On Locking-free Finite Element Schemes for three-dimensional elasticity*

Qi He Wang Lie-heng Zheng Weiying†

State Key Laboratory of Scientific Engineering Computing,

Institute of Computational Mathematics and Scientific/Engeering Computing,

Academy of Mathematics and System Sciences,

Chinese Academy of Sciences, P. O. Box 2719, Beijing 100080, China

Abstract

In the present paper, the authors discuss the locking phenomenon of the finite element method for three-dimensional elasticity as the Lamé constant $\lambda \to \infty$. Three kinds of finite elements are proposed and analyzed to approximate the three-dimensional elasticity with pure displacement boundary condition. Optimal order error estimates which are uniform with respect to $\lambda \in (0, +\infty)$ are obtained for three schemes. Furthermore, numerical results are presented to show that, our schemes are locking-free and and the trilinear conforming finite element scheme is locking.

Keywords: three-dimensional elasticity, locking-free, nonconforming finite element.

AMS subject classification: 65N30, 73V05

1 Introduction

For the linear isotropic elasticity, it is well known that many numerical methods suffer deteriorations in performance as the Lame constant $\lambda \to \infty$, i.e., as the material becomes incompressible[1]. This is the so-called locking phenomenon. Many literatures concerning the planar elasticity have appeared to be locking-free[2] [3] [9] [13] [14] [15]. In 1983,

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†The third author is responsible for the paper.

M. Vogelius [18] considered conforming finite element approximations to the linear planar elasticity as $\lambda \to \infty$. He showed that the piecewise linear conforming finite element scheme did not converge any more. For higher order conforming finite element schemes, optimal error estimates could not be obtained. To overcome the locking, we need to construct some finite element schemes whose optimal error estimates are uniform with respect to $\lambda \in (0, \infty)$. They are nonconforming in general. In [2],[3],[14] and [15], some nonconforming finite finite elements are constructed and analyzed, to be locking-free. The authors obtained optimal error estimates uniform for $\lambda \in (0, \infty)$, by virtue of the variational formula of pure displacement boundary value problem, based on the minimization of the energy functional. The pure traction boundary value problem was considered in [9], [12] and [16] by triangular element approximations, in [21] by quadrilateral element approximations and [13] by the NRQ_1 element approximations following the argument of [21] by the mixed finite element analysis.

To the best of our knowledge, no paper deals with the locking phenomenon of three-dimensional elasticity by finite element methods. Since discrete variational formulas, based on the minimization of the energy functional, are easier to be solved than the mixed formula, we consider this formula with pure displacement boundary condition. In the present paper, the three-dimensional Crouzeix-Raviart element is showed to be locking-free and the optimal error estimate is obtain. We construct two kinds of nonconforming cuboidal finite elements showed to be locking-free and obtain optimal error estimates of them. The order of one of our schemes is the lowest. We also present some numerical experiments to show the locking phenomenon of the trilinear conforming finite element and the locking-free of our lowest order nonconforming finite element. The conforming element converges well when λ is small, but loses convergency when $\lambda \to \infty$. Our lowest locking-free scheme converges very well and uniformly for $\lambda \in (0, \infty)$.

The paper is arranged as follows: In section 2, we present the preliminary consideration of three-dimensional linear elasticity with pure displacement boundary condition, the locking phenomenon of conforming finite element method and the construction of locking-free finite element method. In section 3, we present the Crouziex-Raviart tetrahedral finite element and construct two kinds of nonconforming cuboidal elements first, then show that they satisfy some general conditions required to be locking-free. In section 4, three finite element schemes are presented and showed to be locking-free; optimal error estimates of them are obtained, uniformly for $\lambda \in (0, \infty)$. We end this paper with some numerical examples in the last section.

2 Preliminary

For isotropic and homogeneous materials, we consider the pure displacement boundary value problem of three-dimensional linear elasticity. Let $\Omega \in R^3$ be a bounded convex polyhedron with the boundary $\partial\Omega$. The displacement $\vec{u}(x) = (u_1(x), u_2(x), u_3(x))^T$ satisfies the following partial differential equation:

$$\begin{cases} -div\,\sigma(\vec{u}) = \vec{f}, & \text{in} \quad \Omega, \\ \vec{u} = \vec{0}, & \text{on} \quad \partial\,\Omega, \end{cases} \tag{2.1}$$

where $\vec{f} \in L^2(\Omega)^3$, and

$$\sigma(\vec{u}) = \mu\left(\nabla\vec{u} + (\nabla\vec{u})^T\right) + \lambda\, div\vec{u}\, I, \qquad div\vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z},$$

and I is the identity. (2.1) is equivalent to the following boundary value problem

$$\begin{cases} -\mu\Delta\vec{u} - (\mu+\lambda)\nabla(div\vec{u}) = \vec{f} & \text{in} \quad \Omega, \\ \vec{u} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

whose equivalent weak form is

$$\begin{cases} \text{Find} \quad \vec{u} \in V, & \text{such that} \\ a(\vec{u},\vec{v}) = (\vec{f},\vec{v}) & \forall\vec{v} \in V, \end{cases} \tag{2.3}$$

where $V = H_0^1(\Omega)^3$,

$$a(\vec{u},\vec{v}) = \int_\Omega \Big\{\mu \sum_{i=1}^{3} \nabla u_i \cdot \nabla v_i + (\mu+\lambda)(div\vec{u})(div\vec{v})\Big\} dx, \tag{2.4}$$

$$(\vec{f},\vec{v}) = \int_\Omega \vec{f}\cdot\vec{v}dx, \tag{2.5}$$

and $\lambda \in (0,\infty)$, $\mu \in [\mu_1,\mu_2]$, $0 < \mu_1 < \mu_2$ are Lamé constants.

It is easy to see that the bilinear form in (2.4) is symmetric, coercive and continuous on V by Poincáre's inequality and the Lax-Milgram Theorem. So there exists a unique solution of (2.3). To analyze the convergence of our finite element schemes, we need the following assumption:

Proposition 2.1 Assume $\Omega \subset R^3$ is a convex polyhedron. $\vec{u}$ is the solution of (2.1) or (2.2). Then the following regularity of $\vec{u}$ is true:

$$||\vec{u}||_{2,\Omega} + \lambda||div\vec{u}||_{1,\Omega} \le C||\vec{f}||_{0,\Omega}, \tag{2.6}$$

where C is a positive constant independent of λ.

Remark 2.2 Proposition 2.1 is true for the planar elasticity(see [2][3]). But as for the three-dimensional case, we have not found the result to the best our knowledge. Since the proof of (2.6) is far more difficult than that of the planar case and beyond the object of this paper, we use it as an assumption and do not attempt to prove it. A rougher result is (see Theorem 6.3-6 of [6], pp. 262): $\vec{u} \in H^2(\Omega)^3$ in the case of $\partial\Omega \in C^2$. Furthermore, we give a numerical example in the last section to show our error estimate coincides with real computation very well.

