Journal of Computational Mathematics
Vol.xx, No.x, 200x, 1–18.

http://www.global-sci.org/jcm
doi:??

Monolithic multigrid for reduced magnetohydrodynamic equations*

Xiaodi Zhang
LSEC, NCMIS, Institute of Computational Mathematics and Scientific Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China.
School of Mathematical Science, University of Chinese Academy of Sciences, Beijing, 100049, China.
Email: zhangxiaodi@lsec.cc.ac.cn
Weiying Zheng
LSEC, NCMIS, Institute of Computational Mathematics and Scientific Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China.
School of Mathematical Science, University of Chinese Academy of Sciences, Beijing, 100049, China.
Email: zwy@lsec.cc.ac.cn

Abstract

In this paper, the monolithic multigrid method is investigated for reduced magnetohydrodynamic equations. We propose a diagonal Braess-Sarazin smoother for the finite element discrete system and prove the uniform convergence of the MMG method with respect to mesh sizes. A multigrid-preconditioned FGMRES method is proposed to solve the magnetohydrodynamic equations. It turns out to be robust for relatively large physical parameters. By extensive numerical experiments, we demonstrate the optimality of the monolithic multigrid method with respect to the number of degrees of freedom.

Mathematics subject classification: 65M60, 76W05
Key words: Monolithic multigrid, magnetohydrodynamic equations, diagonal Braess-Sarazin smoother, finite element method.

1. Introduction

The incompressible magnetohydrodynamic (MHD) equations governs the dynamics of a charged fluid in the presence of electromagnetic fields. It has broad applications in technology and engineering, such as aluminum electrolysis, electromagnetic pumping, stirring of liquid metals, and flow-quantity measurements based on magnetic induction [1, 9, 12]. The governing model is a coupled system of Navier-Stokes equations and Maxwell's equations. When the magnetic field tends to be saturated or the electric conductivity is relatively small, the model is usually simplified to the reduced MHD (RMHD) equations [19, 22, 27]. In dimensionless form, the stationary RMHD model is given by

$$-\frac{1}{R_e}\triangle \boldsymbol{u} + \nabla p + N(\nabla\phi - \boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B} = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$-\triangle\phi + \nabla\cdot(\boldsymbol{u}\times\boldsymbol{B}) = \chi \quad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{u} = 0, \quad \phi = \xi \quad \text{on } \Gamma, \tag{1.1d}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with Lipschitz-continuous boundary $\Gamma := \partial\Omega$. The unknowns are the velocity of fluid $\boldsymbol{u}$, the hydrodynamic pressure p, and the electric potential ϕ.

* Received xxx / Revised version received xxx / Accepted xxx /

The right-hand sides $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $\chi \in L^2(\Omega)$ and the applied magnetic field $\boldsymbol{B} \in \boldsymbol{H}^1(\Omega)$ are assumed to be given. The non-dimensional parameters R_e and N are the Reynolds number and coupling parameter. The boundary value for ϕ satisfies $\xi \in H^{1/2}(\Gamma)$. Here the momentum equation (1.1a) does not contain the convection term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$. The model can be interpreted as the coupling between the Stokes equations and the electric potential Poisson equation or as the Stokes linearization of time-dependent Navier-Stokes equations where the convective term is treated explicitly in time (cf. e.g. [16, 20, 24]).

There are many papers in the literature on numerical solutions of RMHD equations. In [27], Peterson proved the existence and uniqueness of weak solutions of RMHD model and studied its finite element approximation. Layton et al [19] proved L^2-error estimates and proposed a two-level method to deal with the nonlinearity. Ervin et al studied a posteriori error estimates for both standard finite element method as well as a two-level Newton finite element method [11]. For time-dependent problems, Yuksel and Ingram gave a comprehensive error analysis for both semi-discrete and fully discrete approximate [35, 36]. In 2007, Ni et al developed consistent and charge-conservative schemes for inductionless MHD equations on both structured and unstructured meshes [25,26]. For theoretical and numerical studies of other MHD models, we refer to [14, 17, 18, 22, 30, 32] and references therein.

After linearization and discretization, the MHD equations usually result in a large and indefinite linear system which is very hard to solve. The study for robust and efficient preconditioners is an important research topic. Among various existing solvers for these types of systems, one may distinguish between block preconditioners and monolithic multigrid (MMG). Block preconditioners exploit inherent block structure of the fully coupled system and utilize existing Poisson solvers as building blocks. The key ingredient is to construct approximate Schur complements (cf. e.g. [21, 23, 24, 28, 29]).

The MMG method solves the fully-coupled system not only on the finest level of finite element meshes, but also on coarse levels. Over the past three decades, great success has been achieved in MMG for large-scale multi-physics problems. We refer to [7, 38] for MMG methods for Stokes equations and Navier-Stokes equations and to [2] for MMG method for resistive MHD equations. In [33, 34], Salah et al proposed a fully-coupled multilevel preconditioner for incompressible resistive MHD in the context of fully-implicit time integration and direct-to-steady-state solution. Recently, Adler et al proposed an MMG-preconditioned GMRES method for vector-potential formulation of two-dimensional resistive MHD equations [3]. However, to the authors' knowledge, there are few papers in the literature on the convergence of multigrid (MG) method for MHD equations.

The objective of this paper is to investigate the MMG method for solving the discrete problem of (1.1). Inspired by Braess and Sarazin, we propose a diagonal Braess-Sarazin (DBS) smoother for solving the discrete system. Different from classical Braess-Sarazin smoother for Stokes equations [7, 38], our smoother uses damped Jacobi relaxation for each variable. It is easy to implement and economic in practical computations. A rigorous convergence analysis is presented for the MMG method by verifying its smoothing and approximation properties. Furthermore, we also propose a MMG-preconditioned FGMRES method for the discrete problem. Numerical experiments show that the method is robust for large physical parameters and quasi-optimal with respect to mesh sizes.

The paper is organized as follows. In Section 2, we introduce the variational formulation and present mixed finite element approximation to the RMHD equations. Some preliminary estimates are also presented. In Section 3, we present the MMG algorithm with DBS smoother.

In Section 4, we prove the convergence of the MMG method by virtue of smoothing property and approximation property. In Section 5, we present some numerical experiments to demonstrate the uniform convergence of the MMG method and the robustness of MMG-preconditioned FGMRES method with respect to large physical parameters. Section 6 concludes the main result of the paper.

2. Mixed finite element method for the RMHD equations

Let $L^2(\Omega)$ be the space of square-integrable functions and define $L_0^2(\Omega) := L^2(\Omega)/\mathbb{R}$. The inner product and norm on $L^2(\Omega)$ are given by

$$(u, v) := \int_\Omega uv\mathrm{d}x, \qquad ||u||_{L^2(\Omega)} := (u, u)^{1/2}.$$

Define $H^k(\Omega) := \left\{v \in L^2(\Omega) : D^\xi v \in L^2(\Omega),\ |\xi| \le k\right\}$ where ξ represents non-negative triple index. Let $H_0^1(\Omega) \subset H^1(\Omega)$ be the subspace whose functions have zero traces on Γ. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. Vector-valued quantities will be denoted by boldface notations, such as $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^3$.

2.1. Weak formulation and the well-posedness

For convenience, we introduce some notations for function spaces

$$\boldsymbol{V} = \boldsymbol{H}_0^1(\Omega), \qquad Q = L_0^2(\Omega), \qquad S = H_0^1(\Omega).$$

Define the following linear and bilinear forms:

$$\begin{aligned}
&a(\boldsymbol{u}, \boldsymbol{v}) := R_e^{-1}(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + N(\boldsymbol{u} \times \boldsymbol{B}, \boldsymbol{v} \times \boldsymbol{B}), \qquad b(\boldsymbol{v}, q) := -(\nabla \cdot \boldsymbol{v}, q),\\
&c(\phi, \psi) := N(\nabla\phi, \nabla\psi), \qquad d(\boldsymbol{v}, \psi) := -N(\boldsymbol{v} \times \boldsymbol{B}, \nabla\phi),\\
&l_{\boldsymbol{u}}(\boldsymbol{v}) := (\boldsymbol{f}, \boldsymbol{v}), \qquad l_\phi(\psi) := N(\chi, \psi).
\end{aligned}$$

Since we are studying MG method for solving the discrete problem, the boundary conditions are assumed to be homogeneous namely, $\boldsymbol{g} = \boldsymbol{0}$ and $\xi = 0$, for simplicity. A weak formulation of (1.1) reads: Find $\boldsymbol{u} \in \boldsymbol{V}$, $p \in Q$, and $\phi \in S$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + d(\boldsymbol{v}, \phi) = l_{\boldsymbol{u}}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.1a}$$

$$b(\boldsymbol{u}, q) = 0 \qquad \forall\, q \in Q, \tag{2.1b}$$

$$c(\phi, \psi) + d(\boldsymbol{u}, \psi) = l_\phi(\psi) \quad \forall\, \psi \in S. \tag{2.1c}$$

The weak formulation can be casted into one in the product space $\boldsymbol{W} = \boldsymbol{V} \times Q \times S$: Find $\boldsymbol{x} := (\boldsymbol{u}, p, \phi) \in \boldsymbol{W}$ such that

$$\mathcal{B}(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{F}(\boldsymbol{y}) \quad \forall\, \boldsymbol{y} := (\boldsymbol{v}, q, \psi) \in \boldsymbol{W}, \tag{2.2}$$

where $\mathcal{B}(\cdot,\cdot)$ and $\mathcal{F}(\cdot)$ are defined by

$$\begin{aligned}
\mathcal{B}(\boldsymbol{x}, \boldsymbol{y}) &= a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + d(\boldsymbol{v}, \phi) + b(\boldsymbol{u}, q) + c(\phi, \psi) + d(\boldsymbol{u}, \psi),\\
\mathcal{F}(\boldsymbol{y}) &= l_{\boldsymbol{u}}(\boldsymbol{v}) + l_\phi(\psi).
\end{aligned}$$

In [27], Peterson proved the existence and uniqueness of weak solution to (2.1) with nonlinear convection term. Here we only summarize the well-posedness of (2.1) without proofs.

