A POSTERIORI ERROR ANALYSIS AND ADAPTIVE METHODS FOR PARABOLIC PROBLEMS*

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Abstract We report the recent progress in deriving sharp a posteriori error estimates for linear and nonlinear parabolic problems. We show how to use special properties of a linear dual problem in non-divergence form with vanishing diffusion and strong advection to derive L^1L^1 norm estimate for the continuous casting problem. This estimate exhibits a mild explicit dependence on velocity. We next use a direct energy estimate method to develop an efficient and reliable a posteriori error estimator for linear parabolic equations which does not depend on any regularity assumption on the underlying elliptic operator. A convergent adaptive algorithm with variable time-step sizes and space meshes is proposed and studied which, at each time step, delays the mesh coarsening until the final iteration of the adaptive procedure, allowing only mesh and time-step size refinements before. The key ingredient in the convergence analysis is a new coarsening strategy.

Keywords: A posteriroi error estimates, parabolic, finite element

1. Introduction

A posteriori error estimates are computable quantities in terms of the discrete solution and data that measure the actual discrete errors without the knowledge of limit solutions. They are essential in designing algorithms for mesh and time-step size modification which equi-distribute the computational effort and optimize the computation. Ever since the pioneering work of Babuška and Rheinboldt [3], the adaptive finite el-

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ement methods based on a posteriori error estimates have become a central theme in scientific and engineering computations. There are considerable efforts in the literature devoted to the development of efficient adaptive algorithms for various linear and nonlinear parabolic partial differential equations. In particular, a posteriori error estimates are derived in Bieterman and Babuška [1], [2] and Moore [14] for one dimensional and in Eriksson and Johnson [12], [13], Verfürth [20] for multi-dimensional linear and mildly nonlinear parabolic problems. Efficient adaptive procedures based on a posteriori error estimates are also developed in Chen and Dai [4], Chen, Nochetto and Schmidt [9], Nochetto, Schmidt and Verdi [16] for solving nonlinear partial differential equations arising from physical and industrial processes.

The continuous casting problem is a convection-dominated nonlinearly degenerate diffusion problem. It is discretized implicitly in time via the method of characteristics, and in space via continuous piecewise linear finite elements. The first objective of this paper is to show how to derive an a posteriori error estimate for the L^1L^1 norm of temperature which exhibits a mild explicit dependence on velocity. The analysis is based on special properties of a linear dual problem in non-divergence form with vanishing diffusion and strong advection.

The main tool in deriving a posteriori error estimates in [4], [9], [12], [13], [16] is the analysis of linear dual problems of the corresponding *error* equations. The derived a posteriori error estimates, however, depend on the H^2 regularity assumption on the underlying elliptic operator. The next objective of the paper is to remove such a rather restrictive assumption and derive an a posteriori error estimate which bounds the full energy error of the approximate solution for a linear parabolic problem with discontinuous coefficients. Moreover, a local lower bound is proved for the associated local a posteriori error estimator. The method to prove the upper bound is a direct energy estimate argument which has been used in Chen and Nochetto [8] for elliptic obstacle problem and in Chen, Nochetto and Schmidt [10] for the phase relaxation model, a system of one parabolic equation coupled with one variational inequality. Based on this error estimator, we develop a convergent adaptive algorithm with variable time-step sizes and space meshes which, at each time step, delays the mesh coarsening until the final iteration of the adaptive procedure, allowing only mesh and time-step size refinements before.

2. Sharp a posteriori error estimate for the continuous casting problem

2.1 The setting

Let the ingot occupy a cylindrical domain Ω with large aspect ratio. Let $0 < L < +\infty$ be the length of the ingot and $\Gamma \subset \mathbf{R}^d$ for $d = 1$ or 2 be its (polygonal) cross section. We show $\Omega = \Gamma \times (0, L)$ in Figure 1, and hereafter write $x = (y, z) \in \Omega$ with $y \in \Gamma$ and $0 < z < L$.

