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# On Friedrichs–Poincaré-type inequalities ☆

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**Abstract**

Friedrichs- and Poincaré-type inequalities are important and widely used in the area of partial differential equations and numerical analysis. Most of their proofs appearing in references are the argument of reduction to absurdity. In this paper, we give direct proofs of Friedrichs-type inequalities in $H^1(\Omega)$ and Poincaré-type inequalities in some subspaces of $W^{1,p}(\Omega)$. The dependencies of the inequality coefficients on the domain $\Omega$ and some sub-domains are illustrated explicitly.


*Keywords:* Friedrichs-type inequality; Poincaré-type inequality; Direct proof; Sobolev space

## 1. Introduction

Friedrichs-type inequalities and Poincaré-type inequalities are very important tools and widely used in the area of partial differential equations and numerical analysis. They are frequently used in proving the existence of the solution of partial differential equation and in finite element error estimates. These inequalities ensure that the solution is in a

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more suitable space from a numerical viewpoint than the solution space itself. Most proofs of them in references are by reduction to absurdity [1,3,6]. The method of reduction to absurdity produces an controlling constant depending on the domain implicitly. It is not convenient in application to numerical analysis.

J.C. Nédélec [4] proved directly the Poincaré inequality for functions in $H_0^1(\Omega)$. S. Chen et al. [2], A. Ženišek, and M. Vanmaele [5] proved the Friedrichs inequality for quadrilateral domains. To the best of our knowledge, we have not found other direct proofs for Friedrichs- or Poincaré-type inequalities. Nearly all existing proofs are by reduction to absurdity. In this paper, we are going to prove these inequalities by a direct argument. The constraints which ensure these inequalities on $W^{1,p}(\Omega)$ vary from body constraints to boundary constraints.

Let $A \in \mathbf{R}^n$, we denote the closed ball of radius $R$ and centering at $A$ by $B(A,R)$. $B(0,1)$ is the unit ball centering at the origin. Denote $r = \sqrt{\sum_{i=1}^n x_i^2}$. We define the following exterior cutoff function $\varphi \in C^\infty(\mathbf{R}^n)$:

$$\varphi_{0,1}(x) = \begin{cases} 0, & \text{in } B(0,1/2); \\ e^{\frac{1-r^2}{1-4r^2}}, & \text{in } B(0,1) \setminus B(0,1/2); \\ 1, & \text{in } \mathbf{R}^n \setminus B(0,1); \end{cases} \tag{1.1}$$

$$\varphi_{A,R}(x) = \varphi_{0,1}\left(\frac{x-A}{R}\right). \tag{1.2}$$

Let $\Omega \subset \mathbf{R}^n$ be a bounded and connected open domain. For any multiple index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geqslant 0$, $i = 1, \ldots, n$, define $|\alpha| := \sum_{i=1}^n \alpha_i$ and

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x).$$

We assume $p > 1$ throughout this paper. The usual Sobolev space $W^{m,p}(\Omega)$ is defined as

$$W^{m,p}(\Omega) := \left\{ v \,\middle|\, \int_\Omega \left|D^\alpha v(x)\right|^p dx < \infty,\ \forall |\alpha| \leqslant m \right\}.$$

It is equipped with the following norm and semi-norm:

$$\|v\|_{m,p,\Omega} := \left( \sum_{|\alpha| \leqslant m} \int_\Omega \left|D^\alpha v(x)\right|^p dx \right)^{1/p},$$

$$|v|_{m,p,\Omega} := \left( \sum_{|\alpha| = m} \int_\Omega \left|D^\alpha v(x)\right|^p dx \right)^{1/p}.$$

We also define $L^p(\Omega) := W^{0,p}(\Omega)$ for convenience. In the rest of this paper, we only concern the results in $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

Let $\omega$ be a sub-domain of $\Omega$ with positive measure. Define the following function spaces as

$$W_\omega^{1,p}(\Omega) := \left\{ v \in W^{1,p}(\Omega) \,\middle|\, v|_\omega = 0 \right\}, \qquad C_\omega^\infty(\bar{\Omega}) := \left\{ v \in C^\infty(\bar{\Omega}) \,\middle|\, v|_\omega = 0 \right\}.$$

Obviously, $C_\omega^\infty(\bar{\Omega})$ is dense in $W_\omega^{1,p}(\Omega)$. Denote the diameter of $\Omega$ by $d_\Omega$, the radius of the largest inscribed sphere in $\Omega$ as $r_\Omega$. Hence $d_\Omega \geqslant 2r_\Omega$.