Now we consider conforming finite element approximations to the problem (2.3). Let $\mathcal{T}_h$ be a regular subdivision of Ω and $V_h^c \subset V = H_0^1(\Omega)^3$ be the conforming finite element space. Then the approximation of (2.3) is

$$\begin{cases} \text{Find} \quad \vec{u}_h \in V_h^c, & \text{such that} \\ a(\vec{u}_h, \vec{v}_h) = (\vec{f}, \vec{v}_h), & \forall \vec{v}_h \in V_h^c. \end{cases} \tag{2.7}$$

We can establish the following error estimate:

Theorem 2.3(see [3] and [15]) Let $\vec{u}$ and $\vec{u}_h$ are solutions of (2.3) and (2.7) respectively, then

$$\|\vec{u} - \vec{u}_h\|_{1,\Omega} \leq C\sqrt{2\mu + \lambda} \cdot h|\vec{u}|_{2,\Omega}, \tag{2.8}$$

where $C > 0$ is a generic constant and independent of h and λ.

Remark 2.4 From Theorem 2.3, it can be seen that the solution $\vec{u}_h$ of the conforming finite element approximation (2.5) converges to the solution $\vec{u}$ of the problem (2.2), as $h \to 0$, for each fixed λ; but we can not say anything for convergency of $\vec{u}_h$ when $\lambda \to \infty$. In fact, in the last section, we will show by numerical example that the trilinear conforming finite element solution of (2.2) does not converge to the true solution any more. By (2.6) and the argument of [15], we will bound

$$\lambda\|div(\vec{u} - \Pi_h\vec{u})\|_{0,\Omega}^2, \tag{2.9}$$

by a quantity independent of λ to overcome the locking phenomenon. If we can construct a finite element space V_h, and an interpolation operator $\Pi_h : H^2(\Omega)^3 \to V_h$, such that

$$div\Pi_h\vec{u} = \gamma_h div\vec{u}, \tag{2.10}$$

where the operator $\gamma_h : L^2(\Omega) \to W_h$ and W_h is a piecewise polynomial space of lower order than that of V_h; and the following error estimate is true:

$$\|div\vec{u} - \gamma_h(div\vec{u})\|_{0,\Omega} \leq Ch|div\vec{u}|_{1,\Omega}. \tag{2.11}$$

Combining (2.6) and (2.9)-(2.11) gives an uniformly optimal error estimate with respect to $\lambda \in (0, \infty)$.

This idea coincides with the Commuting diagram property:

$$\begin{array}{lcl} U \subset H(div;\Omega) & \stackrel{div}{\longrightarrow} & L^2(\Omega) \\ \downarrow \Pi_h & & \downarrow \gamma_h \\ V_h & \stackrel{div}{\longrightarrow} & W_h \end{array} \tag{2.12}$$

Following this, we introduce three kind of finite element interpolation operators giving our desired locking-free schemes.

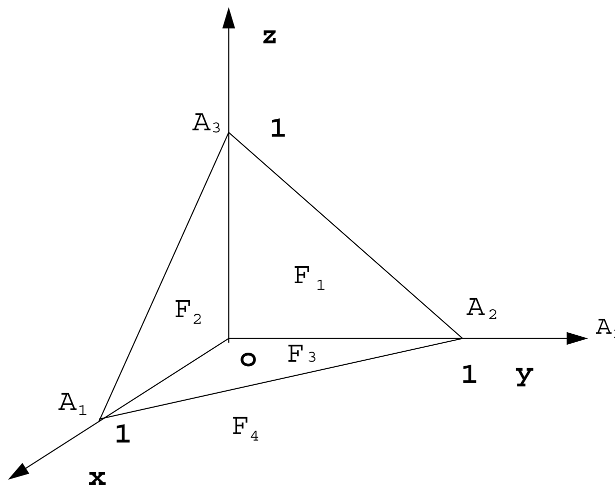

Fig 3.1: A tetrahedral element.

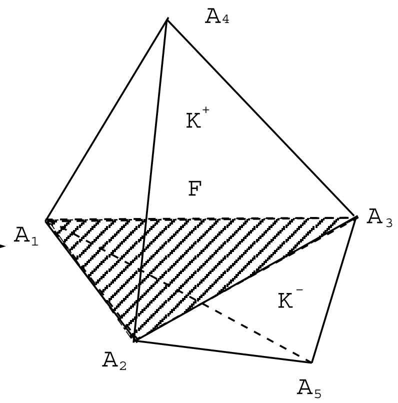

Fig 3.2: Adjacent tetrahedral elements.

3 Locking-free nonconforming finite element schemes

We define nonconforming finite element spaces as follows:

$$V_h = \Big\{ \vec{v}_h \in L^2(\Omega)^3 \mid \vec{v}_h \in P_K, \int_F \vec{v}_h^+ ds = \int_F \vec{v}_h^- ds, \forall F \subset \partial K^+ \cap \partial K^-, \\ \text{and } F \not\subset \partial\Omega, \int_F \vec{v}_h ds = 0, \forall F \subset \partial\Omega \Big\}, \tag{3.1}$$

where P_K are three kinds of shape function spaces defined in the rest of this section. We define the discrete bilinear form and energy norm as

$$a_h(\vec{u}_h, \vec{v}_h) := \sum_{K \in \mathcal{T}_h} \int_K \{\mu \nabla \vec{u}_h \cdot \nabla \vec{v}_h + (\mu + \lambda)(div\vec{u}_h)(div\vec{v}_h)\} dx, \tag{3.2}$$

$$\|\vec{u}\|_h := \sqrt{a_h(\vec{u}_h, \vec{u}_h)}. \tag{3.3}$$

Then the approximation of (2.3) is

$$\begin{cases} \text{Find} \quad \vec{u}_h \in V_h, & \text{such that} \\ a_h(\vec{u}_h, \vec{v}_h) = (\vec{f}, \vec{v}_h), & \forall \vec{v}_h \in V_h. \end{cases} \tag{3.4}$$

For the sake of simplicity in notation, we denote Π_K, γ_K, Π_h and γ_h as the corresponding finite element interpolation operators, in all the following three cases.

