A DELTA-REGULARIZATION FINITE ELEMENT METHOD FOR A DOUBLE CURL PROBLEM WITH DIVERGENCE-FREE CONSTRAINT

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Abstract. To deal with the divergence-free constraint in a double curl problem: $\operatorname{curl}\mu^{-1}\operatorname{curl} u = f$ and $\operatorname{div}\varepsilon u = 0$ in Ω, where μ and ε represent the physical properties of the materials occupying Ω, we develop a δ-regularization method: $\operatorname{curl}\mu^{-1}\operatorname{curl} u_\delta + \delta\varepsilon u_\delta = f$ to completely ignore the divergence-free constraint $\operatorname{div}\varepsilon u = 0$. It is shown that u_δ converges to u in $H(\operatorname{curl};\Omega)$ norm as $\delta \to 0$. The edge finite element method is then analyzed for solving u_δ. With the finite element solution $u_{\delta,h}$, quasi-optimal error bound in $H(\operatorname{curl};\Omega)$ norm is obtained between u and $u_{\delta,h}$, including a uniform (with respect to δ) stability of $u_{\delta,h}$ in $H(\operatorname{curl};\Omega)$ norm. All the theoretical analysis is done in a general setting, where μ and ε may be discontinuous, anisotropic and inhomogeneous, and the solution may have a very low piecewise regularity on each material subdomain Ω_j with $u, \operatorname{curl} u \in (H^r(\Omega_j))^3$ for some $0 < r < 1$, where r may be not greater than $1/2$. To establish the uniform stability and the error bound for $r \le 1/2$, we have respectively developed a new theory for the $\mathscr{K}_h$ ellipticity (related to mixed methods) and a new theory for the Fortin interpolation operator. Numerical results presented confirm the theory.

Key words. Double curl problem, divergence-free constraint, regularization, edge element, uniform stability, $\mathscr{K}_h$ ellipticity, error bound, Fortin operator

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1. Introduction. Given a simply-connected Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$, with a connected boundary $\partial\Omega$. Let $\mu, \varepsilon : \Omega \mapsto \mathbb{R}^{3\times 3}$ be given matrix functions, representing the physical properties (such as permeability and permittivity) of the material occupying Ω. We assume that μ and ε are piecewise smooth with respect to a finite partition $\mathscr{P}$ of Ω, $\mathscr{P} = \{\Omega_j, j = 1, 2, \cdots, J\}$, where every Ω_j is a simply-connected Lipschitz polyhedron with connected boundary. Let $\mathscr{S}_{\rm int}$ and $\mathscr{S}_{\rm ext}$ denote the set of the faces of $\mathscr{P}$ contained in Ω and the set of the faces of $\mathscr{P}$ contained in $\partial\Omega$, respectively. Let $[q]|_S$ denote the jump of q across $S \in \mathscr{S}_{\rm int}$. Given $f : \Omega \mapsto \mathbb{R}^3$, satisfying $\operatorname{div} f = 0$. Consider the double curl problem as follows:

$$\operatorname{curl}\mu^{-1}\operatorname{curl} u = f \quad \text{in } \Omega_j, 1 \le j \le J, \tag{1.1}$$

$$\operatorname{div}\varepsilon u = 0 \quad \text{in } \Omega_j, 1 \le j \le J, \tag{1.2}$$

$$[u \times n]|_S = 0, \quad [\mu^{-1}\operatorname{curl} u \times n]|_S = 0, \quad [\varepsilon u \cdot n]|_S = 0 \quad \forall S \in \mathscr{S}_{\rm int}, \tag{1.3}$$

$$u \times n|_S = 0 \quad \forall S \in \mathscr{S}_{\rm ext}. \tag{1.4}$$

Problem (1.1)-(1.4) arises from computational electromagnetism [10][47][43][44]. An example is the vector potential method [31] for some divergence-free unknown which may be expressed as the curl of u(the vector potential), where the divergence-free constraint (1.2) is set up to ensure the uniqueness of the solution. Otherwise, there would exist infinitely many solutions, due to the infinite dimensional kernel of the curl operator consisting of the form ∇p where $p \in H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_S = 0, \forall S \in \mathscr{S}_{\rm ext}\}$. Another example [47] is from the stabilization of the time-harmonic Maxwell's equation $\operatorname{curl}\mu^{-1}\operatorname{curl} u - \kappa^2 u = f$ with a very low frequency number κ, where the divergence-free constraint (1.2) may be introduced to play the stabilization role so that a unique solution can exist even for $\kappa = 0$. By introducing Hilbert spaces $H(\operatorname{div};\Omega) = \{v \in (L^2(\Omega))^3 : \operatorname{div} v \in L^2(\Omega)\}$, $H(\operatorname{curl};\Omega) = \{v \in (L^2(\Omega))^3 : \operatorname{curl} v \in (L^2(\Omega))^3\}$, $H_0(\operatorname{curl};\Omega) = \{v \in H(\operatorname{curl};\Omega) : u \times n|_S = 0 \quad \forall S \in \mathscr{S}_{\rm ext}\}$, $H(\operatorname{div};\varepsilon;\Omega) = \{v \in (L^2(\Omega))^3 : \operatorname{div}\varepsilon v \in L^2(\Omega)\}$, $H(\operatorname{div}^0;\varepsilon;\Omega) = \{v \in H(\operatorname{div};\varepsilon;\Omega) : \operatorname{div}\varepsilon v = 0\}$, and $H(\operatorname{div}^0;\Omega) := H(\operatorname{div}^0;1;\Omega)$, $H_0(\operatorname{div};\Omega) = \{v \in H(\operatorname{div};\Omega) : v \cdot n|_S = 0 \quad \forall S \in \mathscr{S}_{\rm ext}\}$. Let $(\cdot,\cdot)$ denote the L^2-inner product. Corresponding to (1.1)-(1.4), we may state a variational problem as follows: Find $u \in H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div}^0;\varepsilon;\Omega)$ such that

$$(\mu^{-1}\operatorname{curl} u, \operatorname{curl} v) = (f, v) \quad \forall v \in H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div}^0;\varepsilon;\Omega), \tag{1.5}$$

When discretized by a finite element method for solving problem (1.5), we would naturally seek the finite element solution in a finite element subspace of $H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div}^0;\varepsilon;\Omega)$. However, as is well-known, it is quite difficult to construct a finite element space consisting of lower-order piecewise polynomials to satisfy

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the divergence-free constraint. With rather restrictive finite element triangulations (e.g., multiply-refined composite elements) for lower-order (e.g., quadratic) elements or with higher-order (at least sextic) elements together with some relatively less restrictive but still quite structured finite element triangulations, one could construct divergence-free elements in the case where ε itself is piecewise polynomial [54].

In practice, the rule for dealing with the divergence-free constraint is to let it be satisfied weakly. This could be done by including the divergence-free constraint directly into the variational formulation and by seeking the solution in some bigger Hilbert space U without the divergence-free constraint other than the restricted $H_0(\mathrm{curl}\,;\Omega)\cap H(\mathrm{div}^0;\varepsilon;\Omega)$ with the divergence-free constraint. Two general ways are the divergence-regularization method and the mixed method. The former is to find $u\in U$ such that $(\mu^{-1}\mathrm{curl}\,u,\mathrm{curl}\,v)+\langle \mathrm{div}\,\varepsilon u,\mathrm{div}\,\varepsilon v\rangle=(f,v)$ for all $v\in U$, where there are several choices for defining $\langle\cdot,\cdot\rangle$ and U, see [36][25][29][11]. When $\langle\cdot,\cdot\rangle$ is simply taken as the L^2 inner product, the method is referred to as the plain regularization (PR) method [36][24], with a second-order $U=H_0(\mathrm{curl}\,;\Omega)\cap H(\mathrm{div}\,;\varepsilon;\Omega)$-elliptic problem. For smooth ε, one may consider the classical continuous finite element method for the PR method. But, when the solution is only in H^r for some $r<1$, the continuous finite element method cannot give a correct solution [37][47][10][25][24]. Meanwhile, the PR formulation is not suitable for discontinuous ε; otherwise, by the introduction of amounts of jumps, one may consider the discontinuous Galerkin method. But, likewise, this method cannot accommodate the nonsmooth solution with a low-regularity in H^r with $r<1$ [50]. Some of the combination of the nonconforming element method and the discontinuous Galerkin method may lead to a correct approximation of the nonsmooth solution [14]. One may still consider to use the continuous element if adopting the recently developed L^2 projected method [29][28], or the H^{-1} method [11] and the weighted method [25][16][46]. All these methods involve sophisticated modifications.

The mixed method [47] for dealing with the divergence-free constraint is to find $u\in H_0(\mathrm{curl}\,;\Omega)$ and $p\in H_0^1(\Omega)$ such that

$$\begin{aligned}(\mu^{-1}\mathrm{curl}\,u,\mathrm{curl}\,v)+\quad(\nabla p,\varepsilon v)&=(f,v)\quad\forall v\in H_0(\mathrm{curl}\,;\Omega),\\ (\varepsilon u,\nabla q)&=0\quad\forall q\in H_0^1(\Omega).\end{aligned}\tag{1.6}$$

With the help of the Lagrange multiplier p, the solution is only required to belong to the Hilbert space $H_0(\mathrm{curl}\,;\Omega)$. The Lagrange multiplier $p\in H_0^1(\Omega)$ satisfies the following weak problem

$$(\varepsilon\nabla p,\nabla q)=(f,\nabla q)\quad\forall q\in H_0^1(\Omega).\tag{1.7}$$

Note that p is actually equal to zero with a compatible $f\in H(\mathrm{div}^0;\Omega)$. For the mixed problem (1.6) one may consider to use the edge element for u and the continuous element for p, or the discontinuous Galerkin method [41]. The difficulty would be the verification of $\mathscr{K}$-ellipticity in the classical theory for saddle-point problems [15]. In the literature [3][47], the verification has been to rely on the following continuous embedding: there is $s>1/2$ such that

$$H_0(\mathrm{curl}\,;\Omega)\cap H(\mathrm{div}^0;\Omega)\hookrightarrow (H^s(\Omega))^3\quad\text{where } ||v||_s\le C||\mathrm{curl}\,v||_0 \text{ for } v\in H_0(\mathrm{curl}\,;\Omega)\cap H(\mathrm{div}^0;\Omega).\tag{1.8}$$

But, in some cases, such $s>1/2$ does not exist for (1.8) to hold true, e.g., when Ω is only Lipschitz, we can only find $s=1/2$, see [23]. We shall address this point again later in the paper. The most difficult would be of course the saddle-point structure of the mixed problem, since the indefiniteness of the saddle-point system would thwart many classical iterative algorithms, such as conjugate gradient algorithm. The preconditioning step is necessary to have a good iterative algorithm for solving the saddle-point system [6].

