MAXWELL'S EQUATIONS IN AN UNBOUNDED STRUCTURE

PEIJUN LI*, GAOFENG ZHENG †, AND WEIYING ZHENG ‡

Abstract. This paper is concerned with the mathematical analysis of the electromagnetic wave scattering by an unbounded dielectric medium, which is mounted on a perfectly conducting infinite plane. By introducing a transparent boundary condition on a plane surface confining the medium, the scattering problem is modeled as a boundary value problem of Maxwell's equations. Based on a variational formulation, the problem is shown to have a unique weak solution for a wide class of dielectric permittivity by using the generalized Lax-Milgram theorem.

Key words. Maxwell's equations, generalized Lax-Milgram theorem, unbounded rough surface

AMS subject classifications. 35Q60, 35Q61, 35J20

1. Introduction. Consider the electromagnetic wave scattering by an unbounded dielectric medium which is mounted on a perfectly conducting infinite plane. The free space above the medium is filled with a homogeneous material; while the medium iteself may be inhomogeneous with variable dielectric permittivity and magnetic permeability. The interface, which separates the free space and the inhomogeneous medium, may be represented by an infinite rough surface. It is referred to as a surface which is a nonlocal perturbation of an infinite plane surface such that the whole surface lies within a finite distance of the original plane. By introducing a transparent boundary condition on a planar surface confining the medium, the scattering problem is formulated as a boundary value problem of Maxwell's equations. This paper is concerned with the mathematical analysis of the solution for its variational problem. The main theorem indicates that there is a unique weak solution in a suitable functional space for a wide class of dielectric permittivity. The proof is based on a Hodge decomposition and the generalized Lax-Milgram theorem. A crucial step is to establish a priori estimate of the solution.

The scattering by unbounded structures has significant applications such as modeling acoustic and electromagnetic wave propagation over outdoor ground and sea surfaces or, at a very different scale, optical scattering from the surface of materials in near-field optics or nano-optics, detection of underwater mines, especially those buried in soft sediments. These problems are extensively studied and a considerable amount of information is available concerning their solutions [3–8, 12, 17, 18].

These scattering problems are quite challenging due to the unbounded nature of the domains. The usual Silver-Müler radiation condition is no longer valid and the Fredholm alternative argument does not apply either due to the lack of compactness result. In particular, rigorous mathematical analysis for the three-dimensional Maxwell equations is very rare. In [13], the electromagnetic scattering by unbounded rough surfaces was considered under the assumption that the medium was lossy in

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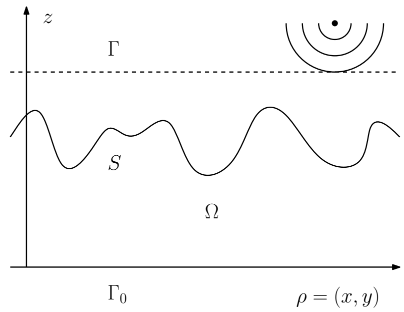

FIG. 2.1. *Problem geometry.*

the entire space. The well-posedness of the solution was established by a direct application of the Lax-Milgram theorem after showing that the sesquilinear form was coercive. In [11], the authors considered the electromagnetic wave scattering from rough penetrable layers. The problem was shown to have a unique weak solution from a priori estimates and the limiting absorption principle. As mentioned in the paper, the source term was assumed to be divergence free, and the assumptions were quite restrictive for the dielectric permittivity and might be hard to be satisfied in practice.

In this work, we adopt the generalized Lax-Milgram theorem to establish the well-posedness of the solution for the electromagnetic scattering problem. Our method is optimal in the sense that the generalized Lax-Milgram theorem gives the sufficient and necessary condition on the existence and uniqueness of the solution. Therefore, the divergence free condition is removed for the source term, and a larger class of dielectric permittivity is allowed since the assumptions are much less restrictive. Related work can be found in [1, 2, 10] for the electromagnetic scattering problems by a cavity and a periodic grating. We refer to [16] for the electromagnetic scattering in unbounded domains, and [9, 14, 15] for an account of electromagnetic wave scattering problem in bounded domains.

The outline of this paper is as follows. In Section 2, the model problem and some functional spaces are introduced. A Hodge decomposition is discussed in Section 3. Section 4 is devoted to the study of the variational formulation of the scattering problem. In Section 5, an a priori estimate is derived for the solution and the well-posedness of the solution is established. The paper is concluded with some general remarks and directions for future research in Section 6.

2. A model problem. In this section we introduce a mathematical model and define some notation for the electromagnetic scattering by an unbounded structure. Let us first specify the problem geometry. As seen in Figure 2.1, let S be a Lipschitz continuous surface imbedded in the strip

$$\Omega = \{(\rho, z) \in \mathbb{R}^3 : 0 < z < h\} = \mathbb{R}^2 \times (0, h),$$

where $\rho = (x, y) \in \mathbb{R}^2$ and $h > 0$ is a constant. The free space is filled with some homogeneous material above S; while the medium may be inhomogeneous in the region below S. The unbounded medium is assumed to be deposited on a perfectly conducting infinite (x, y)-plane. Define two boundaries $\Gamma_0 = \{(\rho, z) \in \mathbb{R}^3 : z = 0\}$ and $\Gamma = \{(\rho, z) \in \mathbb{R}^3 : z = h\}$.

Specifically, we assume that the space $\mathbb{R}^3_+ = \{(\rho, z) \in \mathbb{R}^3 : z > 0\}$ is filled with a material with the dielectric permittivity ε and the magnetic permeability μ. The electromagnetic wave propagation is governed by the time-harmonic Maxwell equations with time dependence $e^{-\mathrm{i}\omega t}$:

$$\nabla \times \mathbf{E} - \mathrm{i}\omega\mu\mathbf{H} = 0, \quad \nabla \times \mathbf{H} + \mathrm{i}\omega\varepsilon\mathbf{E} = \mathbf{J} \quad \text{in } \mathbb{R}^3_+, \tag{2.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric field and the magnetic field, ω is the angular frequency, and $\mathbf{J}$ is the electric current density. Throughout the paper, we assume that $\varepsilon \in W^{1,\infty}(\mathbb{R}^3_+)$ and $\mu \in L^\infty(\mathbb{R}^3_+)$ and they satisfy

$$0 < \varepsilon_0 \le \mathrm{Re}\varepsilon \le \varepsilon_1, \quad \mathrm{Im}\varepsilon \ge 0, \quad 0 < \mu_0 \le \mu \le \mu_1,$$

where $\varepsilon_0, \varepsilon_1, \mu_0, \mu_1$ are constants, and ε_0, μ_0 are known as the dielectric permittivity and the magnetic permeability in the free space, respectively.

Eliminating $\mathbf{H}$ from (2.1), we obtain a decoupled equation for $\mathbf{E}$:

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - \omega^2\varepsilon\mathbf{E} = \mathbf{f} \quad \text{in } \Omega, \tag{2.2}$$

where $\kappa = \omega(\varepsilon\mu)^{1/2}$ with $\mathrm{Im}\kappa \ge 0$ is the wavenumber and $\mathbf{f} = \mathrm{i}\omega\mathbf{J}$. Since the unbounded medium is mounted on the perfectly electrical conducting plane Γ_0, the electric field $\mathbf{E}$ satisfies

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \Gamma_0, \tag{2.3}$$

where $\mathbf{n} = (0, 0, 1)$.

To describe the boundary value problem and derive its variational formulation, we introduce some Sobolev spaces. For $u \in L^2(\mathbb{R}^2)$, we denote $\hat{u}$ the Fourier transform of u by

$$\hat{u}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} u(\rho)e^{-\mathrm{i}\rho\cdot\xi}\mathrm{d}\rho,$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Denote by $L^2(\Omega)$ the space of complex square integrable functions on Ω with the norm

$$\|u\|_{L^2(\Omega)} = \left[\int_0^h \int_{\mathbb{R}^2} |u(\rho, z)|^2 \mathrm{d}\rho\mathrm{d}z\right]^{1/2} = \left[\int_0^h \int_{\mathbb{R}^2} |\hat{u}(\xi, z)|^2 \mathrm{d}\xi\mathrm{d}z\right]^{1/2}.$$

We define the Sobolev space: $H^s(\Omega) = \{D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \le s\}$ which is a Banach space for the norm:

$$\|u\|_{H^s(\Omega)} = \left[\int_0^h \sum_{n+m\le s}\left(\int_{\mathbb{R}^2}(1+|\xi|^2)^n |D_z^m\hat{u}(\xi, z)|^2\mathrm{d}\xi\right)\mathrm{d}z\right]^{1/2},$$

where $n, m \in \mathbb{N}$ and D_z^m is the m-th derivative with respect to z. These norms, given in the spatial-frequency domain, are equivalent to the usual Sobolev norms in the entire spatial domain due to the Parseval identity.