3.1 Crouzeix-Raviart Element Approximation

Now we introduce the three-dimensional Crouzeix-Raviart finite element interpolation operators Π_K and Π_h, and show that $div\Pi_K$ is in the form of (2.10). Let $\mathcal{T}_h$ be a regular subdivision. For any $K \in \mathcal{T}_h$, by notations in [7], the triple (K, Σ_K, P_K) is defined as follows:

$$\begin{cases} K \text{ — tetrahedron;} \\ \Sigma_K = \left\{ \frac{1}{|e_i|} \int_{e_i} v_j ds,\ 1 \le j \le 3;\ 1 \le i \le 4 \right\}, \quad \forall \vec{v} = (v_1, v_2, v_3)^T \in P_K; \\ P_K = P_1(K) \times P_1(K) \times P_1(K). \end{cases} \tag{3.5}$$

$P_k(K)$ is the polynomial space of order k on K. Choose a basis of $P_1(K)$ to be

$$p_1 = 1 - 3\lambda_1, \quad p_2 = 1 - 3\lambda_2, \quad p_3 = 1 - 3\lambda_3, \quad p_4 = 1 - 3\lambda_4. \tag{3.6}$$

For any function $v \in H^1(K)$, define the interpolation operator on K as

$$\Pi_K v = \sum_{i=1}^{4} \frac{1}{|e_i|} \int_{e_i} v ds\ p_i. \tag{3.7}$$

For any vector-valued function $\vec{u} = (u_1, u_2, u_3)^T$, define the interpolation operators as: $\Pi_K \vec{u} = (\Pi_K u_1, \Pi_K u_2, \Pi_K u_3)^T$, $\Pi_h \vec{u}|_K = \Pi_K \vec{u}$, $\forall K \in \mathcal{T}_h$.

Lemma 3.1 For any $\vec{v} \in H^1(\Omega)^3$, $\Pi_h \vec{v} \in V_h$.

Proof: In fact, for two adjacent elements K^+ and K^- with a common face F, by direct calculation, we have

$$\int_F \Pi_{K^+} \vec{v} ds = \int_F \vec{v} ds = \int_F \Pi_{K^-} \vec{v} ds.$$

The proof is finished.

Lemma 3.2 There exists a positive constant C independent of the mesh size h, such that

$$\|\vec{v} - \Pi_h \vec{v}\|_{0,\Omega} + h|\vec{v} - \Pi_h \vec{v}|_{1,\Omega} \le Ch^2 \|\vec{v}\|_{2,\Omega}, \quad \forall \vec{v} \in H^2(\Omega)^3. \tag{3.8}$$

Proof: Since $P_K = (P_1(K))^3$, Lemma 3.1 can be proved by the Bramble-Hilbert Theorem[5] and the technique of affine transformation. The proof is finished.

We choose γ_h to be the simplest piecewise projection operator defined as follows. Let γ_K be the L^2-projection from $L^2(K)$ onto $P_0(K)$, i.e.

$$\gamma_K(div\vec{v}) = \frac{1}{|K|} \int_K div\vec{v} dxdy, \tag{3.9}$$

where $|K| = \int_K 1 dxdy$. Denote W_h as the piecewise constant space defined on Ω, i.e.

$$W_h = \{w \mid w|_K = Const.,\ \forall K \in \mathcal{T}_h\}. \tag{3.10}$$

and $\gamma_h : L^2(\Omega) \to W_h$ defined as

$$\gamma_h w|_K = \gamma_K w \quad \forall\, w \in L^2(\Omega), \quad \forall K \in \mathcal{T}_h. \tag{3.11}$$

Lemma 3.3 There exists a constant C independent of the mesh size h, such that

$$div\Pi_K\vec{v} = \gamma_K div\vec{v}, \quad \forall K \in \mathcal{T}_h, \quad \forall \vec{v} \in H^1(\Omega)^3, \tag{3.12}$$

$$\|w - \gamma_h w\|_{0,\Omega} \leq Ch\|w\|_{1,\Omega}, \quad \forall w \in H^1(\Omega). \tag{3.13}$$

Proof: For any $\vec{v} \in H^1(\Omega)^3$ and $K \in \mathcal{T}_h$, by Green's formula,

$$\begin{aligned} div\Pi_K\vec{v} &= \frac{1}{|K|}\int_K div\Pi_K\vec{v}dx = \frac{1}{|K|}\sum_{i=1}^{4}\int_{e_i}(\Pi_K\vec{v})\cdot\vec{n}ds \\ &= \frac{1}{|K|}\sum_{i=1}^{4}\int_{e_i}\vec{v}\cdot\vec{n}ds = \frac{1}{|K|}\int_K div\vec{v}dx \\ &= \gamma_K(div\vec{v}), \end{aligned} \tag{3.14}$$

where $\vec{n}$ is the outer normal of ∂K. By the definition of γ_h and the technique of affine transformation, (3.11) can be easily gotten. The proof is finished.

3.2 Locking-free cuboidal finite element of the lowest order

Assume $\Omega \subset R^3$ be a cuboidal domain and $\mathcal{T}_h$ be one of its cuboidal regular partition. h is the mesh length. The triple $(\hat{K}, \hat{\Sigma}, \hat{P})$ is defined as:

$$\begin{cases} \hat{K} & := [-1,1]^3\ ; \\ \hat{P} & := \text{span}\{\vec{p}_i = (p_{i1}, p_{i2}, p_{i3}),\ 1 \leq i \leq 6\}\ ; \\ \hat{\Sigma} & := \left\{\dfrac{1}{|\hat{F}_i|}\displaystyle\int_{\hat{F}_i} v_j d\hat{s},\ 1 \leq j \leq 3;\ 1 \leq i \leq 6\right\}, \quad (v_1, v_2, v_3)^T \in \hat{P}, \end{cases} \tag{3.15}$$

where $\hat{F}_i$, $1 \leq i \leq 6$ are faces of $\hat{K}$ defined as

$$\begin{array}{ll} \hat{F}_1 = \{(\xi,\eta,\zeta) \in \hat{K} \mid \xi = 1\}, & \hat{F}_2 = \{(\xi,\eta,\zeta) \in \hat{K} \mid \eta = 1\}, \\ \hat{F}_3 = \{(\xi,\eta,\zeta) \in \hat{K} \mid \zeta = 1\}, & \hat{F}_4 = \{(\xi,\eta,\zeta) \in \hat{K} \mid \xi = -1\}, \\ \hat{F}_5 = \{(\xi,\eta,\zeta) \in \hat{K} \mid \eta = -1\}, & \hat{F}_6 = \{(\xi,\eta,\zeta) \in \hat{K} \mid \zeta = -1\}, \end{array}$$