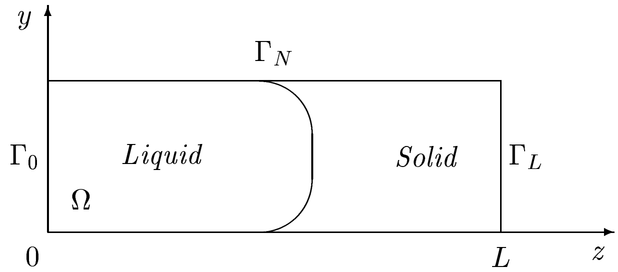

Figure 1: The domain Ω

We study the following *convection-dominated nonlinearly degenerate diffusion* problem

$$
\begin{aligned}
\partial_t u + v(t)\partial_z u - \Delta\theta = 0 \quad & \text{in} \quad Q_T, & (2.1)\\
\theta = \beta(u) \quad & \text{in} \quad Q_T, & (2.2)\\
\theta = g_D \quad & \text{on} \quad \Gamma_0 \times (0,T), & (2.3)\\
\partial_\nu\theta + p(\theta - \theta_{\mathrm{ext}}) = 0 \quad & \text{on} \quad \Gamma_N \times (0,T), & (2.4)\\
u(x,0) = u_0(x) \quad & \text{in} \quad \Omega, & (2.5)
\end{aligned}
$$

where $Q_T = \Omega \times (0, T)$,

$$\Gamma_0 = \Gamma \times \{0\}, \quad \Gamma_L = \Gamma \times \{L\}, \quad \Gamma_N = \partial\Gamma \times (0, L),$$

and $\theta + \theta_c$ is the absolute temperature, θ_c is the melting temperature, u is the enthalpy, $v(t) > 0$ is the extraction velocity of the ingot, ν is the unit outer normal to $\partial\Omega$, and θ_{ext} is the external temperature. The mapping $\beta : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and monotone increasing; since β is not strictly increasing, (2.1) is *degenerate* parabolic. The missing *outflow* boundary condition on Γ_L is unclear because the ingot moves at the casting speed and is cut shorter from time to time. It is thus evident that any standard boundary condition could only be an approximation. For simplicity, we impose a Dirichlet outflow condition

$$\theta = g_D < 0 \quad \text{on} \quad \Gamma_L \times (0, T). \qquad (2.6)$$

A Robin type boundary condition on Γ_L is also possible (see the discussion in [9]). It is convenient to denote by Γ_D the Dirichlet part of $\partial\Omega$, that is $\Gamma_0 \cup \Gamma_L$.

The importance of simulating and controlling the continuous casting process in the production of steel, copper, and other metals is recognized in industry. The extraction velocity $v(t)$ as well as the cooling conditions on the mold and water spray region are known to be decisive in determining material properties of the ingot. Avoiding excessive thermal stresses and material defects is an essential, and rather empirical, aspect of the continuous casting process.

The system (2.1)-(2.5) is a special case of general Stefan problems with prescribed convection Rulla [18]. An outflow Dirichlet condition together with an inflow Neumann condition is assumed in [18] to guarantee uniqueness of weak solutions; our more realistic boundary data (2.3) and (2.6) violate this restriction. Under the additional assumption that the free boundary does not touch the inflow boundary Γ_0, uniqueness of weak solutions to (2.1)-(2.5) and (2.6) is shown in Rodrigues and Yi [17].

We now introduce the fully discrete problem, which combines continuous piecewise linear finite elements in space with characteristic finite differences in time. In fact, we use the method of characteristics to discretize the convection. We denote by τ_n the n-th time step and set

$$t^n := \sum_{i=1}^{n} \tau_i, \quad \varphi^n(\cdot) := \varphi(\cdot, t^n)$$

for any function φ continuous in $(t^{n-1}, t^n]$. Let N be the total number of time steps, that is $t^N \geq T$.

Let $\mathbf{V}_0 = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ and $\mathbf{V}^*$ the dual space of $\mathbf{V}_0$. We denote by $\mathcal{M}^n$ a uniformly regular partition of Ω into simplexes. The mesh $\mathcal{M}^n$ is obtained by refinement/coarsening of $\mathcal{M}^{n-1}$, and thus $\mathcal{M}^n$ and $\mathcal{M}^{n-1}$ are *compatible*. Let $\mathbf{V}^n$ indicate the usual space of C^0 piecewise linear finite elements over $\mathcal{M}^n$ and $\mathbf{V}_0^n = \mathbf{V}^n \cap \mathbf{V}_0$. Let $\{x_k^n\}_{k=1}^{K^n}$ denote the interior nodes of $\mathcal{M}^n$. Let $I^n : C(\bar{\Omega}) \to \mathbf{V}^n$ be the usual Lagrange interpolation operator, namely $(I^n\varphi)(x_k^n) = \varphi(x_k^n)$ for all $1 \leq k \leq K^n$. Finally, let the discrete inner products $\langle\cdot,\cdot\rangle^n$ and $\langle\!\langle\cdot,\cdot\rangle\!\rangle_E^n$ be the sum over $S \in \mathcal{M}^n$ of the element scalar products

