Commun. Comput. Phys.
doi: 10.4208/cicp.040215.280815a

Vol. **19**, No. 2, pp. 534-558
February 2016

An Adaptive Perfectly Matched Layer Method for Multiple Cavity Scattering Problems

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Received 4 February 2015; Accepted (in revised version) 28 August 2015

Abstract. A uniaxial perfectly matched layer (PML) method is proposed for solving the scattering problem with multiple cavities. By virtue of the integral representation of the scattering field, we decompose the problem into a system of single-cavity scattering problems which are coupled with Dirichlet-to-Neumann maps. A PML is introduced to truncate the exterior domain of each cavity such that the computational domain does not intersect those for other cavities. Based on the a posteriori error estimates, an adaptive finite element algorithm is proposed to solve the coupled system. The novelty of the proposed method is that its computational complexity is comparable to that for solving uncoupled single-cavity problems. Numerical experiments are presented to demonstrate the efficiency of the adaptive PML method.

AMS subject classifications: 65N30, 65N50

Key words: Uniaxial perfectly matched layer, multiple cavity scattering, adaptive finite element, a posteriori error estimate.

1 Introduction

We propose and study a uniaxial perfectly matched layer (PML) method for solving the multiple cavity scattering problems

$$\Delta u+k^2u=0 \qquad \text{in } D\cup\mathbb{R}^2_+, \tag{1.1a}$$

$$u=0 \qquad \text{on } \Gamma^c\cup S, \tag{1.1b}$$

$$[u]=\left[\frac{\partial u}{\partial x_2}\right]=0 \quad \text{on } \Gamma_D, \tag{1.1c}$$

$$\lim_{r=|x|\to\infty}\sqrt{r}\left(\frac{\partial u^s}{\partial r}-\mathbf{i}k_0u^s\right)=0, \tag{1.1d}$$

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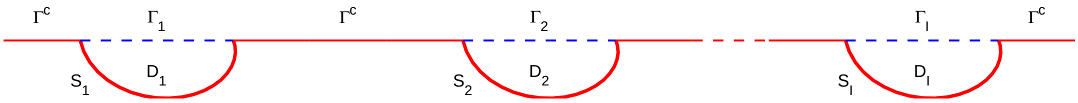

Figure 1: An illustration for the setting of multiple cavity scattering problem.

where u and u^s are the total field and the scattering field respectively and $\mathbf{i}$ is the imaginary unit. The radiation condition (1.1d) is imposed in the upper half space $\mathbb{R}^2_+$ where

$$\mathbb{R}^2_\pm = \{(x_1,\pm x_2)\,|\, x_1\in\mathbb{R},\, x_2>0\}, \qquad \Gamma=\partial\mathbb{R}^2_+ = \{(x_1,0)\,|\,x_1\in\mathbb{R}\}.$$

The region $D\subset\mathbb{R}^2_-$ consists of well-separated cavities, $D=\cup_{i=1}^I D_i$, which are bounded and have Lipschitz boundaries. For each cavity, say D_i, S_i denotes the cavity wall and Γ_i denotes the cavity aperture (see Fig. 1 for a simple illustration), namely,

$$\partial D_i = \Gamma_i\cup S_i, \qquad 1\le i\le I.$$

For convenience we let

$$S=\cup_{i=1}^I S_i, \qquad \Gamma_D=\cup_{i=1}^I\Gamma_i, \qquad \Gamma^c=\Gamma\backslash\Gamma_D.$$

The wavenumber $k(\boldsymbol{x})$ is assumed to be constant in the upper half plane

$$k=k_0 \qquad \text{in } \mathbb{R}^2_+, \tag{1.2}$$

but may be inhomogeneous inside cavities. Let $u_\pm$ denote the limits of u as the argument goes to Γ from above and below respectively. Then the jump of u across Γ is defined by

$$[u]=u_+-u_- \quad \text{on } \Gamma.$$

The scattering of cavities in the infinite ground plane is of great importance for its industrial and military applications. There are plenty of papers that study the scattering problems by cavities both in the engineering community and the mathematical community. In [21, 25], Jin et al. studied high-order finite element approximations to the scattering problem by deep cavities. Based on Fourier's transform, Ammari et al. [1, 2], Bao and Sun [6], Van and Wood [29] studied nonlocal transparent boundary conditions on the open aperture of the cavity. A mode matching method was proposed by Bao et al. [5, 8] for solving electromagnetic scattering problem by large cavities. For scattering problems by overfilled cavities, one can not restrict the computational domain to cavities any more. Wood [31] and Li et al. [24] introduced an artificial boundary condition on a semicircle or hemisphere and developed numerical methods for dealing with the scattering by overfilled cavities. We also refer to [4, 16, 18, 19, 22, 30, 32–34] and the references therein for various numerical investigations into cavities scattering problems, such as finite difference method, finite element method, boundary element method and hybrid methods.

However, one usually deals with multiple scattering problems in practical applications, namely, the scattering by multiple well-separated obstacles or cavities. In these cases, one can not truncate the exterior domain simply by a large domain which comprises all material inhomogeneities. Otherwise, the numerical approximation will leads to a massive system of algebraic equations and will be very hard to solve. One natural idea is to treat the obstacles or cavities separately such that each scatterer is surrounded by one respective truncation boundary close to it. We refer to Grote and Kirsch [17] and Jiang and Zheng [20] for multiple obstacle scattering problems, where the scattering field is split into the addition of single-obstacle scatting waves. Li and Wood [23] studied the transparent boundary condition of multiple cavity scattering problems. By using the total field, the global transparent boundary condition is first introduced on all apertures. Then the multiple cavity scattering problem is transformed into a coupled system of boundary value problems of the Helmholtz equation, each of which is restricted to one cavity.

The main theme of this paper is to study the PML method for (1.1). The perfectly matched layer (PML) method, which was first proposed by Bérenger [9], is an efficient technique for solving the wave propagating problems. Various constructions of PML absorbing layers have been proposed and studied in the literature (cf., e.g., [27, 28] for the reviews). The basic idea of the PML method is to surround the computational domain by a layer of finite thickness with a specially designed model medium that absorbs all the waves propagating from inside the computational domain. For practical applications, Chen and coauthors [11–14] proposed the adaptive PML method for solving acoustic and electromagnetic scattering problems. For multiple cavity scattering problems, one can not simply surround each cavity by a PML and solve the truncated problems individually. The total field consists of the incident waves and the scattering waves by all cavities. The scattering waves by one cavity may be incoming waves for all other cavities and can not be absorbed by the PML. To overcome this difficulty, we decompose the scattering problem (1.1) into a coupled system of I single-cavity scattering problems by using the integral representation of the field in the upper half plane. The single-cavity scattering problems are coupled by Dirichlet-to-Neumann (DtN) maps on the apertures. Then we construct the PML for each single cavity scattering problem and obtain a system of PML problems which are coupled by the DtN operators. We proved that the solution of the PML problem converges exponentially to the exact solution of (1.1) as either the thickness of the layers or the medium properties increase. For the conforming finite element approximation of the PML problem, we propose an APML algorithm based on reliable a posteriori error estimates.

Because of the DtN operators, the stiffness matrix has dense blocks for the finite element approximation of each single-cavity scattering problem. We propose a Block Gauss-Seidel method in the APML algorithm such that the stiffness matrices of the APML method are sparse and independent of the DtN operators. The computational complexity for solving the coupled system is proportional to that for solving I uncoupled single-cavity scattering problems. In the last section, we present some numerical experiments to demonstrate the efficiency of the APML method. Our numerical results show that the approximation error and the a posteriori error estimate decrease quasi-optimally as the number of degrees of freedom increase. All through this paper, the vector-valued quantities will be denoted by boldface symbols, such as $\boldsymbol{x}=(x_1,x_2)$.

2 Decomposition of the multiple scattering problem

In this section, we shall decompose the multiple scattering problem into a system of single-cavity problems by using the integral representation on the apertures. Let

$$u^i=e^{\mathbf{i}k_0(x_1\sin\theta-x_2\cos\theta)}$$

be the incident field on the cavity from above, and

$$u^r=-e^{\mathbf{i}k_0(x_1\sin\theta+x_2\cos\theta)}$$

be the reflective field, where $-\pi/2<\theta<\pi/2$ denotes the incident angle. Denote the scattering field by $u^s=u-u^i-u^r$. Then the scattering problem (1.1) can be rewritten into the partial differential equation for u^s

$$\Delta u^s+k^2u^s=f \qquad \text{in } D\cup\mathbb{R}^2_+, \tag{2.1a}$$
$$u^s=g \qquad \text{on } \Gamma^c\cup S, \tag{2.1b}$$
$$[u^s]=\left[\frac{\partial u^s}{\partial x_2}\right]=0 \quad \text{on } \Gamma_D, \tag{2.1c}$$
$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u^s}{\partial r}-\mathbf{i}k_0u^s\right)=0, \tag{2.1d}$$

where $f=(k_0^2-k^2)(u^i+u^r)$ and $g=-(u^i+u^r)$. From (1.2) we know that

$$\operatorname{supp}(f)\subset\mathbb{R}^2_-, \qquad \operatorname{supp}(g)\subset S. \tag{2.2}$$