and the basis of $\hat{P}$ is defined as

$$\begin{array}{lll} p_{11} = 1 + \frac{1}{2}\xi - \frac{3}{4}\eta^2 - \frac{3}{4}\zeta^2 & p_{12} = \frac{1}{4} + \frac{1}{2}\eta - \frac{3}{4}\eta^2 & p_{13} = 1 - \frac{1}{2}\xi - \frac{3}{4}\eta^2 - \frac{3}{4}\zeta^2 \\ p_{14} = -\frac{1}{4} + \frac{1}{2}\eta + \frac{3}{4}\eta^2 & p_{15} = \frac{1}{4} + \frac{1}{2}\zeta - \frac{3}{4}\zeta^2 & p_{16} = -\frac{1}{4} - \frac{1}{2}\zeta + \frac{3}{4}\zeta^2 \\ p_{21} = \frac{1}{4} + \frac{1}{2}\xi - \frac{3}{4}\xi^2 & p_{22} = 1 + \frac{1}{2}\eta - \frac{3}{4}\xi^2 - \frac{3}{4}\zeta^2 & p_{23} = -\frac{1}{4} + \frac{1}{2}\xi + \frac{3}{4}\xi^2 \\ p_{24} = 1 - \frac{1}{2}\eta - \frac{3}{4}\xi^2 - \frac{3}{4}\zeta^2 & p_{25} = -\frac{1}{4} + \frac{1}{2}\zeta + \frac{3}{4}\zeta^2 & p_{26} = -\frac{1}{4} - \frac{1}{2}\zeta + \frac{3}{4}\zeta^2 \\ p_{31} = -\frac{1}{4} + \frac{1}{2}\xi + \frac{3}{4}\xi^2 & p_{32} = -\frac{1}{4} + \frac{1}{2}\eta + \frac{3}{4}\eta^2 & p_{33} = -\frac{1}{4} - \frac{1}{2}\xi + \frac{3}{4}\xi^2 \\ p_{34} = -\frac{1}{4} - \frac{1}{2}\eta + \frac{3}{4}\eta^2 & p_{35} = 1 + \frac{1}{2}\zeta - \frac{3}{4}\xi^2 - \frac{3}{4}\eta^2 & p_{36} = 1 - \frac{1}{2}\zeta - \frac{3}{4}\xi^2 - \frac{3}{4}\eta^2. \end{array} \tag{3.16}$$

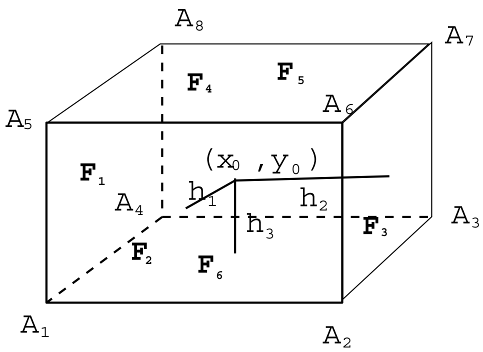

Fig 3.3: A cuboidal element.

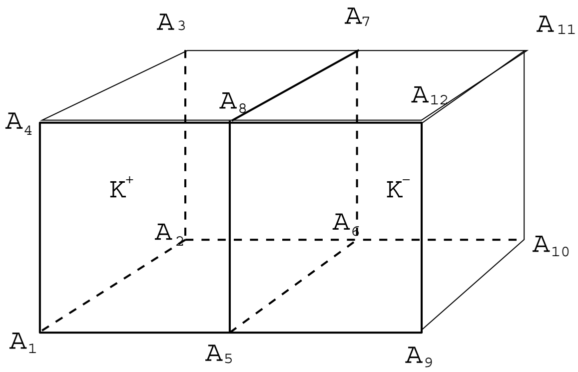

Fig 3.4: Adjacent cuboidal elements.

It is easy to show that every function in $\hat{P}$ is uni-solvent with respect to $\hat{\Sigma}$ and

$$\int_{\hat{F}_i} p_{mj}\, d\hat{s} = \delta_{ij}, \quad 1 \le m \le 3, \quad 1 \le i, j \le 6. \tag{3.17}$$

We will define an 18-freedom cuboidal finite element interpolation operator for vector-valued functions in the following. It has similar properties to the Crouzeix-Raviart interpolation operator and $div\Pi_K$ satisfies the commuting diagram (2.1). In next section, we will show that it is locking-free.

First we define $\hat{\Pi}$ on the reference element as: for any $\hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3) \in H^1(\hat{K})^3$,

$$\hat{\Pi}\vec{v} \;:=\; \left(\sum_{i=1}^{6} \frac{1}{|\hat{F}_i|} \int_{\hat{F}_i} \hat{v}_1\, d\hat{s}\; p_{1i}, \quad \sum_{i=1}^{6} \frac{1}{|\hat{F}_i|} \int_{\hat{F}_i} \hat{v}_2\, d\hat{s}\; p_{2i}, \quad \sum_{i=1}^{6} \frac{1}{|\hat{F}_i|} \int_{\hat{F}_i} \hat{v}_3\, d\hat{s}\; p_{3i} \right)^T. \tag{3.18}$$

For any $K \in \mathcal{T}_h$, let (x_0, y_0, z_0) be the center of K and h_{1K}, h_{2K}, h_{3K} be edge lengths of K(see fig 3.3). The affine transformation $F_K^{-1} : \; K \to \hat{K}$ is defined as:

$$\xi = \frac{x - x_0}{h_{1K}}, \quad \eta = \frac{y - y_0}{h_{2K}}, \quad \zeta = \frac{z - z_0}{h_{3K}}. \tag{3.19}$$

For any $\vec{v} \in H^1(K)^3$, Piola's transformation(page 100 of [4]) reads

$$\hat{v} = \begin{pmatrix} h_{2K}h_{3K} & 0 & 0 \\ 0 & h_{1K}h_{3K} & 0 \\ 0 & 0 & h_{1K}h_{2K} \end{pmatrix} \vec{v} \circ F_K. \tag{3.20}$$

Hence the shape function space on K is

$$P_K = \left\{ \vec{v} \mid (h_2 h_3\; v_1 \circ F_K,\; h_1 h_3\; v_2 \circ F_K,\; h_1 h_2\; v_3 \circ F_K)^T \in \hat{P} \right\}. \tag{3.21}$$

For any $\vec{v} \in H^1(\Omega)^3$, the local and global finite element interpolation operators Π_K and Π_h are defined as

$$\widehat{\Pi_K \vec{v}} = \hat{\Pi}\hat{v}, \quad (\Pi_h \vec{v})|_K = \Pi_K(\vec{v}|_K)\;. \tag{3.22}$$

Lemma 3.4 $\hat{\Pi}$ is a bounded linear operator on $H^1(\hat{K})^3$. For any $\vec{v} \in H^1(\Omega)^3$, $\Pi_h \vec{v} \in V_h$.

