NUMERICAL HOMOGENIZATION OF WELL SINGULARITIES IN THE FLOW TRANSPORT THROUGH HETEROGENEOUS POROUS MEDIA

ZHIMING CHEN* AND XINYE YUE†

Abstract. We consider the steady flow transport through highly heterogeneous porous media driven by extraction wells. We develop a new upscaling technique which lumps the small-scale details of the medium property into a few representative macroscopic parameters on a coarse scale that preserve the large-scale behavior of the medium and are more appropriate for simulations. The method is based on the recently introduced over-sampling multiscale finite element method and the introduction of new base functions that locally resolve the well singularities. The modeling error which reduces the original problem having wells into problems with Dirac sources is carefully analyzed. We also provide a detailed multiscale convergence analysis of the method under the assumption that the oscillating coefficients are locally periodic. While such a simplifying assumption is not required by our method, it allows us to use homogenization theory to obtain the asymptotic structure of the solutions. New homogenization results for Green functions are derived. A new convergent algorithm approximating Green functions in the vicinity of singularities is proposed which extends the well-known Peaceman technique in the engineering literature. Numerical experiments are carried out for flow transport in both periodic and randomly generated log-normal permeabilities to demonstrate the efficiency and accuracy of the proposed method.

Key words. Numerical homogenization, well singularity, heterogeneous porous media

AMS subject classifications. 65F10 65F30

1. Introduction. The central difficulty in the modeling of subsurface flow and transport is the accounting for the spatial variability in the parameters used to characterize the relevant physical properties of the natural porous media. In realistic situations, the precise spatial distribution of the parameters required to characterize the problem is never available due to the lack of enough data. Thus sophisticated geological and geostatistical modeling tools are used in practice to generate highly detailed medium parameters based on some site-specific measurements and experience from other sites. There exists a vast literature on the upscaling or homogenization techniques that lump the small-scale details of the medium into a few representative macroscopic parameters on a coarse scale which preserve the large-scale behavior of the medium and are more appropriate for reservoir simulations. We refer to the book of Christakos [5] for more information on the random field modeling of the natural porous medium parameters and the recent review paper [26] on the existent upscaling techniques in the engineering literature.

The study of steady flow through highly heterogeneous porous medium driven by extraction wells is of great importance in hydrology, petroleum reservoir engineering or soil venting techniques (vapor extraction of volatile and semi-volatile organic compounds from the contaminated soil [11]). It is observed in the engineering literature (cf. e.g. [8] and [21] and the references therein) that in the near-well region, many of the existing upscaling methods do not provide satisfactory results. The reason may be explained as the standard upscaling methods generally assume the pressure field

*LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, Beijing 100080, China. This author was supported in part by China NSF under the grant 10025102 and by China MOS under the grant G1999032802. (zmchen@lsec.cc.ac.cn).

†LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, Beijing 100080, China. (xyyue@lsec.cc.ac.cn). *Current Address:* Department of Mathematics, Suzhou University, Suzhou 215006, China.

is slowly varying, that is clearly not true in the vicinity of the flowing wells [8]. This fact may also be explained mathematically from the homogenization theory for the second order elliptic equations with periodic coefficients. In the homogenization theory, multiscale convergence is ensured under the assumption that the source should be at least in H^{-1} so that the solution is bounded in Sobolev space H^1. As we will see below, however, well singularities can be modeled as Dirac sources and thus in the near-well region, the solution behaves like Green function which is not uniformly bounded in H^1.

The recently introduced multiscale finite element method [15, 16] for solving elliptic equations with oscillating coefficients provides an effective way to capture the large scale structures of the solutions on a coarse mesh without resolving all the fine scale structures. The central idea of the method is to incorporate the local small scale information of the leading order differential operator into the finite element bases. It is through these multiscale bases and the finite element formulation that the effect of small scales on the large scales is correctly captured. The convergence of the method for the locally periodic coefficients can be established based on the homogenization theory. The other related numerical homogenization techniques based on representation formulas for the solution can be found in [6, 23].

