current problem (2.7) numerically.

We let $\mathcal{T}_h$ be a tetrahedral triangulation of Ω which subdivides each Ω_i into the union of tetrahedra, and let $\{t_n=n\tau:\ n=0,1,\cdots,N\}$, $\tau=T/N$ be the partition of the time interval $[0,T]$ for integer $N>0$. Let P_k be the space of polynomials of degree $k\ge 0$. First we introduce the third-order Lagrange finite element space

$$V_h=\left\{v\in H^1_0(\Omega):\ v|_K\in P_3(K),\ \forall\, K\in\mathcal{T}_h\right\},$$

and the second-order Nédélec edge element space in the second family [16]

$$\boldsymbol{U}_h=\left\{\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega):\ \boldsymbol{v}|_K\in\left(P_2(K)\right)^3,\ \forall\, K\in\mathcal{T}_h\right\}.$$

Similar to (3.10), we define a subspace of $\boldsymbol{U}_h$ by

$$\boldsymbol{U}_{\text{odd}}=\left\{\boldsymbol{v}\in\boldsymbol{U}_h:\ (\boldsymbol{v},\nabla\varphi)=0\quad\forall\,\varphi\in V_h,\ \varphi=\text{Const. in }\ \tilde{\Omega}_{\text{odd}}\right\}.$$

Then the discrete test function space is defined by

$$\tilde{\boldsymbol{X}}_h:=\boldsymbol{U}_{\text{odd}}+\chi\nabla V_h.$$

The fully discrete finite element approximation to problem (3.15) reads: Given $\tilde{\boldsymbol{u}}_0=0$, find $\tilde{\boldsymbol{u}}_n\in\tilde{\boldsymbol{X}}_h$ such that

$$\int_\Omega\left[\tilde{\sigma}\frac{\tilde{\boldsymbol{u}}_n-\tilde{\boldsymbol{u}}_{n-1}}{\tau}\cdot\boldsymbol{v}_h+\tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_n)\cdot\widetilde{\mathbf{curl}}\,\boldsymbol{v}_h\right]=\int_\Omega\boldsymbol{J}_n\cdot\boldsymbol{v}_h\qquad\forall\,\boldsymbol{v}_h\in\tilde{\boldsymbol{X}}_h, \tag{6.1}$$

where $\boldsymbol{J}_n$ is the temporal average of $\boldsymbol{J}_s$ over (t_{n-1}, t_n) and defined in (4.1).

Since this paper mainly focuses on the mathematical modeling of GO silicon steel laminations, we do not elaborate much on the numerical analysis for (6.1). We refer to [1, 5, 9, 22] for further studies on finite element methods for eddy current problems. Our implementation is based on the adaptive finite element package PHG [21] and the computations are carried out on the cluster LSEC-III of Chinese Academy of Sciences. The numerical experiment is performed for the TEAM Workshop Problem 21^c–M1 where the magnetic shield is made of 20 silicon steel laminations. We refer to [7] for more details of the model.

TABLE 6.1
Iron loss in the laminations and the magnetic plate (W).

Experimental value	3.72 [7]	
Calculated value	Total loss	3.73
	Loss in the laminations	2.789
	Loss in the magnetic plate	0.941

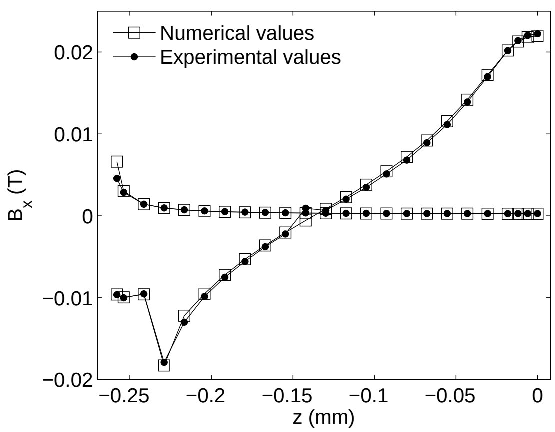

FIG. 6.1. *Numerical and experimental values of the magnetic flux density [7]. The couple of curves between* $B_x = 0\,\mathrm{T}$ *and* $B_x = 0.01\,\mathrm{T}$ *show the numerical and experimental values along the line* $\{(x, y, z) : x = -5.76\,\mathrm{mm},\ y = 0\,\mathrm{mm}\}$*, and the other couple of curves show the numerical and experimental values along the line* $\{(x, y, z) : x = 11.76\,\mathrm{mm},\ y = 0\,\mathrm{mm}\}$*.*