In our paper, we shall study a much simpler method, as originally appeared in [51], to deal with the divergence-free constraint. Just completely neglecting the divergence-free constraint, instead, we consider a δ-perturbed problem: with $\delta>0$ decreasing to zero, finding a family of $u_\delta\in H_0(\mathrm{curl}\,;\Omega)$ such that

$$(\mu^{-1}\mathrm{curl}\,u_\delta,\mathrm{curl}\,v)+\delta(\varepsilon u_\delta,v)=(f,v)\quad\forall v\in H_0(\mathrm{curl}\,;\Omega).\tag{1.9}$$

The δ-perturbed method will be called the δ-regularization method, since problem (1.9) is free of the divergence-free constraint and since problem (1.9) is $H_0(\mathrm{curl}\,;\Omega)$-elliptic. In comparisons with previous existing methods, there are several obvious features of the present method: a) it is no longer subject to the divergence-free constraint; b) it is more suitable for discontinuous ε, since no $\mathrm{div}\,\varepsilon v$ appears; c) it only involves a space $H_0(\mathrm{curl}\,;\Omega)$ which can be discretized by edge elements composing of lower-order piecewise polynomials, since no Lagrange multiplier is introduced; d) it is always well-posed and results in a symmetric, positive definite system in the finite element discretization, so the resultant algebraic system may be implemented more readily. In fact, since it results in a symmetric, positive definite system, the δ-regularization problem may be conveniently solved by any direct or iterative methods [33]. Moreover, nowadays there are highly efficient multigrid methods and preconditioning techniques available for solving (1.9) where multigrid convergence and preconditioned conditioning are uniform with respect to the parameter δ [4][38].

With $f \in H(\operatorname{div}^0;\Omega)$, for all $\delta > 0$, we can verify that u_δ satisfies the divergence-free condition:

$$\operatorname{div} \varepsilon u_\delta = 0. \tag{1.10}$$

Introduce $H(\operatorname{curl};\Omega)$-norm $||v||^2_{0,\operatorname{curl}} := ||v||_0^2 + ||\operatorname{curl} v||_0^2$ and $H(\operatorname{curl};\Omega) \cap H(\operatorname{div};\varepsilon;\Omega)$-norm $||v||^2_{0,\operatorname{curl},\operatorname{div},\varepsilon} := ||v||_0^2 + ||\operatorname{curl} v||_0^2 + ||\operatorname{div} \varepsilon v||_0^2$, where $||\cdot||_0$ represents the L^2-norm. We show that u_δ converges to the original u as follows:

$$||u - u_\delta||_{0,\operatorname{curl},\operatorname{div},\varepsilon} = ||u - u_\delta||_{0,\operatorname{curl}} \le C\delta||u||_0. \tag{1.11}$$

We then analyze the edge finite element method for the δ-regularization problem in the finite element space $U_h \subset H_0(\operatorname{curl};\Omega)$, under the general setting where μ, ε may be discontinuous, anisotropic and inhomogeneous and $u, \operatorname{curl} u$ are nonsmooth and may have very low regularity only in $\prod_{j=1}^J (H^r(\Omega_j))^3$ for some $0 < r < 1$. Assume that U_h allows the usual both L^2 and $H(\operatorname{curl})$-orthogonal decomposition [47]. For the lowest-order edge/Nédélec element of first-family [48][47][37], we establish the following error estimates for the finite element solution $u_{\delta,h}$ of the δ-regularization problem:

$$||u - u_{\delta,h}||_{0,\operatorname{curl}} \le C(\delta + h^r)||f||_0. \tag{1.12}$$

From (1.12) we may choose $\delta \le h^r$ to have the optimal error bound in the usual sense. A ready choice is $\delta = h$. Note that if the solution and its curl are more regular, higher-order edge elements can be employed to obtain higher-order error bounds. At the same time, a δ-uniform stability for any given compatible $f \in H(\operatorname{div}^0;\Omega)$ is obtained, where C does not depend on δ, as follows:

$$||u_{\delta,h}||_{0,\operatorname{curl}} \le C||f||_0. \tag{1.13}$$

Interestingly, the theory for the uniform stability (1.13) is closely related to the well-posedness of the mixed problem (1.6). In fact, (1.13) is essentially the consequence of the $\mathscr{K}_h$-ellipticity (which, together with the Inf-Sup condition, ensures the well-posedness of the mixed problem (1.6)). In the literature [47][3], the $\mathscr{K}_h$-ellipticity was only shown under the assumption (1.8) with $s > 1/2$. In this paper, we shall establish the $\mathscr{K}_h$-ellipticity using Assumption A2) (a type of regular-singular decomposition) in section 4, instead of (1.8). The Assumption A2) is much weaker than (1.8), because the former generally holds but the latter may not, see Remarks 3.3, 4.1, 5.4 and 6.4. In addition, the error bound in $H(\operatorname{curl};\Omega)$-norm in (1.12) is obtained with the help of the Fortin operator [9]. Likewise, in the literature, the well-posedness and the error estimate of the Fortin operator rely on the assumption (1.8) with $s > 1/2$. Under the much weaker Assumption A2) again, in this paper we shall provide a new theory for the Fortin operator, so that we can establish the error estimate (1.12) in $H(\operatorname{curl};\Omega)$ norm for very low regular solution, i.e., $r \le 1/2$ in (1.12). This is in sharp contrast to the numerous existing literature, where r and s are usually assumed to be greater than 1/2, e.g., see [2][9][18][39][47][40][41][49], just to name a few. Note that for interface problem, not only the global regularity of the solution is very low (this is a well-known fact), but also its piecewise regularity over each material subdomain Ω_j may still be possibly very low, see [26].

Also, there exists the so-called bounded co-chain or smoothed projector [5][19][52] for a low regularity solution. Such projector may be applied to problem (1.9), but we cannot obtain the δ-uniform optimal convergence, since it does not preserve the divergence-free constraint. By contrast, the Fortin operator trivially does, as it is indeed a finite element solution solver, and it is for this reason that we can obtain δ-uniform convergence $||u_\delta - u_{\delta,h}||_{0,\operatorname{curl}} \le Ch^r||f||_0$ from which and (1.11), we have (1.12). In addition, as mentioned earlier, the low regularity convergence is also known in the continuous element methods where the regular-singular decomposition is also often used as a fundamental tool in the error estimates, e.g., [11].

The rest of this paper is organized as follows. In section 2, we obtain the convergence of the solution of problem (1.9) to the solution of problem (1.5). In section 3, the edge finite element method is defined and the uniform stability (1.13) is obtained under the $\mathscr{K}_h$ ellipticity. In section 4, a general $\mathscr{K}_h$ ellipticity is established without the assumption (1.8), instead under Assumption A2) the regular-singular decomposition. In section 5, the δ-uniform error estimates of the finite element solution of problem (1.9) is established, with the application of the Fortin operator. In section 6, under Assumption A2), we present the general theory for the Fortin operator without the assumption (1.8). In section 7, numerical results are presented to illustrate the proposed method. In the last section, some concluding remarks are given, and an extension of the proposed method to a more general problem arising in computational electromagnetism is briefly discussed.

2. Convergence for continuous problem. Throughout the paper, we shall use the standard Hilbert and Sobolev spaces $H^s(\Omega)$ for any $s \in \mathbb{R}$ and $L^p(\Omega)$ for any $p \geq 2$, see [1][35]. Assume that μ and ε are symmetric, uniformly positive definite, satisfying $\xi'\mu(x)\xi, \xi'\varepsilon(x)\xi \geq C|\xi|^2$ for all $\xi \in \mathbb{R}^3$ almost everywhere over $\bar{\Omega}$, $\mu, \varepsilon \in (L^\infty(\Omega))^{3\times 3}$, satisfying $\mu|_{\Omega_j}, \varepsilon|_{\Omega_j} \in (W^{1,\infty}(\Omega_j))^{3\times 3}, 1 \leq j \leq J$, where μ, ε are required to be piecewise smooth so that we could obtain the regularity of the solution of problem (1.1)-(1.4). Let $||v||^2_{0,\mu^{-1}} := (\mu^{-1}v, v)$ and $||v||^2_{0,\varepsilon} := (\varepsilon v, v)$ be the μ^{-1}-weighted and ε-weighted L^2 norms respectively.

We first give a lemma about the divergence of u_δ.

Lemma 2.1 *For compatible $f \in H(\mathrm{div}^0; \Omega)$, the solution $u_\delta \in H_0(\mathrm{curl}; \Omega)$ of problem (1.9), in fact, belongs to $H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}; \varepsilon; \Omega)$, and satisfies the equation*

$$\mathrm{div}\, \varepsilon u_\delta = 0. \tag{2.1}$$

Proof. Taking any $\varphi \in C_0^\infty(\Omega)$ (the linear space of infinitely differentiable functions, with compact support on Ω) and inserting $v = \nabla\varphi \in H_0(\mathrm{curl}; \Omega)$ into (1.9) immediately yields $(\varepsilon u_\delta, \nabla\varphi) = 0$ which proves the lemma. $\Box$

We next recall a Poincaré-Friedrichs' inequality over $H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}^0; \varepsilon; \Omega)$.

Proposition 2.1 [30] *For any Lipschitz domain Ω and for any ε as assumed earlier, we have*

$$||v||_0 \leq C||\mathrm{curl}\, v||_0 \quad \forall v \in H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}^0; \varepsilon; \Omega). \tag{2.2}$$

We are now in a position to investigate the existence and uniqueness and the convergence of u_δ.

Theorem 2.1 *For any given $\delta > 0$ and for any $f \in (L^2(\Omega))^3$ problem (1.9) has a unique solution u_δ. Moreover, for the compatible $f \in H(\mathrm{div}^0; \Omega)$, there exists a constant $C > 0$, independent of δ, such that*

$$||u_\delta||_{0,\mathrm{curl}} \leq C||f||_0. \tag{2.3}$$

Proof. From the classical Lax-Milgram lemma [13][20], existence and uniqueness of the solution u_δ to problem (1.9) is obvious. Note that u_δ is in fact in $H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}^0; \varepsilon; \Omega)$ from Lemma 2.1. Taking $v = u_\delta$ in problem (1.9), from (2.2) in Proposition 2.1 we can obtain $||\mathrm{curl}\, u_\delta||_0 \leq C||\mathrm{curl}\, u_\delta||_{0,\mu^{-1}} \leq C||f||_0$. From (2.2) again, (2.3) follows. $\Box$

Remark 2.1 With the compatible $f \in H(\mathrm{div}^0; \Omega)$, from Proposition 2.1 we have a stability for the original u, the solution to problem (1.5), as follows:

$$||u||_{0,\mathrm{curl}} \leq C||f||_0. \tag{2.4}$$

Theorem 2.2 *Let u and u_δ denote the solutions of problem (1.5) and problem (1.9), respectively. For a compatible $f \in H(\mathrm{div}^0; \Omega)$, there exists C, independent of δ, such that*

$$||u - u_\delta||_{0,\mathrm{curl}} = ||u - u_\delta||_{0,\mathrm{curl},\mathrm{div},\varepsilon} \leq C\delta||u||_0. \tag{2.5}$$