Introduce the following space:

$$H(\mathrm{curl}, \Omega) = \{\mathbf{u} \in L^2(\Omega)^3,\ \nabla \times \mathbf{u} \in L^2(\Omega)^3\},$$

which is a Hilbert space for the norm:

$$\|\mathbf{u}\|_{H(\mathrm{curl},\,\Omega)} = \left(\|\mathbf{u}\|^2_{L^2(\Omega)^3} + \|\nabla\times\mathbf{u}\|^2_{L^2(\Omega)^3}\right)^{1/2}.$$

For any vector field $\mathbf{u} = (u_1, u_2, u_3)$, denote the tangential component on Γ by

$$\mathbf{u}_\Gamma = -\mathbf{n}\times(\mathbf{n}\times\mathbf{u}) = (u_1(\rho, h), u_2(\rho, h), 0).$$

For any smooth vector $\mathbf{u} = (u_1, u_2, u_3)$ defined on Γ, denote by $\mathrm{div}_\Gamma\mathbf{u} = \partial_x u_1 + \partial_y u_2$ and $\mathrm{curl}_\Gamma\mathbf{u} = \partial_x u_2 - \partial_y u_1$ the surface divergence and the surface scalar curl of the field $\mathbf{u}$, respectively. For a smooth function u, denote by $\nabla_\Gamma u = (\partial_x u, \partial_y u, 0)$ the surface gradient.

To describe the Calderón operator and the transparent boundary condition in formulation of the boundary value problem, we introduce some trace functional spaces. Denote by $H^s(\Gamma)$ the standard Sobolev space, the completion of $L^2(\mathbb{R}^2)$ in the norm characterized by

$$\|u\|_{H^s(\Gamma)} = \left[\int_{\mathbb{R}^2}(1+|\xi|^2)^s|\hat{u}|^2\mathrm{d}\xi\right]^{1/2}.$$

Let $\mathscr{S}(\mathbb{R}^2)$ and $\mathscr{S}_{\mathrm{w}}(\mathbb{R}^2)$ be the Schwartz space and the weighted Schwartz space, i.e.,

$$\mathscr{S}(\mathbb{R}^2) = \left\{u\in C^\infty(\mathbb{R}^2) : \sup_{\rho\in\mathbb{R}^2}|(1+|\rho|^\alpha)\frac{\partial^{|\beta|}}{\partial\rho^\beta}u(\rho)| < \infty \text{ for all } \alpha\in\mathbb{N}_0, \beta\in\mathbb{N}_0^2\right\}$$

and

$$\mathscr{S}_{\mathrm{w}}(\mathbb{R}^2) = \left\{u\in\mathscr{S}(\mathbb{R}^2) : |\kappa_0^2 - |\xi|^2|^{-1/2}\hat{u}\in\mathscr{S}(\mathbb{R}^2)\right\},$$

where $\kappa_0 = \omega(\varepsilon_0\mu_0)^{1/2}$ is the wavenumber in the free space. Denote by $\mathscr{S}^*_{\mathrm{w}}(\mathbb{R}^2)$ the dual space of $\mathscr{S}_{\mathrm{w}}(\mathbb{R}^2)$. Define

$$H^s_\alpha(\mathbb{R}^2) = \left\{u\in\mathscr{S}^*_{\mathrm{w}}(\mathbb{R}^2) : (1+|\xi|^2)^{s/2}\left|\frac{1+|\xi|^2}{\kappa_0^2-|\xi|^2}\right|^{\alpha/2}\hat{u}\in L^2(\mathbb{R}^2)\right\}.$$

and

$$H^s_\alpha(\mathrm{curl};\Gamma) = \left\{\mathbf{u}\in H^s(\mathbb{R}^2)^3 : \mathrm{curl}_\Gamma\mathbf{u}\in H^s_\alpha(\mathbb{R}^2), \mathrm{div}_\Gamma\mathbf{u}\in H^{s-1}_{\alpha+1}(\mathbb{R}^2)\right\}.$$

In this paper, we only need to choose $\alpha = s = -1/2$. Denote $W = H^{-1/2}_{-1/2}(\mathrm{curl};\Gamma)$ and its norm is characterized by

$$\|\mathbf{u}\|^2_W = \int_{\mathbb{R}^2}(1+|\xi|^2)^{-1/2}\left(|\hat{u}_1|^2+|\hat{u}_2|^2\right)\mathrm{d}\xi + \int_{\mathbb{R}^2}\frac{(1+|\xi|^2)^{-1}}{|\kappa_0^2-|\xi|^2|^{-1/2}}|\xi_1\hat{u}_2-\xi_2\hat{u}_1|^2\mathrm{d}\xi$$
$$+\int_{\mathbb{R}^2}\frac{(1+|\xi|^2)^{-1}}{|\kappa_0^2-|\xi|^2|^{1/2}}|\xi_1\hat{u}_1+\xi_2\hat{u}_2|^2\mathrm{d}\xi.$$

Define a subspace of $H(\mathrm{curl},\Omega)$:

$$X = \{\mathbf{u}\in H(\mathrm{curl},\Omega) : \mathbf{u}\times\mathbf{n} = 0 \text{ on } \Gamma_0,\ \mathbf{u}_\Gamma\in W\}.$$

The norm is given by

$$\|\mathbf{u}\|_X = \left(\|\mathbf{u}\|^2_{L^2(\Omega)^3} + \|\nabla\times\mathbf{u}\|^2_{L^2(\Omega)^3} + \|\mathbf{u}_\Gamma\|^2_W\right)^{1/2}.$$

To reduce the problem into the bounded domain Ω, we introduce a transparent boundary condition by using a Calerón operator, which maps the value of the tangential component of the electric field to the value of the tangential trace of the magnetic field.

For any tangential vector $\mathbf{u} = (u_1, u_2, 0)$ on Γ, define

$$T\mathbf{u} = (v_1, v_2, 0),$$

where

$$\begin{bmatrix}\hat{v}_1\\ \hat{v}_2\end{bmatrix} = \frac{1}{\omega\mu_0(\kappa_0^2 - |\xi|^2)^{1/2}}\begin{bmatrix}\kappa_0^2-\xi_2^2 & \xi_1\xi_2\\ \xi_1\xi_2 & \kappa_0^2-\xi_1^2\end{bmatrix}\begin{bmatrix}\hat{u}_1\\ \hat{u}_2\end{bmatrix}.$$

It follows from [13] that the following transparent boundary condition can be imposed:

$$(\nabla\times\mathbf{E})\times\mathbf{n} = \mathrm{i}\omega\mu_0 T\mathbf{E}_\Gamma \quad \text{on } \Gamma.$$

It is shown in [16] (cf. Lemma 5.4.7) that T is a bounded linear operator, i.e., there exists a positive constant C such that

$$|\langle T\mathbf{u}, \mathbf{v}\rangle| \le C\|\mathbf{u}\|_W\|\mathbf{v}\|_W \quad \text{for any } \mathbf{u}, \mathbf{v}\in W. \tag{2.4}$$

We mention that C and C_j are positive constants throughout the paper, whose precise values are not required and may change line by line but should be always clear from the context.

3. Hodge decomposition. We present a version of Hodge decomposition, which is important in the proof of our theorems on the existence and uniqueness of the scattering problem.

Introduce a subspace of $H^1(\Omega)$:

$$H^1_\Gamma(\Omega) = \{u\in H^1(\Omega) : u = 0 \text{ on } \Gamma_0, \nabla_\Gamma u\in W\},$$

which is a Hilbert space for the norm

$$\|u\|_{H^1_\Gamma(\Omega)} = \left(\|u\|^2_{H^1(\Omega)} + \|\nabla_\Gamma u\|^2_W\right)^{1/2}.$$

LEMMA 3.1. *Given $f\in L^2(\Omega)$ and $g\in H^{-1/2}(\Gamma)$, the boundary value problem*

$$\begin{cases}\nabla\cdot(\varepsilon\nabla u) = f & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma_0,\\ \varepsilon_0\partial_z u = \mathrm{i}\omega^{-1}\mathrm{div}_\Gamma T\nabla_\Gamma u + g & \text{on } \Gamma,\end{cases}$$

has a unique solution in $H^1_\Gamma(\Omega)$.

Proof. We examine the variational formulation of the problem: find $u\in H^1_\Gamma(\Omega)$ such that

$$b(u,v) = \int_\Gamma g\bar{v} - \int_\Omega f\bar{v} \quad \text{for all } v\in H^1_\Gamma(\Omega),$$

where the sesquilinear form

$$b(u,v)=\int_\Omega \varepsilon\nabla u\cdot\nabla\bar v+\mathrm{i}\omega^{-1}\int_\Gamma T\nabla_\Gamma u\cdot\nabla_\Gamma\bar v.$$

It follows directly from the Cauchy-Schwarz inequality that b is a continuous on $H^1_\Gamma(\Omega)\times H^1_\Gamma(\Omega)$. Hence, it suffices to show that b is coercive on $H^1_\Gamma(\Omega)$.