Theorem 2.1. *Assume that* $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ *and* $\chi \in H^{-1}(\Omega)$*, the variational problem* (2.1) *has a unique solution* $(\boldsymbol{u}, p, \phi) \in \boldsymbol{V} \times Q \times S$ *which satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\phi\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \le C\big(\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\chi\|_{H^{-1}(\Omega)} \big). \tag{2.3}$$

For the forthcoming analysis, we make the assumption on the regularity of solutions. It holds when Ω is a convex polygonal domain or $\partial\Omega$ is smooth (see [1] for more details).

Assumption 2.1. *For any* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\chi \in L^2(\Omega)$*, the solutions* $\boldsymbol{u}, p, \phi$ *satisfy the regularities*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\phi\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \le C(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\chi\|_{L^2(\Omega)}). \tag{2.4}$$

2.2. Finite element approximation

Let $\mathcal{T}_h$ be a quasi-uniform and shape-regular tetrahedral mesh of Ω. The mesh size of $\mathcal{T}_h$ is denoted by $h = \max_{T\in\mathcal{T}_h} h_T$. For any $T \in \mathcal{T}_h$, let $P_k(T)$ be the space of polynomials of degree k on T. Let $\boldsymbol{V}_h \times Q_h \times S_h \subset \boldsymbol{V} \times Q \times S$ be finite element spaces which satisfy the discrete inf-sup condition

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{H}^1(\Omega)}} \ge \beta \|q_h\|_{L^2(\Omega)} \qquad \forall\, q_h \in Q_h, \tag{2.5}$$

where the constant $\beta > 0$ does not depend on h. A wide variety of spaces $\boldsymbol{V}_h, Q_h$ satisfying (2.5) have been proposed in the literature. We refer to [5,8,13] and the references therein. Here we employ the well-known Hood-Taylor $\boldsymbol{P}_2 - P_1$ finite elements to discretize the velocity and the pressure

$$\boldsymbol{V}_h = \left\{\boldsymbol{v}_h \in \boldsymbol{H}_0^1(\Omega):\ \boldsymbol{v}_h|_T \in \boldsymbol{P}_2(T)\ \ \forall T \in \mathcal{T}_h\right\},$$
$$Q_h = Q \cap \left\{q \in H^1(\Omega):\ q|_T \in P_1(T)\ \ \forall T \in \mathcal{T}_h\right\}.$$

The finite element space for the electric potential is given by

$$S_h = \left\{\psi \in H_0^1(\Omega):\ \psi|_T \in P_2(T)\ \ \forall T \in \mathcal{T}_h\right\}.$$

The finite element discretization of (2.1) leads to a coupled discrete system: Find $\boldsymbol{u}_h \in \boldsymbol{V_h}$, $p_h \in Q_h$ and $\phi_h \in S_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) + d(\boldsymbol{v}_h, \phi_h) = l_{\boldsymbol{u}}(\boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.6a}$$
$$b(\boldsymbol{u}_h, q_h) = 0 \qquad \forall\, q_h \in Q_h, \tag{2.6b}$$
$$c(\phi_h, \psi_h) + d(\boldsymbol{u}_h, \psi_h) = l_\phi(\psi_h) \quad \forall\, \psi_h \in S_h. \tag{2.6c}$$

Equivalently, the problem can also be written into a compact form: Find $\boldsymbol{x}_h \in \boldsymbol{W}_h$,

$$\mathcal{B}(\boldsymbol{x}_h, \boldsymbol{y}_h) = \mathcal{F}(\boldsymbol{y}_h) \qquad \forall\, \boldsymbol{y}_h \in \boldsymbol{W}_h := \boldsymbol{V}_h \times Q_h \times S_h. \tag{2.7}$$

In [25], Peterson proved the optimal error estimate in energy norm. In [1], Layton et al proved the L_2-error estimates. For brevity, we only present the results.

Theorem 2.2. *The problem* (2.1) *admits a unique solution. There is a constant* C *independent of* h *such that*

$$\|\boldsymbol{u}_h\|_{\boldsymbol{H}^1(\Omega)} + \|\phi_h\|_{H^1(\Omega)} + \|p_h\|_{L^2(\Omega)} \le C\left(\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\chi\|_{H^{-1}(\Omega)}\right). \tag{2.8}$$

Let $(\boldsymbol{u}, p, \phi)$ *be the solutions of* (1.1) *and let assumption* (2.4) *be satisfied. Then*

$$\begin{aligned} &\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}^j(\Omega)} + \|\phi-\phi_h\|_{H^j(\Omega)} + h^{1-j}\|p-p_h\|_{L^2(\Omega)} \\ \le\, & Ch^{2-j}\left(\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|p\|_{H^1(\Omega)} + \|\phi\|_{H^2(\Omega)}\right), \qquad j=0,1. \end{aligned} \tag{2.9}$$

2.3. The discrete system

We write the discrete problem (2.2) into an algebraic saddle-point problem

$$\mathbb{K}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}, \tag{2.10}$$

where the quantities have the following block forms

$$\mathbb{K} = \begin{pmatrix} \mathbb{X} & \mathbb{Y}^\top \\ \mathbb{Y} & 0 \end{pmatrix}, \quad \mathbb{X} = \begin{pmatrix} \mathbb{A} & \mathbb{D}^\top \\ \mathbb{D} & \mathbb{C} \end{pmatrix}, \quad \mathbb{Y} = \begin{pmatrix} \mathbb{B} & 0 \end{pmatrix},$$
$$\underline{\mathbf{x}} = \begin{pmatrix} \underline{\mathbf{x}}_{\boldsymbol{w}} \\ \underline{\mathbf{x}}_p \end{pmatrix}, \quad \underline{\mathbf{b}} = \begin{pmatrix} \underline{\mathbf{b}}_{\boldsymbol{w}} \\ \underline{\mathbf{b}}_p \end{pmatrix}, \quad \underline{\mathbf{x}}_{\boldsymbol{w}} = \begin{pmatrix} \underline{\mathbf{x}}_{\boldsymbol{u}} \\ \underline{\mathbf{x}}_\phi \end{pmatrix}, \quad \underline{\mathbf{b}}_{\boldsymbol{w}} = \begin{pmatrix} \underline{\mathbf{b}}_{\boldsymbol{u}} \\ \underline{\mathbf{b}}_\phi \end{pmatrix}.$$

The vectors $\underline{\mathbf{x}}_{\boldsymbol{u}}$, $\underline{\mathbf{x}}_p$, $\underline{\mathbf{x}}_\phi$ represent degrees of freedom (DOFs) for $\boldsymbol{u}$, p, ϕ respectively and $\underline{\mathbf{b}}_{\boldsymbol{u}}$, $\underline{\mathbf{b}}_p$, $\underline{\mathbf{b}}_\phi$ are the corresponding load vectors. Let $\boldsymbol{v}_i$, q_i, ψ_i be basis functions of $\boldsymbol{V}_h$, Q_h and S_h respectively. The entries of block matrices are given by

$$\begin{aligned} &\mathbb{A}_{i,j} = R_e^{-1}(\nabla\boldsymbol{v}_j, \nabla\boldsymbol{v}_i) + N(\boldsymbol{v}_j\times\boldsymbol{B}, \boldsymbol{v}_i\times\boldsymbol{B}), && \mathbb{B}_{i,j} = -(\nabla\cdot\boldsymbol{v}_j, q_i), \\ &\mathbb{C}_{i,j} = N(\nabla\psi_j, \nabla\psi_i), && \mathbb{D}_{i,j} = -N(\boldsymbol{v}_j\times\boldsymbol{B}, \nabla\psi_i). \end{aligned}$$

3. Monolithic multigrid method

The purpose of this section is to construct MMG method for solving the discrete problem. First we present the MMG algorithm. The DBS smoother will be presented next.

3.1. Multigrid algorithm

Let $\mathcal{T}_0$ be an initial triangulation of Ω and let the triangulation $\mathcal{T}_k$, $k\ge 1$, be generated from $\mathcal{T}_{k-1}$ through uniform refinement so that the mesh sizes satisfy $h_{k+1} = h_k/2$. The finite element spaces on $\mathcal{T}_k$ satisfy

$$\boldsymbol{V}_k \subset \boldsymbol{V}_{k+1}, \quad Q_k \subset Q_{k+1}, \quad S_k \subset S_{k+1}, \quad k = 0,\cdots,L-1.$$

Let $\boldsymbol{x}_k \in \boldsymbol{W}_k := \boldsymbol{V}_k \times Q_k \times S_k$ solve the discrete problem on level k

$$\mathcal{B}(\boldsymbol{x}_k, \boldsymbol{y}_k) = \mathcal{F}_k(\boldsymbol{y}_k) \qquad \forall \boldsymbol{y}_k \in \boldsymbol{W}_k. \tag{3.1}$$

Let $\mathcal{I}_{k-1}^k$ denote the natural injection $\boldsymbol{W}_{k-1} \hookrightarrow \boldsymbol{W}_k$. Let $\mathcal{S}_k : \boldsymbol{W}_k \to \boldsymbol{W}_k$ denote the DBS smoother which will be specified in the next subsection.

Given an approximate solution $\boldsymbol{x}_k^{(0)} \in \boldsymbol{W}_k$ of (3.1) on level $k\ge 1$, the approximate solution $\mathbf{MG}(k,\gamma,\boldsymbol{x}_k^{(0)},\nu_1,\nu_2) := \boldsymbol{x}_k^{(\nu_1+\nu_2+1)}$ by one MG iteration is computed recursively in three steps.