Proof. From Lemma 2.1 and (1.2) we have $||u - u_\delta||_{0,\mathrm{curl},\mathrm{div},\varepsilon} = ||u - u_\delta||_{0,\mathrm{curl}}$. Taking $v := u - u_\delta \in H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}^0; \varepsilon; \Omega)$, from (1.5) and (1.9) we can obtain (2.5) following the same argument for (2.3). $\Box$

Remark 2.2 A similar result was obtained in [51]. From (2.4) and (2.5) we have the convergence

$$||u - u_\delta||_{0,\mathrm{curl}} = ||u - u_\delta||_{0,\mathrm{curl},\mathrm{div},\varepsilon} \leq C\delta||f||_0. \tag{2.6}$$

3. Edge finite element method. For any given $h > 0$, let $\mathcal{T}_h$ denote the shape-regular conforming triangulation of Ω into tetrahedra [20][13], where $h := \max_{T\in\mathcal{T}_h} h_T$, and h_T denotes the diameter of T. We assume that $\mathcal{T}_h$ is also conforming along every interface $S \in \mathscr{S}_{\mathrm{int}}$ and every boundary face $S \in \mathscr{S}_{\mathrm{ext}}$. In our analysis we shall use the first family of edge/Nédélec elements of $H_0(\mathrm{curl}; \Omega)$, the Raviart-Thomas elements of $H_0(\mathrm{div}; \Omega)$ and the Lagrange elements of $H_0^1(\Omega)$, see [48][47][32]. Over every $T \in \mathcal{T}_h$, let $\mathcal{P}_l(T)$ and $\widetilde{\mathcal{P}}_l(T)$ denote the space of polynomials of total degree not greater than the integer $l \geq 0$ and the subspace of homogeneous polynomials of total degree l, respectively. On every $T \in \mathcal{T}_h$, for $l \geq 1$, we introduce the Nédélec element of order l: $\mathcal{N}_l(T) := \mathrm{span}\{a + b, a \in (\mathcal{P}_{l-1}(T))^3, b \in (\widetilde{\mathcal{P}}_l(T))^3, b \cdot x = 0\}$ and the Raviart-Thomas element of order l: $\mathcal{RT}_l(T) = \mathrm{span}\{a + bx, a \in (\mathcal{P}_{l-1}(T))^3, b \in \widetilde{\mathcal{P}}_{l-1}(T)\}$. Define

$$U_h = \{v \in H_0(\mathrm{curl}; \Omega) : v|_T \in \mathcal{N}_l(T), \forall T \in \mathcal{T}_h\}, \tag{3.1}$$

$$X_h = \{v \in H_0(\mathrm{div}; \Omega) : v|_T \in \mathcal{RT}_l(T), \forall T \in \mathcal{T}_h\}, \tag{3.2}$$

$$Q_h = \{q \in H_0^1(\Omega) : q|_T \in \mathcal{P}_l(T), \forall T \in \mathcal{T}_h\}. \tag{3.3}$$

We state the finite element problem corresponding to problem (1.9): To find $u_{\delta,h} \in U_h$ such that

$$(\mu^{-1}\operatorname{curl} u_{\delta,h}, \operatorname{curl} v) + \delta(\varepsilon u_{\delta,h}, v) = (f, v) \quad \forall v \in U_h. \tag{3.4}$$

Since $U_h \subset H_0(\operatorname{curl};\Omega)$, from Theorem 2.1, we have the well-posedness of (3.4), as stated below.

Theorem 3.1 *For any $f \in (L^2(\Omega))^3$, for any $\delta > 0$ and for any h, problem (3.4) is well-posed.* □

However, simply from the trivial $H_0(\operatorname{curl};\Omega)$-ellipticity of the bilinear form

$$\mathscr{A}_\delta(u, v) := (\mu^{-1}\operatorname{curl} u, \operatorname{curl} v) + \delta(\varepsilon u, v), \tag{3.5}$$

we cannot obtain a δ-uniform stability on the finite element solution $u_{\delta,h}$. In order to have a uniform stability like (2.3) for $u_{\delta,h}$, we recall the well-known decomposition of U_h. Let $(\cdot,\cdot)_{0,\varepsilon} = (\varepsilon\cdot,\cdot)$ denote the ε-weighted L^2 inner product.

Proposition 3.1 [47] *For U_h and Q_h defined by (3.1) and (3.3), defining $Z_h(\varepsilon) = \{v \in U_h : (\varepsilon v, \nabla q) = 0, \forall q \in Q_h\}$, we have the following L^2-orthogonal decomposition with respect to $(\cdot,\cdot)_{0,\varepsilon}$:*

$$U_h = Z_h(\varepsilon) + \nabla Q_h. \tag{3.6}$$

□

Remark 3.1 In fact, for any given $v \in U_h$, we first let $p_h \in Q_h$ uniquely solve $(\varepsilon\nabla p_h, \nabla q) = (\varepsilon v, \nabla q)$ for all $q \in Q_h$. We have $(\varepsilon(v - \nabla p_h), \nabla q) = 0$ for all $q \in Q_h$. Putting $z_h := v - \nabla p_h \in U_h$, we see that $v = z_h + \nabla p_h$ is the desired, with $z_h \in Z_h(\varepsilon)$. Note that $\nabla Q_h \subset U_h$.

Lemma 3.1 *Let $u_{\delta,h} \in U_h$ be the solution of problem (3.4). Then, for a compatible $f \in H(\operatorname{div}^0;\Omega)$, we have $u_{\delta,h} \in Z_h(\varepsilon)$.*

Proof. From Proposition 3.1, we have the ε-weighted L^2-orthogonal decomposition: $u_{\delta,h} = z_{\delta,h} + \nabla p_{\delta,h}$, where $z_{\delta,h} \in Z_h(\varepsilon)$ and $p_{\delta,h} \in Q_h$. From Proposition 3.1 again, problem (3.4) may be reformulated into a mixed problem (in fact, two decoupled subproblems): Find $z_{\delta,h} \in Z_h(\varepsilon)$ and $p_{\delta,h} \in Q_h$ such that

$$(\mu^{-1}\operatorname{curl} z_{\delta,h}, \operatorname{curl} z) + \delta(\varepsilon z_{\delta,h}, z) = (f, z) \quad \forall z \in Z_h(\varepsilon), \tag{3.7}$$

$$\delta(\varepsilon\nabla p_{\delta,h}, \nabla q) = (f, \nabla q) \quad \forall q \in Q_h. \tag{3.8}$$

But, $f \in H(\operatorname{div}^0;\Omega)$, we find that $p_{\delta,h} \equiv 0$, and we conclude that $u_{\delta,h} = z_{\delta,h} \in Z_h(\varepsilon)$, solving (3.7). □

Before giving the δ-uniform stability of the finite element solution $u_{\delta,h} \in U_h$ of problem (3.4), we shall make an assumption.

Assumption A1) We assume that the following $\mathscr{K}_h$-ellipticity holds:

$$||v||_0 \le C||\operatorname{curl} v||_0 \quad \forall v \in \mathscr{K}_h := Z_h(\varepsilon). \tag{3.9}$$

Remark 3.2 We refer (3.9) as the $\mathscr{K}_h$-ellipticity using the terminology in the classical theory for the mixed problem (1.6), with $\mathscr{K}_h := Z_h(\varepsilon)$, since there is some relationship between the δ-regularization problem and the mixed problem (1.6). Nevertheless, problem (3.4) does not need the $\mathscr{K}_h$-ellipticity to ensure the well-posedness, but problem (1.6) does.

Theorem 3.2 *Let $u_{\delta,h} \in U_h$ be the solution to problem (3.4). Assume that Assumption A1), i.e., the $\mathscr{K}_h$-ellipticity (3.9) holds. There exists $C > 0$, independent of δ, such that*

$$||u_{\delta,h}||_{0,\operatorname{curl}} \le C||f||_0. \tag{3.10}$$

Proof. Since $u_{\delta,h}$ solves (3.7) in Lemma 3.1, from (3.9) and (3.7), (3.10) easily follows. □

Remark 3.3 Under the continuous embedding (1.8) for $s > 1/2$, for $\varepsilon = 1$, (3.9) is proven in [3], and for a general ε, (3.9) is essentially proven in [47]. For Ω being Lipschitz polyhedron, we have $s > 1/2$ in (1.8), see [3], but for general Lipschitz domains, $s = 1/2$, see [23]. Possibly, even $s < 1/2$, e.g., for non-Lipschitz, non-simply-connected domains with screening parts [24]. We are not aware of any work in the literature in which the $\mathscr{K}_h$ ellipticity (3.9) was shown without using the continuous embedding (1.8) for $s > 1/2$.

In the next section,we shall develop a different and new argument for showing (3.9), no longer resorting to (1.8), i.e., the continuous embedding $H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div}^0;\Omega) \hookrightarrow (H^s(\Omega))^3$.

4. A general verification of $\mathscr{K}_h$-ellipticity. In order to establish the $\mathscr{K}_h$-ellipticity (3.9) without involving the continuous embedding (1.8), we make the following assumption.

Assumption 2) We assume that for any $v \in H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div}^0;\Omega)$ it admits a regular-singular decomposition in the following:

$$v = v_0 + \nabla p_0, \tag{4.1}$$

where $v_0 \in H_0(\operatorname{curl};\Omega) \cap (H^t(\Omega))^3$ for some $t > 1/2$ and $p_0 \in H_0^1(\Omega)$, satisfying

$$||v_0||_t + ||p_0||_1 \le C||\operatorname{curl} v||_0. \tag{4.2}$$

Remark 4.1 For Lipschitz domains, in [8] it is shown that $t = 1$. In this paper, we only need $t > 1/2$.
Lemma 4.1 *Assuming that Assumption A2) holds, we have the $\mathscr{K}_h$ ellipticity (3.9), i.e.*

$$||v||_0 \le C||\mathrm{curl}\, v||_0 \quad \forall v \in Z_h(\varepsilon). \tag{4.3}$$

Before proving Lemma 4.1, we recall the L^2 orthogonal decomposition for $(L^2(\Omega))^3$ and recall the finite element interpolation theory of U_h.

Proposition 4.1 [30] *For any $v \in (L^2(\Omega))^3$ and for any Lipschitz domain Ω, it can be written as the following L^2-orthogonal decomposition:*

$$v = v_1 + \nabla p_1, \tag{4.4}$$

where $v_1 \in H(\mathrm{div}^0; \Omega)$, $p_1 \in H_0^1(\Omega)$.