Taking the real and imaginary parts of b yields

$$\begin{aligned}\mathrm{Re}b(u,u)&=\int_\Omega \mathrm{Re}(\varepsilon)|\nabla u|^2-\omega^{-1}\mathrm{Im}\int_\Gamma T\nabla_\Gamma u\cdot\nabla_\Gamma\bar u,\\ \mathrm{Im}b(u,u)&=\int_\Omega \mathrm{Im}(\varepsilon)|\nabla u|^2+\omega^{-1}\mathrm{Re}\int_\Gamma T\nabla_\Gamma u\cdot\nabla_\Gamma\bar u.\end{aligned}$$

Let $\nabla_\Gamma u=\mathbf{w}=(w_1,w_2,0)$. Noting $\mathrm{curl}_\Gamma(\nabla_\Gamma u)=0$, we have $\xi_2\hat w_1-\xi_1\hat w_2=0$. It follows from $|\xi|^2|\hat{\mathbf{w}}|^2=|\xi_1\hat w_2-\xi_2\hat w_1|^2+|\xi_1\hat w_1+\xi_2\hat w_2|^2$ that we have

$$\begin{aligned}\int_\Gamma T\nabla_\Gamma u\cdot\nabla_\Gamma\bar u&=\frac{1}{\omega\mu_0}\int_{\mathbb{R}^2}\frac{1}{(\kappa_0^2-|\xi|^2)^{1/2}}(\kappa_0^2|\hat{\mathbf{w}}|^2-|\xi_1\hat w_2-\xi_2\hat w_1|^2)\mathrm{d}\xi\\ &=\frac{1}{\omega\mu_0}\left[\int_{|\xi|^2<\kappa_0^2}\frac{\kappa_0^2|\hat{\mathbf{w}}|^2}{(\kappa_0^2-|\xi|^2)^{1/2}}\mathrm{d}\xi-\mathrm{i}\int_{|\xi|^2>\kappa_0^2}\frac{\kappa_0^2|\hat{\mathbf{w}}|^2}{(|\xi|^2-\kappa_0^2)^{1/2}}\mathrm{d}\xi\right].\end{aligned}$$

Simple calculation gives

$$\begin{aligned}\mathrm{Re}b(u,u)&=\int_\Omega \mathrm{Re}(\varepsilon)|\nabla u|^2+\varepsilon_0\int_{|\xi|^2>\kappa_0^2}\frac{|\hat{\mathbf{w}}|^2}{(|\xi|^2-\kappa_0^2)^{1/2}}\mathrm{d}\xi,\\ \mathrm{Im}b(u,u)&=\int_\Omega \mathrm{Im}(\varepsilon)|\nabla u|^2+\varepsilon_0\int_{|\xi|^2<\kappa_0^2}\frac{|\hat{\mathbf{w}}|^2}{(\kappa_0^2-|\xi|^2)^{1/2}}\mathrm{d}\xi,\end{aligned}$$

which yield

$$\begin{aligned}2|b(u,u)|&\geq \mathrm{Re}b(u,u)+\mathrm{Im}b(u,u)\\ &\geq\int_\Omega|\nabla u|^2+\varepsilon_0\int_{\mathbb{R}^2}\frac{|\hat{\mathbf{w}}|^2}{||\xi|^2-\kappa_0^2|^{1/2}}\mathrm{d}\xi.\end{aligned}\tag{3.1}$$

Using the fact that $\xi_2\hat w_1-\xi_1\hat w_2=0$ again, we have

$$\begin{aligned}\|\nabla_\Gamma u\|_W^2&=\int_{\mathbb{R}^2}\frac{|\hat{\mathbf{w}}|^2}{(1+|\xi|^2)^{1/2}}\mathrm{d}\xi+\int_{\mathbb{R}^2}\frac{|\xi_1\hat w_1+\xi_2\hat w_2|^2}{(1+|\xi|^2)|\kappa_0^2-|\xi|^2|^{1/2}}\mathrm{d}\xi\\ &\leq\int_{\mathbb{R}^2}\frac{|\hat{\mathbf{w}}|^2}{(1+|\xi|^2)^{1/2}}\mathrm{d}\xi+\int_{\mathbb{R}^2}\frac{|\hat{\mathbf{w}}|^2}{||\xi|^2-\kappa_0^2|^{1/2}}\mathrm{d}\xi\\ &\leq C\int_{\mathbb{R}^2}\frac{|\hat{\mathbf{w}}|^2}{||\xi|^2-\kappa_0^2|^{1/2}}\mathrm{d}\xi,\end{aligned}\tag{3.2}$$

where C depends only on κ_0. Since $u=0$ on Γ_0, it follows from the Poincaré inequality and (3.1)–(3.2) that we obtain

$$|b(u,u)|\geq C\left(\|u\|^2_{H^1(\Omega)}+\|\nabla_\Gamma u\|_W^2\right)=C\|u\|^2_{H^1_\Gamma(\Omega)}.$$

The lemma is proved by a direct application of the Lax-Milgram theorem. ∎

Define two subspaces of X:

$$Y = \left\{\mathbf{u} \in X : \nabla \cdot (\varepsilon \mathbf{u}) = 0 \text{ in } \Omega,\ \varepsilon_0 \mathbf{u} \cdot \mathbf{n} = \mathrm{i}\omega^{-1} \mathrm{div}_\Gamma T \mathbf{u}_\Gamma \text{ on } \Gamma\right\},$$

and

$$Y^\perp = \{\mathbf{u} : \mathbf{u} = \nabla u, u \in H^1_\Gamma(\Omega)\}.$$

Lemma 3.2. *The spaces Y and $Y^\perp$ are closed subspaces of X, which is the direct sum of Y and $Y^\perp$, i.e.,*

$$X = Y \oplus Y^\perp.$$

Proof. Take $\mathbf{u}_n = \nabla u_n \in Y^\perp$, and it follows from $\mathbf{u}_n \to \mathbf{u}$ in X that

$$\begin{aligned}\|\mathbf{u}_n - \mathbf{u}\|_X^2 &= \|\mathbf{u}_n - \mathbf{u}\|^2_{L^2(\Omega)^3} + \|\nabla \times \mathbf{u}_n - \nabla \times \mathbf{u}\|^2_{L^2(\Omega)^3} + \|\mathbf{u}_n - \mathbf{u}\|_W^2 \\ &= \|\nabla u_n - \mathbf{u}\|^2_{0,\Omega} + \|\nabla \times \mathbf{u}\|^2_{0,\Omega} + \|\mathbf{u}_n - \mathbf{u}\|^2_W \to 0\end{aligned}$$

as $n \to \infty$, which shows that $\|\nabla \times \mathbf{u}\|_{L^2(\Omega)^3} = 0$ and then $Y^\perp$ is closed.

Equivalently, the subspace Y can be represented as

$$Y = \left\{\mathbf{u} \in X : \int_\Omega \varepsilon \mathbf{u} \cdot \nabla \bar{v} - \mathrm{i}\omega^{-1} \int_\Gamma \mathrm{div}_\Gamma T \mathbf{u}_\Gamma \bar{v} = 0 \text{ for all } v \in H^1_\Gamma(\Omega)\right\}.$$

For fixed $v \in H^1_\Gamma(\Omega)$, define a linear functional on Y:

$$\begin{aligned}l(\mathbf{u}) &= \int_\Omega \varepsilon \mathbf{u} \cdot \nabla \bar{v} - \mathrm{i}\omega^{-1} \int_\Gamma \mathrm{div}_\Gamma T \mathbf{u}_\Gamma \bar{v} \\ &= \int_\Omega \varepsilon \mathbf{u} \cdot \nabla \bar{v} + \mathrm{i}\omega^{-1} \int_\Gamma T \mathbf{u}_\Gamma \cdot \nabla_\Gamma \bar{v} \quad \text{ for all } \mathbf{u} \in Y.\end{aligned}$$

Applying the Cauchy-Schwarz inequality and (2.4), we get

$$\begin{aligned}|l(\mathbf{u})| &\leq C\left(\|v\|_{H^1(\Omega)} \|\mathbf{u}\|_{L^2(\Omega)^3} + \|\mathbf{u}\|_W \|\nabla_\Gamma v\|_W\right) \\ &\leq C\left(\|v\|_{H^1(\Omega)} + \|\nabla_\Gamma v\|_W\right) \|\mathbf{u}\|_X = C(v) \|\mathbf{u}\|_X.\end{aligned}$$

Let $\mathbf{u}_n \in Y$ and $\mathbf{u}_n \to \mathbf{u}$ in X. We have

$$|l(\mathbf{u})| = |l(\mathbf{u} - \mathbf{u}_n) + l(\mathbf{u}_n)| = |l(\mathbf{u} - \mathbf{u}_n)| \leq C(v) \|\mathbf{u}_n - \mathbf{u}\|_X \to 0$$

as $n \to \infty$, which implies that $\mathbf{u} \in Y$ and thus the closedness of the space Y.