(I) *Pre-smoothing*: Compute $\boldsymbol{x}_k^{(\nu_1)}$ by ν_1 iterations starting from $\boldsymbol{x}_k^{(0)}$,

$$\boldsymbol{x}_k^{(\nu_1)} = \mathcal{S}_k^{\nu_1}\boldsymbol{x}_k^{(0)}. \tag{3.2}$$

1. *Coarse-grid correction*: $\boldsymbol{x}_k^{(\nu_1+1)} = \boldsymbol{x}_k^{(\nu_1)} + \mathcal{I}_{k-1}^k \tilde{\boldsymbol{e}}_{k-1}$ where $\tilde{\boldsymbol{e}}_{k-1}$ is the approximate solution of the error equation

$$\mathcal{B}(\boldsymbol{e}_{k-1}, \boldsymbol{y}_{k-1}) = \mathcal{R}(\mathcal{I}_{k-1}^k \boldsymbol{y}_{k-1}) \qquad \forall\, \boldsymbol{y}_{k-1} \in \boldsymbol{W}_{k-1}, \tag{3.3}$$

and $\mathcal{R}(\boldsymbol{y}_k) := \mathcal{F}(\boldsymbol{y}_k) - \mathcal{B}(\boldsymbol{x}_k^{(\nu_1)}, \boldsymbol{y}_k)$ is the residual functional on $\boldsymbol{W}_k$. For $k=1$, $\tilde{\boldsymbol{e}}_0 = \boldsymbol{e}_0$ is the exact soluiton of (3.3), while for $k>1$, $\tilde{\boldsymbol{e}}_{k-1} := \mathbf{MG}(k-1, \gamma, \mathbf{0}, \nu_1, \nu_2)$ is computed by the MG algorithm on $\mathcal{T}_{k-1}$.

2. *Post-smoothing*: Compute $\boldsymbol{x}_k^{(\nu_1+\nu_2+1)}$ by ν_2 iterations starting from $\boldsymbol{x}_k^{(\nu_1+1)}$,

$$\boldsymbol{x}_k^{(\nu_1+\nu_2+1)} = \mathcal{S}_k^{\nu_2} \boldsymbol{x}_k^{(\nu_1+1)}. \tag{3.4}$$

The parameter γ is called cycle index which yields the V-cycle algorithm by setting $\gamma = 1$ and the W-cycle algorithm by setting $\gamma = 2$. The F-cycle MG algorithm $\mathbf{MGF}(k, 1, \boldsymbol{x}_k^{(0)}, \nu_1, \nu_2)$ is defined by computing $\tilde{\boldsymbol{e}}_{k-1}$ as follows

$$\check{\boldsymbol{e}}_{k-1} = \mathbf{MGF}(k-1, 1, \mathbf{0}, \nu_1, \nu_2), \quad \tilde{\boldsymbol{e}}_{k-1} = \mathbf{MG}(k-1, 1, \check{\boldsymbol{e}}_{k-1}, \nu_1, \nu_2), \quad k > 1.$$

3.2. Diagonal Braess-Sarazin smoother

Smoothing method plays the key role in MG method. Here we propose a Braess-Sarazin-type smoother for the saddle-point RMHD system. For convenience, we drop off the subscript k of all quantities.

Following Bank et al [4], we propose an iterative method for solving (2.10)

$$\hat{\underline{\mathbf{x}}}_{\boldsymbol{u}}^{(m)} = \underline{\mathbf{x}}_{\boldsymbol{u}}^{(m)} + \hat{\mathbb{A}}^{-1}\left(\underline{\mathbf{b}}_{\boldsymbol{u}} - \mathbb{A}\underline{\mathbf{x}}_{\boldsymbol{u}}^{(m)} - \mathbb{D}^\top \underline{\mathbf{x}}_\phi^{(m)} - \mathbb{B}^\top \underline{\mathbf{x}}_p^{(m)}\right), \tag{3.5a}$$

$$\underline{\mathbf{x}}_p^{(m+1)} = \underline{\mathbf{x}}_p^{(m)} - \hat{\mathbb{S}}^{-1}\left(\underline{\mathbf{b}}_p - \mathbb{B}\hat{\underline{\mathbf{x}}}_{\boldsymbol{u}}^{(m)}\right), \tag{3.5b}$$

$$\underline{\mathbf{x}}_{\boldsymbol{u}}^{(m+1)} = \underline{\mathbf{x}}_{\boldsymbol{u}}^{(m)} + \hat{\mathbb{A}}^{-1}\left(\underline{\mathbf{b}}_{\boldsymbol{u}} - \mathbb{A}\underline{\mathbf{x}}_{\boldsymbol{u}}^{(m)} - \mathbb{D}^\top \underline{\mathbf{x}}_\phi^{(m)} - \mathbb{B}^\top \underline{\mathbf{x}}_p^{(m+1)}\right), \tag{3.5c}$$

$$\underline{\mathbf{x}}_\phi^{(m+1)} = \underline{\mathbf{x}}_\phi^{(m)} + \hat{\mathbb{C}}^{-1}\left(\underline{\mathbf{b}}_\phi - \mathbb{D}\underline{\mathbf{x}}_u^{(m)} - \mathbb{C}\underline{\mathbf{x}}_\phi^{(m)}\right), \tag{3.5d}$$

where $\hat{\mathbb{A}}^{-1}$, $\hat{\mathbb{C}}^{-1}$, $\hat{\mathbb{S}}^{-1}$ are, respectively, preconditioners for $\mathbb{A}$, $\mathbb{C}$, and the inexact Schur complement $\mathbb{S} := \mathbb{B}\hat{\mathbb{A}}^{-1}\mathbb{B}^\top$. For simplicity, we define

$$\hat{\mathbb{A}} := \alpha_{\mathbb{A}} \mathrm{diag}(\mathbb{A}), \quad \hat{\mathbb{C}} := \alpha_{\mathbb{C}} \mathrm{diag}(\mathbb{C}), \quad \hat{\mathbb{S}} := \alpha_{\mathbb{S}} \mathrm{diag}(\mathbb{S}). \tag{3.6}$$

Here $\alpha_{\mathbb{A}} > 0$, $\alpha_{\mathbb{C}} > 0$, $\alpha_{\mathbb{S}} > 0$ are scaling parameters whose values will be discussed in more details in Section 4.5. Since $\hat{\mathbb{A}}$ is diagonal, $\mathrm{diag}(\mathbb{S})$ can be easily computed. Therefore, each step of (3.5) is economic in computations.

Equivalently, (3.5) can be written into a preconditioned Richardson algorithm

$$\underline{\mathbf{x}}^{(m+1)} = \underline{\mathbf{x}}^{(m)} + \hat{\mathbb{K}}^{-1}\left(\underline{\mathbf{b}} - \mathbb{K}\underline{\mathbf{x}}^{(m)}\right), \qquad m = 0, 1, \cdots, \tag{3.7}$$

where the pre-conditioner is defined by

$$\hat{\mathbb{K}}^{-1} = \begin{pmatrix} \hat{\mathbb{X}} & \mathbb{Y}^\top \\ \mathbb{Y} & \mathbb{S} - \hat{\mathbb{S}} \end{pmatrix}^{-1}, \qquad \hat{\mathbb{X}} := \begin{pmatrix} \hat{\mathbb{A}} & 0 \\ 0 & \hat{\mathbb{C}} \end{pmatrix}.$$

Since $\hat{\mathbb{X}}$ and $\hat{\mathbb{S}}$ are diagonal matrices, the $\hat{\mathbb{K}}^{-1}$ is easy to compute. Therefore, the smoothing algorithm (3.7) is referred to *diagonal Braess-Sarazin smoother* hereafter. Accordingly, (3.7) defines the DBS smoothing operator $\mathcal{S}_k$: $\boldsymbol{W}_k \to \boldsymbol{W}_k$ such that

$$\boldsymbol{x}^{(m+1)} = \mathcal{S}_k \boldsymbol{x}^{(m)}, \qquad m = 0, 1, \cdots . \tag{3.8}$$

Remark 3.1. *One can also consider block upper triangular or block lower triangular approximation to $\mathbb{X}$, which gives*

$$\hat{\mathbb{X}}_{\text{upper}} := \begin{pmatrix} \hat{\mathbb{A}} & \mathbb{D}^\top \\ 0 & \hat{\mathbb{C}} \end{pmatrix}, \qquad \hat{\mathbb{X}}_{\text{lower}} := \begin{pmatrix} \hat{\mathbb{A}} & 0 \\ \mathbb{D} & \hat{\mathbb{C}} \end{pmatrix}.$$

Since we are only interested in the case that $\hat{\mathbb{X}}$ is SPD, the details on these two cases are omitted here.

Remark 3.2. *For large coupling number N, the MMG method may become less efficient due to strong coupling between fluid and electricity. In this case, we recommend to use the MMG method as a pre-conditioner of FGMRES method for solving* (2.10)*. Numerical experiments in Section 5 show that the MG-preconditioned FGMRES method is robust to large coupling number and Reynolds number.*

4. Convergence of MMG method

The purpose of this section is to prove the convergence of the MMG method. The main technique follows the framework in [6] by verifying two vital properties, the approximation property and the smoothing property of the MG algorithm.

4.1. Error representation of two-grid algorithm

First we present the lemma on the continuity of the bilinear form $\mathcal{B}$.