Remark 4.2 With the L^2-orthogonal decomposition in Proposition 4.1, for any $z \in H_0(\mathrm{curl}\,; \Omega)$, from Assumption A2) we can obtain the following regular-singular decomposition:

$$z = z_0 + \nabla p, \tag{4.5}$$

where $z_0 \in H_0(\mathrm{curl}\,; \Omega) \cap (H^t(\Omega))^3$, with the same t in Assumption A3), and $p \in H_0^1(\Omega)$, satisfying

$$||z_0||_t \le C||\mathrm{curl}\, z||_0, \quad ||p||_1 \le C||z||_{0,\mathrm{curl}}. \tag{4.6}$$

Let Π_h and Υ_h respectively denote the canonical finite element interpolation operator onto U_h and X_h, with $\Pi_h u|_T = \Pi_T u$ and $\Upsilon_h u|_T = \Upsilon_T u$ for all $T \in \mathcal{T}_h$, where $\Pi_T u \in \mathcal{N}_l(T)$ and $\Upsilon_T u \in \mathcal{RT}_l(T)$ are defined with respect to degrees of freedom on $T \in \mathcal{T}_h$. Let u be given, making $\Pi_h u \in U_h$ and $\Upsilon_h \mathrm{curl}\, u \in X_h$ well-defined. Then $\mathrm{curl}\, \Pi_h u = \Upsilon_h \mathrm{curl}\, u$, the well-known commuting diagram property. Note that $\mathrm{curl}\, U_h \subset X_h$. It is also well-known that if $\Pi_h \nabla q \in U_h$ is well-defined, then there exists a $q_h \in Q_h$ such that $\Pi_h \nabla q = \nabla q_h$ and that if $v_h \in U_h$ satisfies $\mathrm{curl}\, v_h = 0$ then there exists $p_h \in Q_h$ such that $v_h = \nabla p_h$. See [48][47][15][32].

Remark 4.3 Note that if $u \in H(\mathrm{div}\,; T) \cap (L^p(T))^3$ for $p > 2$, then $\Upsilon_T u$ is well-defined, see [15], while if $u \in \{u \in (L^p(T))^3 : \mathrm{curl}\, u \in (L^p(T))^3, v \times n \in (L^p(\partial T))^3\}$, where $p > 2$, then $\Pi_T u$ is well-defined, see [3].

Throughout this paper, we shall focus on the lowest-order edge element method, namely, $l = 1$, since we are interested in the low regular solution of H^r function for some $r \le 1$. It is of course straightforward to consider higher-order elements if the solution is more regular. Below we recall the finite element interpolation property that we shall use.

Proposition 4.2 [47] *For $u \in (H^s(\Omega))^3$ for $s > 1/2$ and $\mathrm{curl}\, u \in X_h$, $\Pi_T u$ is well-defined, satisfying*

$$||\Pi_T u - u||_{0,T} \le C h_T^s (||u||_{s,T} + ||\mathrm{curl}\, u||_{0,T}) \quad \forall T \in \mathcal{T}_h. \tag{4.7}$$

For $u \in (H^{s_1}(\Omega))^3$ for $s_1 > 1/2$ and $\mathrm{curl}\, u \in (H^{s_2}(\Omega))^3$ for $s_2 > 0$, $\Pi_T u$ is well-defined, satisfying

$$||\Pi_T u - u||_{0,T} \le C h_T^{s_1} (||u||_{s_1,T} + ||\mathrm{curl}\, u||_{s_2,T}) \quad \forall T \in \mathcal{T}_h, \tag{4.8}$$

$$||\mathrm{curl}\,(\Pi_T u - u)||_{0,T} \le C h_T^{s_2} ||\mathrm{curl}\, u||_{s_2,T} \quad \forall T \in \mathcal{T}_h. \tag{4.9}$$

Proof of Lemma 4.1. For $v \in Z_h(\varepsilon) \subset U_h$, it admits a both L^2 and $H_0(\mathrm{curl}\,; \Omega)$ orthogonal decomposition (see Proposition 3.1 with $\varepsilon = 1$): $v = \widetilde{v} + \nabla p_h$, where $\widetilde{v} \in Z_h(1)$ and $p_h \in Q_h$. Since $v \in Z_h(\varepsilon)$, we have $(\varepsilon v, v) = (\varepsilon v, v - \nabla p_h) = (\varepsilon v, \widetilde{v})$. We have $||v||_0^2 \le C||v||_{0,\varepsilon}^2 = C(\varepsilon v, v) = C(\varepsilon v, \widetilde{v}) \le C||v||_0 ||\widetilde{v}||_0$, i.e.

$$||v||_0 \le C||\widetilde{v}||_0. \tag{4.10}$$

Note that $\mathrm{curl}\, v = \mathrm{curl}\, \widetilde{v}$. Hence, if we can show

$$||\widetilde{v}||_0 \le C||\mathrm{curl}\, \widetilde{v}||_0 \tag{4.11}$$

then estimate (4.3) holds. In what follows, we prove (4.11). From Remark 4.2, we have the regular-singular decomposition of $\widetilde{v} \in Z_h(1) \subset U_h \subset H_0(\mathrm{curl}\,; \Omega)$ as follows:

$$\widetilde{v} = \widetilde{v}_0 + \nabla p, \quad \widetilde{v}_0 \in H_0(\mathrm{curl}\,; \Omega) \cap (H^t(\Omega))^3, t > 1/2, p \in H_0^1(\Omega), \tag{4.11}$$

$$||\widetilde{v}_0||_t \le C||\mathrm{curl}\, \widetilde{v}||_0, \quad \mathrm{curl}\, \widetilde{v}_0 = \mathrm{curl}\, \widetilde{v} \in X_h, \quad ||p||_1 \le C||\widetilde{v}||_{0,\mathrm{curl}}. \tag{4.12}$$

From Proposition 4.2 we know that $\Pi_h \widetilde{v}_0 \in U_h$ is well-defined, satisfying

$$||\Pi_h \widetilde{v}_0 - \widetilde{v}_0||_0 \le C h^t (||\widetilde{v}_0||_t + ||\mathrm{curl}\, \widetilde{v}_0||_0) \le C h^t ||\mathrm{curl}\, \widetilde{v}||_0, \tag{4.13}$$

from which we have

$$||\Pi_h \widetilde{v}_0||_0 \le C||\mathrm{curl}\, \widetilde{v}||_0. \tag{4.14}$$

Consequently, $\Pi_h \nabla p$ is also well-defined, and from Remark 4.3 we know that $\Pi_h \nabla p = \nabla q_h$ for some $q_h \in Q_h$.

Therefore,

$$||\widetilde{v}||_0^2 = (\widetilde{v}, \widetilde{v}) = (\widetilde{v}, \widetilde{v}_0 + \nabla p) = (\widetilde{v}, \widetilde{v}_0 + \nabla p - \Pi_h(\widetilde{v}_0 + \nabla p)) + (\widetilde{v}, \Pi_h \widetilde{v}_0 + \nabla q_h), \tag{4.15}$$

where, since $\Pi_h v_h = v_h$ for all $v_h \in U_h$, we have

$$(\widetilde{v}, \widetilde{v}_0 + \nabla p - \Pi_h(\widetilde{v}_0 + \nabla p)) = (\widetilde{v}, \widetilde{v} - \Pi_h \widetilde{v}) = 0, \tag{4.16}$$

and since $\widetilde{v} \in Z_h(1)$, we have

$$(\widetilde{v}, \Pi_h \widetilde{v}_0 + \nabla q_h) = (\widetilde{v}, \Pi_h \widetilde{v}_0) \leq ||\widetilde{v}||_0 ||\Pi_h \widetilde{v}_0||_0 \leq C||\widetilde{v}||_0 ||\mathrm{curl}\, \widetilde{v}||_0. \tag{4.17}$$

Thus, (4.11) follows from (4.15)-(4.17). $\square$

5. Error estimates. In this section, we will establish the convergence of $u_{\delta,h}$.

From Proposition 4.2 we first recall the finite element interpolation theory of Π_h in U_h for a u with a piecewise regularity with respect to the material subdomains.

Proposition 5.1 *Let $u, \mathrm{curl}\, u \in \prod_{j=1}^J (H^r(\Omega_j))^3$ with $r > 1/2$. Then, $\Pi_h u$ is well-defined and satisfies*

$$||u - \Pi_h u||_{0,\mathrm{curl}} \leq Ch^r (\sum_{j=1}^J ||u||_{r,\Omega_j} + ||\mathrm{curl}\, u||_{r,\Omega_j}). \tag{5.1}$$

To relate the right-hand side of (5.1) to the source function f, we need to make the following assumptions.

Assumption A3) We assume that there exists a $r > 0$ such that $H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}^0;\varepsilon;\Omega) \hookrightarrow \prod_{j=1}^J (H^r(\Omega_j))^3$ is a continuous embedding, satisfying for all $v \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}^0;\varepsilon;\Omega)$, $\sum_{j=1}^J ||v||_{r,\Omega_j} \leq C||\mathrm{curl}\, v||_0$.

Assumption A4) Introduce $H_0(\mathrm{div}^0;\mu;\Omega) = \{v \in (L^2(\Omega))^3 : \mathrm{div}\, \mu v = 0, \mu v \cdot n|_S = 0, \forall S \in \mathscr{S}_{\mathrm{ext}}\}$. We assume that there exists a $r > 0$, the same as in Assumption A3), such that $H(\mathrm{curl}\,;\Omega) \cap H_0(\mathrm{div}^0;\mu;\Omega) \hookrightarrow \prod_{j=1}^J (H^r(\Omega_j))^3$ is a continuous embedding, satisfying for all $v \in H(\mathrm{curl}\,;\Omega) \cap H_0(\mathrm{div}^0;\mu;\Omega)$, $\sum_{j=1}^J ||v||_{r,\Omega_j} \leq C||\mathrm{curl}\, v||_0$.

Remark 5.1 Note that Assumption A3) reduces to (1.8) for $\varepsilon = 1$ and $r > 1/2$. The regularity in Assumption A3) and Assumption A4) may be different, but here we assume the same r. In fact, the r in Assumption A3) and Assumption A4) is mainly related to Dirichlet and Neumann boundary value problems for Laplace operator, respectively. See [3] [26][24]. This fact indicates that r depends on the regularity of ε, μ and the geometric regularity of Ω.