The rest of the paper is arranged as follows. In Section 2, Poincaré-type inequalities are proved for functions in $W^{1,p}(\Omega)$ which vanish on the boundary $\partial\Omega$ or in $\omega$. In Section 3, Friedrichs-type inequalities are proved in $W^{1,p}(\Omega)$ with respect to two integral functionals.

## 2. Poincaré-type inequalities

In the rest of the paper, we will make use of the cutoff function $\varphi_{A,R}$ to prove our main results. The following lemma is easy to prove by direct calculations and the scaling technique.

**Lemma 2.1.** *Let $A \in \Omega$ and $B(A,R) \subset \Omega$. For any $v \in W^{1,p}(\Omega)$,*

$$|\varphi_{A,R}|_{1,\infty,\mathbf{R}^n} = C_0 R^{-1}, \tag{2.3}$$

$$|v\varphi|_{1,p,\Omega} \leqslant 2^{1+\frac{1}{p}}\bigl\{|v|_{1,p,\Omega} + C_0 R^{-1}\|v\|_{0,p,B(A,R)}\bigr\}, \tag{2.4}$$

*where $C_0 := \|\nabla\varphi_{0,1}\|_{0,\infty,B(0,1)}$.*

**Lemma 2.2.** *For any $v \in W_{B(0,\rho)}^{1,p}(B(0,R))$, the following estimate is true*:

$$\bigl\|v(x)\bigr\|_{0,p,B(0,R)}^p < \begin{cases} \frac{R^n(\log R-\log\rho)^{n-1}}{n}\|w_n\nabla v\|_{0,n,B(0,R)}^n - \frac{n-1}{n} \\ \quad\times\int_\rho^R r^{n-1}\bigl(\log\frac{r}{\rho}\bigr)^{n-2}\|\nabla v\|_{0,n,B(0,r)}^n\,dr, & p=n,\\ \bigl(\frac{p-1}{p-n}\bigr)^{p-1}\frac{R^p-\rho^p}{p}\|\nabla v\|_{0,p,B(0,R)}^p, & p>n,\\ \bigl(\frac{p-1}{n-p}\bigr)^{p-1}\frac{\rho^{p-n}R^n}{n}\|\nabla v\|_{0,p,B(0,R)}^p, & 1<p<n, \end{cases} \tag{2.5}$$

*where $r > \rho$, $n = 2$ or $3$ is the dimension of $B(0,R)$, and*

$$w_n(x) = \left[1 - \frac{r^n(\log r - \log\rho)^{n-1}}{R^n(\log R - \log\rho)^{n-1}}\right]^{\frac{1}{n}} < 1\,. \tag{2.6}$$

**Proof.** By the density of $C_{B(0,\rho)}^\infty(B(0,R))$ in $W_{B(0,\rho)}^{1,p}(B(0,R))$, we only need to prove (2.5) for functions $v \in C_{B(0,\rho)}^\infty(B(0,R))$. For convenience, we only give the proof in the case of $n = 2$ here. The case of $n = 3$ can be proved by similar argument. Since $v$ vanishes in $B(0,\rho)$, we have

$$\int\limits_{B(0,R)} \bigl|v(x)\bigr|^p\,dx = \int\limits_{B(0,R)} \left|\int\limits_0^x \nabla v\cdot\vec{\tau}\,dt\right|^p dx \leqslant \int\limits_{B(0,R)} \left(\int\limits_\rho^r |\nabla v|\,dt\right)^p dx$$

$$\leqslant \int\limits_{B(0,R)} \left( \int\limits_{\rho}^{r} t^{\frac{1}{1-p}}\, dt \right)^{p-1} \left( \int\limits_{\rho}^{r} |\nabla v|^p t\, dt \right) dx$$