$$\langle\varphi,\chi\rangle_S^n = \int_S I^n\langle\varphi\chi\rangle dx, \quad \langle\!\langle\varphi,\chi\rangle\!\rangle_S^n = \int_{S\cap E} I^n(\varphi\chi) d\sigma,$$

for any piecewise uniformly continuous functions φ, χ. The discrete initial enthalpy $U^0 \in \mathbf{V}^0$ is defined at nodes x_k^0 of $\mathcal{M}^0 = \mathcal{M}^1$ to be

$$U^0(x_k^0) := u_0(x_k^0) \quad \forall\ x_k^0 \in \Omega \backslash F_0, \quad U^0(x_k^0) := 0 \quad \forall\ x_k^0 \in F_0.$$

Hence, U^0 is easy to evaluate in practice.

Discrete Problem. *Given $U^{n-1}, \Theta^{n-1} \in \mathbf{V}^{n-1}$, then $\mathcal{M}^{n-1}$ and τ^{n-1} are modified as described below to get $\mathcal{M}^n$ and τ_n and thereafter $U^n, \Theta^n \in \mathbf{V}^n$ computed according to the following prescription*, for any $\varphi \in \mathbf{V}_0^n$,

$$\Theta^n = I^n \beta(U^n), \quad \Theta^n - I^n g_D^n \in \mathbf{V}_0^n, \quad \bar{U}^{n-1} := U^{n-1}(\bar{x}^{n-1}),$$

$$\frac{1}{\tau_n}\langle U^n - I^n \bar{U}^{n-1}, \varphi\rangle^n + \langle \nabla \Theta^n, \nabla \varphi\rangle + \langle\!\langle p^n(\Theta^n - \theta_{\mathrm{ext}}^n), \varphi\rangle\!\rangle_{\Gamma_N}^n = 0,$$

where $\bar{x}^{n-1} = x - v^{n-1}\tau_n e_z$.

In view of the constitutive relation $\Theta^n = I^n\beta(U^n)$ being enforced only at the nodes, and the use of mass lumping, the discrete problem yields a monotone operator in $\mathbf{R}^{K^n}$ which is easy to implement and solve by nonlinear SOR [16], for example.

2.2 A posteriori error analysis

The a posteriori error analysis depends on the parabolic duality argument which we now discuss. We consider a linear backward parabolic problem in non-divergence form, which can be viewed as the adjoint formal derivative of (2.1). For any $U \in BV(0,T; L^2(\Omega))$, we define

$$b(x,t) = \begin{cases} \dfrac{\beta(u) - \beta(U)}{u - U} & \text{if } u \neq U, \\ \beta_1 & \text{otherwise.} \end{cases} \tag{2.7}$$

We assume that $\beta(s) = 0$ for $s \in [0,\lambda]$ and $0 < \beta_1 \leq \beta'(s) \leq \beta_2$ for a.e. $s \in \mathbf{R}\backslash[0,\lambda]$; $\lambda > 0$ is the latent heat. Thus $0 \leq b(x,t) \leq \beta_2$, for a.e. $(x,t) \in Q_T$. Let $b_\delta \in C^2(\bar{Q}_T)$ be a regularization of b satisfying

$$b_\delta \geq \delta > 0, \quad 0 \leq b_\delta - b \leq \delta \quad \text{a.e. in } Q_T, \tag{2.8}$$

where $0 < \delta \leq 1$ is a parameter to be chosen later. For arbitrary $t_* \in (0,T]$ and $\chi \in L^\infty(Q_T)$, let ψ be the solution of the following linear backward parabolic problem