Since we are investigating the error between the solution of (2.7) and the solution of (3.15), to reduce the numerical error sufficiently, we adopt a fine mesh of Ω with 9×10^6 tetrahedra and 1.26×10^8 degrees of freedom. Table 6.1 shows that the calculated iron loss is close to the experimental value. And Figure 6.1 shows that the calculated values of the magnetic flux agree well with the experimental values [7]. Thus we conclude that the new eddy current model (3.15) provides an accurate approximation to the original problem (2.7).

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Appendix A.

The purpose of this appendix is to establish the wellposedness of the semi-discrete problem (4.1). Now we present the proof of Lemma 4.2.

Proof. First we write (4.1) as: Find $\tilde{\boldsymbol{u}}_n \in \tilde{\boldsymbol{X}}$ such that

$$\big(\tilde{\sigma}\tilde{\boldsymbol{u}}_n, \boldsymbol{v}\big) + \tau\big(\tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_n), \widetilde{\mathbf{curl}}\,\boldsymbol{v}\big) = (\tilde{\sigma}\tilde{\boldsymbol{u}}_{n-1} + \tau\boldsymbol{J}_n, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \tilde{\boldsymbol{X}}. \tag{A.1}$$

From Lemma 3.1, for any $\boldsymbol{w} \in \tilde{\boldsymbol{X}}$, there exists a unique solution $\mathcal{L}_n(\boldsymbol{w}) \in \tilde{\boldsymbol{X}}$ of the variational problem

$$\big(\mathcal{L}_n(\boldsymbol{w}), \boldsymbol{v}\big)_{\tilde{\boldsymbol{X}}} = \big(\tilde{\sigma}\boldsymbol{w}, \boldsymbol{v}\big) + \tau\big(\tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\,\boldsymbol{w}), \widetilde{\mathbf{curl}}\,\boldsymbol{v}\big) \qquad \forall\, \boldsymbol{v} \in \tilde{\boldsymbol{X}}. \tag{A.2}$$

Let $\boldsymbol{f}_n \in \tilde{\boldsymbol{X}}$ be the unique solution of the variational problem

$$(\boldsymbol{f}_n, \boldsymbol{v})_{\tilde{\boldsymbol{X}}} = (\tilde{\sigma}\tilde{\boldsymbol{u}}_{n-1} + \tau\boldsymbol{J}_n, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \tilde{\boldsymbol{X}}.$$

Clearly (A.1) is equivalent to the operator equation

$$\mathcal{L}_n(\tilde{\boldsymbol{u}}_n) = \boldsymbol{f}_n \qquad \text{in } \tilde{\boldsymbol{X}}. \tag{A.3}$$

From (H2) we infer that the operator $\mathcal{L}_n\colon \tilde{\boldsymbol{X}} \to \tilde{\boldsymbol{X}}$ is Lipschitz continuous. Moreover, the strict monotonicity of $\mathcal{L}_n$ comes directly from (H1)-(H2): for any $\boldsymbol{w}, \boldsymbol{v} \in \tilde{\boldsymbol{X}}$,

$$\begin{aligned}
&\big(\mathcal{L}_n(\boldsymbol{w}) - \mathcal{L}_n(\boldsymbol{v}), \boldsymbol{w} - \boldsymbol{v}\big)_{\tilde{\boldsymbol{X}}} \\
&\quad = \big(\tilde{\sigma}(\boldsymbol{w} - \boldsymbol{v}), \boldsymbol{w} - \boldsymbol{v}\big) + \tau\big(\tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\,\boldsymbol{w}) - \tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\,\boldsymbol{v}), \widetilde{\mathbf{curl}}\,(\boldsymbol{w} - \boldsymbol{v})\big) \\
&\quad \geq \min\big(\sigma_{\min}, \tau\nu_{\min}\big)\, \|\boldsymbol{w} - \boldsymbol{v}\|^2_{\tilde{\boldsymbol{X}}}\,.
\end{aligned}$$

By [20, Theorem 25.B], we know that (A.3) has a unique solution $\tilde{\boldsymbol{u}}_n$ for each $n \geq 1$.