For any $\mathbf{u} \in X$, define $u \in H^1_\Gamma(\Omega)$ by the solution to the variational problem

$$a(\nabla u, \nabla v) = a(\mathbf{u}, \nabla v) \text{ for all } v \in H^1_\Gamma(\Omega),$$

which has an equivalent differential form:

$$\begin{cases} \nabla \cdot (\varepsilon \nabla u) = \nabla \cdot (\varepsilon \mathbf{u}) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \varepsilon_0 \partial_z u = \mathrm{i}\omega^{-1} \mathrm{div}_\Gamma T \nabla_\Gamma u + g & \text{on } \Gamma, \end{cases}$$

where $g = \varepsilon_0 \mathbf{u}\cdot\mathbf{n} - \mathrm{i}\omega^{-1}\mathrm{div}_\Gamma T\mathbf{u}_\Gamma$. It follows from Lemma 3.1 that there exists a unique solution u in $H^1_\Gamma(\Omega)$.

Denote $\mathbf{v} = \mathbf{u} - \nabla u$. Then $a(\mathbf{v}, \nabla v) = 0$ for all $v \in H^1_\Gamma(\Omega)$, i.e.,

$$\int_\Omega \nabla\cdot(\varepsilon\mathbf{v})\bar{v} - \int_\Gamma \left(\varepsilon_0\mathbf{v}\cdot\mathbf{n} - \mathrm{i}\omega^{-1}\mathrm{div}_\Gamma\mathbf{v}_\Gamma\right)\bar{v} = 0 \quad \text{for all } v \in H^1_\Gamma(\Omega),$$

which shows that $\mathbf{v} \in Y$.

Finally, we show that $Y \cap Y^\perp$ consists of the trivial function only. In fact, if $\mathbf{u} = \nabla u \in Y \cap Y^\perp$, then

$$\begin{cases} \nabla\cdot(\varepsilon\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \varepsilon_0\partial_z u = \mathrm{i}\omega^{-1}\mathrm{div}_\Gamma T\nabla_\Gamma u & \text{on } \Gamma, \end{cases}$$

which implies that $\mathbf{u} = \nabla u = 0$ from Lemma 3.1. □

4. Reduced variational problem. Multiplying the complex conjugate of a test function $\psi \in X$ on (2.2), integrating over Ω, and using the integration by parts, we arrive at the variational form for the scattering problem: find $\mathbf{E} \in X$ such that

$$a(\mathbf{E}, \psi) = (\mathbf{f}, \mathbf{v}) \quad \text{for all } \psi \in X, \tag{4.1}$$

where the sesquilinear form

$$a(\mathbf{E}, \psi) = \int_\Omega \mu^{-1}\nabla\times\mathbf{E}\cdot\nabla\times\bar{\psi} - \int_\Omega \omega^2\varepsilon\mathbf{E}\cdot\bar{\psi} - \mathrm{i}\omega\int_\Gamma T\mathbf{u}_\Gamma\cdot\bar{\psi}_\Gamma, \tag{4.2}$$

and the linear functional

$$(\mathbf{f}, \psi) = \int_\Omega \mathbf{f}\cdot\bar{\psi}.$$

Using the Hodge decomposition in Lemma 3.2, we take $\mathbf{E} = \mathbf{u} + \nabla u$ and $\psi = \mathbf{v} + \nabla v$ for any $\mathbf{v} \in Y$ and $v \in H^1_\Gamma(\Omega)$. Observing that for $\mathbf{u} \in Y$ and $v \in H^1_\Gamma(\Omega)$, we have

$$\begin{aligned} a(\mathbf{u}, \nabla v) &= -\omega^2\int_\Omega \varepsilon\mathbf{u}\cdot\nabla\bar{v} - \mathrm{i}\omega\int_\Gamma T\mathbf{u}_\Gamma\cdot\nabla_\Gamma\bar{v} \\ &= \omega^2\int_\Omega \bar{v}\nabla\cdot(\varepsilon\mathbf{u}) - \omega^2\varepsilon_0\int_\Gamma \bar{v}(\mathbf{u}\cdot\mathbf{n}) + \mathrm{i}\omega\int_\Gamma \bar{v}\mathrm{div}_\Gamma T\mathbf{u}_\Gamma \\ &= \omega^2\int_\Omega \bar{v}\nabla\cdot(\varepsilon\mathbf{u}) - \omega^2\int_\Gamma \bar{v}\left(\varepsilon_0\mathbf{u}\cdot\mathbf{n} - \mathrm{i}\omega^{-1}\mathrm{div}_\Gamma T\mathbf{u}_\Gamma\right) = 0. \end{aligned} \tag{4.3}$$

By Lemma 3.1, there exists a unique solution $u \in H^1_\Gamma(\Omega)$ to the problem

$$a(\nabla u, \nabla v) = (\mathbf{f}, \nabla v) \quad \text{for any } v \in H^1_\Gamma(\Omega). \tag{4.4}$$

It follows from (4.3) and (4.4) that the variational problem (4.1) can be equivalently reformulated into the following problem: find $\mathbf{u} \in Y$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a(\nabla u, \mathbf{v}) \quad \text{for all } \mathbf{v} \in Y. \tag{4.5}$$

Notice that u is determined by $\mathbf{f}$ from (4.4). Hence, there exists a functional $\mathcal{F} \in Y^*$, the dual space of Y, such that

$$\mathcal{F}(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a(\nabla u, \mathbf{v}) \quad \text{for all } \mathbf{v} \in Y.$$

The variational problem (4.5) can be further reduced to the problem: find $\mathbf{u} \in Y$ such that

$$a(\mathbf{u}, \mathbf{v}) = \mathcal{F}(\mathbf{v}) \quad \text{for any } \mathbf{v} \in Y. \tag{4.6}$$

In the rest of the paper, we shall apply the generalized Lax-Milgram theorem to prove that the variational problem (4.6) has a unique solution.

First, we show that Y has an equivalent norm defined by

$$\|\mathbf{u}\|_Y = \|\mathbf{u}\|_{H(\mathrm{curl},\Omega)} \quad \text{for any } \mathbf{u} \in Y.$$

The following two trace regularity results in $H^{-1/2}(\Gamma)$ and W are useful in subsequent analysis. The first lemma is proved in [13] (cf. Lemma 2.5).

LEMMA 4.1. *For any $\eta > 0$, there exists a positive constant $C(\eta)$ depending on η and h such that*

$$\|\mathbf{u}_\Gamma\|^2_{H^{-1/2}(\Gamma)} \le \eta \|\nabla \times \mathbf{u}\|^2_{L^2(\Omega)^3} + C(\eta)\|\mathbf{u}\|^2_{L^2(\Omega)^3} \quad \textit{for any } \mathbf{u} \in H(\mathrm{curl}, \Omega).$$

LEMMA 4.2. *It holds the estimate*

$$\|\mathbf{u}_\Gamma\|_W \le C\|\mathbf{u}\|_{H(\mathrm{curl},\Omega)} \quad \textit{for any } \mathbf{u} \in Y,$$

where C depends on ε, κ_0, and h.

Proof. For any $\mathbf{u} \in Y$, we have $\nabla \cdot (\varepsilon \mathbf{u}) = 0$, which gives

$$\nabla \cdot \mathbf{u} = -\left(\frac{\nabla \varepsilon}{\varepsilon}\right) \cdot \mathbf{u}. \tag{4.7}$$

Denote $\mathbf{u} = (u_1, u_2, u_3)$. Since $\varepsilon_0 \mathbf{u} \cdot \mathbf{n} = \mathrm{i}\omega^{-1}\mathrm{div}_\Gamma T\mathbf{u}_\Gamma$ on Γ, we take the Fourier transform with respect to ρ and obtain

$$\begin{aligned}\hat{u}_3 &= \frac{\mathrm{i}}{\omega\varepsilon_0}\widehat{\mathrm{div}_\Gamma T\mathbf{u}_\Gamma} = -\frac{1}{\omega\varepsilon_0}[\xi_1\ \xi_2]\begin{bmatrix}\hat{v}_1\\ \hat{v}_2\end{bmatrix}\\ &= \frac{-1}{\kappa_0^2\left(\kappa_0^2 - |\xi|^2\right)^{1/2}}[\xi_1\ \xi_2]\begin{bmatrix}\kappa_0^2 - \xi_2^2 & \xi_1\xi_2\\ \xi_1\xi_2 & \kappa_0^2 - \xi_1^2\end{bmatrix}\begin{bmatrix}\hat{u}_1\\ \hat{u}_2\end{bmatrix} = \frac{-\xi \cdot \hat{\mathbf{u}}_\Gamma}{\left(\kappa_0^2 - |\xi|^2\right)^{1/2}}.\end{aligned} \tag{4.8}$$