$$\partial_t \psi + v(t)\partial_z \psi + b_\delta \Delta \psi = -b^{1/2}\chi \quad \text{in} \quad \Omega \times (0,t_*), \tag{2.9}$$

$$\psi = 0 \quad \text{on} \quad \Gamma_D \times (0,t_*), \tag{2.10}$$

$$\partial_\nu \psi + p\psi = 0 \quad \text{on} \quad \Gamma_N \times (0,t_*), \tag{2.11}$$

$$\psi(x,t_*) = 0 \quad \text{in} \quad \Omega. \tag{2.12}$$

We set $Q_* = \Omega \times (0,t_*)$. Existence of a unique solution $\psi \in W_q^{2,1}(Q_*)$ for any $q \geq 2$ of (2.9)-(2.12) follows from the theory of nonlinear strictly parabolic problems. Note that we impose a Dirichlet outflow boundary condition on Γ_0, which yields a boundary layer for ψ.

The key point in deriving sharp a posteriori error estimate now is to derive sharp stability estimates for this dual problem. We start with a simple, but essential, non-degeneracy property first proved in [7, Lemma 3.2].

Lemma 2.1 *Let $\xi, \rho \in \mathbf{R}$ satisfy $|\beta(\xi)| \geq \rho > 0$. Then we have*

$$|\xi - \eta| \leq \Big(\frac{1}{\beta_1} + \frac{\lambda}{\rho}\Big)|\beta(\xi) - \beta(\eta)|, \quad \forall\, \eta \in \mathbf{R}.$$

Now we introduce the uniquess condition: $\exists\ \varepsilon_0, \rho_0 > 0$ such that $\theta \geq \rho_0$ a.e. in $\Gamma \times [0, \varepsilon_0] \times [0, T]$. Under this condition we know from Lemma 2.1 that there exists $r > 0$ depending on $= \rho_0$ such that

$$b(x,t) \geq r \quad \text{in } \Gamma \times ([0, \varepsilon_0] \cup [L - \varepsilon_1, L]) \times [0, T].$$

We set $A = \|\chi\|_{L^\infty(Q_*)}$ and assume $0 < v_0 V \leq v(t) \leq V$ for $t \in [0, T]$ and $|v'(t)| \leq v_1 V$ a.e. $t \in [0, T]$, with $v_0, v_1 > 0$ constants. We also let $V \geq 1$.

Lemma 2.2 *The following a priori bound is valid for all $x \in \bar{\Omega}$ and $0 \leq t \leq t_* \leq T$*

$$|\psi(x,t)| \leq \frac{\beta_2^{1/2} L}{v_0 V} \|\chi\|_{L^\infty(Q_*)}.$$

Proof. We consider the barrier function $\Lambda(z) = (\beta_2^{1/2} A / v_0 V)(L - z)$ for $0 \leq z \leq L$. $\diamondsuit$

Lemma 2.3 *There exists $C > 0$ independent of V and t_* such that for all $0 \leq t_* \leq T$*

$$|\partial_\nu \psi| \leq C \|\chi\|_{L^\infty(Q_*)} \quad \text{on } \Gamma_0 \times (0, t_*).$$

Proof. We consider the barrier function

$$\sigma(z) = k\Big(1 - e^{-Vz/r}\Big) - \Big(\frac{\beta_2^{1/2} A}{v_0 V}\Big) z, \quad 0 \leq z \leq \varepsilon_0,$$

with $k = \beta_2^{1/2} A (L + \varepsilon_0) / v_0 V (1 - e^{-V\varepsilon_0/r})$. $\diamondsuit$

Based on above two estimates, we can prove the following stability estimates required in our a posteriori error analysis.

Lemma 2.4 *There exists $C > 0$ independent of V and t_* such that for all $0 \leq t_* \leq T$*

$$\max_{0\leq t\leq t_*} \| \nabla\psi(\cdot,t) \|^2_{L^2(\Omega)} + \int_0^{t_*}\int_\Omega b_\delta |\Delta\psi|^2 dxdt \leq CVt_* \| \chi \|^2_{L^\infty(Q_*)},$$
$$\int_0^{t_*}\int_\Omega \Big(|\partial_t\psi + v(t)\partial_z\psi|^2 + \delta|D^2\psi|\Big) dxdt \leq CVt_* \| \chi \|^2_{L^\infty(Q_*)}.$$

With these stability estimates, we can derive the following a posteriori error estimate which depends mildly on the casting velocity V.