Lemma 4.1. *There exists a constant $C > 0$ depending on N, R_e, $\boldsymbol{B}$ such that*

$$|\mathcal{B}(\boldsymbol{x}, \boldsymbol{y})| \le C \, \|\boldsymbol{x}\|_{\boldsymbol{W}} \, \|\boldsymbol{y}\|_{\boldsymbol{W}} \qquad \forall \, \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{W},$$

where

$$\|\boldsymbol{x}\|_{\boldsymbol{W}} := \Big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|p\|^2_{L^2(\Omega)} + \|\phi\|^2_{H^1(\Omega)} \Big)^{1/2}, \qquad \boldsymbol{x} = (\boldsymbol{u}, p, \phi).$$

Proof. By Schwarz's inequality and the injection of $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we have

$$\|\boldsymbol{u} \times \boldsymbol{B}\|_{\boldsymbol{L}^2(\Omega)} \le \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \, \|\boldsymbol{B}\|_{\boldsymbol{L}^3(\Omega)} \le C \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.$$

The proof is finished by applying Schwarz's inequality to each term of $\mathcal{B}(\boldsymbol{x}, \boldsymbol{y})$.

Now we define the mesh-dependent inner product and norm on $\boldsymbol{W}_k$

$$(\boldsymbol{x}, \boldsymbol{y})_{0,k} := (\boldsymbol{u}, \boldsymbol{v}) + (\phi, \psi) + h_k^2 (p, q), \qquad |||\boldsymbol{x}|||_{0,k} := (\boldsymbol{x}, \boldsymbol{x})_{0,k}^{1/2}. \tag{4.1}$$

By the continuity of $\mathcal{B}$ and inverse inequalities on finite element spaces, the norm of the bilinear form $\mathcal{B}$ satisfies

$$|||\mathcal{B}||| := \sup_{0\neq \boldsymbol{x},\boldsymbol{y}\in \boldsymbol{W}_k} \frac{|\mathcal{B}(\boldsymbol{x},\boldsymbol{y})|}{|||\boldsymbol{x}|||_{0,k}|||\boldsymbol{y}|||_{0,k}} \leq Ch_k^{-2}. \tag{4.2}$$

Associated with this norm, a mesh-dependent 2-norm is defined by

$$|||\boldsymbol{x}|||_{2,k} = \sup_{0\neq \boldsymbol{y}\in \boldsymbol{W}_k} \frac{|\mathcal{B}(\boldsymbol{x},\boldsymbol{y})|}{|||\boldsymbol{y}|||_{0,k}}. \tag{4.3}$$

To study the two-grid algorithm, we need the so-called coarse-to-fine operator $\mathcal{I}_{k-1}^k$ and the fine-to-coarse operator $\mathcal{I}_k^{k-1}$. Here $\mathcal{I}_{k-1}^k$: $\boldsymbol{W}_{k-1}\to\boldsymbol{W}_k$ is the natural injection defined in Section 3.1 and $\mathcal{I}_k^{k-1}$: $\boldsymbol{W}_k\to\boldsymbol{W}_{k-1}$ is the duality of $\mathcal{I}_{k-1}^k$ with respect to the mesh-dependent inner product

$$(\mathcal{I}_k^{k-1}\boldsymbol{x}_k, \boldsymbol{y}_{k-1})_{0,k-1} = (\boldsymbol{x}_k, \mathcal{I}_{k-1}^k\boldsymbol{y}_{k-1})_{0,k} \quad \forall\, \boldsymbol{y}_{k-1}\in\boldsymbol{W}_{k-1},\ \boldsymbol{x}_k\in\boldsymbol{W}_k.$$

Since $\mathcal{I}_{k-1}^k$ is the natural injection, we immediately get

$$\left|\left|\left|\mathcal{I}_{k-1}^k\boldsymbol{y}_{k-1}\right|\right|\right|_{0,k} \leq \left|\left|\left|\boldsymbol{y}_{k-1}\right|\right|\right|_{0,k-1}, \qquad \left|\left|\left|\mathcal{I}_k^{k-1}\boldsymbol{x}_k\right|\right|\right|_{0,k-1} \leq |||\boldsymbol{x}_k|||_{0,k}.$$

Let $\mathcal{P}_k^{k-1}$: $\boldsymbol{W}_k\to\boldsymbol{W}_{k-1}$ be the duality of $\mathcal{I}_{k-1}^k$ with respect to $\mathcal{B}(\cdot,\cdot)$, namely,

$$\mathcal{B}(\mathcal{P}_k^{k-1}\boldsymbol{x}_k, \boldsymbol{y}_{k-1}) = \mathcal{B}(\boldsymbol{x}_k, \mathcal{I}_{k-1}^k\boldsymbol{y}_{k-1}) \qquad \forall\, \boldsymbol{y}_{k-1}\in\boldsymbol{W}_{k-1}, \boldsymbol{x}_k\in\boldsymbol{W}_k. \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\left|\left|\left|\mathcal{P}_k^{k-1}\boldsymbol{x}_k\right|\right|\right|_{2,k-1} \leq C\,|||\boldsymbol{x}_k|||_{2,k} \qquad \forall\, \boldsymbol{x}_k\in\boldsymbol{W}_k.$$

Let $\mathcal{I}_k$: $\boldsymbol{W}_k\to\boldsymbol{W}_k$ be the identity operator and let $\mathcal{S}_k$: $\boldsymbol{W}_k\to\boldsymbol{W}_k$ be the DBS smoothing operator defined in (3.8). Since the two-grid algorithm solves the coarse-grid problem exactly, the error between the exact solution $\boldsymbol{x}_k$ and the approximate solution $\boldsymbol{x}_k^{(\nu_1+\nu_2+1)}$ by one MG iteration can be represented as follows

$$\boldsymbol{x}_k - \boldsymbol{x}_k^{(\nu_1+\nu_2+1)} = \mathcal{S}_k^{\nu_2}(\mathcal{I}_k - \mathcal{P}_k^{k-1})\mathcal{S}_k^{\nu_1}(\boldsymbol{x}_k - \boldsymbol{x}_k^{(0)}).$$

For simplicity, we restrict the convergence analysis to the MMG method only having pre-smoothing, namely, $\nu_1 = m$ and $\nu_2 = 0$. Then the error satisfies

$$\boldsymbol{x}_k - \mathbf{MG}(k,\gamma,\boldsymbol{x}_k^{(0)},m,0) = \boldsymbol{x}_k - \boldsymbol{x}_k^{(m+1)} = (\mathcal{I}_k - \mathcal{P}_k^{k-1})\mathcal{S}_k^m(\boldsymbol{x}_k - \boldsymbol{x}_k^{(0)}). \tag{4.5}$$

The theory can be extended to the case of $\nu_2 \neq 0$ straightforwardly.

4.2. Approximation property

First we prove some preliminary results which are important for proving the approximation property.

Lemma 4.2. *Given $\boldsymbol{\xi}\in\boldsymbol{L}^2(\Omega)$ and $\varphi\in L^2(\Omega)$, let $\boldsymbol{x}_j\in\boldsymbol{W}_j$ be the solutions of*

$$\mathcal{B}(\boldsymbol{x}_j,\boldsymbol{y}_j) = (\boldsymbol{\xi},\boldsymbol{v}_j) + (\varphi,\psi_j) \quad \forall\, \boldsymbol{y}_j = (\boldsymbol{v}_j,q_j,\psi_j)\in\boldsymbol{W}_j, \quad j = k-1,\, k.$$

There is a constant C independent of h_k such that

$$|||\boldsymbol{x}_k - \boldsymbol{x}_{k-1}|||_{0,k} \leq Ch_k^2\big(\,\|\boldsymbol{\xi}\|_{\boldsymbol{L}^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}\,\big).$$

Proof. Let $\boldsymbol{x} \in \boldsymbol{W}$ be the solution to

$$\mathcal{B}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{\xi}, \boldsymbol{v}) + (\varphi, \psi) \qquad \forall \boldsymbol{x} \in \boldsymbol{W}.$$

From Theorem 2.2 and the regularities in (2.4), we have

$$|||\boldsymbol{x} - \boldsymbol{x}_j|||_{0,k} \le C h_j^2 \big(\|\boldsymbol{\xi}\|_{\boldsymbol{L}^2(\Omega)} + \|\varphi\|_{L^2(\Omega)} \big), \qquad j = k-1,\, k.$$

The proof is finished by using the triangle inequality.

Lemma 4.3. *Given $w \in L^2(\Omega)$, let $\boldsymbol{x}_j \in \boldsymbol{W}_j$, $j = k-1, k$, be the solutions to*

$$\mathcal{B}(\boldsymbol{x}_j, \boldsymbol{y}_j) = (w, q_j), \quad \forall \boldsymbol{y}_j = (\boldsymbol{v}_j, q_j, \psi_j) \in \boldsymbol{W}_j. \tag{4.6}$$

There is a constant C independent of h_k such that

$$|||\boldsymbol{x}_k - \boldsymbol{x}_{k-1}|||_{0,k} \le C h_k \|w\|_{L^2(\Omega)}.$$

Proof. By Theorem 2.1, there is a unique solution $\boldsymbol{\theta} = (\boldsymbol{w}, r, \varphi) \in \boldsymbol{W}$ satisfying

$$\mathcal{B}(\boldsymbol{\theta}, \boldsymbol{y}) = (\boldsymbol{u}_k - \boldsymbol{u}_{k-1}, \boldsymbol{v}) + (\phi_k - \phi_{k-1}, \psi) \qquad \forall \boldsymbol{y} = (\boldsymbol{v}, q, \psi) \in \boldsymbol{W}.$$

Let $\boldsymbol{\theta}_j \in \boldsymbol{W}_j$, $j = k-1, k$, be the solutions to

$$\mathcal{B}(\boldsymbol{\theta}_j, \boldsymbol{y}_j) = (\boldsymbol{u}_k - \boldsymbol{u}_{k-1}, \boldsymbol{v}_j) + (\phi_k - \phi_{k-1}, \psi_j) \quad \forall \boldsymbol{y}_j = (\boldsymbol{v}_j, q_j, \psi_j) \in \boldsymbol{W}_j. \tag{4.7}$$