$$= \begin{cases} \int_0^{2\pi} \int_\rho^R r \log \frac{r}{\rho} \int_\rho^r |\nabla v|^2 t\, dt\, dr\, d\theta, & p = 2, \\ \left(\frac{p-1}{p-2}\right)^{p-1} \int_0^{2\pi} \int_\rho^R r \left(r^{\frac{p-2}{p-1}} - \rho^{\frac{p-2}{p-1}}\right)^{p-1} \int_\rho^r |\nabla v|^p t\, dt\, dr\, d\theta, & p > 2, \\ \left(\frac{p-1}{2-p}\right)^{p-1} \int_0^{2\pi} \int_\rho^R r \left(\rho^{\frac{p-2}{p-1}} - r^{\frac{p-2}{p-1}}\right)^{p-1} \int_\rho^r |\nabla v|^p t\, dt\, dr\, d\theta, & 1 < p < 2. \end{cases} \tag{2.7}$$

If $p = 2$, by the formula of integration by part, we have,

$$\int\limits_\rho^R r \log \frac{r}{\rho} \int\limits_\rho^r |\nabla v|^2 t\, dt\, dr$$

$$\leqslant \int\limits_0^R r \log \frac{r}{\rho} \int\limits_0^r |\nabla v|^2 t\, dt\, dr$$

$$= \frac{R^2}{2} \log \frac{R}{\rho} \int\limits_0^R |\nabla v|^2 r\, dr - \frac{1}{2} \int\limits_0^R r^2 \log \frac{r}{\rho} |\nabla v|^2 r\, dr - \frac{1}{2} \int\limits_0^R r \int\limits_0^r |\nabla v|^2 t\, dt\, dr$$

$$= \frac{R^2}{2} \log \frac{R}{\rho} \int\limits_\rho^R \left[ 1 - \frac{r^2(\log r - \log \rho)}{R^2(\log R - \log \rho)} \right] |\nabla v|^2 r\, dr - \int\limits_0^R \frac{r}{2} \int\limits_0^r |\nabla v|^2 t\, dt\, dr. \tag{2.8}$$

If $p > 2$, clearly we have

$$\int\limits_\rho^R r\left(r^{\frac{p-2}{p-1}} - \rho^{\frac{p-2}{p-1}}\right)^{p-1} \int\limits_\rho^r |\nabla v|^p t\, dt\, dr$$

$$< \int\limits_\rho^R r^{p-1} \int\limits_\rho^r |\nabla v|^2 t\, dt\, dr < \frac{R^p - \rho^p}{p} \int\limits_\rho^R |\nabla v|^p r\, dr. \tag{2.9}$$

If $1 < p < 2$, by the formula of integration by part, we have

$$\int\limits_\rho^R r\left(\rho^{\frac{p-2}{p-1}} - r^{\frac{p-2}{p-1}}\right)^{p-1} \int\limits_\rho^r |\nabla v|^p t\, dt\, dr$$

$$> \rho^{p-2} \int\limits_\rho^R r \left[ 1 - \left(\frac{\rho}{r}\right)^{\frac{2-p}{p-1}} \right] \int\limits_\rho^r |\nabla v|^p t\, dt\, dr \sim O\left(\rho^{p-2}\right), \quad \text{as } \rho \to 0. \tag{2.10}$$

Hence we have the following sharp upper bound in terms of the order of $\rho$ for the left-hand side of (2.10):

$$\int_{\rho}^{R} r\left(\rho^{\frac{p-2}{p-1}} - r^{\frac{p-2}{p-1}}\right)^{p-1} \int_{\rho}^{r} |\nabla v|^p t\,dt\,dr < \frac{\rho^{p-2}R^2}{2} \int_{\rho}^{R} |\nabla v|^p r\,dr. \tag{2.11}$$

Substituting (2.8), (2.9), and (2.11) into (2.7) leads to

$$\int_{B(0,R)} \left|v(x)\right|^p dx < \begin{cases} \frac{R^2(\log R-\log\rho)}{2}\|w_2\nabla v\|_{0,B(0,R)}^2 & \\ \quad -\frac{1}{2}\int_{\rho}^{R} r\|\nabla v\|_{0,B(0,r)}^2\,dr, & p=2, \\ \left(\frac{p-1}{p-2}\right)^{p-1}\frac{R^p-\rho^p}{p}\|\nabla v\|_{0,p,B(0,R)}^p, & p>2, \\ \left(\frac{p-1}{2-p}\right)^{p-1}\frac{\rho^{p-2}R^2}{2}\|\nabla v\|_{0,p,B(0,R)}^p, & 1<p<2. \end{cases}$$