The half-plane Green function of the Helmholtz equation reads

$$G(\boldsymbol{x},\boldsymbol{y})=\frac{\mathbf{i}}{4}\left[H_0^{(1)}(k_0|\boldsymbol{x}-\boldsymbol{y}|)-H_0^{(1)}(k_0|\boldsymbol{x}-\boldsymbol{y}'|)\right], \tag{2.3}$$

where $H_0^{(1)}(z)$ is the zero-order Hankel function of the first kind and $\boldsymbol{x}'=(x_1,-x_2)$ is the image of point $\boldsymbol{x}=(x_1,x_2)$ with respect to the horizontal axis. Let $\delta_{\boldsymbol{y}}(\boldsymbol{x})$ be the Dirac source at $\boldsymbol{y}\in\mathbb{R}^2_+$,

$$\delta_{\boldsymbol{y}}(\boldsymbol{x})=\delta(|x_1-y_1|)\delta(|x_2-y_2|).$$

Then $G(\boldsymbol{x},\boldsymbol{y})$ satisfies

$$\Delta_x G(\boldsymbol{x},\boldsymbol{y})+k_0^2G(\boldsymbol{x},\boldsymbol{y})=-\delta_{\boldsymbol{y}}(\boldsymbol{x}) \qquad \text{in } \mathbb{R}^2_+, \tag{2.4a}$$
$$G(\boldsymbol{x},\boldsymbol{y})=0 \qquad \text{on } \Gamma, \tag{2.4b}$$
$$\lim_{r=|x|\to\infty}\sqrt{r}\left[\frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial r}-\mathbf{i}k_0G(\boldsymbol{x},\boldsymbol{y})\right]=0. \tag{2.4c}$$

Theorem 2.1. *The scattering problem* (2.1) *admits a unique solution* $u^s\in H^1_{\rm loc}(\mathbb{R}^2_+\cup D)$. *Furthermore,* u^s *can be split into the addition of outgoing waves*

$$u^s=\sum_{i=1}^{I}u_i \qquad in\ \mathbb{R}^2_+, \tag{2.5}$$

where u_i *solves the scattering problem in the upper half-plane*

$$\Delta u_i+k^2u_i=f \quad in\ \mathbb{R}^2_+, \tag{2.6a}$$
$$u_i=u^s \quad on\ \Gamma_i, \qquad u_i=0 \quad on\ \Gamma_i^c:=\Gamma\backslash\Gamma_i, \tag{2.6b}$$
$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u_i}{\partial r}-\mathbf{i}k_0u_i\right)=0. \tag{2.6c}$$

Proof. First we refer to [31] for the unique existence of the solution.

Now we are going to prove the decomposition of u^s into the addition of single-cavity scattering solutions. Using (2.1)-(2.2), (2.4), and the formula of integral by part, we find that, for any $\boldsymbol{x}\in\mathbb{R}^2_+$,

$$\begin{aligned}
u^s(\boldsymbol{x})&=\int_{\mathbb{R}^2_+}u^s(\boldsymbol{y})\delta_{\boldsymbol{x}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\\
&=-\int_{\mathbb{R}^2_+}u^s(\boldsymbol{y})\left[\Delta_y G(\boldsymbol{x},\boldsymbol{y})+k^2G(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}\boldsymbol{y}\\
&=\int_\Gamma\left[\frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial y_2}u_s(\boldsymbol{y})-\frac{\partial u^s(\boldsymbol{y})}{\partial y_2}G(\boldsymbol{x},\boldsymbol{y})\right]\mathrm{d}s_{\boldsymbol{y}}.
\end{aligned}$$

Since $G(\boldsymbol{x},\cdot)$ vanishes on Γ and u^s vanishes on Γ^c, the above representation can be written as

$$u^s=\sum_{i=1}^{I}u_i, \qquad u_i(\boldsymbol{x})=\int_{\Gamma_i}\frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial y_2}u^s(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}} \quad \forall\boldsymbol{x}\in\mathbb{R}^2_+. \tag{2.7}$$

By (2.4), it is easy to see that u_i satisfies the Helmholtz equation (2.6a) and the radiation condition (2.6c). Notice the facts that

$$G(\boldsymbol{x},\boldsymbol{y})=0 \quad \forall\boldsymbol{x}\in\Gamma,\boldsymbol{y}\neq\boldsymbol{x} \qquad \text{and} \qquad u^s=0 \quad \text{on } \Gamma^c.$$

This indicates the boundary conditions in (2.6b). ☐

Now we define the Dirichlet-to-Neumann (DtN) operator T_i: $H^{\frac12}(\Gamma_i)\mapsto H^{-\frac12}(\Gamma_i)$ as follows

$$T_i(\xi):=\int_{\Gamma_i}\frac{\partial^2G(\boldsymbol{x},\boldsymbol{y})}{\partial x_2\partial y_2}\xi(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}} \qquad \forall\xi\in H^{\frac12}(\Gamma_i). \tag{2.8}$$

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From [23, Lem. 2.1], T_i is well-defined and continuous. And from (2.6b) we find that

$$T_i(u_j)(\boldsymbol{x})=0 \qquad \forall \boldsymbol{x}\in\Gamma_i, \quad j\neq i.$$

This indicates that for any $\boldsymbol{x}\in\Gamma_i$,

$$\frac{\partial u_i}{\partial x_2}(\boldsymbol{x})=\int_{\Gamma_i}\frac{\partial^2 G(\boldsymbol{x},\boldsymbol{y})}{\partial x_2\partial y_2}u^s(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}}=\sum_{j=1}^{I}T_i(u_j)(\boldsymbol{x})=T_i(u_i)(\boldsymbol{x}), \tag{2.9}$$

$$\frac{\partial u^s}{\partial x_2}(\boldsymbol{x})=\sum_{j=1}^{I}\frac{\partial u_j}{\partial x_2}(\boldsymbol{x})=\frac{\partial u_i}{\partial x_2}(\boldsymbol{x})+\sum_{\substack{j=1\\ j\neq i}}^{I}T_j(u_j)(\boldsymbol{x}). \tag{2.10}$$

Let the extension of u_i into D_i be defined as follows

$$\tilde{u}_i=\begin{cases} u_i & \text{in } \mathbb{R}^2_+,\\ u^s & \text{in } D_i. \end{cases}$$

In the rest of the paper, we shall refer to $\tilde{u}_i$ for the scattering solution by the aperture Γ_i, or formally by the cavity D_i. In view of the jump conditions in (2.1c), $\tilde{u}_i$ satisfies the following interface problems

$$\Delta\tilde{u}_i+k^2\tilde{u}_i=f \quad \text{in } D_i\cup\mathbb{R}^2_+, \tag{2.11a}$$

$$\tilde{u}_i=g \quad \text{on } S_i, \qquad u_i=0 \quad \text{on } \Gamma_i^c, \tag{2.11b}$$

$$[\tilde{u}_i]=0, \qquad \left[\frac{\partial\tilde{u}_i}{\partial x_2}\right]=-\sum_{j\neq i}T_j(\tilde{u}_j) \quad \text{on } \Gamma_i, \tag{2.11c}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial\tilde{u}_i}{\partial r}-\mathbf{i}k_0\tilde{u}_i\right)=0. \tag{2.11d}$$

3 Uniaxial PML method for single-cavity scattering problems

Notice that (2.11) is a system of single-cavity scattering problem with interface conditions. In this section we shall focus on the uniaxial PML method for this single scattering problem. For convenience in notation, we omit the subscripts and rewrite (2.11) into the following problem:

$$\Delta w+k^2 w=f \qquad \text{in } D\cup\mathbb{R}^2_+, \tag{3.1a}$$

$$w=g \quad \text{on } \Gamma^c\cup S, \tag{3.1b}$$

$$[w]=0, \ \left[\frac{\partial w}{\partial x_2}\right]=-q \quad \text{on } \Gamma_D, \tag{3.1c}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial w}{\partial r}-\mathbf{i}k_0 w\right)=0, \tag{3.1d}$$

where D stands for a single cavity, $\Gamma_D\subset\Gamma$ stands for the aperture, $S=\partial D\backslash\Gamma_D$ stands for the cavity wall, and $\Gamma^c=\Gamma\backslash\Gamma_D$. Furthermore, we assume that $q\in H^{-\frac{1}{2}}(\Gamma_D)$ and

$$\begin{aligned}&\operatorname{supp}(g)\subset\bar{S}\text{ such that }g\in H^{\frac{1}{2}}(S),\\&\operatorname{supp}(f)\subset\bar{D}\text{ such that }f\in L^2(D),\\&\operatorname{supp}(k-k_0)\subset\bar{D}\text{ such that }k-k_0\in L^\infty(D).\end{aligned}$$

3.1 A weak formulation

First we shall derive a weak formulation of (3.1). Let $H_0^{\frac{1}{2}}(\Gamma_D)$ be the closure of $C_0^\infty(\Gamma_D)$ under the norm $\|\cdot\|_{H^{\frac{1}{2}}(\Gamma_D)}$. Then for any $\xi\in H_0^{\frac{1}{2}}(\Gamma_D)$, its extension by zero to Γ^c yields a function $\tilde{\xi}\in H^{\frac{1}{2}}(\Gamma)$. Without confusion, $H_0^{\frac{1}{2}}(\Gamma_D)$ also denotes the subspace of $H^{\frac{1}{2}}(\Gamma)$ whose functions are supported in Γ_D.