Theorem 2.1 *There exists a constant $C > 0$ independent of V and t^m such that the following a posteriori error estimate holds for all $t^m \in [0,T]$,*

$$\int_0^{t^m} \| \beta(u) - \beta(U) \|_{L^1(\Omega)} dt \leq C(Vt^m)^{1/2}\Big(\mathcal{E}_0 + \sum_{i=5}^{10}\mathcal{E}_i + \Big(\Lambda_m \sum_{i=1}^{4}\mathcal{E}_i\Big)^{1/2}\Big),$$

where

$$\Lambda_m = \Big(\sum_{n=1}^{m} \tau_n(1 + \lambda|\Omega| + \| \Theta^n \|^2_{L^2(\Omega)})\Big)^{1/2}$$

and the error indicators $\mathcal{E}_i$ depending only the discrete solutions U^n, Θ^n and the known data.

The detailed proof of this theorem as well as extensive numerical experiments can be found in [9].

3. Reliable and efficient a posteriori error estimate for a linear parabolic problem

Let Ω be a polyhedron domain in $\mathbf{R}^d (d = 1, 2, 3)$, $\Gamma = \partial\Omega$ and $T > 0$, In this section we consider the following linear parabolic equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \mathrm{div}(a(x)\nabla u) = f \quad \text{in } \Omega\times(0,T), \\ &u = 0 \quad \text{on } \Gamma\times(0,T), \quad u(x,0) = u_0(x) \quad \text{in } \Omega, \end{aligned} \tag{3.1}$$

where $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and $a(x)$ is a piecewise constant function. Let τ_n be the n-th time-step size and $t^n = \sum_{i=1}^n \tau_i$. Denote by $\mathcal{M}^n$ a uniformly regular partition of Ω into simplices such that $a(x)$ is a constant on each element $K \in \mathcal{M}^n$. We use refinement/coarsening procedures based on the bisection algorithm, which lead to compatible consecutive meshes whose minimum angles are bounded uniformly away from zero. Let V^n indicate the usual space of conforming linear finite elements over $\mathcal{M}^n$ and $V_0^n = V^n \cap H_0^1(\Omega)$. In this section, we consider the following simple fully discrete finite element scheme for solving (3.1): For $n = 1, 2, \cdots$, find $U_h^n \in V_0^n$ such that

$$\Big\langle \frac{U_h^n - U_h^{n-1}}{\tau_n}, v \Big\rangle + \langle a\nabla U_h^n, \nabla v\rangle = \langle \bar{f}^n, v\rangle \qquad \forall v \in V_0^n, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product on $L^2(\Omega)$, and

$$\bar{f}^n = \frac{1}{\tau_n} \int_{t^{n-1}}^{t^n} f(x,t)\; dt.$$

Let $U^n \in H_0^1(\Omega)$ be the solution of the following semi-discrete problem

$$\Big\langle \frac{U^n - U^{n-1}}{\tau_n}, \varphi \Big\rangle + \langle a\nabla U^n, \nabla \varphi\rangle = \langle \bar{f}^n, \varphi\rangle \quad \forall \varphi \in H_0^1(\Omega). \tag{3.3}$$

We observe that by modifying the time-step size τ_n, we are essentially controlling the error between $u^n = u(x, t^n)$ and U^n, not between u^n and U_h^n. Moreover, let $U_*^n \in H_0^1(\Omega)$ be the solution of the following auxiliary problem

$$\Big\langle \frac{U_*^n - U_h^{n-1}}{\tau_n}, \varphi \Big\rangle + \langle a\nabla U_*^n, \nabla \varphi\rangle = \langle \bar{f}^n, \varphi\rangle \quad \forall \varphi \in H_0^1(\Omega), \tag{3.4}$$

then we also observe that for fixed time-step size τ_n, by adapting the mesh $\mathcal{M}^n$ we are essentially controlling the error between U_h^n and U_*^n, not between U_h^n and U^n (or the real solution u). These two observations play an important role in the subsequent analysis to prove the local lower bound for the space a posteriori error estimator and in the development of a convergent refinement/coarsening strategy.