We complete the proof. □

**Theorem 2.3.** *Let $\Omega \subset \mathbb{R}^n$ $(n=2,3)$ be a bounded domain, the measure of $\omega \subset \Omega$ be positive, and $1 < P < \infty$. Assume $\Omega$ is star-shaped* [1] *with respect to $\omega$. Then for any $v \in W_{\omega}^{1,p}(\Omega)$,*

$$\|v\|_{0,p,\Omega} \leqslant \begin{cases} d_{\Omega}(\log d_{\Omega} - \log r_{\omega})^{\frac{n-1}{n}}\,|v|_{1,n,\Omega}, & p=n, \\ p^{-\frac{1}{p}}\left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}} d_{\Omega}|v|_{1,p,\Omega}, & p>n, \\ \left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}} r_{\omega}^{1-\frac{n}{p}} d_{\Omega}^{\frac{n}{p}}|v|_{1,p,\Omega}, & 1<p<n. \end{cases} \tag{2.12}$$

**Proof.** Since $\operatorname{meas}(\omega) > 0$, without loss of generality we assume $B(0, r_{\omega}) \subset \omega$ and $r_{\omega} > 0$. Extend $\nabla v$ by zero to the exterior of $\Omega$ and denote the extension by $\mathbf{w} \in \mathbf{L}^p(B(0, d_{\Omega}))$. Then we have

$$\mathbf{w} = \nabla v, \quad \text{in } \Omega; \qquad \|\mathbf{w}\|_{0,p,B(0,d_{\Omega})} = \|\nabla v\|_{0,p,\Omega}.$$

By Lemma 2.1 and its proof, it is easy to reach (2.12). □

**Remark 2.4.** The proof of Theorem 2.3 depends much on the extension of $v \in W_{\omega}^{1,p}(\Omega)$ to a larger ball. Hence the theorem is true for all convex domains $\Omega$.

**Remark 2.5.** (2.12) is the so-called Poincaré-type inequality:

$$\|v\|_{0,p,\Omega} \leqslant C|v|_{1,p,\Omega}, \quad \forall v \in W_{\omega}^{1,p}(\Omega),\ 1<p. \tag{2.13}$$

It gives the explicit dependence of the constant $C$ on $\Omega$ and $\omega$. An interesting result is that both (2.5) and (2.12) are independent of $\rho = r_{\omega}$ when $p > n$. In fact, since $W^{1,p}(\Omega) \Subset C^0(\bar{\Omega})$ for $p > n$, the point-value functional $A: W^{1,p}(\Omega) \mapsto \mathbb{R}^1$,

$$A(v) = v(A), \quad \forall v \in W^{1,p}(\Omega), \tag{2.14}$$

is linear and continuous on $W^{1,p}(\Omega)$ for any $A \in \bar{\Omega}$. Hence if $\omega = \{A\}$, (2.13) is also true and can be proved by the standard argument of reduction to absurdity (see the proof of [3, Theorem 3.1.1, p. 115]).

If $1 < p \leqslant n$, the Poincaré constant in the left-hand side of (2.13) increases when $\omega$ shrinks. In fact, when $\omega$ shrinks to a point, (2.13) is by no means valid. The following counterexample supports this proclamation.

**Counterexample 2.6.** Let $n = 2$, $\Omega = B(0,1)$, and $v = r^s$ with $0 < s < 1$. Obviously, $v(0) = 0$ and $v \in H^1(\Omega)$. By direct calculations, it is easy to see that

$$\|v\|_{0,\Omega}^2 = \frac{\pi}{s+1}, \qquad |v|_{1,\Omega}^2 = \pi s. \tag{2.15}$$

Setting $s \to 0^+$ in (2.15) leads to the desired contradiction with (2.13).