To derive the weak formulation, we need the DtN operator T: $H_0^{\frac{1}{2}}(\Gamma_D)\mapsto H^{-\frac{1}{2}}(\Gamma_D)$ defined as follows: for any $\xi\in H_0^{\frac{1}{2}}(\Gamma_D)$, $T\xi:=\frac{\partial v}{\partial y_2}\big|_{\Gamma_D}$, where v solves the Dirichlet problem of Helmholtz equation in half upper plane

$$\begin{aligned}&\Delta v+k_0^2v=0\quad\text{in }\mathbb{R}_+^2,\\&v=\xi\quad\text{on }\Gamma_D,\qquad v=0\quad\text{on }\Gamma^c,\\&\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial v}{\partial r}-\mathbf{i}k_0v\right)=0.\end{aligned}$$

From (2.6) and (2.7), the operator T can be written as

$$T\xi(\boldsymbol{x})=\int_{\Gamma_D}\frac{\partial^2G(\boldsymbol{x},\boldsymbol{y})}{\partial x_2\partial y_2}\xi(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}},\tag{3.3}$$

which is well-defined and continuous (cf. e.g. [23, Lem. 2.1]).

Let $a:H^1(D)\times H^1(D)\to\mathbb{C}$ be the sesquilinear form

$$a(\varphi,\psi)=\int_D(\nabla\varphi\cdot\nabla\bar{\psi}-k^2\varphi\bar{\psi})-\langle T\varphi,\psi\rangle_{\Gamma_D},\tag{3.4}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_D}$ stands for the duality pairing between $H^{-\frac{1}{2}}(\Gamma_D)$ and $H_0^{\frac{1}{2}}(\Gamma_D)$. The scattering problem (3.1) is equivalent to the following weak formulation: Find $w\in H^1(D)$ such that $w=g$ on S and

$$a(w,\varphi)=-(f,\varphi)_D+\langle q,\varphi\rangle_{\Gamma_D}\qquad\forall\varphi\in H_S^1(D),\tag{3.5}$$

where $(\cdot,\cdot)_D$ denotes the L^2-inner product on $L^2(D)$ and

$$H_S^1(D)=\{\psi\in H^1(D):\psi=0\text{ on }S\}.$$

The existence of a unique solution of the scattering problem (3.5) is well-known (cf., e.g., [23], [33]). Then the general theory in Babuška and Aziz [3, Chap. 5] implies that there exists a constant $\mu>0$ such that the following inf-sup condition is satisfied

$$\sup_{0\neq\psi\in H_S^1(D)}\frac{|a(\varphi,\psi)|}{\|\psi\|_{H^1(D)}}\geq\mu\|\varphi\|_{H^1(D)},\quad\forall\varphi\in H_S^1(D).\tag{3.6}$$

3.2 The PML problem

Now we turn to the PML method for the single-cavity scattering problem. For convenience in notation, we let $\Gamma_D=\{(x_1,0):|x_1|<L\}$ be the aperture and define the domain for PML by (see Fig. 2)

$$\Omega^{\rm PML}=\{(x_1,x_2):|x_1|<L+d,\,0<x_2<d\}.$$

The wave-absorbing layer is defined by the complex coordinate stretching

$$\tilde{x}_1=x_1+\mathbf{i}\int_0^{x_1}\sigma(|t|-L)\mathrm{d}t,\qquad\tilde{x}_2=x_2+\mathbf{i}\int_0^{x_2}\sigma(t)\mathrm{d}t.$$

Let $\sigma_0>0$ be a constant and $m\geq0$ be an integer. The model medium property is defined by

$$\sigma(t)=0,\quad\text{if } t\leq0;\qquad\sigma(t)=\sigma_0\left(\frac{t}{d}\right)^m,\quad\text{if } t>0.\tag{3.7}$$

Here for simplicity we assume that the depths of the layer are equal in both directions. For convenience we write $\alpha_1(t)=1+\mathbf{i}\sigma(|t|-L)$ and $\alpha_2(t)=1+\mathbf{i}\sigma(t)$ such that

$$\tilde{x}_j=\int_0^{x_j}\alpha_j(t)\mathrm{d}t,\qquad j=1,2.$$

It is easy to see that the scattering field propagates as follows in the PML

$$w(\tilde{\boldsymbol{x}})=\int_\Gamma\frac{\partial G(\tilde{\boldsymbol{x}},\boldsymbol{y})}{\partial y_2}w(\boldsymbol{y})\mathrm{d}s_{\boldsymbol{y}}\qquad\forall\boldsymbol{x}\in\mathbb{R}_+^2.$$

Define $\tilde{w}(\boldsymbol{x}):=w(\tilde{\boldsymbol{x}})$ for $\boldsymbol{x}\in\Omega^{\rm PML}$, it is obvious that $\tilde{w}$ satisfies

$$\frac{\partial^2\tilde{w}}{\partial\tilde{x}_1^2}+\frac{\partial^2\tilde{w}}{\partial\tilde{x}_2^2}+k_0^2\tilde{w}=0\quad\text{in }\mathbb{R}_+^2,$$

which yields the desired PML equation in real coordinates

$$\nabla\cdot(A\nabla\tilde{w})+k_0^2J\tilde{w}=0\quad\text{in }\mathbb{R}_+^2,$$

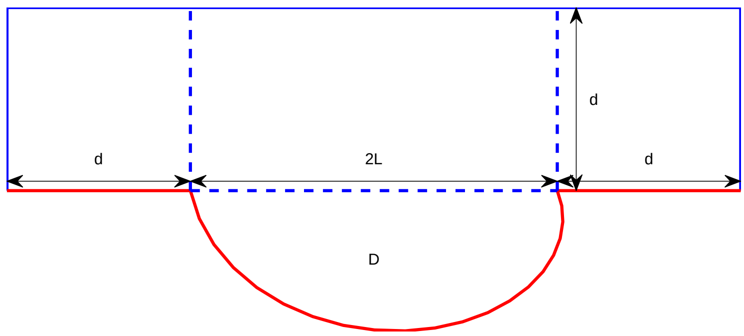


Figure 2: Setting of single-cavity scattering problem with PML layer

where

$$A(\boldsymbol{x})=\operatorname{diag}\left(\frac{\alpha_2(x_2)}{\alpha_1(x_1)},\frac{\alpha_1(x_1)}{\alpha_2(x_2)}\right),\qquad J(\boldsymbol{x})=\alpha_1(x_1)\alpha_2(x_2).$$

The PML solution $\hat{w}(\boldsymbol{x})$ in $\Omega=D\cup\Omega^{\mathrm{PML}}$ is defined by imposing homogeneous Dirichlet boundary condition on the outer boundary

$$\nabla\cdot(A\nabla\hat{w})+k^2J\hat{w}=f \qquad \text{in } \Omega, \tag{3.8a}$$

$$\hat{w}=g \qquad \text{on } \partial\Omega, \tag{3.8b}$$

$$[\hat{w}]=0,\quad \left[\frac{\partial\hat{w}}{\partial x_2}\right]=-q \qquad \text{on } \Gamma_D, \tag{3.8c}$$

where g vanishes on $\partial\Omega\backslash S$ (or $\partial\Omega^{\mathrm{PML}}\backslash\Gamma_D$). Now we define the approximate DtN operator $\hat{T}$: $H_0^{\frac{1}{2}}(\Gamma)\to H^{-\frac{1}{2}}(\Gamma)$ by $\hat{T}w=\frac{\partial\zeta}{\partial x_2}|_{\Gamma_D}$, where ζ is the solution of the PML problem in the layer

$$\nabla\cdot(A\nabla\zeta)+k_0^2J\zeta=0 \qquad \text{in } \Omega^{\mathrm{PML}}, \tag{3.9a}$$

$$\zeta=w \quad \text{on } \Gamma_D,\quad \zeta=0 \qquad \text{on } \Gamma^{\mathrm{PML}}, \tag{3.9b}$$

where $\Gamma^{\mathrm{PML}}=\partial\Omega^{\mathrm{PML}}\backslash\Gamma_D$. The spectral theory of compact operators implies that (3.9) has a unique solution for every real k_0 except possibly for a discrete set of values of k_0. The well-posedness of the PML problems with circular layer has been studied in [7,12]. For uniaxial PML methods, the stability estimates were proved in [10] for a special designed medium property and also in [15] for the piecewise constant case. In this paper we will not elaborate on this issue and simply make the following assumption

(**H1**) There exists a unique solution to the PML problem (3.9) in the layer.