Proof: Thanks to (3.17), (3.18), and (3.20), the Lemma can be proved by means of scaling techniques and the trace inequality. □

The definitions of the L^2-orthogonal projection γ_K, piecewise constant L^2-projection γ_h, and piecewise constant function space W_h are same to (3.9)–(3.11), but based on cuboidal partitions of Ω. We have the following results by similar arguments in Lemma 3.2 and 3.3 .

Lemma 3.5 There exists a positive constant C independent of h such that

$$div \Pi_K \vec{v} = \gamma_K div \vec{v}, \quad \forall K \in \mathcal{T}_h, \quad \forall \vec{v} \in H^1(\Omega)^3; \tag{3.23}$$

$$\|w - \gamma_h w\|_{0,\Omega} \leq Ch \|w\|_{1,\Omega}, \quad \forall w \in H^1(\Omega); \tag{3.24}$$

$$\|\vec{v} - \Pi_h \vec{v}\|_{0,\Omega} + h|\vec{v} - \Pi_h \vec{v}|_{1,\Omega} \leq Ch^2 \|\vec{v}\|_{2,\Omega}, \quad \forall \vec{v} \in H^2(\Omega)^3. \tag{3.25}$$

3.3 21-freedom interpolation operator

Let the cuboidal domain Ω and partition $\mathcal{T}_h$ be defined same to those in §3.2. We define reference finite element triple $(\hat{K}, \hat{\Sigma}, \hat{P})$ as:

$$\left\{ \begin{array}{ll} \hat{K} & := [-1,1]^3; \\ \hat{P} & := \text{span}\{\vec{p}_i, \ 1 \leq i \leq 21\}; \\ \hat{\Sigma} & := \left\{ \dfrac{1}{|\hat{F}_i|} \displaystyle\int_{\hat{F}_i} v_j \, d\hat{s}, \quad 1 \leq j \leq 3; \quad 1 \leq i \leq 6; \right. \\ & \qquad \left. \displaystyle\int_{\hat{K}} div\vec{v} \, \xi \, d\xi d\eta d\zeta, \int_{\hat{K}} div\vec{v} \, \eta \, d\xi d\eta d\zeta, \int_{\hat{K}} div\vec{v} \, \zeta \, d\xi d\eta d\zeta \right\} \\ & := \{l_i(\vec{v}), \ 1 \leq i \leq 21\}, \qquad \forall \vec{v} = (v_1, v_2, v_3)^T \in \hat{P}. \end{array} \right. \tag{3.26}$$

where the basis functions $\vec{p}_i$ are defined as

$$\vec{p}_1 = (-\frac{1}{16} + \frac{1}{8}\xi + \frac{3}{16}\xi^2, \ 0, \ -\frac{3}{8}\xi\zeta)^T, \qquad \vec{p}_2 = (-\frac{3}{8}\xi\eta, \ \frac{1}{4} + \frac{1}{8}\xi - \frac{3}{16}\eta^2 - \frac{3}{16}\zeta^2, \ 0)^T,$$

$$\vec{p}_3 = (0, \ 0, \ -\frac{1}{16} + \frac{1}{8}\xi + \frac{3}{16}\xi^2)^T, \qquad \vec{p}_4 = (-\frac{1}{16} + \frac{1}{8}\eta + \frac{3}{16}\eta^2, \ 0, \ 0)^T,$$

$$\vec{p}_5 = (-\frac{3}{8}\xi\eta, \ -\frac{1}{16} + \frac{1}{8}\eta + \frac{3}{16}\eta^2, \ 0)^T, \qquad \vec{p}_6 = (0, \ \frac{3}{8}\eta\zeta, \ \frac{1}{4} + \frac{1}{8}\eta - \frac{3}{16}\xi^2 - \frac{3}{16}\zeta^2)^T,$$

$$\vec{p}_7 = (\frac{1}{4} + \frac{1}{8}\zeta - \frac{3}{16}\xi^2 - \frac{3}{16}\eta^2, \ 0, \ \frac{3}{8}\xi\zeta)^T, \qquad \vec{p}_8 = (0, \ -\frac{1}{16} + \frac{1}{8}\zeta + \frac{3}{16}\zeta^2, \ 0)^T,$$

$$\vec{p}_9 = (0, \ -\frac{3}{8}\eta\zeta, \ -\frac{1}{16} + \frac{1}{8}\zeta + \frac{3}{16}\zeta^2)^T, \qquad \vec{p}_{10} = (-\frac{1}{16} - \frac{1}{8}\xi + \frac{3}{16}\xi^2, \ 0, \ -\frac{3}{8}\xi\zeta)^T,$$

$$\vec{p}_{11} = (\frac{3}{8}\xi\eta, \ \frac{1}{4} - \frac{1}{8}\xi - \frac{3}{16}\eta^2 - \frac{3}{16}\zeta^2, \ 0)^T, \qquad \vec{p}_{12} = (0, \ 0, \ -\frac{1}{16} - \frac{1}{8}\xi + \frac{3}{16}\xi^2)^T,$$

$$\vec{p}_{13} = (-\frac{1}{16} - \frac{1}{8}\eta + \frac{3}{16}\eta^2, \ 0, \ 0)^T, \qquad \vec{p}_{14} = (-\frac{3}{8}\xi\eta, \ -\frac{1}{16} - \frac{1}{8}\zeta + \frac{3}{16}\zeta^2, \ 0)^T,$$

$$\vec{p}_{15}=(0,\ \frac{3}{8}\eta\zeta,\ \frac{1}{4}-\frac{1}{8}\eta-\frac{3}{16}\xi^2-\frac{3}{16}\zeta^2)^T,\quad \vec{p}_{16}=(\frac{1}{4}-\frac{1}{8}\zeta-\frac{3}{16}\xi^2-\frac{3}{16}\eta^2,\ 0,\ \frac{3}{8}\xi\zeta)^T,$$
$$\vec{p}_{17}=(0,\ -\frac{1}{16}-\frac{1}{8}\zeta+\frac{3}{16}\zeta^2,\ 0)^T,\qquad \vec{p}_{18}=(0,\ -\frac{3}{8}\eta\zeta,\ -\frac{1}{16}-\frac{1}{8}\zeta+\frac{3}{16}\zeta^2)^T,$$
$$\vec{p}_{19}=(0,\ 0,\ \frac{3}{8}\xi\zeta)^T,\qquad \vec{p}_{20}=(\frac{3}{8}\xi\eta,\ 0,\ 0)^T,\qquad \vec{p}_{21}=(0,\ \frac{3}{8}\eta\zeta,\ 0)^T.$$