Assume $\Omega, \Omega_1, \Omega_2 \subset \mathbf{R}^n$ and $\Omega_1 \subset \Omega$; define

$$\begin{aligned} &C(\Omega_1, \Omega_2) := \big\{ y \in \mathbf{R}^n \bigm| y = tx_1 + (1-t)x_2,\ \forall t \in [0,1],\ x_1 \in \Omega_1,\ x_2 \in \Omega_2 \big\}, \\ &S(\Omega_1, \Omega) := \big\{ x \in \Omega \bigm| C\big(\{x\}, \Omega_1\big) \subset \Omega \big\}. \end{aligned} \tag{2.16}$$

Clearly, $C(\{x\}, \Omega_1)$ is the cone with vertex $x$ and bottom $\Omega_1$, $S(\Omega_1, \Omega)$ is the maximal star-shaped subset of $\Omega$ with respect to $\Omega_1$.

**Definition 2.7.** *$\Omega$ is $M$-ball star-shaped with respect to $B_1, B_2, \ldots, B_M$,* if there exist at least $M$ balls $B_1, \ldots, B_M$ such that

- $\Omega = \bigcup_{i=1}^M S(B_i, \Omega)$;
- for any $B_i$, there exists $B_j \neq B_i$ such that $B_i \subset S(B_j, \Omega)$.

Obviously, if $\Omega$ is star-shaped with respect to $B$, it is 1-ball star-shaped with respect to $B$.

**Theorem 2.8.** *Let $\Omega$ be a bounded domain. The measure of $\omega \subset \Omega$ is positive. $B_1$ is the maximal inscribed ball of $\omega$. Assume $\Omega$ is $M$-ball star-shaped with respect to $B_1, B_2, \ldots, B_M$. Then there exists a positive constant $C$ depending only on $n$ and $C_0$ such that for any $v \in W_\omega^{1,p}(\Omega)$,*

$$\|v\|_{0,p,\Omega} \leqslant \begin{cases} C d_\Omega \sum_{i=1}^M \big(\log \frac{2d_\Omega}{r_i}\big)^{\frac{n-1}{n}} \sum_{k=1}^i \alpha_{ik} |v|_{1,n,\Omega}, & p = n, \\ C \sum_{i=1}^M \sum_{k=1}^i \beta_{ik} |v|_{1,p,\Omega}, & 1 < p < n, \end{cases} \tag{2.17}$$

*where for $i = 1, \ldots, M$, $r_i$ is the radius of $B_i$, and all coefficients are defined to be*

$$\alpha_{ii} := 1, \qquad \beta_{ii} = \left( \frac{p-1}{n-p} \right)^{1-\frac{1}{p}} r_i^{1-\frac{n}{p}} d_\Omega^{\frac{n}{p}}, \tag{2.18}$$

$$\alpha_{ik} := \left( \log \frac{2d_\Omega}{r_k} \right)^{\frac{n-1}{n}} \prod_{m=k+1}^{i} \frac{d_\Omega}{r_m}, \tag{2.19}$$

$$\beta_{ik} := \left(\frac{p-1}{n-p}\right)^{(i+1-k)(1-1/p)} r_k^{1-\frac{n}{p}} d_\Omega^{\frac{n}{p}} \prod_{m=k+1}^{i} \left(\frac{d_\Omega}{r_m}\right)^{\frac{n}{p}}, \qquad (2.20)$$
$$1 \leqslant k \leqslant i-1.$$

**Proof.** Denote $r_i := r_{B_i}$ and $\varphi_i := \varphi_{B_i,r_i}$ for convenience. We will prove the theorem by the argument of induction. Without loss of generality, we may assume $B_i \subset S(B_{i+1}, \Omega)$. Hence $B_{i+1} \subset S(B_i, \Omega)$, $1 \leqslant i \leqslant M-1$.