Define the sesquilinear form $\hat{a}:H^1(D)\times H^1(D)\to\mathbb{C}$ by

$$\hat{a}(\varphi,\psi)=\int_D(\nabla\varphi\cdot\nabla\bar{\psi}-k^2\varphi\bar{\psi})-\langle\hat{T}\varphi,\psi\rangle_{\Gamma_D}. \tag{3.10}$$

Then the equivalent weak formulation of the PML problem (3.8) is: Find $\hat{w}\in H^1(D)$ such that $\hat{w}=g$ on S and

$$\hat{a}(\hat{w},\varphi)=-(f,\varphi)_D+\langle q,\varphi\rangle_{\Gamma_D} \qquad \forall \varphi\in H^1_S(D). \tag{3.11}$$

Theorem 3.1. *Let (H1) be satisfied. Let w be the solution of* (2.11)*. Then for sufficiently large $\sigma_0>0$, the PML problem* (3.8) *has a unique solution $\hat{w}\in H^1(\Omega)$. And there exists a constant $C_1>0$ depends on k_0, σ_0 and d at most polynomially such that*

$$\|w-\hat{w}\|_{H^1(D)}\le C_1 e^{-\gamma k_0\bar{\sigma}}\|\hat{w}\|_{H^{\frac{1}{2}}(\Gamma_D)},$$

where

$$\gamma=\frac{1}{\sqrt{1+(1+2L/d)^2}},\quad \bar{\sigma}=\int_0^d \sigma(t)\mathrm{d}t=\frac{\sigma_0 d}{m+1}.$$

Proof. The proof is similar to the arguments as in the proof of [14, Theorem 3.1]. For the sake of simplicity, we omit the details. □

3.3 Finite element approximation

In this subsection we introduce the finite element approximation of the PML problem (3.8). Let $b: H^1(\Omega)\times H^1(\Omega)\to\mathbb{C}$ be the sesquilinear form given by

$$b(\varphi,\psi)=\int_\Omega \big(A\nabla\varphi\cdot\nabla\bar{\psi}-k^2 J\varphi\bar{\psi}\big). \tag{3.12}$$

Then we propose another weak formulation of (3.8): Find $\hat{w}\in H^1(\Omega)$ such that $\hat{w}=g$ on S, $w=0$ on $\Gamma^{\rm PML}$, and

$$b(\hat{w},\psi)=-(f,\psi)_\Omega+\langle q,\psi\rangle_{\Gamma_D} \qquad \forall\psi\in H^1_0(\Omega). \tag{3.13}$$

Let $\mathcal{M}_h$ be a regular triangulation of Ω so that $\Omega=\cup_{K\in\mathcal{M}_h}K$. Let $V_h\subset H^1(\Omega)$ be the conforming linear finite element space over $\mathcal{M}_h$ and $\mathring{V}_h=V_h\cap H^1_0(\Omega)$. The finite element approximation to the PML problem (3.8) reads as follows: Find $w_h\in V_h$ such that $w_h=g_h$ on S, $w_h=0$ on $\Gamma^{\rm PML}$, and

$$b(w_h,\psi_h)=-(f,\psi_h)_\Omega+\langle q_h,\psi_h\rangle_{\Gamma_D} \qquad \forall\psi_h\in\mathring{V}_h, \tag{3.14}$$

where $g_h=I_h g\in V_h|_S$ is the canonical interpolation of g and q_h is some discrete approximation of q. By the general theory in [3], the existence of a unique solution of the discrete problem (3.14) and the finite element convergence analysis depend on the following inf-sup condition

$$\sup_{0\neq\psi_h\in\mathring{V}_h}\frac{|b(\varphi_h,\psi_h)|}{\|\psi_h\|_{H^1(\Omega)}}\ge\hat{\mu}\|\varphi_h\|_{H^1(\Omega)}, \quad \forall\varphi_h\in\mathring{V}_h, \tag{3.15}$$

where the constant $\hat{\mu}>0$ is independent of the finite element mesh size. Since the continuous problem (3.8a)-(3.8c) has a unique solution by Theorem 3.1, the sesquilinear form $b: H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{C}$ satisfies the continuous inf-sup condition, namely,

$$\sup_{0\neq\psi\in H_0^1(\Omega)} \frac{|b(\varphi_h,\psi)|}{\|\psi\|_{H^1(\Omega)}} \geq \mu_1\|\varphi_h\|_{H^1(\Omega)} \qquad \forall\varphi_h\in \mathring{V}_h.$$

Since the finite element space $\mathring{V}_h$ is dense in $H_0^1(\Omega)$ as $h\to 0$, the general argument of Schatz [26] implies that, for sufficiently small mesh size $h<h^*$, (3.15) holds for some $0<\hat{\mu}<\mu_1$. Here the threshold h^* depends on μ_1 and thus depends on the wavenumber k_0 and the PML parameters.

For any $K\in\mathcal{M}_h$ we denote by h_K its diameter. Let $\mathcal{B}_h$ be the collection of all sides in $\mathcal{M}_h$ which do not lie on $\partial\Omega$. For any $e\in\mathcal{B}_h$, h_e stands for its length. To derive the a posteriori error estimate for (3.14), we introduce the residual:

$$R_K := -f+\nabla\cdot(A\nabla w_h|_K)+k^2 J w_h|_K \tag{3.16}$$

for any $K\in\mathcal{M}_h$. For any interior side $e\in\mathcal{B}_h$ which is the common side of $K_1,K_2\in\mathcal{M}_h$, we define the jump residual across e:

$$J_e := -[A\nabla w_h]|_e\cdot\boldsymbol{n} = -A\nabla w_h|_{K_1}\cdot\boldsymbol{n}_1 - A\nabla w_h|_{K_2}\cdot\boldsymbol{n}_2, \tag{3.17}$$

if e does not lie on the interface Γ_D. Otherwise, we define the jump residual:

$$J_e := -[A\nabla w_h]|_e\cdot\boldsymbol{n}+q_h, \tag{3.18}$$

if e lies on Γ_D. Here $\boldsymbol{n}_i$ is the unit outer normal of ∂K_i restricted to e. Then the local error estimator η_K for any $K\in\mathcal{M}_h$ is defined as

$$\eta_K^2 := \|h_K R_K\|_{L^2(K)}^2 + \frac{1}{2}\sum_{e\subset\partial K\backslash\partial\Omega} h_e\|J_e\|_{L^2(e)}^2. \tag{3.19}$$

Using the similar arguments as in [14], we obtain the following a posteriori error estimate.

Theorem 3.2. *Let (H1) be satisfied. Let w be the solution of* (2.11) *and w_h be the solution of finite element problem* (3.14)*. Then for sufficiently large σ_0, there exists a constant $C_2>0$ which depends on k_0, σ_0 and d at most polynomially and depends on the minimal angle of the mesh such that*

$$\begin{aligned}\|w-w_h\|_{H^1(D)} \leq C_2\Bigg\{ &\|g-g_h\|_{H^{\frac{1}{2}}(S)} + \Big(\sum_{K\in\mathcal{M}_h}\eta_K^2\Big)^{\frac{1}{2}} \\ &+\|q-q_h\|_{H^{-\frac{1}{2}}(\Gamma_D)} + e^{-\gamma k_0\bar{\sigma}}\|w_h\|_{H^{\frac{1}{2}}(\Gamma_D)}\Bigg\}.\end{aligned}$$

4 Multiple cavity scattering problems

In this section we turn back to the multiple cavity scattering problem. Notice that the image of the DtN operator T_j is smooth on Γ_i for any $i \neq j$, the estimate for the Green function implies the following lemma.

Lemma 4.1. *There exists a constant $C>0$ only depending on k such that*

$$\|T_j(w)\|_{L^\infty(\Gamma_i)} \leq C d_{\min}^{-1} \|w\|_{H^1(D_j)} \qquad \forall w \in H^1(D_j),\ i \neq j,$$

where

$$d_{\min} = \min_{1\leq i<j\leq I} \mathrm{dist}(\Gamma_i, \Gamma_j).$$

Similar to the single cavity problem, each interface problem in (2.11) is equivalent to the following weak formulation: Find $\tilde{u}_i \in H^1(D_i)$ such that $\tilde{u}_i = g$ on S and

$$a_i(\tilde{u}_i, \psi) = -(f, \psi)_{D_i} + \langle q_i, \psi \rangle_{\Gamma_i} \qquad \forall \psi \in H^1_{S_i}(D_i), \tag{4.1}$$

where $q_i = \sum_{j\neq i} T_j(w_j)$, and the sesquilinear form a_i is defined by

$$a_i(\varphi, \psi) = \int_{D_i} (\nabla\varphi \cdot \nabla\bar{\psi} - k^2 \varphi\bar{\psi}) - \langle T_i \varphi, \psi \rangle_{\Gamma_i}.$$

The uniqueness of the scattering problems implies that there exist positive constants $\mu_1, \cdots, \mu_I$ such that

$$\sup_{0\neq\psi\in H^1_{S_i}(D_i)} \frac{|a(\varphi,\psi)|}{\|\psi\|_{H^1(D_i)}} \geq \mu_i \|\varphi\|_{H^1(D_i)} \qquad \forall \varphi \in H^1_{S_i}(D_i),\ 1\leq i\leq I. \tag{4.2}$$

4.1 Uniaxial PML method for multiple cavity scattering problems

Let $\Gamma_i = \{(x_1, 0) : |x_1 - c_i| < L_i\}$ and $\Omega_i^{\mathrm{PML}} = \{(x_1, x_2) : |x_1 - c_i| < L_i + d, 0 < x_2 < d\}$ be the corresponding PML layer on the cavity aperture. Then the wave-absorbing layer in Ω_i^{PML} is defined by the complex coordinate stretching $\boldsymbol{x} \to \tilde{\boldsymbol{x}}$:

$$\tilde{x}_1 = x_1 + \mathbf{i}\int_{c_i}^{x_1} \sigma(|t - c_i| - L_i)\,\mathrm{d}t, \qquad \tilde{x}_2 = x_2 + \mathbf{i}\int_0^{x_2} \sigma(t)\,\mathrm{d}t, \tag{4.3}$$

where σ is defined as in (3.7). Similarly we write $\alpha_{i,1}(t) = 1 + \mathbf{i}\sigma(|t - c_i| - L_i)$ and $\alpha_{i,2}(t) = 1 + \mathbf{i}\sigma(t)$. Then the coordinate stretching has the form

$$\tilde{x}_1 = c_i + \int_{c_i}^{x_1} \alpha_{i,1}(t)\,\mathrm{d}t, \quad \tilde{x}_2 = \int_0^{x_2} \alpha_{i,2}(t)\,\mathrm{d}t \qquad \forall \boldsymbol{x} \in \Omega_i^{\mathrm{PML}}.$$