Using (4.8) and Lemma 4.1, we have

$$
\begin{aligned}
\|\mathbf{u}_\Gamma\|_W^2 = &\int_{\mathbb{R}^2} \frac{|\hat{\mathbf{u}}_\Gamma|^2}{(1+|\xi|^2)^{1/2}}\mathrm{d}\xi + \int_{\mathbb{R}^2} \frac{|\kappa_0^2-|\xi|^2|^{1/2}}{1+|\xi|^2}|\xi_1\hat{u}_2-\xi_2\hat{u}_1|^2\mathrm{d}\xi \\
&+ \int_{\mathbb{R}^2} \frac{|\xi_1\hat{u}_1+\xi_2\hat{u}_2|^2}{(1+|\xi|^2)|\kappa_0^2-|\xi|^2|^{1/2}}\mathrm{d}\xi \\
\le& \|\mathbf{u}_\Gamma\|_{H^{-1/2}(\Gamma)}^2 + \int_{\mathbb{R}^2}\left(|\kappa_0^2-|\xi|^2|^{1/2}|\hat{\mathbf{u}}_\Gamma|^2 + \frac{|\xi\cdot\hat{\mathbf{u}}_\Gamma|^2}{|\kappa_0^2-|\xi|^2|^{1/2}}\right)\mathrm{d}\xi \\
=& \|\mathbf{u}_\Gamma\|_{H^{-1/2}(\Gamma)}^2 + \int_{\mathbb{R}^2}\left(|\kappa_0^2-|\xi|^2|^{1/2}|\hat{\mathbf{u}}_\Gamma|^2 + |\kappa_0^2-|\xi|^2|^{1/2}|\hat{u}_3|^2\right)\mathrm{d}\xi \\
\le& C\|\mathbf{u}\|_Y^2 + \int_{\mathbb{R}^2}|\kappa_0^2-|\xi|^2|^{1/2}|\hat{\mathbf{u}}|^2\mathrm{d}\xi \\
\le& C\|\mathbf{u}\|_Y^2 + \max\{1,\kappa_0\}\int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{1/2}|\hat{\mathbf{u}}|^2\mathrm{d}\xi. \qquad (4.9)
\end{aligned}
$$

For any $u \in H^1(\Omega)$, we have

$$
|\hat{u}(\xi,h)|^2 \leqslant |\hat{u}(\xi,z)|^2 + 2\int_0^h |\hat{u}(\xi,z)\hat{u}'(\xi,z)|\mathrm{d}z, \quad z\in(0,h).
$$

Multiplying $(1+|\xi|^2)^{1/2}$ on both sides of the above inequality and integrating with z on $(0,h)$, we obtain

$$
\begin{aligned}
h(1+|\xi|^2)^{1/2}|\hat{u}(\xi,h)|^2 \leqslant&(1+|\xi|^2)^{1/2}\int_0^h|\hat{u}(\xi,z)|^2\mathrm{d}z \\
&+ 2h\int_0^h(1+|\xi|^2)^{1/2}|\hat{u}(\xi,z)||\hat{u}'(\xi,z)|\mathrm{d}z \\
\leqslant&(1+|\xi|^2)^{1/2}\int_0^h|\hat{u}(\xi,z)|^2\mathrm{d}z + h\int_0^h(1+|\xi|^2)|\hat{u}(\xi,z)|^2\mathrm{d}z \\
&+ h\int_0^h|\hat{u}'(\xi,z)|^2\mathrm{d}z,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\int_{\mathbb{R}^2}(1+|\xi|^2)^{1/2}|\hat{u}(\xi,h)|^2\mathrm{d}\xi \le& \frac{1+h}{h}\int_{\mathbb{R}^2}\int_0^h(1+|\xi|^2)|\hat{u}(\xi,z)|^2\mathrm{d}\xi\mathrm{d}z \\
&+ \int_{\mathbb{R}^2}\int_0^h|\hat{u}'(\xi,z)|^2\mathrm{d}\xi\mathrm{d}z. \qquad (4.10)
\end{aligned}
$$

Using (4.10) and

$$
\begin{aligned}
|\xi|^2|\hat{\mathbf{u}}(\xi,z)|^2 + |\hat{\mathbf{u}}'(\xi,z)|^2 = &|\xi_1\hat{u}_2-\xi_2\hat{u}_1|^2 + |\mathrm{i}\xi_1\hat{u}_3-\hat{u}_1'|^2 \\
&+|\mathrm{i}\xi_2\hat{u}_3-\hat{u}_2'|^2 + |\hat{u}_3' + \mathrm{i}(\xi_1\hat{u}_1+\xi_2\hat{u}_2)|^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
\int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^{1/2}|\hat{\mathbf{u}}|^2\mathrm{d}\xi \le& C\left[\int_\Omega|\mathbf{u}|^2 + \int_{\mathbb{R}^2}\int_0^h\left(|\xi|^2|\hat{\mathbf{u}}(\xi,z)|^2 + |\hat{\mathbf{u}}'(\xi,z)|^2\right)\mathrm{d}\xi\mathrm{d}z\right] \\
=& C\left(\int_\Omega|\mathbf{u}|^2 + \int_\Omega|\nabla\times\mathbf{u}|^2 + \int_\Omega|\nabla\cdot\mathbf{u}|^2\right) \le C\|\mathbf{u}\|_{H(\mathrm{curl},\Omega)}^2, \qquad (4.11)
\end{aligned}
$$

where (4.7) is used in the last inequality. The proof is completed by combining (4.9) and (4.11). ∎

The following corollary is a direct consequence of Lemma 4.2.

COROLLARY 4.3. *In the subspace Y of X, the norm $\|\cdot\|_X$ is equivalent to the norm $\|\cdot\|_Y$.*

Therefore, for the sake of simplicity, we will use $\|\cdot\|_Y$ instead of $\|\cdot\|_X$ to equip the function space Y.

Next, we examine the conditions in the generalized Lax-Milgram theorem. It follows from Lemma 4.2 and the Cauchy-Schwarz inequality that

$$\begin{aligned}|a(\mathbf{u},\mathbf{v})| &\le C_1\int_\Omega |\nabla\times\mathbf{u}\cdot\nabla\times\bar{\mathbf{v}}| + C_2\int_\Omega |\mathbf{u}\cdot\bar{\mathbf{v}}| + C_3\int_\Gamma |T\mathbf{u}_\Gamma\cdot\bar{\mathbf{v}}_\Gamma|\\ &\le C_1\|\nabla\times\mathbf{u}\|_{L^2(\Omega)^3}\|\nabla\times\mathbf{v}\|_{L^2(\Omega)^3} + C_2\|\mathbf{u}\|_{L^2(\Omega)^3}\|\mathbf{v}\|_{L^2(\Omega)^3} + C_3\|\mathbf{u}\|_W\|\mathbf{v}\|_W\\ &\le C\|\mathbf{u}\|_Y\|\mathbf{v}\|_Y,\end{aligned}$$

which shows that the sesquilinear form a is bounded. It is easy to show that

$$a(\mathbf{v},\mathbf{u}) = a(\bar{\mathbf{u}},\bar{\mathbf{v}}) \quad \text{for all } \mathbf{u},\mathbf{v}\in Y. \tag{4.12}$$

In order to apply the generalized Lax-Milgram theorem, we need to show the inf-sup condition:

$$\gamma = \inf_{0\ne\mathbf{u}\in Y}\sup_{0\ne\mathbf{v}\in Y}\frac{|a(\mathbf{u},\mathbf{v})|}{\|\mathbf{u}\|_Y\|\mathbf{v}\|_Y} > 0 \tag{4.13}$$

and the "transposed" inf-sup condition, which follows from (4.12) if (4.13) holds.

LEMMA 4.4. *If* (4.13) *holds, then the variational problem* (4.6) *has exactly one solution $\mathbf{u}\in Y$ for $\mathcal{F}\in Y^*$. Furthermore, it satisfies*

$$\|\mathbf{u}\|_Y \le \gamma^{-1}\|\mathcal{F}\|_{Y^*}. \tag{4.14}$$

In order to get (4.13), we need to establish an a priori estimate for any solution to (4.6), from which the inf-sup condition will be satisfied by the following lemma.