Then Lemma 4.2 implies that

$$\|r_k - r_{k-1}\|_{L^2(\Omega)} \le C h_k \big(\|\boldsymbol{u}_k - \boldsymbol{u}_{k-1}\|_{\boldsymbol{L}^2(\Omega)} + \|\phi_k - \phi_{k-1}\|_{L^2(\Omega)} \big). \tag{4.8}$$

From (4.7), we find that

$$\|\boldsymbol{u}_k - \boldsymbol{u}_{k-1}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\phi_k - \phi_{k-1}\|_{L^2(\Omega)}^2 = \mathcal{B}(\boldsymbol{\theta}_k, \boldsymbol{x}_k) - \mathcal{B}(\boldsymbol{\theta}_{k-1}, \boldsymbol{x}_{k-1}).$$

Combining (4.6) and (4.8) yields

$$\|\boldsymbol{u}_k - \boldsymbol{u}_{k-1}\| + \|\phi_k - \phi_{k-1}\| \le C h_k \|w\|. \tag{4.9}$$

From (2.8), the stabilities of discrete solutions lead to

$$\|\boldsymbol{u}_j\|_{\boldsymbol{H}^1(\Omega)} + \|\phi_j\|_{H^1(\Omega)} + \|p_j\|_{L^2(\Omega)} \le C \|w\|_{L^2(\Omega)}, \quad j = k-1,\, k. \tag{4.10}$$

The proof is completed by (4.9), (4.10), and the definition of $|||\cdot|||_{0,k}$.

Lemma 4.4. *There is a constant C independent of h_k such that, for $k \ge 1$,*

$$\left|\left|\left| (\mathcal{I}_k - \mathcal{P}_k^{k-1}) \boldsymbol{x}_k \right|\right|\right|_{0,k} \le C h_k^2 |||\boldsymbol{x}_k|||_{2,k} \qquad \forall \boldsymbol{x}_k \in \boldsymbol{W}_k. \tag{4.11}$$

Proof. Write $\mathcal{P}_k^{k-1} \boldsymbol{x}_k = \hat{\boldsymbol{x}} = (\hat{\boldsymbol{u}}_k, \hat{p}_k, \hat{\phi}_k)$ for convenience. It suffices to estimate $\|\boldsymbol{u}_k - \hat{\boldsymbol{u}}_k\|_{\boldsymbol{L}^2(\Omega)}$, $\|\phi_k - \hat{\phi}_k\|_{L^2(\Omega)}$, and $\|p_k - \hat{p}_k\|_{L^2(\Omega)}$. We estimate the first two terms by dual techniques. Let $\boldsymbol{\zeta}_j \in \boldsymbol{W}_j$, $j = k-1, k$, satisfy

$$\mathcal{B}(\boldsymbol{\zeta}_j, \boldsymbol{y}_j) = (\boldsymbol{u}_k - \hat{\boldsymbol{u}}_k, \boldsymbol{v}_j) + (\phi_k - \hat{\phi}_k, \psi_j) \quad \forall \boldsymbol{y}_j = (\boldsymbol{v}_j, q_j, \psi_j) \in \boldsymbol{W}_j. \tag{4.12}$$

Taking $\boldsymbol{y}_k = \boldsymbol{u}_k - \hat{\boldsymbol{u}}_k$ leads to

$$\begin{aligned}\|\boldsymbol{u}_k - \hat{\boldsymbol{u}}_k\|^2_{\boldsymbol{L}^2(\Omega)} + \left\|\phi_k - \hat{\phi}_k\right\|^2_{L^2(\Omega)} &= \mathcal{B}(\boldsymbol{\zeta}_k, \boldsymbol{x}_k) - \mathcal{B}(\boldsymbol{\zeta}_{k-1}, \mathcal{P}_k^{k-1}\boldsymbol{x}_k) \\ = \mathcal{B}(\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_{k-1}, \boldsymbol{x}_k) \leq \left|\left|\left|\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_{k-1}\right|\right|\right|_{0,k} |||\boldsymbol{x}_k|||_{2,k}\,. \end{aligned}$$

Estimating $\left|\left|\left|\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_{k-1}\right|\right|\right|_{0,k}$ by Lemma 4.2, we obtain

$$\|\boldsymbol{u}_k - \hat{\boldsymbol{u}}_k\|_{\boldsymbol{L}^2(\Omega)} + \left\|\phi_k - \hat{\phi}_k\right\|_{L^2(\Omega)} \leq Ch_k^2 \, |||\boldsymbol{x}_k|||_{2,k}\,. \tag{4.13}$$

The third term can be estimated similarly. Let $\boldsymbol{\xi}_j \in \boldsymbol{W}_j$ solve

$$\mathcal{B}(\boldsymbol{\xi}_j, \boldsymbol{y}_j) = (p_k - \hat{p}_k, q_j) \quad \forall \boldsymbol{y}_j \in \boldsymbol{W}_j, \quad j = k-1,\, k. \tag{4.14}$$

By Lemma 4.3 we have $\left|\left|\left|\boldsymbol{\xi}_k - \boldsymbol{\xi}_{k-1}\right|\right|\right|_{0,k} \leq Ch_k \|p_k - \hat{p}_k\|_{L^2(\Omega)}$. So (4.14) shows

$$\|p_k - \hat{p}_k\|^2_{L^2(\Omega)} = \mathcal{B}(\boldsymbol{\xi}_k - \boldsymbol{\xi}_{k-1}, \boldsymbol{x}_k) \leq Ch_k \|p_k - \hat{p}_k\|_{L^2(\Omega)} \, |||\boldsymbol{x}|||_{2,k}\,.$$

This gives $\|p_k - \hat{p}_k\|_{L^2(\Omega)} \leq Ch_k \, |||\boldsymbol{x}|||_{2,k}$. The proof is finished by using (4.13).

Lemma 4.5 (Approximation property) *Let assumption* (2.4) *be satisfied. There exists a constant C such that*

$$\left|\left|\left|\boldsymbol{x}_k^{(m+1)} - \boldsymbol{x}_k\right|\right|\right|_{0,k} \leq Ch_k^2 \left|\left|\left|\boldsymbol{x}_k^{(m)} - \boldsymbol{x}_k\right|\right|\right|_{2,k}. \tag{4.15}$$

Proof. This is a direct consequence of (4.5) and (4.11).

4.3. The smoothing property

For two symmetric matrices $\mathbb{A}$ and $\mathbb{B}$, "$\mathbb{A} > \mathbb{B}$" means that $\mathbb{A} - \mathbb{B}$ is positive definite and "$\mathbb{A} \geq \mathbb{B}$" means that $\mathbb{A} - \mathbb{B}$ is positive semi-definite. Let $\mathbb{M} \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite (SPD) matrix. For a vector $\underline{\mathbf{v}} \in \mathbb{R}^n$ and a matrix $\mathbb{A} \in \mathbb{R}^{n\times n}$, define

$$\|\underline{\mathbf{v}}\|_{\mathbb{M}} = \left(\underline{\mathbf{v}}^\top \mathbb{M}\underline{\mathbf{v}}\right)^{1/2}, \qquad \|\mathbb{A}\|_{\mathbb{M}} = \sup_{0\neq \underline{\mathbf{v}},\underline{\mathbf{w}}\in\mathbb{R}^n} \frac{|\underline{\mathbf{w}}^\top \mathbb{A}\underline{\mathbf{v}}|}{\|\underline{\mathbf{v}}\|_{\mathbb{M}} \|\underline{\mathbf{w}}\|_{\mathbb{M}}}. \tag{4.16}$$

When $\mathbb{M} = \mathbb{I}$ is identity matrix, we also use the notations

$$\|\underline{\mathbf{v}}\| = \|\underline{\mathbf{v}}\|_{\mathbb{I}}\,, \qquad \|\mathbb{A}\| = \|\mathbb{A}\|_{\mathbb{I}}\,.$$

From (3.7), we have the error-propagation equation of the DBS smoother

$$\underline{\mathbf{x}}^{(m)} - \underline{\mathbf{x}} = \mathbb{P}_k\left(\underline{\mathbf{x}}^{(m-1)} - \underline{\mathbf{x}}\right) = \mathbb{P}_k^m\left(\underline{\mathbf{x}}^{(0)} - \underline{\mathbf{x}}\right), \qquad \mathbb{P}_k = \mathbb{I}_k - \hat{\mathbb{K}}_k^{-1}\mathbb{K}_k.$$

So it is convenient to study the smoothing property in an algebraic setting. Define $n_k = \dim(\boldsymbol{V}_k) + \dim(S_k)$ and $m_k = \dim(Q_k)$ and let $\mathbb{I}_{\boldsymbol{w}} \in \mathbb{R}^{n_k\times n_k}$, $\mathbb{I}_p \in \mathbb{R}^{m_k\times m_k}$ be identity matrices. Define

$$\mathbb{L}_k = \begin{pmatrix} h_k^3\mathbb{I}_{\boldsymbol{w}} & 0 \\ 0 & h_k^5\mathbb{I}_p \end{pmatrix}, \qquad \mathbb{D}_k = \begin{pmatrix} \hat{\mathbb{X}}_k - \mathbb{X}_k & 0 \\ 0 & \hat{\mathbb{S}}_k - \mathbb{S}_k \end{pmatrix}.$$

From [10, Theorem 7] and [31, Theorem 3], we obtain the following lemma.