The PML approximation to the multiple cavity scattering problem (2.11) reads

$$\nabla\cdot(A_i\nabla\hat{u}_i)+k^2J_i\hat{u}_i=f \quad \text{in } \Omega_i:=D_i\cup\Omega_i^{\rm PML}, \tag{4.4a}$$

$$\hat{u}_i=g \quad \text{on } S_i, \qquad \hat{u}_i=0 \quad \text{on } \Gamma_i^{\rm PML}, \tag{4.4b}$$

$$[\hat{u}_i]=0, \quad \left[\frac{\partial\hat{u}_i}{\partial x_2}\right]=-\sum_{j\neq i}T_j(\hat{u}_j) \quad \text{on } \Gamma_i, \tag{4.4c}$$

for all $1\le i\le I$, where $\Gamma_i^{\rm PML}:=\partial\Omega_i^{\rm PML}\backslash\Gamma_i$ and

$$A_i(\boldsymbol{x})=\operatorname{diag}\left(\frac{\alpha_{i,2}(x_2)}{\alpha_{i,1}(x_1)},\frac{\alpha_{i,1}(x_1)}{\alpha_{i,2}(x_2)}\right), \quad J_i(\boldsymbol{x})=\alpha_{i,1}(x_1)\alpha_{i,2}(x_2) \qquad \forall\boldsymbol{x}\in\Omega_i.$$

Theorem 4.1. *Let (H1) be satisfied. Let $\tilde{u}_i, 1\le i\le I$ be the solutions to* (2.11)*. Then for sufficiently large σ_0 and $d_{\min}$, the PML problems* (4.4) *have unique solutions $\hat{u}_i$, $1\le i\le I$, and there holds that*

$$\sum_{i=1}^{I}\|\tilde{u}_i-\hat{u}_i\|_{H^1(D_i)}\le 2C_1e^{-\gamma k_0\bar{\sigma}}\sum_{i=1}^{I}\|\hat{u}_i\|_{H^{\frac{1}{2}}(\Gamma_i)}, \tag{4.5}$$

where C_1 is the constant in Theorem 3.1*.*

Proof. First we introduce the auxiliary scattering problems

$$\Delta W_i+k^2W_i=f \quad \text{in } D_i\cup\mathbb{R}^2_+, \tag{4.6a}$$

$$W_i=g \quad \text{on } S_i, \qquad W_i=0 \quad \text{on } \Gamma_i^c, \tag{4.6b}$$

$$[W_i]=0, \quad \left[\frac{\partial W_i}{\partial x_2}\right]=-\sum_{j\neq i}T_j(\hat{u}_j) \quad \text{on } \Gamma_i, \tag{4.6c}$$

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial W_i}{\partial r}-\mathbf{i}k_0W_i\right)=0. \tag{4.6d}$$

Then (4.4) is the PML approximation to (4.6). Theorem 3.1 implies that

$$\|W_i-\hat{u}_i\|_{H^1(D_i)}\le C_1e^{-\gamma k_0\bar{\sigma}}\|\hat{u}_i\|_{H^{\frac{1}{2}}(\Gamma_i)}. \tag{4.7}$$

Denote by $e_i=\tilde{u}_i-W_i$ and $q_i^e=\sum_{j\neq i}T_j(\tilde{u}_j-\hat{u}_j)$, we have

$$\begin{aligned}
&\Delta e_i+k^2e_i=0 \quad \text{in } D_i\cup\mathbb{R}^2_+,\\
&e_i=0 \quad \text{on } \Gamma_i^c\cup S_i,\\
&[e_i]=0, \quad \left[\frac{\partial e_i}{\partial x_2}\right]=-q_i^e \quad \text{on } \Gamma_i,\\
&\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial e_i}{\partial r}-\mathbf{i}k_0e_i\right)=0.
\end{aligned}$$

By the inf-sup condition (4.2) and the weak formulation (4.1), we obtain

$$\mu_i\|e_i\|_{H^1(D_i)}\leq \sup_{0\neq\psi\in H^1_{S_i}(D_i)}\frac{|a_i(e_i,\psi)|}{\|\psi\|_{H^1(D_i)}}=\sup_{0\neq\psi\in H^1_{S_i}(D_i)}\frac{|\langle q_i^e,\psi\rangle_{\Gamma_i}|}{\|\psi\|_{H^1(D_i)}}\leq C\|q_i^e\|_{H^{-\frac{1}{2}}(\Gamma_i)},$$

where we have used the fact that $\|\psi\|_{H^{\frac{1}{2}}(\Gamma_i)}\leq C\|\psi\|_{H^1(D_i)}$ (cf. e.g. [23, Lem. 2.3]). Then we deduce from Lemma 4.1 that

$$\|e_i\|_{H^1(D_i)}\leq \tilde{C}d_{\min}^{-1}\sum_{j\neq i}\|\tilde{u}_j-\hat{u}_j\|_{H^1(D_j)}. \tag{4.8}$$

Combining (4.7) and (4.8) and summing up the inequalities over i, we obtain

$$\sum_{i=1}^{I}\|\tilde{u}_i-\hat{u}_i\|_{H^1(D_i)}\leq C_1 e^{-\gamma k_0\bar{\sigma}}\sum_{i=1}^{I}\|\hat{u}_i\|_{H^{\frac{1}{2}}(\Gamma_i)}+\tilde{C}Id_{\min}^{-1}\sum_{i=1}^{I}\|\tilde{u}_i-\hat{u}_i\|_{H^1(D_j)}.$$

Since the cavities are well-separated, we finish the proof by letting $d_{\min}\geq 2\tilde{C}I$. □

4.2 Finite element approximation for multiple cavity scattering problems

Let $\mathcal{M}_i$ be the regular triangulation of Ω_i so that $\Omega_i=\cup_{K\in\mathcal{M}_i}K$. Let $V_i\subset H^1(\Omega_i)$ be the conforming linear finite element space over $\mathcal{M}_i$ and $\overset{\circ}{V}_i=V_i\cap H^1_0(\Omega_i)$. The finite element approximation to the PML problem (4.4) can be written as: Find $u_i^h\in V_i$ such that $u_i^h=g_h$ on S_i, $u_i^h=0$ on Γ_i^{PML}, and

$$b_i(u_i^h,\psi_h)=-(f,\psi_h)_{\Omega_i}+\langle q_i^h,\psi_h\rangle_{\Gamma_i}\qquad \forall\psi_h\in\overset{\circ}{V}_i, \tag{4.9}$$

where $q_i^h=\sum_{j\neq i}T_j(u_j^h)$ and

$$b_i(\varphi,\psi)=\int_{\Omega_i}\left(A_i\nabla\varphi\cdot\nabla\bar{\psi}-k^2J_i\varphi\bar{\psi}\right)\mathrm{d}x. \tag{4.10}$$

For any $K\in\mathcal{M}_i, 1\leq i\leq I$, the local error indicator is defined as

$$\eta_K^2:=h_K^2\|R_K\|^2_{L^2(K)}+\frac{1}{2}\sum_{e\subset\partial K\backslash\partial\Omega_i}h_e\|J_e\|^2_{L^2(e)}, \tag{4.11}$$

where R_K and J_e are defined as in (3.16)-(3.18) with A,J,w_h,q_h replaced by A_i,J_i,u_i^h,q_i^h.

Theorem 4.2. *Let (H1) be satisfied. Then for sufficiently large σ_0 and $d_{\min}$, there holds that*

$$\sum_{i=1}^{I}\|\tilde{u}_i-u_i^h\|_{H^1(D_i)}\ \leq\ 2C_2\sum_{i=1}^{I}\left\{\|g-g_h\|_{H^{\frac{1}{2}}(S_i)}+\left(\sum_{K\in\mathcal{M}_i}\eta_K^2\right)^{\frac{1}{2}}+e^{-\gamma k_0\bar{\sigma}}\|u_i^h\|_{H^{\frac{1}{2}}(\Gamma_i)}\right\},$$

where C_2 is the constant in Theorem 3.2.

http://dx.doi.org/10.4208/cicp.040215.280815a

Proof. By using Theorem 3.2 for each $1\le i\le I$, we have

$$\|\tilde{u}_i-u_i^h\|_{H^1(D_i)}\le C_2\Bigg\{\|g-g_h\|_{H^{\frac{1}{2}}(S_i)}+\Bigg(\sum_{K\in\mathcal{M}_i}\eta_K^2\Bigg)^{\frac{1}{2}} + \|q_i-q_i^h\|_{H^{-\frac{1}{2}}(\Gamma_i)}+e^{-\gamma k_0\bar{\sigma}}\|u_i^h\|_{H^{\frac{1}{2}}(\Gamma_i)}\Bigg\}.$$

On the other hand, Lemma 4.1 shows that

$$\|q_i-q_i^h\|_{H^{-\frac{1}{2}}(\Gamma_i)}\le \tilde{C}d_{\min}^{-1}\sum_{j\neq i}\|\tilde{u}_j-u_j^h\|_{H_1(D_j)}.$$

Then it completes the proof by letting $d_{\min}\ge 2\tilde{C}I$ and summing up all the inequalities over $1\le i\le I$. □

Remark 4.1. The PML formulation in this section can also be extended to solve the scattering problems by multiple overfilled cavities. We will report this part in a forthcoming paper.