Setting $\boldsymbol{v} = \tilde{\boldsymbol{u}}_n$ in (A.1) shows that

$$\big(\tilde{\sigma}(\tilde{\boldsymbol{u}}_n - \tilde{\boldsymbol{u}}_{n-1}), \tilde{\boldsymbol{u}}_n\big) + \tau\big(\tilde{\boldsymbol{H}}(\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_n), \widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_n\big) = \tau(\boldsymbol{J}_n, \tilde{\boldsymbol{u}}_n). \tag{A.4}$$

Using the initial value $\tilde{\boldsymbol{u}}_0 = 0$ and the inequality

$$2\big(\tilde{\sigma}(\tilde{\boldsymbol{u}}_n - \tilde{\boldsymbol{u}}_{n-1}), \tilde{\boldsymbol{u}}_n\big) \geq \left\|\tilde{\sigma}^{1/2}\tilde{\boldsymbol{u}}_n\right\|^2_{L^2(\Omega)} - \left\|\tilde{\sigma}^{1/2}\tilde{\boldsymbol{u}}_{n-1}\right\|^2_{L^2(\Omega)},$$

we have

$$2\sum_{n=1}^{m}\big(\tilde{\sigma}(\tilde{\boldsymbol{u}}_n - \tilde{\boldsymbol{u}}_{n-1}), \tilde{\boldsymbol{u}}_n\big) \geq \left\|\tilde{\sigma}^{1/2}\tilde{\boldsymbol{u}}_m\right\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{A.5}$$

Inserting (A.5) into (A.4) and using (H2), we find that

$$\left\|\tilde{\sigma}^{1/2}\tilde{\boldsymbol{u}}_m\right\|^2_{\boldsymbol{L}^2(\Omega)} + 2\nu_{\min}\sum_{n=1}^{m}\tau\,\|\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_n\|^2_{\boldsymbol{L}^2(\Omega)} \leq 2\sum_{n=1}^{m}\tau(\boldsymbol{J}_n, \tilde{\boldsymbol{u}}_n). \tag{A.6}$$

Let $\tilde{\boldsymbol{u}}_n = \hat{\boldsymbol{u}}_n + \chi\nabla u_n$ with $\hat{\boldsymbol{u}}_n \in \boldsymbol{X}_{\text{odd}}$ and $u_n \in H^1_0(\Omega)$. Since $\text{supp}(\boldsymbol{J}_s) \cap \tilde{\Omega}_c = \emptyset$ and $\text{div}\,\boldsymbol{J}_n = \tau_n^{-1}\int_{t_{n-1}}^{t_n}\text{div}\,\boldsymbol{J}_s = 0$, an application of Lemma 4.3 and Young's inequality shows that

$$\begin{aligned}
|(\boldsymbol{J}_n, \tilde{\boldsymbol{u}}_n)| &= |(\boldsymbol{J}_n, \hat{\boldsymbol{u}}_n)| \leq C\,\|\boldsymbol{J}_n\|_{\boldsymbol{L}^2(\Omega)}\,\|\mathbf{curl}\,\hat{\boldsymbol{u}}_n\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq \frac{C^2}{2\nu_{\min}}\,\|\boldsymbol{J}_n\|^2_{L^2(\Omega)} + \frac{\nu_{\min}}{2}\,\|\mathbf{curl}\,\hat{\boldsymbol{u}}_n\|^2_{\boldsymbol{L}^2(\Omega)} \\
&= \frac{C^2}{2\nu_{\min}}\,\|\boldsymbol{J}_n\|^2_{L^2(\Omega)} + \frac{\nu_{\min}}{2}\,\|\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_n\|^2_{\boldsymbol{L}^2(\Omega)}\,,
\end{aligned} \tag{A.7}$$

where the constant $C > 0$ only depends on Ω. Inserting (A.7) into (A.6) yields

$$\left\|\tilde{\sigma}^{1/2}\tilde{\boldsymbol{u}}_m\right\|^2_{\boldsymbol{L}^2(\Omega)} + \nu_{\min}\sum_{n=1}^{m}\tau\left\|\widetilde{\mathbf{curl}}\,\tilde{\boldsymbol{u}}_n\right\|^2_{\boldsymbol{L}^2(\Omega)} \le C^2\nu_{\min}^{-1}\sum_{n=1}^{m}\tau\left\|\boldsymbol{J}_n\right\|^2_{L^2(\Omega)}.$$

Then (4.2) comes directly from the definition of $\boldsymbol{J}_n$ and the arbitrariness of m. □

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