Lemma 5.1 *Given a compatible $f \in H(\mathrm{div}^0;\Omega)$. Under Assumption A3) and Assumption A4), the solution u_δ of problem (1.9) and the solution u of problem (1.5), together with $\mathrm{curl}\, u_\delta$ and $\mathrm{curl}\, u$, are in $\prod_{j=1}^J (H^r(\Omega_j))^3$, and the following holds:*

$$\sum_{j=1}^J ||u_\delta||_{r,\Omega_j} + ||u||_{r,\Omega_j} + ||\mathrm{curl}\, u_\delta||_{r,\Omega_j} + ||\mathrm{curl}\, u||_{r,\Omega_j} \leq C||f||_0. \tag{5.2}$$

Proof. Observe that u_δ and u belong to $H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}^0;\varepsilon;\Omega)$ and satisfy $||u_\delta||_{0,\mathrm{curl}}, ||u||_{0,\mathrm{curl}} \leq C||f||_0$ (See Lemma 2.1, Theorem 2.1 and Remark 2.1). At the same time, both $\mu^{-1}\mathrm{curl}\, u_\delta$ and $\mu^{-1}\mathrm{curl}\, u$ belong to $H(\mathrm{curl}\,;\Omega) \cap H_0(\mathrm{div}^0;\mu;\Omega)$, satisfying $||\mathrm{curl}\, \mu^{-1}\mathrm{curl}\, u||_0 = ||f||_0$, $||\mathrm{curl}\, \mu^{-1}\mathrm{curl}\, u_\delta||_0 = ||f - \delta\varepsilon u_\delta||_0 \leq ||f||_0 + C||u_\delta||_0 \leq C||f||_0$. Hence, from Assumption A3) and Assumption A4) we conclude that Lemma 5.1 holds, noting that $\mu|_{\Omega_j}$ is in $(W^{1,\infty}(\Omega_j))^{3\times 3}$. $\square$

Since $\mathscr{A}_\delta(u,v)$ given by (3.5) is coercive over $H_0(\mathrm{curl}\,;\Omega)$ and the consistency property between problem (1.9) and problem (3.4) holds because of the conformity of $U_h \subset H_0(\mathrm{curl}\,;\Omega)$, from Proposition 5.1, Lemma 5.1 and Remark 2.2 it is not difficult to obtain the following error estimates in $||\cdot||_{\mathscr{A}_\delta}^2 := \mathscr{A}_\delta(\cdot,\cdot)$ from the classical Céa's argument in [20].

Theorem 5.1 *Assume that $r > 1/2$ holds in Assumption A3) and Assumption A4). Then, the solution u of problem (1.5) and the solution $u_{\delta,h}$ of problem (3.4) satisfy the following error bound:*

$$||u - u_{\delta,h}||_{\mathscr{A}_\delta} \leq C(\delta + h^r)||f||_0. \tag{5.3}$$

$\square$

Remark 5.2 The inadequacy of the error estimates of (5.3) in Theorem 5.1 is the δ-dependent norm $||\cdot||_{\mathscr{A}_\delta}$. However, we can have uniform error estimates, i.e., $||u - u_{\delta,h}||_{0,\mathrm{curl}} \leq C(\delta + h^r)||f||_0$. We can achieve this by using the Fortin-type finite element interpolation [9].

In what follows, to explore the idea, it suffices to consider the case where $\mu = \varepsilon = 1$ and use the Fortin-operator to obtain the uniform error estimates under the same assumptions in Theorem 5.1. The study for the case with general μ, ε and with very low regular solution will be deferred to the next section.

Let π_h be the Fortin operator defined by seeking $\pi_h u \in Z_h(1) \subset U_h$ for a given $u \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$ such that

$$(\mathrm{curl}\,\pi_h u, \mathrm{curl}\,v) = (\mathrm{curl}\,u, \mathrm{curl}\,v) \quad \forall v \in U_h, \tag{5.4}$$

$$(\pi_h u, \nabla q) = 0 \quad \forall q \in Q_h. \tag{5.5}$$

Proposition 5.2 [4][9][47][37] *Let $u \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$ and let $\pi_h u$ be defined by (5.4) and (5.5). Assuming the same Assumptions as in Theorem 5.1, we have*

$$||\pi_h u - u||_{0,\mathrm{curl}} \le Ch^r(||u||_r + ||\mathrm{curl}\,u||_r). \tag{5.6}$$

Theorem 5.2 *Under the same assumptions in Theorem 5.1, we have the uniform error estimation:*

$$||u - u_{\delta,h}||_{0,\mathrm{curl}} \le C(\delta + h^r)||f||_0. \tag{5.7}$$

Proof. Let $v := u_{\delta,h} - \pi_h u_\delta \in U_h$, where u_δ is the solution of problem (1.9). Note that $\mu = \varepsilon = 1$. From the consistency property between problem (1.9) and problem (3.4) and the definition of the Fortin operator π_h by (5.4) and (5.5), we have

$$\begin{aligned} ||v||^2_{\mathscr{A}_\delta} &= (\mathrm{curl}\,v, \mathrm{curl}\,v) + \delta(v,v) = (\mathrm{curl}\,(u_{\delta,h} - \pi_h u_\delta), \mathrm{curl}\,v) + \delta(u_{\delta,h} - \pi_h u_\delta, v) \\ &= (\mathrm{curl}\,(u_{\delta,h} - u_\delta), \mathrm{curl}\,v) + (\mathrm{curl}\,(u_\delta - \pi_h u_\delta), \mathrm{curl}\,v) + \delta(u_{\delta,h} - u_\delta, v) + \delta(u_\delta - \pi_h u_\delta, v) \\ &= \delta(u_\delta - \pi_h u_\delta, v) \le \delta||u_\delta - \pi_h u_\delta||_0||v||_0 \le \delta^{\frac{1}{2}}||u_\delta - \pi_h u_\delta||_0||v||_{\mathscr{A}_\delta}, \end{aligned} \tag{5.8}$$

that is

$$||\mathrm{curl}\,(u_{\delta,h} - \pi_h u_\delta)||_0 + \delta^{\frac{1}{2}}||u_{\delta,h} - \pi_h u_\delta||_0 \le C\delta^{\frac{1}{2}}||u_\delta - \pi_h u_\delta||_0, \tag{5.9}$$

From Proposition 5.2 with $u_\delta \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$, by the triangle inequality we obtain from (5.9)

$$\begin{aligned} ||u_\delta - u_{\delta,h}||_{0,\mathrm{curl}} &\le ||u_\delta - \pi_h u_\delta||_{0,\mathrm{curl}} + ||u_{\delta,h} - \pi_h u_\delta||_{0,\mathrm{curl}} \\ &\le C||u_\delta - \pi_h u_\delta||_{0,\mathrm{curl}} \le Ch^r(||u_\delta||_r + ||\mathrm{curl}\,u_\delta||_r), \end{aligned} \tag{5.10}$$

and from Lemma 5.1 we further have

$$||u_\delta - u_{\delta,h}||_{0,\mathrm{curl}} \le Ch^r||f||_0. \tag{5.11}$$

Finally, from Remark 2.2 we obtain the desired (5.7). $\Box$

Remark 5.3 The assumption $r > 1/2$ of the regularity of u and $\mathrm{curl}\,u$ over Ω_j is not that restrictive in practice. In fact, such regularity assumption is commonly used in the literature [18][21][39][2][47]. Meanwhile, it has been shown $r > 1/2$ or even $r = 1$ for practical interface problems [17][42].

Remark 5.4 On the other hand, the interface problem from electromagnetism would have a possible very low regularity solution, i.e., $r \le 1/2$, see [26]. In addition, even if $\mu = \varepsilon = 1$, from Remark 3.3, r in both Assumption A3) and Assumption A4) is still possibly less than or equal to 1/2.

Without the requirements $r > 1/2$ in Assumptions A3) and A4) and $s > 1/2$ in (1.8), we shall obtain the uniform error estimates next section. We have seen that the uniform error estimates rely on the Fortin operator, so it suffices that the finite element interpolation property (5.6) for the Fortin operator holds for any $0 < r \le 1$. In addition, we shall deal with general μ and ε as assumed in section 2.

6. A general Fortin operator. The advantage of the Fortin operator over the finite element interpolation operator Π_h is that the former is a projection from (5.4) and (5.5): (5.4) is a commuting property and (5.5) preserves discrete divergence-free constraint. We have seen that it is based on the projection property of π_h that we have established the uniform error bound in the previous section. Although, from the definition (5.4) and (5.5) we infer that the Fortin operator π_h could be well-defined for any $u \in H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,^0;\Omega)$ even if u does not have the regularity with $r > 1/2$. Unfortunately, in the literature, the well-definedness and the interpolation error property of the Fortin operator indeed depend on Assumption A3) with $r > 1/2$ (or (1.8) mentioned in Introduction section with $s > 1/2$) and on the regularity $r > 1/2$ of the interpolated function u.

In this section, we consider the following general Fortin operator: Given $u \in H_0(\mathrm{curl}\,;\Omega)$, to find $\pi_h u \in U_h$ such that

$$(\mu^{-1}\mathrm{curl}\,\pi_h u, \mathrm{curl}\,v) = (\mu^{-1}\mathrm{curl}\,u, \mathrm{curl}\,v) \quad \forall v \in U_h, \tag{6.1}$$

$$(\varepsilon \pi_h u, \nabla q) = (\varepsilon u, \nabla q) \quad \forall q \in Q_h. \tag{6.2}$$

It will be shown that $\pi_h u$ is well-defined and satisfy (5.6) without assuming the regularity index r of the interpolated function u to be greater than 1/2 and without using the continuous embedding (1.8) or the continuous embeddings in Assumptions A3) and A4). In fact, we only assume Assumption A2) which is a regular-singular decomposition already stated in section 4.

Lemma 6.1 *Assume that Assumption A2) holds. For any given $u \in H_0(\mathrm{curl}\,;\Omega)$, $\pi_h u$ is well-defined.*

Proof. Observe that $\pi_h u$ is the first component of the solution pair $(u_h, p_h) \in U_h \times Q_h$ such that

$$(\mu^{-1}\mathrm{curl}\,u_h, \mathrm{curl}\,v) + (\varepsilon v, \nabla p_h) = (\mu^{-1}\mathrm{curl}\,u, \mathrm{curl}\,v) \quad \forall v \in U_h, \tag{6.3}$$

$$(\varepsilon u_h, \nabla q) = (\varepsilon u, \nabla q) \quad \forall q \in Q_h. \tag{6.4}$$

In fact, since $U_h = Z_h(\varepsilon) + \nabla Q_h$ as stated in Proposition 3.1 which is verified in Remark 3.1, p_h is equal to zero in the above mixed problem. If we have shown $\mathscr{K}_h^t \subset \mathscr{K}^t$, $\mathscr{K}_h$-ellipticity and the Inf-Sup condition, then the classical theory in [15] for saddle-point problems yields the well-posedness of the above mixed problem. The inclusion $\mathscr{K}_h^t \subset \mathscr{K}^t$ is verified by noticing that $\mathscr{K}^t = \{q \in H_0^1(\Omega) : (\varepsilon v, \nabla q) = 0, \forall v \in H_0(\mathrm{curl}\,;\Omega)\} = \{0\}$ and $\mathscr{K}_h^t = \{q_h \in Q_h : (\varepsilon v_h, \nabla q_h) = 0, \forall v_h \in U_h\} = \{0\}$, and the verification of the Inf-Sup condition follows from the decomposition $U_h = Z_h(\varepsilon) + \nabla Q_h$, namely,

$$\sup_{0 \neq v \in U_h} \frac{(\varepsilon v, \nabla q)}{||v||_{0,\varepsilon} + ||\mathrm{curl}\,v||_{0,\mu^{-1}}} \geq \frac{(\varepsilon \nabla q, \nabla q)}{||\nabla q||_{0,\varepsilon}} \geq C||q||_1 \quad \forall q \in Q_h. \tag{6.5}$$

We are left to verify the $\mathscr{K}_h$ ellipticity, where $\mathscr{K}_h = Z_h(\varepsilon) = \{v \in U_h : (\varepsilon v, \nabla q) = 0, \forall q \in Q_h\}$. But, this is just the conclusion of Lemma 4.1 since Assumption A2) holds. □