5 Block Gauss-Seidel method

Since the PML problems (4.4) are coupled by the DtN operators T_j on the cavity apertures $\Gamma_1,\cdots,\Gamma_I$, we first introduce an iterative method to solve this coupled system: Given $(\hat{u}_1^{(0)},\cdots,\hat{u}_I^{(0)})$, find $(\hat{u}_1^{(n)},\cdots,\hat{u}_I^{(n)})$ for $n\ge 1$ successively such that

$$\nabla\cdot\left(A_i\nabla\hat{u}_i^{(n)}\right)+k^2J_i\hat{u}_i^{(n)}=f \quad \text{in } \Omega_i, \tag{5.1a}$$

$$\hat{u}_i^{(n)}=g \quad \text{on } S_i, \qquad \hat{u}_i^{(n)}=0 \quad \text{on } \Gamma_i^{\mathrm{PML}}, \tag{5.1b}$$

$$\left[\hat{u}_i^{(n)}\right]=0, \quad \left[\frac{\partial\hat{u}_i^{(n)}}{\partial x_2}\right]=-q_i^{(n)} \quad \text{on } \Gamma_i, \tag{5.1c}$$

for all $1\le i\le I$, where

$$q_i^{(n)}:=\sum_{j=1}^{i-1}T_j\left(\hat{u}_j^{(n)}\right)+\sum_{j=i+1}^{I}T_j\left(\hat{u}_j^{(n-1)}\right).$$

It is easy to see that all terms in $q_i^{(n)}$ have already been calculated in the last iteration or in the previous PML problems for $1\le j\le i-1$. Therefore, (5.1) is just the Dirichlet boundary value problem for one cavity like (3.8).

http://dx.doi.org/10.4208/cicp.040215.280815a

The finite element approximation to the decoupled systems (5.1) reads: Given $(u_1^{0,h},\cdots,u_I^{0,h})$, find $u_i^{n,h}\in V_i$ such that $u_i^{n,h}=g_h$ on S_i, $u_i^{n,h}=0$ on $\Gamma_i^{\rm PML}$, and

$$b_i(u_i^{n,h},\psi_h)=-(f,\psi_h)_{\Omega_i}+\langle q_i^{n,h},\psi_h\rangle_{\Gamma_i} \qquad \forall \psi_h\in \mathring{V}_i, \tag{5.2}$$

for all $1\le i\le I$, where

$$q_i^{n,h}=\sum_{j=1}^{i-1}T_j(u_j^{n,h})+\sum_{j=i+1}^{I}T_j(u_j^{n-1,h}). \tag{5.3}$$

For any $1\le i\le I$, we denote by $\mathcal{V}_i$ the sets of interior vertices of $\mathcal{M}_i$ and by $\mathcal{V}_i'$ the set of vertices on S_i. Let $\phi_p\in V_i$ be the nodal basis function belonging to vertex $p\in\mathcal{V}_i\cup\mathcal{V}_i'$. Then the discrete solution can be written as

$$u_i^h=\sum_{p\in\mathcal{V}_i}u_i^h(p)\phi_p+\sum_{p\in\mathcal{V}_i'}g_h(p)\phi_p.$$

Define the stiffness matrices, DtN matrices and the right hand side vectors as follows

$$\begin{aligned}
&\mathbb{B}_{i;p,p'}=b_i(\phi_{p'},\phi_p) \qquad \forall p,p'\in\mathcal{V}_i,\\
&\mathbb{T}_{i,j;p,p'}=(1-\delta_{i,j})\langle T_j(\phi_{p'}),\phi_p\rangle_{\Gamma_i} \qquad \forall p\in\mathcal{V}_i\cap\Gamma_i,\, p'\in\mathcal{V}_j\cap\Gamma_j,\\
&\mathbb{F}_{i;p}=-(f,\phi_p)_{\Omega_i}-\sum_{p'\in\mathcal{V}_i'}b_i(\phi_{p'},\phi_p)g_h(p') \qquad \forall p\in\mathcal{V}_i.
\end{aligned}$$

Let $\mathbb{W}_i^{(n)}$ be the unknown vector whose entries are $u_i^{n,h}(p)$ for all $p\in\mathcal{V}_i$. Then (5.2) is equivalent to the following block Gauss-Seidel method in matrix version

$$\mathbb{B}_i\mathbb{W}_i^{(n)}=\mathbb{F}_i+\sum_{j=1}^{i-1}\mathbb{T}_{i,j}\mathbb{W}_j^{(n)}+\sum_{j=i+1}^{I}\mathbb{T}_{i,j}\mathbb{W}_j^{(n-1)}. \tag{5.4}$$

Theorem 5.1. *Let (H1) be satisfied and let $\tilde{u}_i,\hat{u}_i^{(n)},1\le i\le I$ be the solutions to* (2.11) *and* (5.1) *respectively. Then for sufficiently large σ_0 and $d_{\min}$, we have*

$$\begin{aligned}
\sum_{i=1}^{I}\|\tilde{u}_i-\hat{u}_i^{(n)}\|_{H^1(D_i)}\le & 2C_1e^{-\gamma k_0\bar{\sigma}}\sum_{i=1}^{I}\|\hat{u}_i^{(n)}\|_{H^{\frac12}(\Gamma_i)}\\
&+\tilde{C}Id_{\min}^{-1}\sum_{i=1}^{I}\|\tilde{u}_i-\hat{u}_i^{(n-1)}\|_{H^1(D_i)},
\end{aligned} \tag{5.5}$$

where C_1 is the constant in Theorem 3.1*, $\tilde{C}$ depends on k and $D_1,\cdots,D_I$ but is independent of $d_{\min}$.*

Proof. As in the proof of Theorem 4.1, we first define the auxiliary problems

$$\begin{aligned}&\Delta W_i+k^2W_i=f \quad \text{in } D_i\cup\mathbb{R}^2_+,\\ &W_i=g \quad \text{on } S_i, \qquad W_i=0 \quad \text{on } \Gamma_i^c,\\ &[W_i]=0, \quad \left[\frac{\partial W_i}{\partial x_2}\right]=-q_i^{(n)} \quad \text{on } \Gamma_i,\\ &\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial W_i}{\partial r}-\mathbf{i}k_0W_i\right)=0.\end{aligned}$$

Then (5.1) is the PML approximation of the above scattering problems. Theorem 3.1 implies that

$$\|W_i-\hat{u}_i^{(n)}\|_{H^1(D_i)}\le C_1e^{-\gamma k_0\bar{\sigma}}\|\hat{u}_i^{(n)}\|_{H^{\frac{1}{2}}(\Gamma_i)}. \tag{5.6}$$

Write $e_i=\tilde{u}_i-W_i$. Then they satisfy

$$\begin{aligned}&\Delta e_i+k^2e_i=0 \qquad \text{in } D_i\cup\mathbb{R}^2_+,\\ &e_i=0 \qquad \text{on } \Gamma_i^c\cup S_i,\\ &[e_i]=0, \quad \left[\frac{\partial e_i}{\partial x_2}\right]=-q_i^e \qquad \text{on } \Gamma_i,\\ &\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial e_i}{\partial r}-\mathbf{i}k_0e_i\right)=0,\end{aligned}$$

where

$$q_i^e=\sum_{j=1}^{i-1}T_j\left(\tilde{u}_j-\hat{u}_j^{(n)}\right)+\sum_{j=i+1}^{I}T_j\left(\tilde{u}_j-\hat{u}_j^{(n-1)}\right).$$

Similar arguments as in the proof of Theorem 4.1 yields that

$$\|e_i\|_{H^1(D_i)}\le\tilde{C}d_{\min}^{-1}\left\{\sum_{j=1}^{i-1}\left\|\tilde{u}_j-\hat{u}_j^{(n)}\right\|_{H^1(D_j)}+\sum_{j=i+1}^{I}\left\|\tilde{u}_j-\hat{u}_j^{(n-1)}\right\|_{H^1(D_j)}\right\}.$$

Combing (5.6) with the above inequalities and summing them up over i, we obtain

$$\begin{aligned}\sum_{i=1}^{I}\left\|\tilde{u}_i-\hat{u}_i^{(n)}\right\|_{H^1(D_i)}\le& C_1e^{-\gamma k_0\bar{\sigma}}\sum_{i=1}^{I}\left\|\hat{u}_i^{(n)}\right\|_{H^{\frac{1}{2}}(\Gamma_i)}\\ &+\tilde{C}Id_{\min}^{-1}\sum_{i=1}^{I}\left(\left\|\tilde{u}_i-\hat{u}_i^{(n)}\right\|_{H^1(D_j)}+\left\|\tilde{u}_i-\hat{u}_i^{(n-1)}\right\|_{H^1(D_j)}\right).\end{aligned}$$

Then we completes the proof by letting $d_{\min}\ge 2\tilde{C}I$. $\square$

Remark 5.1. From Theorem 5.1, it is easy to see that the first term on the right hand side of (5.5) is negligible for sufficiently large σ_0. Thus for well-separated cavities, i.e. $d_{\min}\geq 2\tilde{C}I$, the iterative method will converge exponentially. We also remark that for large wavenumber k_0, the constant $\tilde{C}$ will be much smaller due to the estimate for the Green function and Lemma 4.1, and thus the iterative method will be much more efficient.