Lemma 4.6. *Let the scaling parameters $\alpha_{\mathbb{A}}$, $\alpha_{\mathbb{C}}$, $\alpha_{\mathbb{S}}$ be large enough such that $\hat{\mathbb{X}} \geq \mathbb{X}$ and $\hat{\mathbb{S}} \geq \mathbb{S}$. Then for any $m \geq 2$,*

$$\|\mathbb{K}_k\mathbb{P}_k^m\|_{\mathbb{L}_k} \leq Cm^{-1/2}\|\mathbb{D}_k\|_{\mathbb{L}_k}.$$

Lemma 4.7. *Let the scaling parameters $\alpha_{\mathbb{A}}$, $\alpha_{\mathbb{C}}$, $\alpha_{\mathbb{S}}$ be large enough such that $\hat{\mathbb{X}} \geq \mathbb{X}$ and $\hat{\mathbb{S}} \geq \mathbb{S}$. Then for any $m \geq 2$,*

$$\|\mathbb{K}_k\mathbb{P}_k^m\|_{\mathbb{L}_k} \leq Cm^{-1/2}\|\mathbb{K}_k\|_{\mathbb{L}_k}.$$

Proof. It suffices to prove $\|\mathbb{D}_k\|_{\mathbb{L}_k} \leq C\|\mathbb{K}_k\|_{\mathbb{L}_k}$. Since $\mathbb{X}_k$ is symmetric and semi-positive definite and $\hat{\mathbb{X}}_k$, $\mathbb{S}_k$ are SPD, we have

$$0 \leq \mathbb{D}_k \leq \begin{pmatrix} \hat{\mathbb{X}}_k & 0 \\ 0 & \hat{\mathbb{S}}_k \end{pmatrix}, \qquad \|\mathbb{D}_k\|_{\mathbb{L}_k} \leq h_k^{-3}\max\left(\|\hat{\mathbb{X}}_k\|, h_k^{-2}\|\hat{\mathbb{S}}_k\|\right).$$

Since $\hat{\mathbb{X}}_k$, $\hat{\mathbb{S}}_k$ are diagonal matrices, we have

$$\begin{aligned}\|\mathbb{D}_k\|_{\mathbb{L}_k} &\leq h_k^{-3}\max\left(\alpha_{\mathbb{A}}\|\mathrm{diag}(\mathbb{A}_k)\|, \alpha_{\mathbb{C}}\|\mathrm{diag}(\mathbb{C}_k)\|, \alpha_{\mathbb{S}}h_k^{-2}\|\mathrm{diag}(\mathbb{S}_k)\|\right) \\ &\leq h_k^{-3}\max\left(\alpha_{\mathbb{A}}\|\mathbb{A}_k\|, \alpha_{\mathbb{C}}\|\mathbb{C}_k\|, \alpha_{\mathbb{S}}h_k^{-2}\|\mathbb{S}_k\|\right).\end{aligned}$$

A standard scaling technique as done in [31] shows that

$$\|\mathbb{S}_k\| = \left\|\mathbb{B}_k\hat{\mathbb{A}}_k^{-1}\mathbb{B}_k^\top\right\| \leq C\alpha_{\mathbb{A}}^{-1}\|\mathbb{B}_k\|.$$

Combining the two inequalities above yields

$$\|\mathbb{D}_k\|_{\mathbb{L}_k} \leq Ch_k^{-3}\max\left(\|\mathbb{A}_k\|, \|\mathbb{C}_k\|, h_k^{-2}\|\mathbb{B}_k\|\right) \leq C\|\mathbb{K}_k\|_{\mathbb{L}_k}.$$

This completes the proof.

Lemma 4.8 (Smoothing property) *There is a consant C independent of k, m such that*

$$\left|\!\left|\!\left|\boldsymbol{x}_k^{(m)} - \boldsymbol{x}_k\right|\!\right|\!\right|_{2,k} \leq C_S m^{-1/2}h_k^{-2}\left|\!\left|\!\left|\boldsymbol{x}_k^{(0)} - \boldsymbol{x}_k\right|\!\right|\!\right|_{0,k}. \tag{4.17}$$

Proof. Let $\underline{\mathbf{y}}$ be the vector of DOFs associated with $\boldsymbol{y} \in \boldsymbol{W}_k$. By standard scaling arguments, there are two constants $C_2 > C_1 > 0$ such that

$$C_1\left\|\underline{\mathbf{y}}\right\|_{\mathbb{L}_k} \leq |||\boldsymbol{y}|||_{0,k} \leq C_2\left\|\underline{\mathbf{y}}\right\|_{\mathbb{L}_k}. \tag{4.18}$$

Write $\boldsymbol{e}^{(m)} = \boldsymbol{x}_k^{(m)} - \boldsymbol{x}_k \in \boldsymbol{W}_k$ for convenience and let $\underline{\mathbf{e}}^{(m)}$ be the vector associated with $\boldsymbol{e}^{(m)}$. Then

$$\begin{aligned}\left|\!\left|\!\left|\boldsymbol{e}^{(m)}\right|\!\right|\!\right|_{2,k} &= \sup_{0\neq\boldsymbol{y}\in\boldsymbol{W}_k}\frac{|\mathcal{B}(\boldsymbol{e}^{(m)}, \boldsymbol{y})|}{|||\boldsymbol{y}|||_{0,k}} \leq C\sup_{\underline{\mathbf{y}}\neq 0}\frac{|\underline{\mathbf{y}}^\top\mathbb{K}_k\underline{\mathbf{e}}^{(m)}|}{\left\|\underline{\mathbf{y}}\right\|_{\mathbb{L}_k}} = C\sup_{\underline{\mathbf{y}}\neq 0}\frac{|\underline{\mathbf{y}}^\top(\mathbb{K}_k\mathbb{P}_k^m)\underline{\mathbf{e}}^{(0)}|}{\left\|\underline{\mathbf{y}}\right\|_{\mathbb{L}_k}} \\ &\leq C\|\mathbb{K}_k\mathbb{P}_k^m\|_{\mathbb{L}_k}\left\|\underline{\mathbf{e}}^{(0)}\right\|_{\mathbb{L}_k} \leq Cm^{-1/2}\|\mathbb{K}_k\|_{\mathbb{L}_k}\left|\!\left|\!\left|\boldsymbol{e}^{(0)}\right|\!\right|\!\right|_{0,k}.\end{aligned}$$

By Lemma 4.1 and inverse inequalities on $\boldsymbol{W}_k$, we find that

$$\begin{aligned}\|\mathbb{K}_k\|_{\mathbb{L}_k} &= \sup_{\underline{\mathbf{y}},\underline{\mathbf{z}}\neq 0}\frac{|\underline{\mathbf{y}}^\top\mathbb{K}_k\underline{\mathbf{z}}|}{\left\|\underline{\mathbf{y}}\right\|_{\mathbb{L}_k}\|\underline{\mathbf{z}}\|_{\mathbb{L}_k}} \leq C\sup_{0\neq\boldsymbol{y},\boldsymbol{z}\in\boldsymbol{W}_k}\frac{|\mathcal{B}(\boldsymbol{y}, \boldsymbol{z})|}{|||\boldsymbol{y}|||_{0,k}|||\boldsymbol{z}|||_{0,k}} \\ &\leq C\sup_{0\neq\boldsymbol{y},\boldsymbol{z}\in\boldsymbol{W}_k}\frac{\|\boldsymbol{y}\|_{\boldsymbol{W}}\|\boldsymbol{z}\|_{\boldsymbol{W}}}{|||\boldsymbol{y}|||_{0,k}|||\boldsymbol{z}|||_{0,k}} \leq Ch_k^{-2}.\end{aligned}$$

The proof is finished.

4.4. Convergence

Now we are in the position of presenting the main theorem of this paper.

Theorem 4.1. *Let Assumption 2.1 be satisfied and let the number of smoothing steps m be large enough. Then the two-grid method is uniformly convergent, namely, there exists a constant $\delta \in (0,1)$ independent of mesh size such that*

$$\left|\left|\left| \boldsymbol{x}_k - \mathbf{MG}(k, \gamma, \boldsymbol{x}_k^{(0)}, m, 0) \right|\right|\right|_{0,k} \leq \delta \left|\left|\left| \boldsymbol{x}_k - \boldsymbol{x}_k^{(0)} \right|\right|\right|_{0,k}. \tag{4.19}$$

Proof. Recall from Subsection 3 that the approximate solution by one MG iteration with $\nu_1 = m$, $\nu_2 = 0$ is denoted by $\mathbf{MG}(k, \gamma, \boldsymbol{x}_k^{(0)}, m, 0) := \boldsymbol{x}_k^{(m+1)}$. By the error representation (4.5), Lemma 4.5, and Lemma 4.8, we deduce that

$$\begin{aligned} \left|\left|\left| \boldsymbol{x}_k - \boldsymbol{x}_k^{(m+1)} \right|\right|\right|_{0,k} &= \left|\left|\left| (\mathcal{I} - \mathcal{P}_k^{k-1})(\boldsymbol{x}_k - \boldsymbol{x}_k^{(m)}) \right|\right|\right|_{0,k} \leq C h_k^2 \left|\left|\left| \boldsymbol{x}_k - \boldsymbol{x}_k^{(m)} \right|\right|\right|_{2,k} \\ &\leq C m^{-1/2} \left|\left|\left| \boldsymbol{x}_k - \boldsymbol{x}_k^{(0)} \right|\right|\right|_{0,k}. \end{aligned}$$

The proof is finished by choosing m large enough such that $\delta := C m^{-1/2} < 1$.

Remark 4.1. *Based on two-grid MG method, the convergence of W-cycle and variable V-cycle MG methods can be proven similarly (cf. e.g. [15]). We do not elaborate on the details. In Section 5, we shall also show the uniform convergence of both V-cycle and F-cycle MG methods numerically.*

4.5. Scaling parameters

To end this section, we discuss the choices of the scaling parameters $\alpha_{\mathbb{A}}$, $\alpha_{\mathbb{C}}$, $\alpha_{\mathbb{S}}$ in (3.6) to fulfill the assumptions in Lemma 4.6 and Lemma 4.7.