By direct calculations, we get

$$l_i(\vec{p}_j)=\delta_{ij},\quad 1\le i,j\le 21. \tag{3.27}$$

For any $\hat{v}\in H^1(\hat{K})^3$, we define the interpolation operator $\hat{\Pi}$ as

$$\hat{\Pi}\hat{v}=\sum_{i=1}^{21} l_i(\hat{v})\vec{p}_i. \tag{3.28}$$

Lemma 3.6 $\hat{\Pi}$ is a bounded operator over $H^1(\hat{K})^3$, and there exists a constant C, such that

$$||\hat{\Pi}\vec{v}-\vec{v}||_{1,\hat{K}}\le C|\vec{v}|_{2,\hat{K}},\quad \forall\vec{v}\in H^2(\hat{K})^3. \tag{3.29}$$

Proof: By direct calculations, we have $P_1(\hat{K})^3\subset\hat{P}$. Furthermore, in view of (3.28) and the definition of $l_i(\hat{v})$, it is easy to see that $\hat{\Pi}$ is bounded by the trace theorem. Hence (3.29) is true by the Bramble-Hilbert Theorem. □

Lemma 3.7 The operator defined in (3.28) satisfies

$$\int_{\hat{F}_i}\hat{\Pi}\hat{v}d\hat{s} = \int_{\hat{F}_i}\hat{v}d\hat{s},\quad 1\le i\le 6; \tag{3.30}$$
$$\int_{\hat{K}} div\hat{\Pi}\hat{v}\ p\ d\xi d\eta d\zeta = \int_{\hat{K}} div\hat{v}\ p\ d\xi d\eta d\zeta,\quad \forall p\in P_1(\hat{K}). \tag{3.31}$$

Proof: (3.30) is clearly true by (3.26) — (3.28). Denote $d\hat{x}=d\xi d\eta d\zeta$. Since $div(\hat{\Pi}\hat{v})\in P_1(\hat{K})$, by Green's Formula, we have

$$\int_{\hat{K}} div\hat{\Pi}\vec{\hat{v}}d\hat{x}=\int_{\partial\hat{K}}\hat{\Pi}\vec{\hat{v}}\cdot\vec{n}d\hat{s}=\int_{\partial\hat{K}}\vec{\hat{v}}\cdot\vec{n}d\hat{s}=\int_{\hat{K}} div\vec{\hat{v}}d\hat{x};$$
$$\int_{\hat{K}} div\hat{\Pi}\vec{\hat{v}}\ p\ d\hat{x}=\int_{\hat{K}} div\vec{\hat{v}}\ p\ d\hat{x},\quad p=\xi,\eta.$$

So (3.31) is true. The proof is completed. □

We define γ_K to be the L^2-projection from $L^2(K)$ onto $P_1(K)$, that is:

$$\begin{aligned}\gamma_K w &= \frac{1}{|K|}\int_K wd\mathbf{x}'+\frac{3(x-x_0)}{|K|h_1^2}\int_K(x'-x_0)wd\mathbf{x}'\\ &+ \frac{3(y-y_0)}{|K|h_2^2}\int_K(y'-y_0)wd\mathbf{x}'+\frac{3(z-z_0)}{|K|h_3^2}\int_K(z'-z_0)wd\mathbf{x}'.\end{aligned} \tag{3.32}$$

The operator γ_h is defined by (3.11). The piecewise linear function space of lower order on Ω is defined as

$$W_h = \{w \in L^2(\Omega) \mid w|_K \in P_1(K), \ \forall K \in \mathcal{T}_h\}. \tag{3.33}$$

By similar argument in Lemma 3.2 and 3.3, we can prove the following results.

Lemma 3.8 There exists a constant C independent of the mesh size h such that

$$\Pi_h \vec{v} \in V_h, \quad \forall \vec{v} \in H^1(\Omega)^3; \tag{3.34}$$

$$div \Pi_K \vec{v} = \gamma_K div \vec{v}, \quad \forall K \in \mathcal{T}_h, \quad \forall \vec{v} \in H^1(\Omega)^3; \tag{3.35}$$

$$\|w - \gamma_h w\|_{0,\Omega} \le Ch\|w\|_{1,\Omega}, \quad \forall w \in H^1(\Omega); \tag{3.36}$$

$$\|\vec{v} - \Pi_h \vec{v}\|_{0,\Omega} + h|\vec{v} - \Pi_h \vec{v}|_{1,\Omega} \le Ch^2\|\vec{v}\|_{2,\Omega}, \quad \forall \vec{v} \in H^2(\Omega)^3. \tag{3.37}$$

4 Error estimates for the locking-free schemes

Based on the three kinds of finite element spaces, we consider the convergence analysis of (3.4) in a general frame. Optimal error estimate is obtained uniformly with respect to $\lambda \in (0, +\infty)$.

Theorem 4.1 Assume that $\vec{f} \in L^2(\Omega)^3$, $\vec{u} \in H^2(\Omega)^3 \cap H_0^1(\Omega)^3$ and $\vec{u}_h$ are the solutions of (2.3) and (3.4) respectively. Then there exists a positive constant C independent of λ and h, such that

$$\|\vec{u} - \vec{u}_h\|_h \le Ch\|\vec{f}\|_{0,\Omega}; \tag{4.1}$$

$$\|\vec{u} - \vec{u}_h\|_{0,\Omega} \le Ch^2\|\vec{f}\|_{0,\Omega}. \tag{4.2}$$

Proof: By the second Strang lemma(page 210, Theorem 4.2.2 of [7]), we have

$$\|\vec{u} - \vec{u}_h\|_h \le C\left\{\inf_{\vec{v}_h \in V_h} \|\vec{u} - \vec{v}_h\|_h + \sup_{0 \ne \vec{w}_h \in V_h} \frac{|a_h(\vec{u}, \vec{w}_h) - (\vec{f}, \vec{w}_h)|}{\|\vec{w}_h\|_h}\right\}, \tag{4.3}$$

where $C = Const. > 0$ independent of h and λ. So it is sufficient to estimate the approximate error and nonconforming error. The nonconforming error can be estimated as follows: By Green's formula,

$$\begin{aligned}
E_h(\vec{u}, \vec{w}_h) &:= a_h(\vec{u}, \vec{w}_h) - (\vec{f}, \vec{w}_h) \\
&= \sum_{K \in \mathcal{T}_h} \int_K \{\mu \nabla \vec{u} : \nabla \vec{w}_h + (\mu + \lambda) div \vec{u} \ div \vec{w}_h\} dx - \int_\Omega \vec{f} \cdot \vec{w}_h dx \\
&= -\int_\Omega \{\mu \Delta \vec{u} + (\mu + \lambda) \nabla (div \vec{u})\} \vec{w}_h dx - \int_\Omega \vec{f} \cdot \vec{w}_h dx \\
&\quad + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \{\mu \partial_\nu \vec{u} \cdot \vec{w}_h + (\mu + \lambda) div \vec{u} \ \vec{w}_h \cdot \vec{\nu}\} ds \\
&= \mu \sum_{K \in \mathcal{T}_h} \int_{\partial K} \partial_\nu \vec{u} \cdot \vec{w}_h ds + (\mu + \lambda) \sum_{K \in \mathcal{T}_h} \int_{\partial K} div \vec{u} \ \vec{w}_h \cdot \vec{\nu} \ ds.
\end{aligned} \tag{4.4}$$