For any $K\in\cup_{i=1}^{I}\mathcal{M}_i$, let η_K be the local error indicator in (4.11) with u_i^h,q_i^h replaced by $u_i^{n,h},q_i^{n,h}$, where $q_i^{n,h}$ are defined in (5.3). Similar arguments as in the proof of Theorem 4.2 yield the following theorem.

Theorem 5.2. *Let (H1) be satisfied. Then for sufficiently large σ_0 and $d_{\min}$, we have*

$$\|\tilde{u}_i-u_i^{n,h}\|_{H^1(D_i)}\leq C_2\eta_i+\tilde{C}d_{\min}^{-1}\sum_{j\neq i}\|\tilde{u}_j-u_j^{n-1,h}\|_{H^1(D_j)},$$

where C_2 is the constant in Theorem 3.1, $\tilde{C}$ depends on k and $D_1,\cdots,D_I$ but is independent of $d_{\min}$, and

$$\eta_i=\|g-g_h\|_{H^{\frac{1}{2}}(S_i)}+\left(\sum_{K\in\mathcal{M}_i}\eta_K^2\right)^{\frac{1}{2}}+e^{-\gamma k_0\bar{\sigma}}\|u_i^{n,h}\|_{H^{\frac{1}{2}}(\Gamma_i)}.$$

Remark 5.2. In the hierarchy of adaptive mesh refinements, the coupled problem (4.9) is solved on the initial mesh. Thus we use $q_i^h=\sum_{j\neq i}T_j(u_j^h)$ in computing η_K.

Remark 5.3. Since $\|\tilde{u}_i-u_i^{n,h}\|$ decreases very quickly as n grows, Theorem 5.2 indicates that $\sum_{i=1}^{I}\eta_i$ provides an upper bound for $\sum_{i=1}^{I}\|\tilde{u}_i-u_i^{n,h}\|_{H^1(D_i)}$.

6 Numerical experiments

In this section, we shall propose the adaptive PML finite element algorithm for solving multiple scattering problems. Then we shall present two numerical examples to demonstrate the competitive performance of the adaptive PML method.

To fix the PML region, we first choose the PML parameters d,σ_0 such that the exponential decaying factor

$$\omega=e^{-\gamma k_0\bar{\sigma}_0}<10^{-6}, \tag{6.1}$$

which makes the PML error negligible compared with the finite element discretization error. Then the mesh is refined adaptively according to the a posteriori error estimate

$$\eta_h=\left(\sum_{K\in\mathcal{M}_h}\eta_K^2\right)^{\frac{1}{2}},\qquad \eta_{\max}=\max_{K\in\mathcal{M}_h}\eta_K.$$

On each mesh, the coupled system is solved by using the block Gauss-Seidel method. Now we present the adaptive uniaxial PML (APML) algorithm.

Algorithm 6.1. Given tolerance $\varepsilon > 0$.

1. Set the PML absorbing layers by choosing d and σ_0 such that $\omega < 10^{-6}$.
2. Set the computational domains Ω_i and generate an initial mesh $\mathcal{M}_i$ over Ω_i for each $1 \le i \le I$.
3. Solve the coupled problem (4.9) for $u_1^h, \cdots, u_I^h$ on $\mathcal{M}_h = \cup_{i=1}^I \mathcal{M}_i$ and compute the error indicators $\{\eta_K : K \in \mathcal{M}_h\}$.
4. While $\eta_h > \varepsilon$ do
   - refine the mesh $\mathcal{M}_h$ according to the strategy:

     if $\eta_K > \frac{1}{2}\eta_{\max}$, refine the element $K \in \mathcal{M}_h$;
   - solve the decoupled problems (5.2) for $u_1^h, \cdots, u_I^h$ on mesh $\mathcal{M}_h$;
   - compute the error indicators $\{\eta_K : K \in \mathcal{M}_h\}$;

   end while

Now we report two numerical examples to demonstrate the efficiency of the APML algorithm.

Example 6.1. We consider plane waves incident on the cavities vertically from above, i.e. $\theta = 0$. The cavities consist of three separated half circles. The first and second cavities are assumed to be empty and the third one is filled with homogeneous nonmagnetic medium, i.e.

$$k(x) = k_0 \quad \text{in } D_1 \cup D_2; \qquad k(x) = k_0\sqrt{1+\mathbf{i}} \quad \text{in } D_3,$$

with the wavenumber $k_0 = 4\pi$, i.e. the wavelength $\lambda = 0.5$.

We set the PML parameters by

$$d = 0.4, \quad \sigma_0 = 30, \quad m = 2.$$

The numerical results are obtained by using Algorithm 6.1. Fig. 3 shows the geometry of the cavities and the mesh with 3882 nodes after 18 adaptive mesh refinements. The mesh is rather coarse near the PML outer boundary due to the exponential decay of the solution in the layer.

One of the important quantities in the cavity scattering is the radar cross section (RCS). When the incident and observation directions are the same, it is called the backscatter RCS, which is defined by

$$\mathrm{RCS}(\phi) = \frac{4}{k_0}|P(\phi)|^2,$$

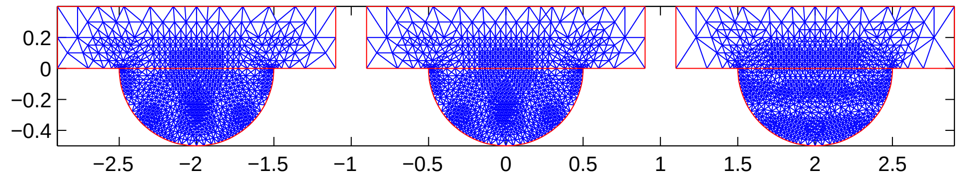
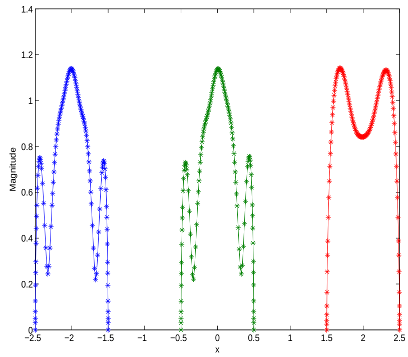


Figure 3: Three cavities and the mesh with 3882 nodes after 18 adaptive refinements (Example 6.1).

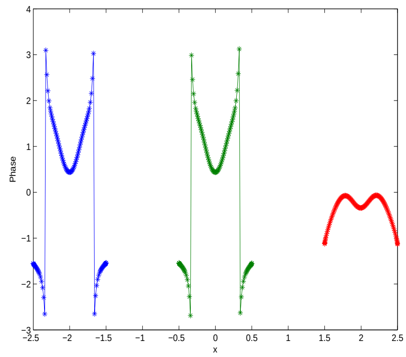
where ϕ is the observation angle and P is the far-field coefficient given by

$$P(\phi)=\frac{k_0}{2}\sin\phi\int_\Gamma u(x_1,0)e^{ik_0x_1\cos\phi}\mathrm{d}x_1.$$

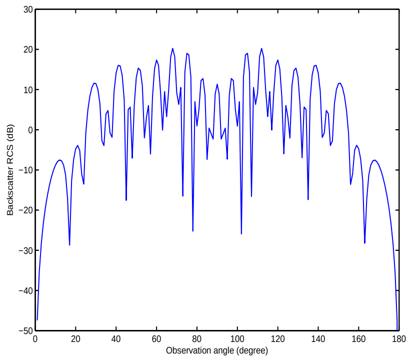
Fig. 4(a)-(c) shows the magnitude and the phase of the scattering field on the apertures



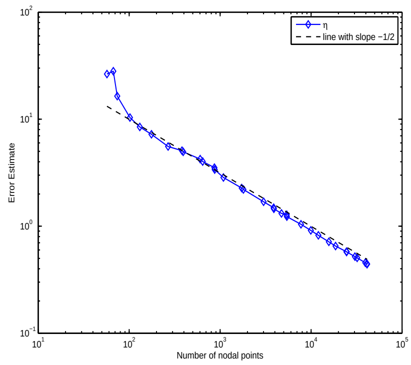
(a)



(b)



(c)



(d)

Figure 4: (a) The magnitude, (b) the phase of the scattering field on the apertures at normal incidence, (c) the backscatter RCS and (d) the a posteriori error estimate (Example 6.1).