We begin the induction from $S(B_1, \Omega)$. By Theorem 2.3, it follows that

$$\|v\|_{0,p,S(B_1,\Omega)} \leqslant \begin{cases} d_\Omega (\log d_\Omega - \log r_1)^{\frac{n-1}{n}} |v|_{1,n,\Omega}, & p = n, \\ \left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}} r_1^{1-\frac{n}{p}} d_\Omega^{\frac{n}{p}} |v|_{1,p,\Omega}, & 1 < p < n. \end{cases} \qquad (2.21)$$

Applying (2.1) and (2.2) to $\varphi_2 v$ leads to

$$\|\varphi_2 v\|_{0,p,S(B_2,\Omega)} \leqslant \begin{cases} C d_\Omega \left(\log \frac{2d_\Omega}{r_2}\right)^{\frac{n-1}{n}} \left(|v|_{1,n,\Omega} + \frac{1}{r_2}\|v\|_{0,n,B_2}\right), & p = n, \\ C\left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}} r_2^{1-\frac{n}{p}} d_\Omega^{\frac{n}{p}} \left(|v|_{1,\Omega} + \frac{1}{r_2}\|v\|_{0,p,B_2}\right), & 1 < p < n, \end{cases}$$
$$\leqslant \begin{cases} C d_\Omega \left(\log \frac{2d_\Omega}{r_2}\right)^{\frac{n-1}{n}} \left[1 + \frac{d_\Omega}{r_2}\left(\log \frac{d_\Omega}{r_1}\right)^{\frac{n-1}{n}}\right] |v|_{1,n,\Omega}, & p = n, \\ C\left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}} r_2^{1-\frac{n}{p}} d_\Omega^{\frac{n}{p}} |v|_{1,p,\Omega} & \\ \quad + C\left(\frac{p-1}{n-p}\right)^{2-\frac{2}{p}} \left(\frac{d_\Omega}{r_2}\right)^{\frac{n}{p}} r_1^{1-\frac{n}{p}} d_\Omega^{\frac{n}{p}} |v|_{1,p,\Omega}, & 1 < p < n, \end{cases}$$

where $C$ is a generic positive constant depending only on $n$ and $C_0$. Since all balls link each other with $S(\cdot, \Omega)$, similarly, we can prove that for $2 \leqslant i \leqslant M$,

$$\|\varphi_i v\|_{0,p,S(B_i,\Omega)} \leqslant \begin{cases} C d_\Omega \left(\log \frac{2d_\Omega}{r_i}\right)^{\frac{n-1}{n}} \sum_{k=1}^{i} \alpha_{ik} |v|_{1,n,\Omega}, & p = n; \\ C \sum_{k=1}^{i} \beta_{ik} |v|_{1,p,\Omega}, & 1 < p < n. \end{cases} \qquad (2.22)$$

Adding (2.21) to the total sum of (2.22) with respect to $i = 2, \ldots, M$ gives (2.17). □

**Remark 2.9.** At the first glance, the estimate (2.17) seems much more complicated and worse than (2.12). In many cases, even if $\Omega$ is not convex, the number of balls $M$ in Theorem 2.8 is very small ($m = 2, 3$), hence (2.17) may have a much simpler form.

Furthermore, if the topology of $\Omega$ is not very complicated, we can chose the radiuses $r_2, \ldots, r_M > \theta d_\Omega$ in Theorem 2.8 with $\theta \gg \frac{r_1}{d_\Omega}$. Therefore, the main contribution to the coefficient in (2.17) is due to $d_\Omega \log \frac{d_\Omega}{r_1}$ ($p = n$) or $r_1^{1-\frac{n}{p}} d_\Omega^{\frac{n}{p}}$ ($1 < p < n$).

**Remark 2.10.** The worst case for (2.17) is that $\Omega$ is a circular ring with very narrow bandwidth. Then all analyses in Remark 2.9 are not true and (2.17) becomes very bad. The improvement of Theorem 2.8 will be our future work.

The proof of the Poincaré inequality in $W_0^{1,p}(\Omega)$ is much easier, since we may make use of the density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ and extend all functions by zero to the exterior of $\Omega$. A similar argument to the proof of [4, Lemma 2.5.5, p. 57 ], shows the following theorem.