Recall that for a SPD matrix $\mathbb{M} \in \mathbb{R}^{n \times n}$, the estimate $\mathbb{M} \leq nnz(\mathbb{M}) \cdot \mathrm{diag}(\mathbb{M})$ always holds, where $nnz(\mathbb{M})$ stands for the maximum number of non-zero entries per row in $\mathbb{M}$. For finite element matrices $\mathbb{X}_k$ and $\mathbb{S}_k$, $nnz(\mathbb{X}_k)$ and $nnz(\mathbb{S}_k)$ are respectively bounded by two constants $N_{\mathbb{X}}$ and $N_{\mathbb{S}}$ which are independent of h_k. It suffices to set $\alpha_{\mathbb{A}} = \alpha_{\mathbb{C}} = N_{\mathbb{X}}$ and $\alpha_{\mathbb{S}} = N_{\mathbb{S}}$.

In practice, the choices of scaling parameters can be less restrictive. An efficient way is to set $\alpha_{\mathbb{A}} = \alpha_{\mathbb{C}} = \lambda_{\mathbb{X}}^{\max}$ and $\alpha_{\mathbb{S}} = \lambda_{\mathbb{S}}^{\max}$, where $\lambda_{\mathbb{X}}^{\max}$ and $\lambda_{\mathbb{S}}^{\max}$ are, respectively, approximate maximal eigenvalues of the generalized eigenvalue problems

$$\mathbb{X}\,\underline{\mathbf{w}} = \lambda_{\mathbb{X}}\,\hat{\mathbb{X}}\,\underline{\mathbf{w}}, \qquad \mathbb{S}\,\underline{\mathbf{p}} = \lambda_{\mathbb{S}}\,\hat{\mathbb{S}}\,\underline{\mathbf{p}}. \tag{4.20}$$

We merely compute $\lambda_{\mathbb{X}}^{\max}$, $\lambda_{\mathbb{S}}^{\max}$ by several iterations of Power methods for solving (4.20) on a coarse mesh $\mathcal{T}_H$.

5. Numerical experiments

In this section, we carry out numerical experiments for the 3D RMHD problem to demonstrate the performance of the MMG method. All numerical tests are carried out on the LSSC-IV cluster at the State Key Laboratory of Scientific and Engineering Computing (LSEC), Chinese Academy of Sciences. The finite element method and the discrete solver are implemented on adaptive finite element package "Parallel Hierarchical Grid" (PHG) [37].

Example 5.1. *This example is to examine the uniform convergence of the MMG method with respect to mesh size. The problem setting is defined by*

$$\Omega = (0,1)^3, \qquad \boldsymbol{B} = (0,0,1)^\top, \qquad R_e = N = 1.$$

The exact solutions are chosen as

$$\boldsymbol{u} = 2\,(\cos 2x\ \sin 2y,\ -\sin 2x\ \cos 2y,\ 0)^\top,$$
$$p = \sin y + \cos 1 - 1, \qquad \phi = \cos 2x\ \cos 2y + x^2 - y^2.$$

Table 5.1: Number of W-cycle MG iterations and convergence rate.

Levels	N_{dofs}	Smoothing steps $\nu_1+\nu_2$ versus $N_{\text{MG}}\,(\rho)$				
		4+4	5+5	6+6	7+7	8+8
2	3,041	21 (0.27)	17 (0.20)	15 (0.14)	14 (0.12)	13 (0.11)
3	20,381	21 (0.27)	18 (0.20)	17 (0.17)	15 (0.14)	14 (0.12)
4	148,661	21 (0.26)	18 (0.20)	16 (0.16)	15 (0.14)	14 (0.12)
5	1,134,437	20 (0.25)	18 (0.20)	16 (0.16)	15 (0.13)	14 (0.11)

Table 5.2: Number of V-cycle MG iterations and convergence rate.

Levels	N_{dofs}	Smoothing steps $\nu_1+\nu_2$ versus $N_{\text{MG}}\,(\rho)$				
		4+4	5+5	6+6	7+7	8+8
2	3,041	21 (0.27)	17 (0.20)	15 (0.14)	14 (0.12)	13 (0.11)
3	20,381	22 (0.28)	18 (0.21)	17 (0.18)	16 (0.15)	14 (0.12)
4	148,661	21 (0.27)	18 (0.21)	17 (0.17)	16 (0.15)	14 (0.12)
5	1,134,437	21 (0.26)	19 (0.22)	17 (0.18)	16 (0.14)	15 (0.12)

Table 5.3: Number of F-cycle MG iterations and convergence rate.

Levels	N_{dofs}	Smoothing steps $\nu_1+\nu_2$ versus $N_{\text{MG}}\,(\rho)$				
		4+4	5+5	6+6	7+7	8+8
2	3,041	21 (0.27)	17 (0.19)	15 (0.14)	14 (0.12)	13 (0.11)
3	20,381	21 (0.27)	18 (0.20)	17 (0.17)	15 (0.14)	14 (0.12)
4	148,661	21 (0.26)	18 (0.21)	17 (0.17)	15 (0.14)	14 (0.12)
5	1,134,437	20 (0.25)	18 (0.20)	16 (0.16)	15 (0.13)	14 (0.11)

The tolerance for relative residual is set by $\varepsilon = 10^{-10}$. The convergence rate ρ is defined by the geometric mean of convergence rates

$$\rho := \left\|\underline{\mathbf{b}} - \mathbb{K}\underline{\mathbf{x}}^{(j)}\right\|^{1/j} \left\|\underline{\mathbf{b}} - \mathbb{K}\underline{\mathbf{x}}^{(0)}\right\|^{-1/j},$$

Table 5.4: Convergence rates of $\boldsymbol{u}_h$ and p_h.

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)}$	order	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}^1(\Omega)}$	order	$\|p-p_h\|_{L^2(\Omega)}$	order
0.433	3.41e-03	—	9.42e-02	—	4.37e-02	—
0.216	4.44e-04	2.94	2.34e-02	2.01	7.08e-03	2.67
0.108	5.65e-05	2.98	5.80e-03	2.01	1.20e-03	2.57
0.054	7.10e-06	2.99	1.44e-03	2.01	2.13e-04	2.49
0.027	8.91e-07	3.00	3.60e-04	2.00	4.00e-05	2.41

Table 5.5: Convergence rates of ϕ_h.

h	$\|\|\phi-\phi_h\|\|$	order	$\|\|\phi-\phi_h\|\|_{1,\Omega}$	order
0.433	1.171e-03	—	3.186e-02	—
0.216	1.514e-04	2.95	7.909e-03	2.01
0.108	1.920e-05	2.98	1.968e-03	2.00
0.054	2.414e-06	2.99	4.905e-04	2.00
0.027	3.027e-07	3.00	1.224e-04	2.00

where $j > 0$ is the number of MG iterations. Table 5.1 shows the convergence rate ρ and the total number N_{MG} of W-cycle MG iterations to reduce the relative residual below ε. Clearly both N_{MG} and ρ are quasi-uniform with respect to the number of mesh levels and the total number of DOFs N_{dofs} for $\boldsymbol{u}$, p, and ϕ. Moreover, the convergence rate satisfies the asymptotic behavior

$$\rho \sim m^{-1}, \qquad m = \nu_1 = \nu_2. \tag{5.1}$$

This is superior to the theoretical estimate, namely, $\rho \le Cm^{-1/2}$.

Tables 5.2 and 5.3 show the convergence rates of V-cycle and F-cycle MG methods respectively. Both methods yield the asymptotic convergence rate in (5.1).

Tables 5.4 and 5.5 show the decay of approximation errors as the mesh is refined uniformly. Optimal convergence rates are obtained for finite element solutions.

Example 5.2. (Driven cavity flow) *This example is to show the optimality and robustness of MG-preconditioned FGMRES method for relatively large Reynolds number and coupling number by a driven cavity flow. The cavity domain is* $\Omega = (0,1)^3$ *and the right-hand side functions are given by* $\boldsymbol{f} = 0$ *and* $\chi = 0$. *Dirichlet boundary conditions are set by* $\boldsymbol{u} = (g_1, 0, 0)^\top$ *and* $\phi = 0$ *on* $\partial\Omega$ *where* $g_1 = g_1(z)$ *is a continuous function and satisfies*

$$g_1(1) = 1 \quad \textit{and} \quad g_1(z) = 0 \quad \textit{if } 0 \le z \le h.$$

Here h *is the mesh size. The applied magnetic field is* $\boldsymbol{B} = (0,0,1)^\top$.

The information on four successively refined meshes are listed in Table 5.6. The tolerance for relative residual is set by $\varepsilon = 10^{-10}$. We present numerical results for both the W-cyle MG solver and the MG-preconditioned FGMRES solver.