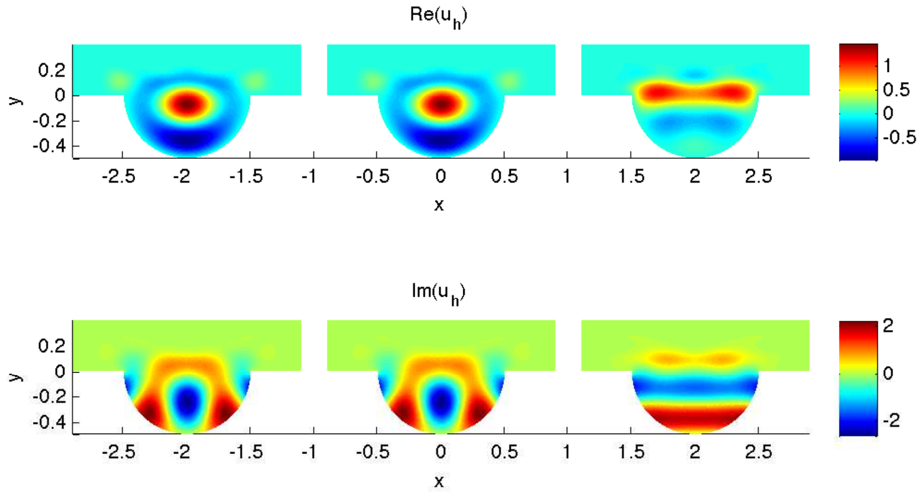


Figure 5: The real and imaginary parts of the computational solution (Example 6.1).

at normal incidence, as well as the backscatter RCS in dB for observation angles ranging from 0 to π. In addition, the $\log N_l$-$\log \eta_l$ curve is also shown in Fig. 4(d), where N_l is the number of nodes of the mesh $\mathcal{M}_l$ and η_l is the corresponding a posteriori error estimate. It is observed that the meshes and the associated numerical complexity are quasi-optimal, namely,

$$\eta_l \approx C N_l^{-\frac{1}{2}}$$

is valid asymptotically. Fig. 5 shows the real and imaginary parts of the computational solution on the finest mesh. It can be found that the solution decays rapidly away from the apertures and the efficiency of the PML method is demonstrated.

Example 6.2. The cavities consist of three separated polygonal domains, and all of them are assumed to be empty. The plane waves are incident on the cavities from above with the incident angle $\theta = \pi/4$. In this example, the wavenumber is set to be $k_0 = 10\pi$, i.e. the wavelength $\lambda = 0.2$.

We set the PML parameters by

$$d = 0.4, \quad \sigma_0 = 15, \quad m = 2.$$

Fig. 6 shows the geometry of the cavities and the mesh with 4994 nodes after 15 adaptive refinements. The mesh is adaptively refined according to the local error indicators computed with the solution.

Fig. 7(a)-(c) shows the magnitude and the phase of the scattering field on the apertures at incident angle $\pi/4$, as well as the backscatter RCS. Moreover, Fig. 7(d) shows the

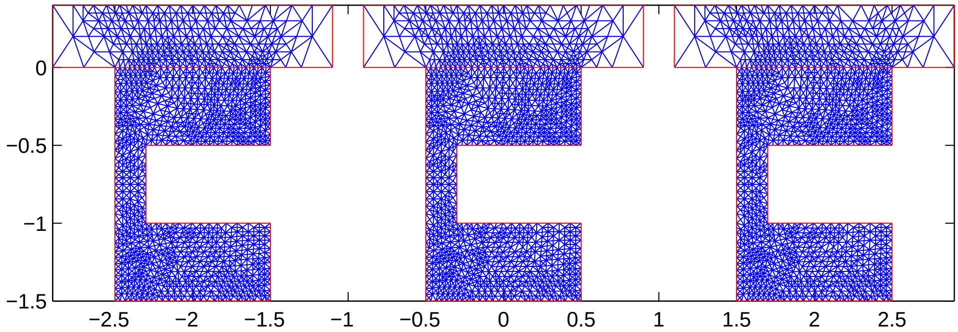


Figure 6: The geometry of the cavities and the mesh of 4994 nodes after 15 adaptive iterations (Example 6.2).

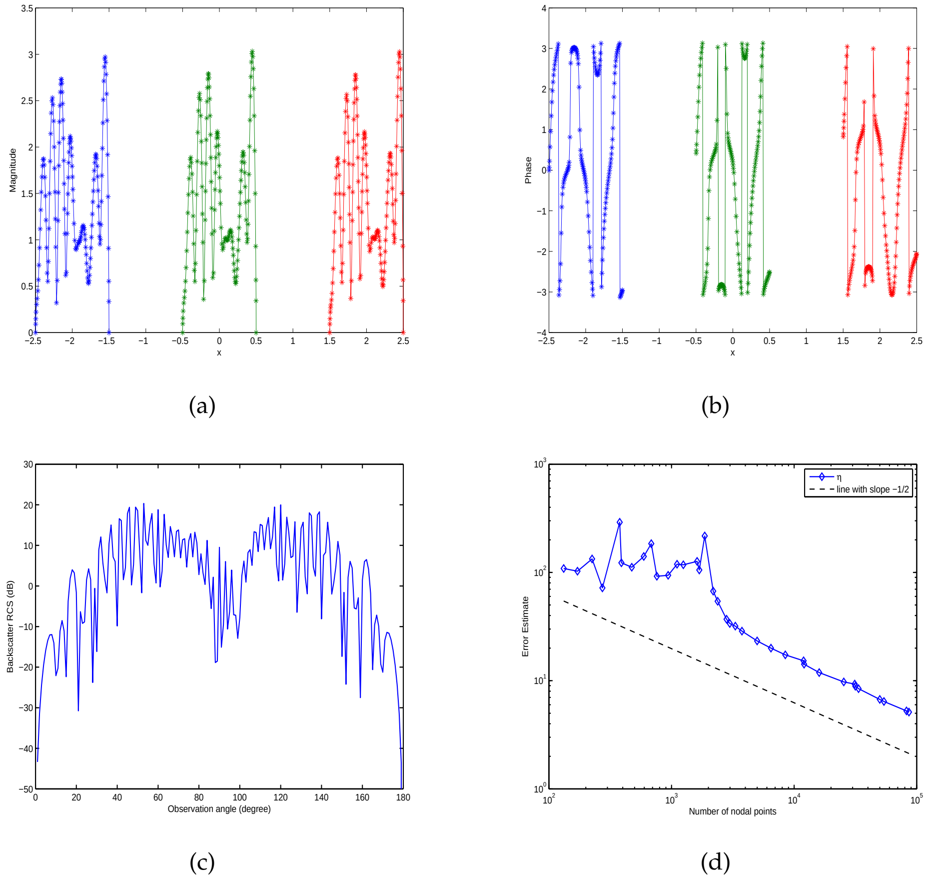


Figure 7: (a) The magnitude, (b) the phase of the scattering filed on the apertures at incident angle $\pi/4$, (c) the backscatter RCS and (d) the a posteriori error estimate (Example 6.2).

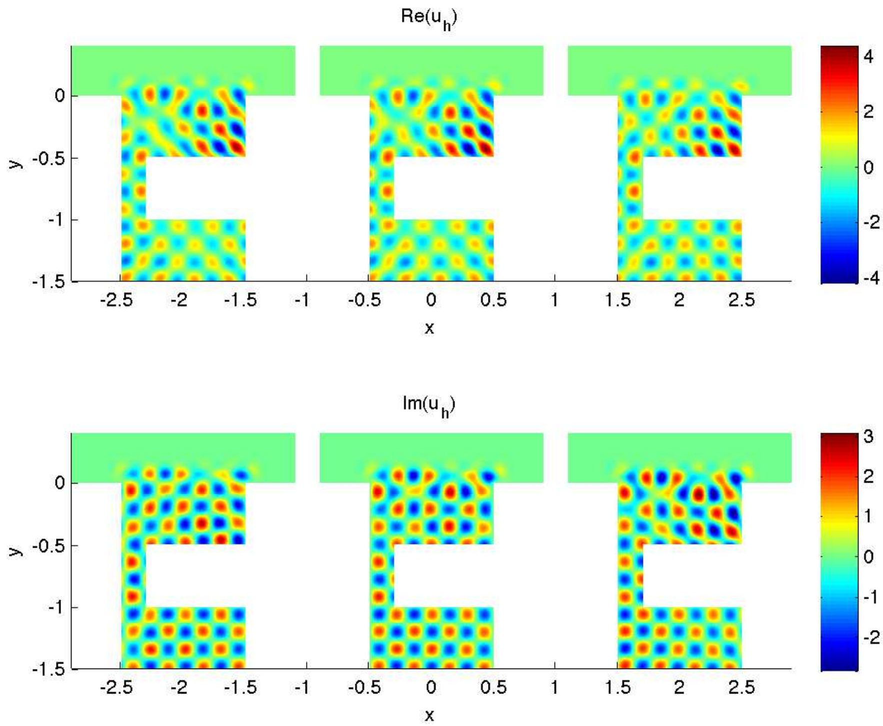


Figure 8: The real and imaginary parts of the computational solution (Example 6.2).

$\log N_l$-$\log\eta_l$ curve and verifies the quasi-optimality of the adaptive PML finite element method. Fig. 8 shows the real and imaginary parts of the solution at the finest mesh. It indicates that the solution decays rapidly away from the cavities.

Acknowledgments

The first author was supported in part by China NSF grants 91330202 and 11301089. The second author was supported in part by China NSF grants 11171334 and 91430215, by the Funds for Creative Research Groups of China 11321061, by National 863 Project of China under the grant 2012AA01A309, and by the National Magnetic Confinement Fusion Science Program 2015GB110003.

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