**Theorem 2.11.** *Let $1 < p < \infty$, then the following inequality is true*:

$$\|v\|_{0,p,\Omega} \leqslant d_\Omega |v|_{1,p,\Omega}, \quad \forall v \in W_0^{1,p}(\Omega). \tag{2.23}$$

## 3. Friedrichs-type inequalities

In this section, we give some direct demonstrations for Friedrichs-type inequalities in $H^1(\Omega)$. Since the extension of our proof is not straightforward, it becomes very tedious in the case of $W^{1,p}(\Omega)$ for general $1 < p < \infty$. We restrict our analysis to $p = 2$ because of its extensive applications in numerical analysis. The following definition is needed first.

**Definition 3.1.** *$\Omega$ is $N$-point connected with respect to $A_1, A_2, \ldots, A_N$*, if $\Omega$ is connected and there exist at least $N$ points such that $\Omega = \bigcup_{i=1}^N S(\{A_i\}, \Omega)$.

**Theorem 3.2.** *$\Omega$ is a bounded and $N$-point connected, then there exists a constant $C > 0$ independent of $\Omega$ and $N$ such that*

$$\|v\|_{0,\Omega} \leqslant (N+1) d_\Omega \sqrt{\frac{n d_\Omega^n}{2|\Omega|}} |v|_{1,\Omega} + |\Omega|^{-\frac{1}{2}} \left| \int_\Omega v(x)\, dx \right|, \quad \forall v \in H^1(\Omega), \tag{3.1}$$

*where $|\Omega|$ is the measure of $\Omega$.*

**Proof.** Without loss of generality, we assume that $\Omega$ is $N$-point connected with respect to $A_1, A_2, \ldots, A_N$ and $\Omega \subset [0, d_\Omega]^n$. We expand $\nabla v$ by 0 to the exterior of $\Omega$, denote the extension by $\mathbf{w} \in \mathbf{L}^2([0, d_\Omega]^n)$. By the argument of density, we only need to prove (3.1) for functions in $C^\infty(\bar{\Omega})$. For the sake of convenience in notation, we refer to $\mathbf{w}(t_j)$ as the function of the $j$th component of $t$ while fixing the others.

For any two points $x, y \in \Omega$, denote the vector $y - x$ by $\overrightarrow{xy}$. Our proof is going to follow a similar argument to that in the proof of [2, Lemma 3.2]. Since $\Omega = \bigcup_{i=1}^N S(\{A_i\}, \Omega)$,

$$\begin{aligned}
&v(x)^2 + v(y)^2 - 2v(x)v(y) \\
&\quad = \bigl[v(x) - v(y)\bigr]^2 = \left( \sum_{i=1}^{N+1} \int_{\overrightarrow{A_{i-1}A_i}} \nabla v \cdot \vec{\tau}\, dt \right)^2 \\
&\quad = \left( \sum_{i=1}^{N+1} \sum_{j=1}^{n} \int_{\overrightarrow{A_{i-1}A_i}} w_j \tau_j\, dt \right)^2 = \left( \sum_{i=1}^{N+1} \sum_{j=1}^{n} \int_{A_{i-1,j}}^{A_{i,j}} w_j(t_j)\, dt_j \right)^2 \\
&\quad \leqslant n(N+1) \sum_{i=1}^{N+1} \sum_{j=1}^{n} |A_{i,j} - A_{i-1,j}| \int_{A_{i-1,j}}^{A_{i,j}} \bigl|w_j(t_j)\bigr|^2\, dt_j,
\end{aligned} \tag{3.2}$$

where $A_0 = x$ and $A_{N+1} = y$. Integrating both sides of (3.2) over $x$ and $y$ on $\Omega$, we have

$$\begin{aligned}
&2|\Omega|\|v\|_{0,\Omega}^2 - 2\left|\int\limits_{\Omega} v(x)\,dx\right|^2 \\
&\quad\leqslant n(N+1)\int\limits_{\Omega} dx \int\limits_{\Omega} dy \sum_{i=1}^{N+1}\sum_{j=1}^{n} |A_{i,j} - A_{i-1,j}| \int\limits_{A_{i-1,j}}^{A_{i,j}} \left|w_j(t_j)\right|^2 dt_j \\
&\quad\leqslant n(N+1)\int\limits_{[0,d_\Omega]^n} dx \int\limits_{[0,d_\Omega]^n} dy \sum_{i=1}^{N+1}\sum_{j=1}^{n} \left|x_j^i - x_j^{i-1}\right| \int\limits_{x_j^{i-1}}^{x_j^i} \left|w_j(t_j)\right|^2 dt_j \\
&\quad\leqslant n(N+1)^2 d_\Omega^{2+n} \|\mathbf{w}\|_{0,[0,d_\Omega]^n}^2 \\
&\quad= n(N+1)^2 d_\Omega^{2+n} |v|_{1,\Omega}^2.
\end{aligned} \tag{3.3}$$