LEMMA 4.5. *If there exists a C such that*

$$\|\mathbf{u}\|_Y \le C\|\mathcal{F}\|_{Y^*} \tag{4.15}$$

for all $\mathbf{u}\in Y, \mathcal{F}\in Y^*$ satisfying (4.6), *then* (4.13) *holds with $C\ge\gamma^{-1}$.*

Proof. Fix $\mathbf{u}\in Y$ and define $\mathcal{F}(\mathbf{v}) = a(\mathbf{u},\mathbf{v})$ for all $\mathbf{v}\in Y$, then $\mathcal{F}\in Y^*$ and $\mathbf{u}$ is the solution to (4.6). We have

$$\|\mathbf{u}\|_Y \le C\|\mathcal{F}\|_{Y^*} = C\sup_{0\ne\mathbf{v}\in Y}\frac{|\mathcal{F}(\mathbf{v})|}{\|\mathbf{v}\|_Y} = C\sup_{0\ne\mathbf{v}\in Y}\frac{|a(\mathbf{u},\mathbf{v})|}{\|\mathbf{v}\|_Y}.$$

Hence we have

$$\gamma = \inf_{0\ne\mathbf{u}\in Y}\sup_{0\ne\mathbf{v}\in Y}\frac{|a(\mathbf{u},\mathbf{v})|}{\|\mathbf{u}\|_Y\|\mathbf{v}\|_Y} \ge C^{-1},$$

which completes the proof. ∎

LEMMA 4.6. *If there exists a constant* C_1 *such that*

$$\|\mathbf{u}\|_Y \le C_1 \|\mathbf{f}\|_{L^2(\Omega)^3} \tag{4.16}$$

for all $\mathbf{u} \in Y, \mathbf{f} \in L^2(\Omega)$ *which satisfy*

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \textit{for all } \mathbf{v} \in Y, \tag{4.17}$$

then for all $\mathbf{u} \in Y, \mathcal{F} \in Y^*$ *which satisfy* (4.6), *there exists a constant* C_2 *such that*

$$\|\mathbf{u}\|_Y \le C_2 \|\mathcal{F}\|_{Y^*}.$$

Proof. Suppose $\mathbf{u} \in Y$ is a solution to the problem

$$a(\mathbf{u}, \mathbf{v}) = \mathcal{F}(\mathbf{v}) \quad \text{for all } \mathbf{v} \in Y. \tag{4.18}$$

Let $a_0 : Y \times Y \to \mathbb{C}$ be defined by

$$a_0(\mathbf{u}, \mathbf{v}) = \int_\Omega \mu^{-1} \nabla \times \mathbf{u} \cdot \nabla \times \bar{\mathbf{v}} + M \int_\Omega \omega^2 \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} - \mathrm{i}\omega \int_\Gamma T \mathbf{u}_\Gamma \cdot \bar{\mathbf{v}}_\Gamma,$$

where $M > 0$ is a constant which will be determined later to ensure that a_0 is coercive in Y. It is easy to show that a_0 is a bounded sesquilinear form. For any $\mathbf{u} \in Y$,

$$\begin{aligned}
\mathrm{Re} a_0(\mathbf{u}, \mathbf{u}) = &\int_\Omega \mu^{-1} |\nabla \times \mathbf{u}|^2 + M\omega^2 \int_\Omega \mathrm{Re}(\varepsilon) |\mathbf{u}|^2 \\
&+ \mu_0^{-1} \int_{|\xi|^2 > \kappa_0^2} \left[\left(|\xi|^2 - \kappa_0^2\right)^{1/2} |\hat{\mathbf{u}}_\Gamma|^2 - \frac{|\xi \cdot \hat{\mathbf{u}}_\Gamma|^2}{\left(|\xi|^2 - \kappa_0^2\right)^{1/2}} \right] \mathrm{d}\xi \\
\geqslant &\mu_1^{-1} \int_\Omega |\nabla \times \mathbf{u}|^2 + M\omega^2 \varepsilon_0 \int_\Omega |\mathbf{u}|^2 + \mu_0^{-1} \int_{|\xi|^2 \geqslant \kappa_0^2 + \tau} \frac{|\xi_1 \hat{u}_2 - \xi_2 \hat{u}_1|^2 - \kappa_0^2 |\hat{\mathbf{u}}_\Gamma|^2}{\left(|\xi|^2 - \kappa_0^2\right)^{1/2}} \mathrm{d}\xi \\
&+ \mu_0^{-1} \int_{\kappa_0^2 < |\xi|^2 < \kappa_0^2 + \tau} \left[\left(|\xi|^2 - \kappa_0^2\right)^{1/2} |\hat{\mathbf{u}}_\Gamma|^2 - \frac{|\xi \cdot \hat{\mathbf{u}}_\Gamma|^2}{\left(|\xi|^2 - \kappa_0^2\right)^{1/2}} \right] \mathrm{d}\xi \\
\geqslant &\mu_1^{-1} \int_\Omega |\nabla \times \mathbf{u}|^2 + M\omega^2 \varepsilon_0 \int_\Omega |\mathbf{u}|^2 - \kappa_0^2 \mu_0^{-1} \int_{|\xi|^2 \geqslant \kappa_0^2 + \tau} \frac{|\hat{\mathbf{u}}_\Gamma|^2}{\left(|\xi|^2 - \kappa_0^2\right)^{1/2}} \mathrm{d}\xi \\
&- \mu_0^{-1} \int_{\kappa_0^2 < |\xi|^2 < \kappa_0^2 + \tau} \frac{|\xi \cdot \hat{\mathbf{u}}_\Gamma|^2}{\left(|\xi|^2 - \kappa_0^2\right)^{1/2}} \mathrm{d}\xi,
\end{aligned} \tag{4.19}$$

where $\tau > 0$ is a constant and we have used $|\xi|^2 |\hat{\mathbf{u}}_\Gamma|^2 = |\xi_1 \hat{u}_2 - \xi_2 \hat{u}_1|^2 + |\xi \cdot \hat{\mathbf{u}}_\Gamma|^2$.

Since $\mathbf{u} \in Y$, it follows from (4.8), (4.10), and (4.11) that we have

$$\begin{aligned}
&\int_{\kappa_0^2 < |\xi|^2 < \kappa_0^2 + \tau} \frac{|\xi \cdot \hat{\mathbf{u}}_\Gamma|^2}{\left(|\xi|^2 - \kappa_0^2\right)^{1/2}} \mathrm{d}\xi = \int_{\kappa_0^2 < |\xi|^2 < \kappa_0^2 + \tau} \left(|\xi|^2 - \kappa_0^2\right)^{1/2} |\hat{u}_3|^2 \mathrm{d}\xi \\
&\leqslant \tau^{1/2} \int_{\kappa_0^2 < |\xi|^2 < \kappa_0^2 + \tau} |\hat{u}_3|^2 \mathrm{d}\xi \leqslant \tau^{1/2} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{1/2} |\hat{u}_3|^2 \mathrm{d}\xi \le C \tau^{1/2} \|\mathbf{u}\|_Y^2.
\end{aligned}$$

We choose τ to be small enough such that

$$\int_{\kappa_0^2<|\xi|^2<\kappa_0^2+\tau} \frac{|\xi\cdot\hat{\mathbf{u}}_\Gamma|^2}{\left(|\xi|^2-\kappa_0^2\right)^{1/2}}\mathrm{d}\xi \le \frac{1}{10}\|\mathbf{u}\|_Y^2. \tag{4.20}$$

Fix such a $\tau>0$, we have from Lemma 4.1 that

$$\begin{aligned}
\kappa_0^2\int_{|\xi|^2\geqslant\kappa_0^2+\tau}\frac{|\hat{\mathbf{u}}_\Gamma|^2}{\left(|\xi|^2-\kappa_0^2\right)^{1/2}}\mathrm{d}\xi &\leqslant \kappa_0^2\max_{|\xi|^2>\kappa_0^2+\tau}\frac{\left(|\xi|^2+1\right)^{1/2}}{\left(|\xi|^2-\kappa_0^2\right)^{1/2}}\int_{|\xi|^2>\kappa_0^2+\tau}\frac{|\hat{\mathbf{u}}_\Gamma|^2}{\left(1+|\xi|^2\right)^{1/2}}\mathrm{d}\xi \\
&\leqslant \kappa_0^2\left(1+\frac{1+\kappa_0}{\tau}\right)^{1/2}\int_{\mathbb{R}^2}\frac{|\hat{\mathbf{u}}_\Gamma|^2}{\left(1+|\xi|^2\right)^{1/2}}\mathrm{d}\xi \\
&\leqslant \kappa_0^2\left(1+\frac{1+\kappa_0}{\tau}\right)^{1/2}\left(\eta\int_\Omega|\nabla\times\mathbf{u}|^2+C(\eta)\int_\Omega|\mathbf{u}|^2\right) \\
&\leqslant \frac{1}{10}\int_\Omega|\nabla\times\mathbf{u}|^2+C_1\int_\Omega|\mathbf{u}|^2, \qquad (4.21)
\end{aligned}$$

if we choose $\eta>0$ to be small enough such that

$$\eta\kappa_0^2\left(1+\frac{1+\kappa_0^2}{\tau}\right)^{1/2}<\frac{1}{10}.$$

Combining (4.19), (4.20) and (4.21), we obtain that

$$\begin{aligned}
\mathrm{Re}a_0(\mathbf{u},\mathbf{u}) &\geqslant \frac{4}{5\mu}\int_\Omega|\nabla\times\mathbf{u}|^2+\left(M\omega^2\varepsilon_0-\frac{1}{10\mu}-C_3\right)\int_\Omega|\mathbf{u}|^2 \\
&\geqslant \frac{4}{5\mu}\left(\int_\Omega|\nabla\times\mathbf{u}|^2+\int_\Omega|\mathbf{u}|^2\right)=\frac{4}{5\mu}\|\mathbf{u}\|_Y^2,
\end{aligned}$$