Table 5.6: Mesh information

Meshes	h	N_{dofs}
$\mathcal{T}_1$	0.216	20,381
$\mathcal{T}_2$	0.108	148,661
$\mathcal{T}_3$	0.054	1,134,437
$\mathcal{T}_4$	0.027	8,861,381

Table 5.7: W-cycle MG method with $\nu_1 = \nu_2 = 4$, $N = 1$.

meshes	R_e	N_{MG}	R_e	N_{MG}	R_e	N_{MG}	R_e	N_{MG}
$\mathcal{T}_1$	100	17	200	18	400	20	800	25
$\mathcal{T}_2$		15		15		16		17
$\mathcal{T}_3$		13		13		13		13
$\mathcal{T}_4$		13		12		12		12

Table 5.8: W-cycle MG method with $\nu_1 = \nu_2 = 4$, $N = 10$.

meshes	R_e	N_{MG}	R_e	N_{MG}	R_e	N_{MG}	R_e	N_{MG}
$\mathcal{T}_1$	100	32	200	40	400	52	800	64
$\mathcal{T}_2$		21		26		35		49
$\mathcal{T}_3$		16		17		19		25
$\mathcal{T}_4$		14		14		14		15

Table 5.9: MG-preconditioned FGMRERS(10) method, $N = 1$.

meshes	R_e	N_{MG}	R_e	N_{MG}	R_e	N_{MG}	R_e	N_{MG}
$\mathcal{T}_1$	100	8	200	8	400	9	800	10
$\mathcal{T}_2$		8		8		8		8
$\mathcal{T}_3$		8		7		7		7
$\mathcal{T}_4$		7		7		6		6

Table 5.10: MG-preconditioned FGMRERS(10) method, $N = 10$.

meshes	R_e	N_{MG}	R_e	N_{MG}	R_e	N_{MG}	R_e	N_{MG}
$\mathcal{T}_1$	100	12	200	13	400	15	800	17
$\mathcal{T}_2$		10		11		12		15
$\mathcal{T}_3$		8		8		9		10
$\mathcal{T}_4$		7		7		7		7

Tables 5.7 and 5.8 shows the number of iterations for the W-cycle MG solver with $\nu_1 = \nu_2 = 4$. We find that the MG solver is not robust to Reynolds number when the coupling number is large, say, $N = 10$. However, it is still optimal with respect to mesh size h for both $N = 1$ and $N = 10$.

Tables 5.9 and 5.10 shows that the number of iterations for the MG-preconditioned FGMRES solver is significantly reduced, compared with that of the MMG solver. Moreover, the preconditioned FGMRES solver is robust even for $N = 10$.

6. Conclusions

In this paper, we investigate the monolithic multigrid method for solving discrete problem of stationary RMHD equations. A diagonal Braess-Sarazin smoother is proposed for the MMG method. By verifying approximation property and smoothing property, we give a rigorous convergence analysis for the MMG method. An MG-preconditioned FGMRES method is proposed and is demonstrated numerically to be robust for relatively large physical parameters.
Acknowledgments. Weiying Zheng was supported in part by the National Science Fund for Distinguished Young Scholars 11725106 and China NSF grant 11831016.

References

[1] Abdou, M. A., Ying, A., Morley, N. et al, *On the exploration of innovative concepts for fusion chamber technology*, Fusion Eng. Des., 54 (2001), 181–247.

[2] Adams, M.F., Samtaney, R., Brandt, A., *Toward textbook multigrid efficiency for fully implicit resistive magnetohy-drodynamics*, Journal of Computational Physics, 229 (2010), 6208–6219.

[3] Adler, J., Benson, T.R., Cyr, E.C., et al, *Monolithic Multigrid Methods for Two-Dimensional Resistive Magnetohydrodynamics*, SIAM Journal on Scientific Computing, 38 (2016), 1–24.

[4] Bank, E.R., Welfert, B.D., Yserentant, H., *A class of iterative methods for solving saddle point problems*, Numerische Mathematik, 56 (1990), 645–666.

[5] Boffi, D., Brezzi, F., Fortin, M., *Mixed Finite Element Methods and Applications*, Springer-Verlag, Berlin, 2013.

[6] Brenner, S.C., *Multigrid methods for parameter dependent problems*, Esaim Mathematical Modelling & Numerical Analysis. 30 (1996), 265–297.

[7] Braess, D., Sarazin, R., *An efficient smoother for the Stokes problem*, Applied Numerical Mathematics, 23 (1997), 3–19.

[8] Brezzi, F., Fortin M., *Mixed and Hybrid Finite Element Methods*, Springer, New York, 1991.

[9] Davidson, P. A., *An Introduction to Magnetohydrodynamics*, Cambridge University Press, United Kingdom, 2001.

[10] Drzisga, D., John, L., Rüde, U., Wohlmuth, B., Zulehner, W., *On the analysis of block smoothers for saddle point problems*, SIAM Journal on Matrix Analysis and Applications, 39 (2018), 932–960.

[11] Ervin, V.J., Layton, W.J., *A posteriori, error estimation for two level discretizations of flows of electrically conducting, incompressible fluids*, Computers Mathematics with Applications, 31 (1996), 105–114.

[12] Gerbeau, J., Bris, C., Lelièvre, T., *Mathematical Methods for the Magnetohydrodynamics of Liquid Metals, Numerical Mathematics and Scientific Computation*, Oxford University Press, New York, 2006.

[13] Girault, V., Raviart, P.A., *Finite Element Methods for Navier-Stokes Equations*, Springer, New York, 1986.

[14] Gunzburger, M.D., Meir, A.J., Peterson, J.S., *On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics*, Mathematics of Computation, 56 (1991), 523–563).

[15] Hackbusch, W., *Multi-Grid Methods and Applications*, Springer-Verlag, New York, 1985.

[16] He, Y., *The Euler implicit/explicit scheme for the 2D time-dependent Navier-Stokes equations with smooth or non-smooth initial data*, Mathematics of Computation, 77 (2008), 2097–2124.

[17] Hiptmair, R., Li, L., Mao, S., Zheng, W., *A fully divergence-free finite element method for magnetohydrodynamic equations*, Mathematical Models and Methods in Applied Sciences (M3AS), 28 (2018), 659–695).

[18] Hu, K., Ma, Y., Xu, J., *Stable finite element methods preserving $\nabla \cdot \boldsymbol{B} = 0$ exactly for MHD models*, Numerische Mathematik, 135 (2017), 371–396.

[19] Layton, W., Lenferink, H.W.J, Peterson, J.S., *A two-level Newton, finite element algorithm for approximating electrically conducting, incompressible fluid flows*, Computers Mathematics with Applications, 28 (1994), 21–31.

[20] Layton, W., Lenferink, W., *Two-level Picard and modified Picard methods for the Navier-Stokes equations*, Applied Mathematics and Computation, 69 (1995), 263–274.

[21] Li, L., Zheng, W., *A robust solver for the finite element approximation of stationary incompressible MHD equations in 3D*, Journal of Computational Physics, 351 (2017), 254–270.

[22] Li, L., Ni, M., Zheng, W., *A charge-conservative finite element method for inductionless MHD equations. Part I: convergence*, SIAM Journal on Scientific Computing, 41 (2019), B796–B815.

[23] Li, L., Ni, M., Zheng, W., *A charge-conservative finite element method for inductionless MHD equations. Part II: a robust solver*, SIAM Journal on Scientific Computing, 41 (2019), B816–B842.

[24] Ma, Y., Hu, K., Hu, X., Xu, J., *Robust preconditioners for incompressible MHD models*, Journal of Computational Physics, 316 (2016), 721–746.

[25] Ni, M., Munipalli, R., Morley, N.B. et al, *A current density conservative scheme for incompressible MHD flows at a low magnetic Reynolds number. Part I: On a rectangular collocated grid system*, Journal of Computational Physics, 227 (2007), 174–204.

[26] Ni, M., Munipalli, R., Huang, P., et al, *A current density conservative scheme for incompressible MHD flows at a low magnetic Reynolds number. Part II: On an arbitrary collocated mesh*, Journal of Computational Physics, 227 (2007), 205–228.

[27] Peterson, J.S., *On the finite element approximation of incompressible flows of an electrically conducting fluid*, Numerical Methods for Partial Differential Equations, 4 (1988), 57–68.

[28] Philips, E.G., Elman, H.C., Cyr, E.C., Shadid, J.N., Pawlowski, R.P., *A block preconditioner for an exact penalty formulation for stationary MHD*, SIAM Journal on Scientific Computing, 36 (2014), 930–951.

[29] Philips, E.G., Shadid, J.N., Cyr, E.C., Elman, H.C., Pawlowski, R.P., *Block preconditioners for stable mixed nodal and edge finite element representations of incompressible resistive MHD*, SIAM Journal on Scientific Computing, 38 (2016), B1009–B1031.

[30] Salah, N.B., Soulaimani, A., Habashi, W.G., *A finite element method for magnetohydrodynamics*, Computer Methods in Applied Mechanics and Engineering, 190 (2001), 5867–5892.

[31] Schöberl, J., Zulehner, W., *On Schwarz-type smoothers for saddle point problems*, Numerische Mathematik, 95 (2003), 377–399.

[32] Schötzau, D., *Mixed finite element methods for stationary incompressible magnetohydrodynamics*, Numerische Mathematik, 96 (2004), 771–800.

[33] Shadid, J., Pawlowski, R., Banks, J., Chacon, L., Lin, P., Tuminaro, R., *Towards a scalable fully-implicit fully-coupled resistive MHD formulation with stabilized FE methods*, Journal of Computational Physics, 229 (2010), 7649–7671.

[34] Shadid, J.N., Pawlowski, R.P., Cyr, E.C., et al, *Scalable implicit incompressible resistive MHD with stabilized FE and fully-coupled Newton-Krylov-AMG*, Computer Methods in Applied Mechanics and Engineering, 304 (2016), 1–25.

[35] Yuksel, G., Ingram, R., *Numerical analysis of a finite element Crank-Nicolson discretization for MHD flows at small magnetic Reynolds numbers*, International Journal of Numerical Analysis Modeling, 10 (2012), 74–98.

[36] Yuksel, G., Isik, O.R., *Numerical analysis of Backward-Euler discretization for simplified magnetohydrodynamic flows*, Applied Mathematical Modelling, 39 (2015), 1889–1898.

[37] Zhang, L., *A Parallel Algorithm for Adaptive Local Renement of Tetrahedral Meshes Using Bisection*, Numerical Mathematics: Theory, Methods and Applications, 2 (2009), 65–89. (Version 0.9.4: http://lsec.cc.ac.cn/phg)

[38] Zulehner, W., *A class of smoothers for saddle point problems*, Computing, 65 (2000), 227–246.