We define the interior residual

$$R^n := \bar{f}^n - \frac{U_h^n - U_h^{n-1}}{\tau_n},$$

along with the jump residual across $e \in \mathcal{B}^n$

$$J_e^n := [\![a\nabla U_h^n]\!]_e \cdot \nu_e = (a\nabla U_h^n|_{K_1} - a\nabla U_h^n|_{K_2}) \cdot \nu_e, \qquad e = \partial K_1 \cap \partial K_2,$$

using the convention that the unit normal vector ν_e to e points from K_2 to K_1. We observe that integration by parts implies

$$\langle a\nabla U_h^n, \nabla\varphi\rangle = -\sum_{e\in\mathcal{B}^n} \int_e J_e^n \varphi ds \qquad \forall \varphi \in H_0^1(\Omega).$$

Theorem 1 (Chen and Jia [6]) *For any integer $1 \le m \le N$, there exists a constant $C > 0$ depending only the minimum angle of meshes $\mathcal{M}^n, n = 1, 2, \cdots, m$, and the coefficient $a(x)$ such that the following a posteriori error estimate holds*

$$\begin{aligned} & \frac{1}{2} \| u^m - U_h^m \|_{L^2(\Omega)}^2 + \sum_{n=1}^{m} \int_{t^{n-1}}^{t^n} \|\!| u - U_h^n |\!\|_\Omega^2 dt \\ \le \quad & \| u_0 - U_h^0 \|_{L^2(\Omega)}^2 + \sum_{n=1}^{m} \tau_n \eta_{\text{time}}^n + C \sum_{n=1}^{m} \tau_n \eta_{\text{space}}^n \\ + \quad & 2 \Big(\sum_{n=1}^{m} \int_{t^{n-1}}^{t^n} \| f - \bar{f}^n \|_{L^2(\Omega)} dt \Big)^2, \end{aligned}$$

where the time error estimator η_{time}^n and space error estimator η_{space}^n are given by

$$\eta_{\text{time}}^n = \frac{1}{3} \|\!| U_h^n - U_h^{n-1} |\!\|_\Omega^2, \qquad \eta_{\text{space}}^n = \sum_{e \in \mathcal{B}^n} \eta_e^n$$

with the local error estimator η_e^n defined as

$$\eta_e^n = \frac{1}{2} \sum_{K \in \Omega_e} h_K^2 \| R^n \|_{L^2(K)}^2 + h_e \| J_e^n \|_{L^2(e)}^2 .$$

Here Ω_e is the collection of two elements sharing the common side $e \in \mathcal{B}^n$.

Let $P^n : L^2(\Omega) \to V_0^n$ be the L^2 projection operator. We remark that the coarsening errors involving $U_h^{n-1} - P^n U_h^{n-1}$ which appeared in previous studies [9], [12], [16] are not present in our a posteriori error estimates. They are hidden in the space error term $\| h_n R^n \|_{L^2(\Omega)}^2$ and time error term $\|\!| U_h^n - U_h^{n-1} |\!\|_\Omega^2$. Here h_n is the piecewise constant function which equals to h_K on each $K \in \mathcal{M}^n$.

Now we consider the following local lower bound of the space error estimator which ensures over-refinement will not occur for the refinement strategy based on our space error estimator. For any $K \in \mathcal{M}^n$ and $\varphi \in L^2(\Omega)$, we define $P_K \varphi = \dfrac{1}{|K|} \displaystyle\int_K \varphi dx$, the average of φ over K.

Theorem 2 (Chen and Jia [6]) *Suppose that there exists a constant $\hat{C} > 0$ such that $h_K^2 \le \hat{C} \tau_n$ for any $K \in \mathcal{M}^n$. Then there exist constants $C_2, C_3 > 0$ depending only on $\hat{C}$, the minimum angle of $\mathcal{M}^n$ and the* *coefficient $a(x)$ such that for any $e \in \mathcal{B}^n$, the following estimate holds*

$$
\begin{aligned}
\eta_e^n &\le C_2 \sum_{K\in\Omega_e} \Big(\frac{1}{\tau_n} \| U_*^n - U_h^n \|_{L^2(K)}^2 + \| U_*^n - U_h^n \|_K^2 \Big) \\
&+ C_3 \sum_{K\in\Omega_e} h_K^2 \| R^n - P_K R^n \|_{L^2(K)}^2 .
\end{aligned}
$$