Remark 6.1 Lemma 6.1 implies the invariance or projection property over U_h: $\pi_h u = u$ on $U_h \subset H_0(\mathrm{curl}\,;\Omega)$ and the $H(\mathrm{curl}\,;\Omega)$-boundedness property $||\pi_h u||_{0,\mathrm{curl}} \leq C||u||_{0,\mathrm{curl}}$ for all $u \in H_0(\mathrm{curl}\,;\Omega)$. Moreover, from (6.1) we have

$$||\mathrm{curl}\,\pi_h u||_0 \leq C||\mathrm{curl}\,u||_0, \quad ||\mathrm{curl}\,(\pi_h u - u)||_0 \leq C \inf_{v_h \in U_h} ||\mathrm{curl}\,(u - v_h)||_0. \tag{6.6}$$

Lemma 6.2 *For any $p \in H_0^1(\Omega)$ with $\nabla p \in H_0(\mathrm{curl}\,;\Omega)$, we have $\pi_h \nabla p = \nabla p_h$, where $p_h \in Q_h$ satisfies*

$$(\varepsilon \nabla p_h, \nabla q) = (\varepsilon \nabla p, \nabla q) \quad \forall q \in Q_h. \tag{6.7}$$

Proof. With $u := \nabla p \in H_0(\mathrm{curl}\,;\Omega)$ and $\pi_h u \in U_h \subset H_0(\mathrm{curl}\,;\Omega)$ at hand, we find from equation (6.1) that $\mathrm{curl}\,\pi_h \nabla p = 0$. Thus, from section 4 we have some $p_h \in Q_h$ satisfies $\pi_h \nabla p = \nabla p_h$, and we obtain (6.7) from equation (6.2). □

Lemma 6.3 *For $p \in H_0^1(\Omega) \cap \prod_{j=1}^{J} H^{1+r}(\Omega_j)$ for some $r > 0$, there exists $I_h p \in Q_h$ such that*

$$||p - I_h p||_1 \leq Ch^r \sum_{j=1}^{J} ||p||_{1+r,\Omega_j}. \tag{6.8}$$

Proof. If $p \in H_0^1(\Omega) \cap H^{1+r}(\Omega)$, then (6.8) is a classical result from the finite element interpolation theory [13][20][22][53]. Now, p is piecewise $H^{1+r}(\Omega_j)$, no immediate literature could be located. Here, we give an approach to obtain an interpolation $I_h p \in Q_h$ such that estimate (6.8) holds. Firstly, take any Ω_j and put

$$Q_{h,\Omega_j} := Q_h|_{\Omega_j}. \tag{6.9}$$

We use the Clément interpolation[22][7] or Scott-Zhang interpolation[53] to find a finite element interpolation $\mathcal{C}_{h,j} p \in Q_{h,\Omega_j}$ of $p|_{\Omega_j} \in H^{1+r}(\Omega_j)$,

$$||\mathcal{C}_{h,j} p - p||_{1,\Omega_j} + \left(\sum_{F \in \mathscr{S}_{\Omega_j}} h_F^{-1} ||\mathcal{C}_{h,j} p - p||_{0,F}^2 \right)^{\frac{1}{2}} \leq Ch^r ||p||_{1+r,\Omega_j}, \tag{6.10}$$

where $\mathscr{S}_{\Omega_j}$ denotes the set of all element faces in $\mathcal{T}_h|_{\Omega_j}$. Note that if $p|_S = 0$ for $S \in \mathscr{S}_{\mathrm{ext}}$, then $\mathcal{C}_{h,j} p|_S = 0$ also, i.e., $\mathcal{C}_{h,j} p$ preserves the homogenous boundary condition. Define a function p_h by

$$p_h|_{\Omega_j} := \mathcal{C}_{h,j} p \quad 1 \leq j \leq J. \tag{6.11}$$

In general, such p_h is discontinuous when crossing any interface S in $\mathscr{S}_{\rm int}$. However, by 'a nodal averaging procedure' (see Remark 6.2 below), from p_h, we can find a new finite element function $I_hp \in Q_h$. Since p_h is continuous over Ω_j and homogeneous on $\mathscr{S}_{\rm ext}$, the nodal average $I_hp \in Q_h$ satisfies $I_hp(a) = p_h(a)$ for all interior nodes inside Ω_j, $1 \le j \le J$, for all boundary nodes on $\mathscr{S}_{\rm ext}$, and

$$\sum_{j=1}^{J} ||I_hp - p_h||^2_{1,\Omega_j} \le C \sum_{F\in\mathscr{S}_{\rm int}} h_F^{-1} \int_F |[p_h]|^2. \tag{6.12}$$

But, $p \in H_0^1(\Omega)$, we have

$$\begin{aligned}\left(\sum_{F\in\mathscr{S}_{\rm int}} h_F^{-1} \int_F |[p_h]|^2\right)^{\frac{1}{2}} &= \left(\sum_{F\in\mathscr{S}_{\rm int}} h_F^{-1} \int_F |[p_h - p]|^2\right)^{\frac{1}{2}} \le C\left(\sum_{j=1}^{J} \sum_{F\in\mathscr{S}_{\Omega_j}} h_F^{-1} ||\mathcal{C}_{h,j}p - p||^2_{0,F}\right)^{\frac{1}{2}} \\ &\le Ch^r \sum_{j=1}^{J} ||p||_{1+r,\Omega_j}.\end{aligned} \tag{6.13}$$

Hence, using the triangle inequality, we obtain (6.8) from (6.10),(6.12) and (6.13). □

Remark 6.2 For linear element, the argument for constructing the finite element function I_hp from the discontinuous p_h through a nodal averaging approach may be referred to [12]. In fact, with the discontinuous p_h, we can construct I_hp from (2.1) in [12] on page 1072, and then from (2.10) in [12] on page 1073 and **Example 2.3** in [12] on page 1074, and from the local inverse estimates (as used in (3.10) in [12] on page 1076), it follows that (6.12) holds. For higher-order elements, readers may refer to [45] for a general nodal averaging approach to construct a continuous finite element function from a discontinuous finite element function, see Theorem 2.2 on page 2378 therein.

Now we state the main result in Theorem 6.1 below for the Fortin operator π_h with the special $u := \nabla p$.

Theorem 6.1 *Let $\nabla p_h = \pi_h \nabla p \in Q_h$ be given by (6.7). For $p \in H_0^1(\Omega) \cap \prod_{j=1}^{J} H^{1+r}(\Omega_j)$, with $r > 0$,*

$$||\pi_h \nabla p - \nabla p||_0 \le Ch^r \sum_{j=1}^{J} ||p||_{1+r,\Omega_j}. \tag{6.14}$$

Proof. Since $p_h \in Q_h \subset H_0^1(\Omega)$ is the finite element solution to (6.7), it follow from (6.8) and the standard analysis in [13][20] that (6.14) holds. □

Lemma 6.4 *Assume that Assumption A2) holds. For $z \in H_0(\mathrm{curl}\,;\Omega) \cap \prod_{j=1}^{J} (H^{r_1}(\Omega_j))^3$ with some $r_1 > 1/2$ and $\mathrm{curl}\, z \in \prod_{j=1}^{J} (H^{r_2}(\Omega_j))^3$ with some $r_2 > 0$, we have*

$$||z - \pi_h z||_0 \le Ch^{r_2} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\mathrm{curl}\, z||_{r_2,\Omega_j}), \tag{6.15}$$

$$||\mathrm{curl}\,(z - \pi_h z)||_0 \le Ch^{r_2} \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.16}$$

Proof. Since Assumption A2) holds, from Lemma 6.1 it follows that $\pi_h z$ is well-defined for $z \in H_0(\mathrm{curl}\,;\Omega)$. We are now ready to estimate the difference between z and $\pi_h z$.

Noticing that $\Pi_h z \in U_h$ is well-defined, since $z \in \prod_{j=1}^{J} (H^{r_1}(\Omega_j))^3$ for some $r_1 > 1/2$ and $\mathrm{curl}\, z \in \prod_{j=1}^{J} (H^{r_2}(\Omega_j))^3$ for some $r_2 > 0$, from Proposition 4.2, we have

$$||z - \Pi_h z||_0 \le Ch^{r_1} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\mathrm{curl}\, z||_{r_2,\Omega_j}), \tag{6.17}$$

$$||\mathrm{curl}\,(z - \Pi_h z)||_0 \le Ch^{r_2} \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.18}$$

We first estimate the difference between $\Pi_h z$ and $\pi_h z$.

From Remark 4.2 we decompose $\Pi_h z - \pi_h z$ as follows:

$$\Pi_h z - \pi_h z = z_0 + \nabla p, \tag{6.19}$$

where $z_0 \in H_0(\mathrm{curl}\,;\Omega) \cap (H^t(\Omega))^3$ for some $t > 1/2$ and $p \in H_0^1(\Omega)$, satisfying

$$||z_0||_t \leq C||\mathrm{curl}\,(\Pi_h z - \pi_h z)||_0, \quad ||p||_1 \leq C||\Pi_h z - \pi_h z||_{0,\mathrm{curl}}\,, \tag{6.20}$$

$$\mathrm{curl}\, z_0 = \mathrm{curl}\,(\Pi_h z - \pi_h z) \in X_h. \tag{6.21}$$

But, from Remark 6.1 and (6.18), we have

$$||\mathrm{curl}\,(\Pi_h z - \pi_h z)||_0 = ||\mathrm{curl}\, \pi_h(\Pi_h z - z)||_0 \leq C||\mathrm{curl}\,(\Pi_h z - z)||_0 \leq Ch^{r_2} \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.22}$$

Thus, it follows from (6.20)-(6.22) that

$$||z_0||_{0,\mathrm{curl}} \leq Ch^{r_2} \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r_2,\Omega_j}, \quad ||z_0||_t \leq Ch^{r_2} \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.23}$$

Note that $\Pi_h z_0$ is well-defined, because $z_0 \in (H^t(\Omega))^3$ for some $t > 1/2$ and $\mathrm{curl}\, z_0 \in X_h$. From Proposition 4.2 and (6.23), we have

$$||z_0 - \Pi_h z_0||_0 \leq Ch^t(||z_0||_t + ||\mathrm{curl}\, z_0||_0) \leq Ch^{t+r_2} \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r_2,\Omega_j}. \tag{6.24}$$

In addition, $\Pi_h \nabla p$ is also well-defined since $\nabla p = \Pi_h z - \pi_h z - z_0$, and we have some $q_h \in Q_h$ such that $\Pi_h \nabla p = \nabla q_h$, see section 4. Thus, we have

$$\Pi_h z - \pi_h z = \Pi_h(\Pi_h z - \pi_h z) = \Pi_h z_0 + \Pi_h \nabla p = \Pi_h z_0 + \nabla q_h. \tag{6.25}$$