if we choose $M>0$ to be large enough such that

$$M\omega^2\varepsilon_0-C_3\ge\frac{9}{10\mu}.$$

This implies that a_0 is coercive by Corollary 4.3. Thus the problem of finding $\mathbf{u}_0\in Y$ such that

$$a_0(\mathbf{u}_0,\mathbf{v})=\mathcal{F}(\mathbf{v}) \quad \text{for all } \mathbf{v}\in Y \tag{4.22}$$

has a unique solution $\mathbf{u}_0$ which satisfies

$$\frac{4}{5\mu}\|\mathbf{u}_0\|_Y\leqslant\|\mathcal{F}\|_{Y^*}. \tag{4.23}$$

Now, defining $\mathbf{w}=\mathbf{u}-\mathbf{u}_0$ and using (4.18) and (4.22), we have

$$\begin{aligned}
a(\mathbf{w},\mathbf{v}) &= a(\mathbf{u},\mathbf{v})-a(\mathbf{u}_0,\mathbf{v})=\mathcal{F}(\mathbf{v})-\left(\mathcal{F}(\mathbf{v})+a(\mathbf{u}_0,\mathbf{v})-a_0(\mathbf{u}_0,\mathbf{v})\right) \\
&= ((M+1)\omega^2\varepsilon\mathbf{u}_0,\mathbf{v}).
\end{aligned}$$

Therefore $\mathbf{w}$ satisfies (4.17) with $\mathbf{f} = (M+1)\omega^2\varepsilon\mathbf{u}_0 \in L^2(\Omega)^3$. By the assumption, we have

$$\begin{aligned}\|\mathbf{w}\|_Y &\leqslant C_1(M+1)\omega^2\|\varepsilon\|_{L^\infty(\Omega)}\|\mathbf{u}_0\|_{L^2(\Omega)^3}\\ &\leqslant C_1(M+1)\omega^2\|\varepsilon\|_{L^\infty(\Omega)}\|\mathbf{u}_0\|_Y\\ &\leqslant \frac{5\mu}{4}C_1(M+1)\omega^2\|\varepsilon\|_{L^\infty(\Omega)}\|\mathcal{F}\|_{Y^*},\end{aligned}$$

where we have used (4.23) in the last inequality. Finally,

$$\|\mathbf{u}\|_Y \leqslant \|\mathbf{w}\|_Y + \|\mathbf{u}_0\|_Y \leqslant \frac{5\mu}{4}\left[1 + C_1(M+1)\omega^2\|\varepsilon\|_{L^\infty(\Omega)}\right]\|\mathcal{F}\|_{Y^*} = C_2\|\mathcal{F}\|_{Y^*},$$

which proves the Lemma. □

Combining Lemma 4.4, Lemma 4.5 and Lemma 4.6, we obtain the main theorem in this section.

THEOREM 4.7. *If it holds* (4.16) *for all* $\mathbf{u} \in Y, \mathbf{f} \in L^2(\Omega)^3$ *satisfying* (4.17)*, then the variational problem* (4.1) *has a unique solution in* X*.*

5. A priori estimate. In the section, we show that the assumption of Theorem 4.7 is valid for a certain class of dielectric permittivity ε, and thus the existence and uniqueness of our original scattering problem.

We shall show (4.16) holds for the solution $\mathbf{u}$ of problem (4.17) for some $\mathbf{f} \in L^2(\Omega)^3$. Since in Y, we have $\nabla\cdot(\varepsilon\mathbf{u}) = 0$, it follows from the equation satisfied by the solution $\mathbf{u}$ to (4.17) that $\nabla\cdot\mathbf{f} = 0$. First, we give some assumptions on ε.

Let $0 < \delta < h/2$ be small. We denote a "tubular domain of thickness δ" of Ω any open domain

$$D_\delta = \left\{(\rho, z) \in \Omega : r(\rho) - \frac{\delta}{2} < z < r(\rho) + \frac{\delta}{2}\right\}$$

where $r : \mathbb{R}^2 \to \mathbb{R}$ is a piecewise Lipschitz continuous function which satisfies $\delta < r(\rho) < h - \delta$.

We assume:

(a) $\varepsilon \in W^{1,\infty}(\Omega)$, $0 < \varepsilon_0 \le \mathrm{Re}(\varepsilon) \le \varepsilon_1$;

(b) There exist $\theta > 0$ and $D_\delta \subset \Omega$ of thickness of $\delta > 0$ such that

$$D_\delta \subset \{(\rho, z) \in \Omega : \mathrm{Im}\varepsilon > \theta\}; \tag{5.1}$$

(c) For any small $\eta > 0$, there exists a constant $C(\eta) > 0$ depending only on η, such that for all $(\rho, z) \in \Omega$ satisfying $\mathrm{Im}\varepsilon(\rho, z) < C(\eta)$ we have $|\nabla\varepsilon(\rho, z)| < \eta$.

To get an a priori estimate for the solutions to problem (4.17), we need the same technique as that used in [11] to derive Rellich identity. General speaking, we choose $z\partial_z\mathbf{E}$ and $\mathbf{E}$ as the test functions in (4.17) and use the integration by parts, we can obtain some identities, which are called Rellich type identity.

LEMMA 5.1. *Let* $\mathbf{u} \in Y$ *be a solution to* (4.17)*. It holds*

$$\begin{aligned}&\int_\Omega \left(2|\partial_z\mathbf{u}|^2 + \omega^2\mu z(\partial_z\mathrm{Re}\varepsilon)|\mathbf{u}|^2\right) + 2\mathrm{Re}\int_\Omega\left(\frac{\nabla\varepsilon}{\varepsilon}\right)\cdot\mathbf{u}\partial_z\bar{u}_3\\ &\qquad + \int_{\mathbb{R}^2}\mathrm{Im}\left(\kappa_0^2 - |\xi|^2\right)^{1/2}|\hat{\mathbf{u}}|^2\mathrm{d}\xi\\ &\le \left(2h\|\partial_z\mathbf{u}\|_{L^2(\Omega)^3} + (1+2\sqrt{2}\kappa_0 h)\|\mathbf{u}\|_{L^2(\Omega)^3}\right)\|\mathbf{g}\|_{L^2(\Omega)^3},\end{aligned} \tag{5.2}$$

where $\mathbf{g} := \mathbf{f} + \omega^2\mu\mathrm{Im}(\varepsilon)\mathbf{u}$. *Moreover,*

$$\int_\Omega \omega^2\mu\mathrm{Im}(\varepsilon)|\mathbf{u}|^2 \leqslant \|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3}. \tag{5.3}$$

Proof. Note that

$$\begin{aligned} 2\mathrm{Re}\int_\Omega \left(\frac{\nabla\varepsilon}{\varepsilon}\right)\cdot \mathbf{u}\partial_z\bar{u}_3 =& 2\mathrm{Re}\int_\Omega \left(\frac{\nabla_\Gamma\varepsilon}{\varepsilon}\right)\cdot \mathbf{u}\partial_z\bar{u}_3 + \int_\Omega \partial_z\log(|\varepsilon|)\partial_z|u_3|^2 \\ &- 2\int_\Omega \mathrm{Im}\left(\partial_z\log(\varepsilon)\right)\mathrm{Im}\left(\partial_z\bar{u}_3 u_3\right). \end{aligned}$$

The results follow directly from [11] (cf. Proposition 5.1) by taking $\alpha = 0$. ◻

Using this lemma, we can obtain the following theorem.