Hence we obtain (3.1) by (3.3). □

**Remark 3.3.** The finite-point connection constant $N$ in Theorem 3.2 is very small for many domains. Obviously, $N = 1$ for convex domains. Hence we obtain the following improved result for convex domains.

**Corollary 3.4.** *Let $\Omega$ be a bounded convex domain, then*

$$\|v\|_{0,\Omega} \leqslant 3 d_\Omega^{1+\frac{n}{2}} |\Omega|^{-\frac{1}{2}} |v|_{1,\Omega} + |\Omega|^{-\frac{1}{2}} \left|\int\limits_{\Omega} v(x)\,dx\right|, \quad \forall v \in H^1(\Omega). \tag{3.4}$$

**Theorem 3.5.** *$\Omega$ is a bounded domain. $\omega \subset \Omega$ is $N$-point connected and $|\omega| > 0$. If $\Omega$ is star-shaped with respect to $\omega$, then there exists a positive constant $C$ independent of $\Omega$ and $\omega$ such that for any $v \in H^1(\Omega)$,*

$$\begin{aligned}
\|v\|_{0,\Omega}^2 &\leqslant C d_\Omega^2 \log\frac{2d_\Omega}{r_\omega} |v|_{1,\Omega}^2 + C(N+1)^2 \frac{d_\omega^4}{|\omega|}\left(\frac{d_\Omega^2}{r_\omega^2}\log\frac{2d_\Omega}{r_\omega} + 1\right)|v|_{1,\omega}^2 \\
&\quad + \frac{C}{|\omega|}\left(\frac{d_\Omega^2}{r_\omega^2}\log\frac{2d_\Omega}{r_\omega} + 1\right)\left|\int\limits_{\omega} v(x)\,dx\right|^2, \quad n = 2
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\|v\|_{0,\Omega}^2 &\leqslant C \frac{d_\Omega^3}{r_\omega} |v|_{1,\Omega}^2 + C(N+1)^2 \frac{d_\omega^5}{|\omega|}\left(\frac{d_\Omega^3}{r_\omega^3} + 1\right)|v|_{1,\omega}^2 \\
&\quad + \frac{C}{|\omega|}\left(\frac{d_\Omega^3}{r_\omega^3} + 1\right)\left|\int\limits_{\omega} v(x)\,dx\right|^2, \quad n = 3.
\end{aligned} \tag{3.6}$$

**Proof.** Without loss of generality, we may assume that $B(0, r_\omega) \subset \omega$. Define $u = v\varphi_{0,r_\omega}$. By (3.3), Theorems 2.3 and 3.2, there exists a constant $C > 0$ independent of $\Omega$ and $\omega$ such that

$$\|u\|_{0,\Omega}^2 \leqslant C d_\Omega^2 \log \frac{2d_\Omega}{r_\omega} \big(|v|_{1,\Omega}^2 + r_\omega^{-2} \|v\|_{0,B(0,r_\omega)}^2\big), \quad n = 2, \tag{3.7}$$

$$\|u\|_{0,\Omega}^2 \leqslant C d_\Omega^3 r_\omega^{-1} \big(|v|_{1,\Omega}^2 + r_\omega^{-2} \|v\|_{0,B(0,r_\omega)}^2\big), \quad n = 3, \tag{3.8}$$

$$\|v\|_{0,\omega}^2 \leqslant \frac{n(N+1)^2}{2} \frac{d_\omega^{2+n}}{|\omega|} |v|_{1,\omega}^2 + |\omega|^{-1} \left| \int_\omega v(x)\,dx \right|^2. \tag{3.9}$$

Substituting (3.9) into (3.7) and (3.8) leads to (3.5) and (3.6). □

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