We next estimate $z - \pi_h z$. We have

$$||z - \pi_h z||_{0,\varepsilon}^2 = (\varepsilon(z - \pi_h z), z - \pi_h z) = (\varepsilon(z - \pi_h z), z - \Pi_h z) + (\varepsilon(z - \pi_h z), \Pi_h z - \pi_h z), \tag{6.26}$$

where, from (6.17), we have

$$(\varepsilon(z - \pi_h z), z - \Pi_h z) \leq C||z - \pi_h z||_{0,\varepsilon}||z - \Pi_h z||_0 \leq C||z - \pi_h z||_{0,\varepsilon} h^{r_1} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\mathrm{curl}\, z||_{r_2,\Omega_j}), \tag{6.27}$$

and from (6.2) in the definition of π_h with $u := z$ here, i.e., $(\varepsilon \pi_h z, \nabla q) = (\varepsilon z, \nabla q)$ holds for all $q \in Q_h$, and from (6.23)-(6.25), we have

$$\begin{aligned}
(\varepsilon(z - \pi_h z), \Pi_h z - \pi_h z) &= (\varepsilon(z - \pi_h z), \Pi_h z_0 + \nabla q_h) = (\varepsilon(z - \pi_h z), \Pi_h z_0) \\
&= (\varepsilon(z - \pi_h z), \Pi_h z_0 - z_0) + (\varepsilon(z - \pi_h z), z_0) \\
&\leq C||z - \pi_h z||_{0,\varepsilon}||z_0 - \Pi_h z_0||_0 + C||z - \pi_h z||_{0,\varepsilon}||z_0||_0 \\
&\leq C||z - \pi_h z||_{0,\varepsilon} h^{r_2} \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r_2,\Omega_j}.
\end{aligned} \tag{6.28}$$

Hence, combining (6.26)-(6.28) we obtain

$$||z - \pi_h z||_0 \leq C||z - \pi_h z||_{0,\varepsilon} \leq Ch^{r_2} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\mathrm{curl}\, z||_{r_2,\Omega_j}). \tag{6.29}$$

This competes the proof of (6.15). Regarding (6.16), it follows from (6.6) in Remark 6.1 and (6.18). $\square$

Remark 6.3 Compared with the error bound (6.17) of $\Pi_h z$, the error bound (6.15) of $\pi_h z$ could be improved, since we would expect r_1 in (6.15), i.e., $||z - \pi_h z||_0 \leq Ch^{r_1} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\mathrm{curl}\, z||_{r_2,\Omega_j})$. We are not aware of any work in the literature that dealt with this issue where z and $\mathrm{curl}\, z$ have different regularity. At the same time, we did not find the way to obtain such estimate. However, if (1.8) holds for some $s > 1/2$ and Assumption A3) and Assumption A4) hold for some $r > 0$, using a different argument, we could obtain $||z - \pi_h z||_0 \leq Ch^{\min(r_1, s+r_2, r+r_2)} \sum_{j=1}^{J} (||z||_{r_1,\Omega_j} + ||\mathrm{curl}\, z||_{r_2,\Omega_j})$, where the order is not the same as r_1 but better than (6.15). Here, we will not deal with this issue any further, since (6.15) and (6.16) are sufficient for the main result in the following.

We now state the main result for the Fortin operator π_h for the general $u := z$ with suitable regularity.

Theorem 6.2 *Assume that Assumption A2) holds for some* $t > 1/2$ *and that* $z \in H_0(\mathrm{curl}\,;\Omega)$ *and* $z, \mathrm{curl}\, z \in \prod_{j=1}^{J} (H^r(\Omega_j))^3$, *where* $0 < r \le t$. *Then*

$$||z - \pi_h z||_0 \le Ch^r(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||z||_{r,\Omega_j} + \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r,\Omega_j}), \tag{6.30}$$

$$||\mathrm{curl}\,(z - \pi_h z)||_0 \le Ch^r \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r,\Omega_j}. \tag{6.31}$$

Proof. From Remark 4.2 we write $z \in H_0(\mathrm{curl}\,;\Omega)$ into the following regular-singular decomposition:

$$z = z_0 + \nabla p, \tag{6.32}$$

where $z_0 \in H_0(\mathrm{curl}\,;\Omega) \cap (H^t(\Omega))^3$ for some $t > 1/2$ and $p \in H_0^1(\Omega)$, satisfying

$$||z_0||_t \le C||\mathrm{curl}\, z||_0, \quad ||p||_1 \le C||z||_{0,\mathrm{curl}}, \quad \mathrm{curl}\, z_0 = \mathrm{curl}\, z. \tag{6.33}$$

Since $z_0 \in (H^t(\Omega))^3$ for some $t > 1/2$ and $\mathrm{curl}\, z_0 = \mathrm{curl}\, z \in \prod_{j=1}^{J} (H^r(\Omega_j))^3$ for some $r > 0$, from Proposition 4.2 and (6.33), we have

$$||z_0 - \Pi_h z_0||_0 \le Ch^t(||z_0||_t + \sum_{j=1}^{J} ||\mathrm{curl}\, z_0||_{r,\Omega_j}) \le Ch^t(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r,\Omega_j}), \tag{6.34}$$

$$||\mathrm{curl}\,(z_0 - \Pi_h z_0)||_0 \le Ch^r \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r,\Omega_j}. \tag{6.35}$$

On the other hand, since $z \in \prod_{j=1}^{J} (H^r(\Omega_j))^3$ for some $r > 0$ and $z_0 \in (H^t(\Omega))^3$ with $t \ge r$, from (6.32) we know that $p \in \prod_{j=1}^{J} H^{1+r}(\Omega_j)$, satisfying

$$\sum_{j=1}^{J} ||p||_{1+r,\Omega_j} \le C(\sum_{j=1}^{J} ||z||_{r,\Omega_j} + ||z_0||_t) \le C(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||z||_{r,\Omega_j}). \tag{6.36}$$

Thus, from Theorem 6.1 we have

$$||\pi_h \nabla p - \nabla p||_0 \le Ch^r \sum_{j=1}^{J} ||p||_{1+r,\Omega_j} \le Ch^r(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||z||_{r,\Omega_j}). \tag{6.37}$$

We are now in a position to estimate the difference $z - \pi_h z$ in the following.
Observe that

$$||z - \pi_h z||_{0,\varepsilon}^2 = (\varepsilon(z - \pi_h z), z - \pi_h z) = (\varepsilon(z - \pi_h z), z_0 - \pi_h z_0) + (\varepsilon(z - \pi_h z), \nabla p - \pi_h \nabla p), \tag{6.38}$$

where, from Lemma 6.4 for this z_0 with $r_1 := t$ and $r_2 := r$ we have

$$\begin{aligned}(\varepsilon(z - \pi_h z), z_0 - \pi_h z_0) &\le C||z - \pi_h z||_{0,\varepsilon} ||z_0 - \pi_h z_0||_0 \\ &\le C||z - \pi_h z||_{0,\varepsilon} h^r(||z_0||_t + \sum_{j=1}^{J} ||\mathrm{curl}\, z_0||_{r,\Omega_j}) \\ &\le C||z - \pi_h z||_{0,\varepsilon} h^r(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||\mathrm{curl}\, z||_{r,\Omega_j}),\end{aligned} \tag{6.39}$$

and from (6.37), we have

$$(\varepsilon(z - \pi_h z), \nabla p - \pi_h \nabla p) \le C||z - \pi_h z||_{0,\varepsilon} ||\nabla p - \pi_h \nabla p||_0 \le C||z - \pi_h z||_{0,\varepsilon} h^r(||\mathrm{curl}\, z||_0 + \sum_{j=1}^{J} ||z||_{r,\Omega_j}), \tag{6.40}$$

It then follows from (6.38)-(6.40) that

$$||z-\pi_h z||_0 \le C||z-\pi_h z||_{0,\varepsilon} \le h^r(||\mathrm{curl}\, z||_0 + \sum_{j=1}^J ||z||_{r,\Omega_j} + \sum_{j=1}^J ||\mathrm{curl}\, z||_{r,\Omega_j}). \tag{6.41}$$

Regarding (6.31), from (6.6) in Remark 6.1 we have

$$||\mathrm{curl}\,(z-\pi_h z)||_0 \le C \inf_{v_h\in U_h} ||\mathrm{curl}\, z - \mathrm{curl}\, v_h||_0, \tag{6.42}$$

but, $\mathrm{curl}\, z = \mathrm{curl}\, z_0$ and $\Pi_h z_0$ is well-defined, from (6.35) we have

$$\inf_{v\in U_h} ||\mathrm{curl}\, z - \mathrm{curl}\, v||_0 \le ||\mathrm{curl}\, z_0 - \mathrm{curl}\,\Pi_h z_0||_0 \le Ch^r \sum_{j=1}^J ||\mathrm{curl}\, z||_{r,\Omega_j}, \tag{6.43}$$

and we obtain (6.31). □

Following the argument in proving Theorem 5.2 we can obtain the following $H(\mathrm{curl}\,)$-error bound for very low regular solution, with r being possibly not greater than 1/2.

Corollary 6.1 *Assume that Assumption A2) holds for some $t > 1/2$ and that Assumption A3) and Assumption A4) hold for some $0 < r \le t$. Given any compatible $f \in H(\mathrm{div}^0;\Omega)$. Let u be the solution of problem (1.5) and $u_{\delta,h} \in U_h$ the finite element solution of problem (3.4). We have*

$$||u-u_{\delta,h}||_{0,\mathrm{curl}} \le C(\delta + h^r)||f||_0. \tag{6.44}$$

Remark 6.4 We have used Assumption A2) to ensure that π_h is well-defined. We also used this assumption to establish the error estimates for the Fortin operator. Assumption A2) states a regular-singular decomposition where $t > 1/2$. This t is different from the s in the continuous embedding (1.8) and the r in the continuous embedding of Assumption A3) and Assumption A4), since where s and r may be not greater than 1/2. For example, for Lipschitz domains we can have $t = 1$, but $s = 1/2$ only. For interface problem, we may still have $t = 1$, but r may be close to zero [26]. In fact, the regular-singular decomposition in Assumption A2) depends little on the domain boundary and on the material occupying Ω, since it has been established mainly from the H^1 existence of the Poisson equation of Laplace operator and the extension of $H_0(\mathrm{curl}\,;\Omega)$ to the $H(\mathrm{curl}\,;\mathbb{R}^3)$ [27][8]. On the contrary, the continuous embedding in (1.8), Assumption A3) and Assumption A4), and the regularity of the solution and its curl counterpart of problem (1.1)-(1.4), are determined by the domain boundary singularities (due to reentrant corners and edges, etc), the regularity of the materials occupying Ω, and the topology of Ω (i.e., simply-connected or multi-connected, etc), see [24][26]. In general, these are profoundly related to the singularities of the solution of the second-order elliptic problem of Laplace operator in nonsmooth domains [35].