THEOREM 5.2. *Let* $\mathbf{u} \in Y$ *be a solution to* (4.17)*. If assumptions (a)–(c) hold, then there exists a positive constant* C *such that*

$$\|\mathbf{u}\|_Y \leqslant C\|\mathbf{f}\|_{L^2(\Omega)^3}.$$

Proof. It follows from Young's inequality and (5.2) that

$$\begin{aligned} &\frac{3}{2}\int_\Omega |\partial_z\mathbf{u}|^2 + \int_{\mathbb{R}^2} \mathrm{Im}\left(\kappa_0^2 - |\xi|^2\right)^{1/2}|\hat{\mathbf{u}}|^2\mathrm{d}\xi \\ &\leqslant \left(2h\|\partial_z\mathbf{u}\|_{L^2(\Omega)^3} + (1+2\sqrt{2}\kappa_0 h)\|\mathbf{u}\|_{L^2(\Omega)^3}\right)\|\mathbf{g}\|_{L^2(\Omega)^3} + C\int_\Omega |\nabla\varepsilon|^2|\mathbf{u}|^2. \end{aligned} \tag{5.4}$$

By assumption (c), for any $\eta > 0$, there exists a $C(\eta) > 0$ such that $|\nabla\varepsilon(\rho, z)| < \eta$ for all $(\rho, z) \in \Omega$ satisfying $\mathrm{Im}\varepsilon(\rho, z) < C(\eta)$. So we have

$$\begin{aligned} \int_\Omega |\nabla\varepsilon|^2|\mathbf{u}|^2 &= \left(\int_{\{\mathrm{Im}\varepsilon < C(\eta)\}} + \int_{\{\mathrm{Im}\varepsilon \geqslant C(\eta)\}}\right)|\nabla\varepsilon|^2|\mathbf{u}|^2 \\ &\leqslant \eta\int_{\{\mathrm{Im}\varepsilon < C(\eta)\}} |\mathbf{u}|^2 + C\int_{\{\mathrm{Im}\varepsilon \geqslant C(\eta)\}} |\mathbf{u}|^2 \\ &\leqslant \eta\int_\Omega |\mathbf{u}|^2 + \frac{C}{\omega^2\mu C(\eta)}\int_{\{\mathrm{Im}\varepsilon \geqslant C(\eta)\}} \omega^2\mu\mathrm{Im}(\varepsilon)|\mathbf{u}|^2 \\ &\leqslant \eta\int_\Omega |\mathbf{u}|^2 + \frac{C}{\omega^2\mu C(\eta)}\|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3}, \end{aligned} \tag{5.5}$$

where (5.3) is used in the last inequality. Applying Lemma 6.4 in [11], we have

$$\delta\|\mathbf{u}\|^2_{L^2(\Omega)} \leqslant 4h\|\mathbf{u}\|^2_{L^2(D_\delta)^3} + 8h^3\|\partial_z\mathbf{u}\|^2_{L^2(\Omega)^3}.$$

By assumption (b), we have

$$\begin{aligned} \delta\|\mathbf{u}\|^2_{L^2(\Omega)^3} &\leqslant 4h\int_{\{\mathrm{Im}\varepsilon > \theta\}} |\mathbf{u}|^2 + 8h^3\|\partial_z\mathbf{u}\|^2_{L^2(\Omega)^3} \\ &\leqslant \frac{4h}{\omega^2\mu\theta}\int_\Omega \omega^2\mu\mathrm{Im}\varepsilon|\mathbf{u}|^2 + 8h^3\|\partial_z\mathbf{u}\|^2_{L^2(\Omega)^3} \\ &\leqslant \frac{4h}{\omega^2\mu\theta}\|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3} + 8h^3\|\partial_z\mathbf{u}\|^2_{L^2(\Omega)^3}. \end{aligned} \tag{5.6}$$

Combining (5.5) with (5.6), we obtain

$$\int_\Omega |\nabla\varepsilon|^2|\mathbf{u}|^2 \leqslant \frac{8h^3}{\delta}\eta\int_\Omega |\partial_z\mathbf{u}|^2 + \frac{1}{\omega^2\mu}\left(\frac{4h}{\theta\delta}+\frac{C}{C(\eta)}\right)\|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3}.$$

Choosing small enough η, we derive from (5.3)–(5.4) and the above inequality that

$$\begin{aligned}\int_\Omega|\partial_z\mathbf{u}|^2 &+ \int_{\mathbb{R}^2}\mathrm{Im}\left(\kappa_0^2-|\xi|^2\right)^{1/2}|\hat{\mathbf{u}}|^2\mathrm{d}\xi\\ &\leqslant \left(2h\|\partial_z\mathbf{u}\|_{L^2(\Omega)^3}+(1+2\sqrt{2}\kappa_0 h)\|\mathbf{u}\|_{L^2(\Omega)^3}\right)\|\mathbf{g}\|_{L^2(\Omega)^3}\\ &\quad + C\|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3}\\ &\leqslant C\left(\|\partial_z\mathbf{u}\|_{L^2(\Omega)^3}+\|\mathbf{u}\|_{L^2(\Omega)^3}\right)\left[\|\mathbf{f}\|_{L^2(\Omega)^3}+\left(\|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3}\right)^{1/2}\right].\end{aligned}\tag{5.7}$$

Using (5.6) and (5.7), we also get

$$\begin{aligned}\int_\Omega|\mathbf{u}|^2 \leqslant{}& C\left(\|\partial_z\mathbf{u}\|_{L^2(\Omega)^3}+\|\mathbf{u}\|_{L^2(\Omega)^3}\right)\\ &\times\left[\|\mathbf{f}\|_{L^2(\Omega)^3}+\left(\|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3}\right)^{1/2}\right].\end{aligned}\tag{5.8}$$

Adding (5.7) and (5.8), we obtain

$$\begin{aligned}\|\partial_z\mathbf{u}\|_{L^2(\Omega)^3}+\|\mathbf{u}\|_{L^2(\Omega)^3}+\left(\int_{\mathbb{R}^2}\mathrm{Im}\left(\kappa_0^2-|\xi|^2\right)^{1/2}|\hat{\mathbf{u}}|^2\mathrm{d}\xi\right)^{1/2}&\\ \leqslant C\left[\|\mathbf{f}\|_{L^2(\Omega)^3}+\left(\|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3}\right)^{1/2}\right].&\end{aligned}$$

Applying Young's inequality again, we have

$$\|\partial_z\mathbf{u}\|_{L^2(\Omega)^3}+\|\mathbf{u}\|_{L^2(\Omega)^3}+\left(\int_{\mathbb{R}^2}\mathrm{Im}\left(\kappa_0^2-|\xi|^2\right)^{1/2}|\hat{\mathbf{u}}|^2\mathrm{d}\xi\right)^{1/2}\leqslant C\|\mathbf{f}\|_{L^2(\Omega)^3}.\tag{5.9}$$

Since $\mathbf{u}\in Y$ is a solution to (4.17), taking $\mathbf{v}=\mathbf{u}$ in (4.17), we have

$$\begin{aligned}\int_\Omega\mu^{-1}|\nabla\times\mathbf{u}|^2-\int_\Omega\omega^2\mathrm{Re}\varepsilon|\mathbf{u}|^2+\mu_0^{-1}\int_{|\xi|^2>\kappa_0^2}\left[\left(|\xi|^2-\kappa_0^2\right)^{1/2}|\hat{\mathbf{u}}_\Gamma|^2-\frac{|\xi\cdot\hat{\mathbf{u}}_\Gamma|^2}{\left(|\xi|^2-\kappa_0^2\right)^{1/2}}\right]\mathrm{d}\xi&\\ =\mathrm{Re}\int_\Omega\mathbf{f}\cdot\bar{\mathbf{u}}.&\end{aligned}$$

By (4.8), we have

$$\begin{aligned}\int_\Omega|\nabla\times\mathbf{u}|^2 &\leqslant C_1\int_\Omega|\mathbf{u}|^2+C_2\int_{|\xi|^2>\kappa_0^2}\left(|\xi|^2-\kappa_0^2\right)^{1/2}|\hat{u}_3|^2\mathrm{d}\xi+C_3\|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3}\\ &= C_1\int_\Omega|\mathbf{u}|^2+C_2\int_{\mathbb{R}^2}\mathrm{Im}\left(|\xi|^2-\kappa_0^2\right)^{1/2}|\hat{u}_3|^2\mathrm{d}\xi+C_3\|\mathbf{f}\|_{L^2(\Omega)^3}\|\mathbf{u}\|_{L^2(\Omega)^3}\\ &\leqslant C_1\|\mathbf{u}\|^2_{L^2(\Omega)^3}+C_2\int_{\mathbb{R}^2}\mathrm{Im}\left(|\xi|^2-\kappa_0^2\right)^{1/2}|\hat{\mathbf{u}}|^2\mathrm{d}\xi+C_3\|\mathbf{f}\|^2_{L^2(\Omega)^3}.\end{aligned}$$

It follows from (5.9) that we get

$$\int_\Omega|\nabla\times\mathbf{u}|^2\leqslant C\|\mathbf{f}\|^2_{L^2(\Omega)^3}.\tag{5.10}$$

Finally, combining (5.9) and (5.10), we derive that

$$\|\mathbf{u}\|_Y \leqslant C\|\mathbf{f}\|_{L^2(\Omega)^3},$$

which completes the proof. □

The conclusion still holds if ε satisfies the assumptions in [11]. Here we give different assumptions on ε, which are valid for a larger class of functions.

Our main result in this paper is the following theorem.

THEOREM 5.3. *If assumptions (a)–(c) hold, then the variational problem* (4.1) *has a unique solution in* X.

Proof. The proof follows directly by combining Theorem 5.2 with Theorem 4.7. □

6. Conclusions. In this paper, we studied the solution for the electromagnetic wave scattering by an unbounded structure, which was mounted on a perfectly conducting infinite plane. A variational formulation was introduced by using a transparent boundary condition. Based on a Hodge decomposition and an a priori estimate, the problem was shown to have a unique weak solution for a wide class of the dielectric permittivity by using the generalized Lax-Milgram theorem. However, what is the best condition is open for the dielectric permittivity, and remains the topic of a future work.

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