Assume K^+ and K^- are adjacent elements with a common face F. Denote $\vec{w}_h^{\pm} = \vec{w}_h|_{K^{\pm}}$. Then $\int_F \vec{w}_h^+ ds = \int_F \vec{w}_h^- ds$. Since $\int_F \vec{w}_h ds = 0$ for $F \subset \partial\Omega$, by the error estimate of nonconforming finite element (see [17] and [19]), let $P_0^K(w) = 1/|K| \int_K w dx$, then

$$\begin{aligned} |\sum_{K\in\mathcal{T}_h} \int_{\partial K} \partial_\nu \vec{u} \cdot \vec{w}_h ds| &\leq \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{3} \|\partial_i \vec{u} - p_0^K(\partial_i \vec{u})\|_{0,\partial K} \cdot \|\vec{w}_h - p_0^K(\vec{w}_h)\|_{0,\partial K} \\ &\leq Ch|\vec{u}|_{2.\Omega} \cdot \|\vec{w}_h\|_h, \end{aligned} \tag{4.5}$$

$$\begin{aligned} |\sum_{K\in\mathcal{T}_h} \int_{\partial K} div\vec{u} \cdot \vec{w}_h \cdot \vec{\nu} ds| &\leq \sum_{K\in\mathcal{T}_h} \|div\vec{u} - p_0^K(div\vec{u})\|_{0,\partial K} \cdot \|\vec{w}_h - p_0^K(\vec{w}_h)\|_{0,\partial K} \\ &\leq Ch|div\vec{u}|_{1,\Omega} \cdot \|\vec{w}_h\|_h. \end{aligned} \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), we have

$$|E_h(\vec{u}, \vec{w}_h)| \leq Ch\{|\vec{u}|_{2,\Omega} + \lambda |div\vec{u}|_{1,\Omega}\} \cdot \|\vec{w}_h\|_h. \tag{4.7}$$

The approximate error can be estimated as follows: By (2.6) and Lemma 3.2, 3.3, 3.5, and 3.8, we have

$$\begin{aligned} \inf_{\vec{v}\in V_h} \|\vec{u} - \vec{u_h}\|_h^2 &\leq \|\vec{u} - \Pi_h\vec{u}\|_h^2 = a_h(\vec{u} - \Pi_h\vec{u}, \vec{u} - \Pi_h\vec{u}) \\ &= \mu \sum_{K\in\mathcal{T}_h} |\vec{u} - \Pi_K\vec{u}|_{1,K}^2 + (\mu+\lambda) \sum_{K\in\mathcal{T}_h} \|div\vec{u} - div\Pi_K\vec{u}\|_{0,K}^2 \\ &\leq Ch^2 \sum_{K\in\mathcal{T}_h} |\vec{u}|_{2,K}^2 + (\mu+\lambda) \sum_{K\in\mathcal{T}_h} \|div\vec{u} - r_K div\vec{u}\|_{0,K}^2 \\ &\leq Ch^2\{\sum_{K\in\mathcal{T}_h} |\vec{u}|_{2,K}^2 + \lambda \sum_{K\in\mathcal{T}_h} |div\vec{u}|_{1,K}^2\} \\ &= Ch^2\{|\vec{u}|_{2,\Omega}^2 + \lambda |div\vec{u}|_{1,\Omega}^2\}. \end{aligned} \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.3), we obtain (4.1) by (2.6).

(4.2) can be proved by the dual technique of the standard nonconforming element error estimate. The proof is completed. □

5 Numerical experiments

To check the convergence of our finite element schemes as $\lambda \to +\infty$, we carry out a numerical experiment in this section. We can see that the present schemes converges very well and uniformly with respect to $\lambda \in (0, \infty)$; but the trilinear conforming element scheme converges well for small λ and is locking when $\lambda \to \infty$.

Let $\Omega = [0,1]\times[0,1]\times[0,1]$, $\vec{f} = (f_1, f_2, f_3)^T$, where

$$\begin{aligned} f_1 &= -400\mu(2y-1)(2z-1)[3(x^2-x)^2(y^2-y+z^2-z) \\ &\quad +(1-6x+6x^2)(y^2-y)(z^2-z)]; \\ f_2 &= 200\mu(2x-1)(2z-1)[3(y^2-y)^2(x^2-x+z^2-z) \\ &\quad +(1-6y+6y^2)(x^2-x)(z^2-z)]; \\ f_3 &= 200\mu(2x-1)(2y-1)[3(z^2-z)^2(x^2-x+y^2-y) \\ &\quad +(1-6z+6z^2)(x^2-x)(y^2-y)]. \end{aligned}$$

Then the exact solution $\vec{u} = \{u_1, u_2, u_3\}$ of (2.2) is:

$$\begin{aligned} u_1 &= 200\mu(x-x^2)^2(2y^3-3y^2+y)(2z^3-3z^2+z); \\ u_2 &= -100\mu(y-y^2)^2(2x^3-3x^2+x)(2z^3-3z^2+z); \\ u_3 &= -100\mu(z-z^2)^2(2y^3-3y^2+y)(2x^3-3x^2+x). \end{aligned}$$

The domain Ω is divided into cubes uniformly. Let h be the edge length of each element. In the following tables, we only list the numerical information of the finite element scheme associated with the 18-freedom nonconforming finite element interpolation. The other schemes are also numerically convergent uniformly with respect to $\lambda \in (0, +\infty)$.

The L^2-norms of approximate solutions $\vec{u}_h$, the exact solution $\vec{u}$ and $\vec{u} - \vec{u}_h$ are listed in the following tables. To show the coincidence of our theoretical analysis with the numerical experiment, we also list theoretical convergence rates(TCR:$=h_1^2/h_2^2$) and numerical convergence rates(NCR:$=\|\vec{u}-\vec{u}_{h_2}\|_{0,\Omega}/\|\vec{u}-\vec{u}_{h_1}\|_{0,\Omega}$) for comparison. h_1 is the grid size of a coarser mesh and h_2 is that of the next refinement mesh.

The L^2-norm of the exact solution of (2.2) is: $\|\vec{u}\|_{0,\Omega} = 0.04647143$.