Remark 6.5 As highlighted in Remark 6.4, $r \le t$ is generally true, since $t = 1$ usually. If r is larger than t and $r > 1/2$, then the theory has already been developed in section 5. For more regular solution, say $r > 1$, we may use higher-order elements, and the theory in section 5 can be easily applied to obtain higher-order error bounds.

7. Numerical test. In this section, some numerical results are reported to support the method and the theory developed in the previous sections. Below, Ω is triangulated into uniform tetrahedra, with the mesh reduction of factor two, i.e., $h = 1/4, 1/8, 1/16,$, etc. In the δ-regularization problem, we use the lowest-order Nédélec element of first-family and choose $\delta = h$.

Given the domain in $\mathbb{R}^3$: $\Omega = ([-1,3]\times[-1,1] \setminus ([0,3]\times[-1,0] \cup \{(x,y)\in\mathbb{R}^2 : 2 < x < 3, y = \frac{1}{2}\})) \times [0,1]$ in the $O-xyz$-coordinates system. There is a reentrant edge originating from the origin $O(0,0,0)$ along the positive z axis with an opening angle $3\pi/2$ and a screen originating from the point $(2,1/2,0)$ along the positive x axis with an opening angle 2π in Ω. We take $\mu = 1$. But, we assume there are two material subdomains in Ω, $\Omega_1 = [1,3]\times[0,1]^2$, $\Omega_2 = [-1,1]^2 \setminus ([0,1]\times[-1,0])\times[0,1]$, which are introduced by the interface $\mathscr{S}_{\mathrm{int}} = \{(x,y,z)\in\mathbb{R}^3 : x = 1, 0 \le y \le 1, 0 \le z \le 1\}$. We consider ε by piecewise constants, where $\varepsilon|_{\Omega_1} = 1$ and $\varepsilon|_{\Omega_2} = 1/2$. We choose f so that the exact solution $u|_{\Omega_j} = (\partial_y p|_{\Omega_j}, -\partial_x p|_{\Omega_j}, 0)$, $\quad 1 \le j \le 2$, where $p|_{\Omega_1^-} = (1-x)^2(1-y)^2 q_1(x,y)$, with $q_1(x,y) = r^{\frac{2}{3}}\cos(\frac{2\theta}{3})$, and $p|_{\Omega_2^-} = (1-x)^2(3-x)^2(\frac{1}{4}-(y-\frac{1}{2})^2)^2 q_2(x-2, y-\frac{1}{2})$, with $q_2(x,y) = r^{\frac{1}{2}}\cos(\frac{\theta}{2})$, where $q_1(x,y) = r^{\frac{2}{3}}\cos(\frac{2\theta}{3})$ and $q_2(x,y) = r^{\frac{1}{2}}\cos(\frac{\theta}{2})$ in the cylindrical coordinates system of $\mathbb{R}^3$, $x = r\cos(\theta), y = r\sin(\theta), z = z$, r is the distance to the origin O and θ is the angular degree between 0 and 2π. The regularity of the exact solution u and its $\mathrm{curl}\, u = (0,0,-\Delta p)$ is the same on each material subdomain Ω_j and $u, \mathrm{curl}\, u \in (H^{r_j}(\Omega_j))^3$, r_j is respectively $2/3-\epsilon, 1/2-\epsilon$ for any $\epsilon > 0$. The solution u and its $\mathrm{curl}\, u$ belong to $\prod_{j=1}^{2}(H^r(\Omega_j))^3$ for r being approximately $1/2$. From the theoretical results, we should expect that the ratio of the error reduction is approximately $2^{1/2} \approx 1.4142$ for the mesh reduction of factor two and that the finite element solution is uniform stable independent of the regularization parameter δ decreasing to zero. The computed results in L^2 semi-norm and curl semi-norm which are listed in Table 1 and Table 2, as expected, are consistent with the theoretical results. Also, we notice that although the solution and its curl have higher regularity in Ω_1, the whole convergence rate is governed by the lower regularity in Ω_2.

TABLE 1 Errors in L^2 semi-norm and curl semi-norm

h	1/4	1/8	1/16	1/32	1/64
$\|\|u-u_{\delta,h}\|\|_0$	0.3955	0.2604	0.1566	0.0981	0.0700
Ratio		1.5188	1.6628	1.5963	1.4014
$\|\|\mathrm{curl}\,(u-u_{\delta,h})\|\|_0$	2.9515	1.5506	0.8720	0.5282	0.3517
Ratio		1.9035	1.7782	1.6509	1.5018

TABLE 2 Stability in L^2 semi-norm and curl semi-norm
for a fixed mesh-size $h = 1/16$ but δ decreases

δ	1/10	1/20	1/30	1/40	1/50
$\|\|u-u_{\delta,h}\|\|_0$	0.1510	0.1434	0.1419	0.1414	0.1412
Ratio		1.0530	1.0106	1.0035	1.0014
$\|\|\mathrm{curl}\,(u-u_{\delta,h})\|\|_0$	0.8541	0.8294	0.8247	0.8230	0.8223
Ratio		1.0300	1.0057	1.0021	1.0009

Keeping $\varepsilon|_{\Omega_1}$ unchanged, we change $\varepsilon|_{\Omega_2}$ so that there is a discontinuity of ε with high ratio/contrast across the interface. We consider two cases for $\varepsilon|_{\Omega_2}$: $\varepsilon|_{\Omega_2} = 1/100$ and $\varepsilon|_{\Omega_2} = 1/1000$. We find that the computed results are accurate to four decimal places as shown in Tables 1 and 2. This may be interpreted as follows. For δ decreasing to zero, the theoretical results show that the finite element solution is uniformly stable, independent of δ, and holds optimal convergence with respect to $\delta+h$, thus, when the combination of $\delta\varepsilon$ with $\delta = h$ and $h\varepsilon_{\max}$ decrease to zero, all the theoretical results are expected to still be valid, where $\varepsilon_{\max}$ represents the upper bound of ε over Ω, and here $\varepsilon_{\max} = 1$. Therefore, the present method appears to cover much wider problems, in particular, the case where the discontinuous materials have high ratio/contrast across material subdomains, although we did not develop the related theory for this situation.

8. Conclusion and extension. In this paper, we have analyzed a general approach, δ-regularization method, for dealing with the divergence-free constraint in a double curl problem which typically arises from computational electromagnetism. With this δ-regularization method, we can completely disregard the divergence-free constraint and instead we introduce a δ perturbation zero-term which couples the curlcurl operator to constitute a well-posed coercive problem for any given δ. Such δ-regularization method is shown to have a uniform stable finite element solution independent of the regularization parameter δ which decreases to zero. For nonsmooth solution, together with its curl, being H^r regularity for some $0 < r < 1$, we have established the optimal error bound $\mathcal{O}(h^r)$ in the natural $H(\mathrm{curl}\,)$ norm (which is independent of δ) for $\delta \leq Ch$ when using the lowest-order Nédélec element of first-family. Higher-order Nédélec elements can be used to yield higher-order accuracy if the exact solution is more regular. Furthermore, we have developed the new theory for the $\mathscr{K}_h$ ellipticity (a discrete Poincaré-Friedrichs' type inequality) and the new theory for the Fortin-interpolation operator. The $\mathscr{K}_h$ ellipticity is one of the two critical conditions (the other is the Inf-Sup condition) for the well-posedness and the optimal convergence for the mixed finite element method, while the Fortin operator is fundamental in the edge finite element method, as is well-known. These two theories generalize the existing ones to cover those problems whose solutions may have very low regularity. In fact, they are established only under the regular-singular decomposition assumption. Such assumption is true for general domains and does not depend on the material properties occupying the domain and the topology of the domain. Numerical results have been presented in three-dimensional cases to illustrate the method and confirm the theory in this paper. Moreover, the proposed δ-regularization method numerically appears to cover the interface problem with high contrast/ratio material coefficients across material subdomains, although we did not have the theory for this situation. These have justified the capability of the δ-regularization method in dealing with divergence-free constraint. Meanwhile, these have exhibited the potential to deal with the discontinuous materials of high contrast/ratio among different material subdomains.

We should point out that although the proposed δ-regularization method is developed, analyzed and performed for the model problem in (1.1)-(1.4), but, in actual fact, it can cover a number of models on computational electromagnetism. To illustrate this point, for example, let us consider the following problem: to find u and p such that

$$\operatorname{curl}\mu^{-1}\operatorname{curl}u+\alpha\varepsilon u+\varepsilon_1\nabla p=f,\quad \operatorname{div}\varepsilon u=g\quad\text{in }\Omega,\quad u\times n=0,\quad p=0\quad\text{on }\partial\Omega,$$

where α is a given real number which may arise from either the time-discretization problems of the time-dependent Maxwell's equations with α inversely proportional to the time-step, or the time-harmonic Maxwell's equations with $-\alpha$ amounting to the angular frequency, and ε_1 is a third material coefficient matrix. Below we simply show how to apply the proposed δ-regularization method to the above problem. This consists of two stages. We first parallel solve two second-order elliptic problems: to find $p^*\in H_0^1(\Omega)$ such that

$$\operatorname{div}\varepsilon\nabla p^*=g\quad\text{in }\Omega,\quad p^*=0\quad\text{on }\partial\Omega,$$

and to find $p\in H_0^1(\Omega)$ such that

$$\operatorname{div}\varepsilon_1\nabla p=\operatorname{div}f-\alpha g\quad\text{in }\Omega,\quad p=0\quad\text{on }\partial\Omega,$$

and then we solve the problem: to find $w\in H_0(\operatorname{curl};\Omega)\cap H(\operatorname{div}^0;\varepsilon;\Omega)$ such that

$$\operatorname{curl}\mu^{-1}\operatorname{curl}w+\alpha\varepsilon w=F:=f-\varepsilon_1\nabla p-\alpha\varepsilon\nabla p^*,\quad \operatorname{div}\varepsilon w=0\quad\text{in }\Omega,\quad w\times n=0\quad\text{on }\partial\Omega.$$

Clearly, we have $u=w+\nabla p^*$. However, we do not directly solve w. Instead, we solve the following δ regularization problem: to find w_δ such that

$$\operatorname{curl}\mu^{-1}\operatorname{curl}w_\delta+\alpha\varepsilon w_\delta+\delta\varepsilon w_\delta=F\quad\text{in }\Omega,\quad w_\delta\times n=0\quad\text{on }\partial\Omega.$$

Noticing that α is known, we choose δ so that $\delta+\alpha\neq 0$, and we can analyze this δ regularization problem following the routine from the previous sections. Hence, first simultaneously solving two symmetric, positive definite problems (second-order elliptic interface problems) in parallel, and then solving a δ-regularization problem, we can obtain the desired solution. In addition, we could generalize the developed theory to those problems with mixed boundary conditions (i.e., $u\times n|_{\Gamma_1}=0,\varepsilon u\cdot n|_{\Gamma_2}=0$, with $\Gamma=\Gamma_1\cup\Gamma_2$), since the two fundamental tools in our analysis are now available for mixed boundary conditions, the regular-singular decomposition [34] and the L^2 orthogonal decomposition [30].

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