Based on the local error indicators, the usual adaptive algorithm solving the parabolic problem (3.1) at the n-th time step reads as follows

$$\text{Solve } \to \text{Estimate } \to \text{Refine/Coarsen.}$$

Here the refinement/coarsening procedure includes both the mesh and time-step size modifications. There are several possibilities proposed in the literature on the strategies of the adaptive control of meshes and time-step sizes. We refer to Schmidt and Siebert [19] for a nice review on various adaptive algorithms and their implementation details. In this paper, at each time step n, we propose the following algorithm to modify the time-step size τ_n and mesh $\mathcal{M}^n$ starting from the initial time-step size $\tau_{n,0} = \tau_{n-1}$ and initial mesh $\mathcal{M}^{n,0} = \mathcal{M}^{n-1}$:

1 Refine the time-step size $\tau_{n.0}$ and the mesh $\mathcal{M}^{n,0}$ so that for the solution $U_h^n \in V_0^n$ of (3.2) on the final mesh $\mathcal{M}^n$ with final time-step size τ_n, the associated space and time error estimators are less than the prescribed tolerances respectively;

2 Coarsen the mesh $\mathcal{M}^n$ so that for the solution U_H^n on the coarsened mesh $\mathcal{M}_H^n$, the associated coarsening error estimator less than some prescribed tolerance;

3 Enlarge the initial time-step size $\tau_{n+1,0}$ for next time step if the current time error estimator is much less than the tolerance.

In [6] we extend the convergence analysis of adaptive finite element methods developed for linear elliptic problems in Dörfler [11] and Morin, Nochetto and Siebert [15] to prove the iteration in Step 1 of above algorithm terminates in finite number of steps. The main difficulty now is the treatment of the so-called data oscillation of the residual $\bar{f}^n - (U_h^n - U_h^{n-1})/\tau_n$ which changes at each refinement procedure.

The choice of the coarsening error estimator and coarsening strategy in Step 2 is a subtle issue. The error incurred due to the over-coarsening can only be reduced by re-refining the coarsened mesh. Thus over-coarsening leads to unnecessary solution of discrete problems, that is usually expensive and undesirable. Our coarsening error indicator is based on a direct control of the error between the coarsened discrete solution and the limit solution U_*^n of (3.4). More precisely, let $\mathcal{M}_H^n$ be a coarsening of $\mathcal{M}^n$, and U_H^n, U_h^n be the corresponding solutions of (3.2) with fixed time-step size τ_n. Since $V_0^{n,H} \subset V_0^n$, we deduce by Galerkin orthogonality

$$\begin{aligned} & \frac{1}{\tau_n}\| U_*^n - U_H^n \|_{L^2(\Omega)}^2 + \|\!| U_*^n - U_H^n |\!\|_\Omega^2 \\ \leq\ & \frac{1}{\tau_n}\| U_*^n - U_h^n \|_{L^2(\Omega)}^2 + \|\!| U_*^n - U_h^n |\!\|_\Omega^2 \\ +\ & \frac{1}{\tau_n}\| U_h^n - I_H^n U_h^n \|_{L^2(\Omega)}^2 + \|\!| U_h^n - I_H^n U_h^n |\!\|_\Omega^2, \end{aligned}$$

where $\|\!|\varphi|\!\|_\Omega = \langle a\nabla\varphi, \nabla\varphi\rangle^{1/2}$ is the energy norm of $\varphi \in H^1(\Omega)$, and $I_H^n : C(\bar{\Omega}) \to V^{n,H}$ is the usual linear finite element interpolant over $\mathcal{M}_H^n$. Our coarsening error indicator is then given by

$$\eta_{\mathrm{coarse}}^n := \frac{1}{\tau_n}\| U_h^n - I_H^n U_h^n \|_{L^2(\Omega)}^2 + \|\!| U_h^n - I_H^n U_h^n |\!\|_\Omega^2,$$

which does not depend on the coarsened solution U_H^n. It is shown in [6] that this direct error control leads to only one coarsening step. The implementation issue of the coarsening startegy is discussed in detail in [6].

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