Table 1: $\lambda = 1.0$ (18-freedom FEM)

h	$\|\vec{u}_h\|_{0,\Omega}$	$\|\vec{u}-\vec{u}_h\|_{0,\Omega}$	$\frac{\|\vec{u}-\vec{u}_h\|_{0,\Omega}}{\|\vec{u}\|_{0,\Omega}}$	TCR	NCR
1/4	0.05748045	0.01620755	0.348764		
1/8	0.04959072	0.00452705	0.097416	0.25	0.27932
1/12	0.04789030	0.002095142	0.045085	0.4444	0.46281

Table 2: $\lambda = 10^3$ (18-freedom FEM)

h	$\|\vec{u}_h\|_{0,\Omega}$	$\|\vec{u}-\vec{u}_h\|_{0,\Omega}$	$\frac{\|\vec{u}-\vec{u}_h\|_{0,\Omega}}{\|\vec{u}\|_{0,\Omega}}$	TCR	NCR
1/4	0.05756476	0.01637972	0.352469		
1/8	0.04964923	0.00461481	0.099304	0.25	0.28174
1/12	0.04792085	0.00209539	0.045090	0.4444	0.45406

Table 3: $\lambda = 10^6$ (18-freedom FEM)

h	$\|\vec{u}_h\|_{0,\Omega}$	$\|\vec{u}-\vec{u}_h\|_{0,\Omega}$	$\frac{\|\vec{u}-\vec{u}_h\|_{0,\Omega}}{\|\vec{u}\|_{0,\Omega}}$	TCR	NCR
1/4	0.05756504	0.01638037	0.352483		
1/8	0.04964946	0.00461520	0.099313	0.25	0.28175
1/12	0.04792097	0.00209559	0.045094	0.4444	0.45406

From the numerical results by the trilinear conforming finite element scheme, it can be found that when $\lambda \to \infty$, $\|\vec{u_h}\|_{0,\Omega} \to 0$, $\|\vec{u} - \vec{u_h}\|_{0,\Omega} \to \|\vec{u}\|_{0,\Omega}$. From table 4 – 6, the relative errors are almost equal to 1 when $\lambda \gg 1$ even for very small h.

Table 4: $\lambda = 1.0$ (trilinear FEM)

h	$\|\vec{u}_h\|_{0,\Omega}$	$\|\vec{u}-\vec{u}_h\|_{0,\Omega}$	$\frac{\|\vec{u}-\vec{u}_h\|_{0,\Omega}}{\|\vec{u}\|_{0,\Omega}}$
1/4	0.03445655	0.01680566	0.362729
1/8	0.04361379	0.00440989	0.094895
1/12	0.04534260	0.0022540	0.048503

Table 5: $\lambda = 10^3$ (trilinear FEM)

h	$\|\vec{u}_h\|_{0,\Omega}$	$\|\vec{u}-\vec{u}_h\|_{0,\Omega}$	$\frac{\|\vec{u}-\vec{u}_h\|_{0,\Omega}}{\|\vec{u}\|_{0,\Omega}}$
1/4	0.0013457	0.0452041	0.972729
1/8	0.0060261	0.0407648	0.877201
1/12	0.0120302	0.035478	0.763437

Table 6: $\lambda = 10^6$ (trilinear FEM)

h	$\|\vec{u}_h\|_{0,\Omega}$	$\|\vec{u}-\vec{u}_h\|_{0,\Omega}$	$\frac{\|\vec{u}-\vec{u}_h\|_{0,\Omega}}{\|\vec{u}\|_{0,\Omega}}$
1/4	0.000001405	0.046470112	0.999972
1/8	0.000007222	0.046464663	0.999854
1/12	0.000007222	0.046464663	0.999854

References

[1] I. Babuska and M. Suri, *Locking Effects in the Finite Element Approximation of Elasticity Problems*, Numer. Math. 62(1992), pp.436-463.

[2] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, 1994.

[3] S. C. Brenner and Li Yungsung, *Linear Finite Element Methods for Planar Linear Elasticity*, Math. Comp. Vol. 59(1992), No. 200, pp.321-330.

[4] F. Brezzi and M. Fortin, *Mixed and Hybrid Element Methods*, Springer-Verlag, New York, 1991.

[5] P. G. Ciarlet, *Basic Error Estimates for Elliptic Problems, Handbook of Numerical Analysis, vol. II, Finite Element Methods(Part 1), General Editiors: P. G. Ciarlet and J. L. Lions*, North-Holland, (1991), pp. 17-352.

[6] P. G. Ciarlet, *Mathematical Elasticity*, Vol. *I, Three-dimensional elasticity*, North-Holland Amsterdam, 1988.

[7] P.G.Ciarlet,*The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, New York, Oxford(1978).

[8] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains*, Smoothness and Asymptotics of Solutions, Lecture Notes in Mathematics, 1341, Spring-Verlag, Berlin, Heidelberg, New York, 1988.

[9] R. S. Falk, *Noncomforming Finite Element Methods for the Equations of Linear Elasticity*, Math. Comp. 57(1991), pp. 529-550.

[10] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Advanced Publishing Program, Boston, 1985.

[11] P. Grisvard, *Problémes aux limites dans les polygones*, Mode d'emploi. Bulletin de la Direction des Etudes et Recherches Series C Mathematiques, Informatique 1(1986), pp. 21-59.

[12] R. Kouhia and R. Stenberg, *A linear nonconforming finite element methods for nearly incompressible elasticity and Stokes flow*, Comput. Meths. Appl. Mech. Engrg., 124(1995), pp. 195-212.

[13] P. Ming, Nonconforming finite element method VS locking problems, Doctor thesis, The Academy of Mathematics and System Sciences, CAS, 2000.

[14] Wang Lie-heng and Qi He, *On locking-free finite element schemes for the planar elasticity*, Mathematica Numerica Sinica, V. 24, No.1(2002).

[15] Wang Lie-heng and Qi He, *A locking-free scheme of nonconforming rectangle finite element for the planar elasticity*, to appear in J. Comp. Math..

[16] Wang Lie-heng and Song Ruowei, On the linear nonconforming finite element method for the planar elasticity, to appear in J. Comp. Math..

[17] F. Stummel, *The Generalized Patch Test*, SIAM. J. Numer. Anal. 16, (1979), pp.449-471.

[18] M. Vogelius, *An Analysis of the p-version of the Finite Element Method for Nearly incompressible Materials. Uniformly Valid, Optimal order Estimates*, Numer. Math. 41(1983).

[19] Shi, Z., *The F-E-M-test for convergence of nonconforming finite elements*, Math. Comp. 49(1987), 391-405.

[20] Li, K., Huang,A. and Huang,H. *The Finite Element Method and Appilcation*, Xian Jiaotong University.

[21] Z. Zhang, *Analysis of some quadrilateral nonconforming elements for incompressible elasticity*, SIAM J. Numer. Anal., 34(1998), pp.640-663.