The purpose of this paper is to develop a complete multiscale coarse grid algorithm for solving steady flow problem involving well singularities in heterogeneous porous medium based on the multiscale finite element method. The additional well singularities of the problem are resolved locally by adding finite element base functions. The final coarse grid model is based on a variational formulation which is different from the heuristic techniques in [8, 21]. This rigorous variational coarse grid model allows us to establish a complete multiscale error analysis for locally periodic medium properties based on the homogenization theory. We emphasize that as pointed out in [9], the spatial periodicity assumption does not a priori restrict the applicability of the results only to media which do exhibit such strict repetitive spatial ordering in the properties of interest. The numerical experiments carried out for random log-normal relative permeabilities demonstrate clearly the applicability of our method beyond the periodic structures.

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with Lipschitz boundary Γ. $B(x_0, r)$ will be denoted as the disk centered at x_0 with radius $r > 0$. Let $B_j = B(\bar{x}_j, \delta), 1 \le j \le N$, be mutually disjoint subdomains inside Ω that are occupied by the wells. Denote by $\Omega_\delta = \Omega \backslash (\cup_{j=1}^N \bar{B}_j)$. We consider the following single phase pressure equation which is formed by combining Darcy's law with the conservation of mass

$$-\mathrm{div}\,(K(x)\nabla u_\delta) = 0 \quad \text{ in } \Omega_\delta, \tag{1.1}$$

where u_δ is pressure, K is the permeability which is typically highly variable in space. We will impose homogeneous Dirichlet boundary condition $u_\delta|_\Gamma = 0$ on the exterior boundary. The other types of boundary conditions can be treated similarly without any essential difficulties. On the well boundary $\Gamma_j = \partial B_j$, two quantities are of particular importance in practical applications: the well bore pressure $u_\delta|_{\Gamma_j}$ and the well flow rate $\displaystyle\int_{\Gamma_j} K \frac{\partial u_\delta}{\partial \nu} ds$, where ν is the unit outer normal to $\partial\Omega_\delta$. The boundary condition to be imposed on Γ_j is either the Dirichlet boundary condition which fixes the well bore pressure α_j (assume to be constant)

$$u_\delta|_{\Gamma_j} = \alpha_j, \tag{1.2}$$

or the following mixed boundary condition which fixes the well flow rate q_j

$$u_\delta|_{\Gamma_j} = c_j = \text{const}\,, \qquad \int_{\Gamma_j} K \frac{\partial u_\delta}{\partial \nu} ds = \mathrm{q}_j. \tag{1.3}$$

The constant c_j in (1.3) is unknown.

Denote by I_D the collection of all indices j such that Dirichlet boundary condition (1.2) is imposed on Γ_j, and I_M the collection of all indices j such that the mixed boundary condition (1.3) is imposed on Γ_j. Let $\Omega_M = \Omega \backslash (\cup_{j\in I_D} \bar{B}_j)$ and define the subspaces

$$V = \{v \in H^1(\Omega_M) : \nabla v|_{B_j} = 0 \ \text{ for } j \in I_M, v|_\Gamma = 0\},$$

and

$$V_0 = \{v \in V : v|_{\Gamma_j} = 0 \ \text{ for } j \in I_D\}.$$

Then following the idea in [17], the existence and uniqueness of weak solutions to the problem (1.1)-(1.3) can be easily proved by considering the following variational form: Find $u_\delta \in V$ such that $u_\delta|_{\Gamma_i} = \alpha_i$ for $i \in I_D$, and

$$\int_{\Omega_M} K(x)\nabla u_\delta \cdot \nabla v dx = \sum_{j\in I_M} \frac{1}{|B_j|} \int_{B_j} \mathrm{q}_j v ds \quad \forall v \in V_0. \tag{1.4}$$

Since the size of the wells δ is negligible in practical situations, the first approximation to be made is to replace (1.4) by the following problem

$$-\mathrm{div}\,(K(x)\nabla u) = \sum_{j=1}^N \mathrm{q}_j \delta_{\bar{x}_j} \quad \text{ in } \Omega \tag{1.5}$$

with the boundary condition $u|_\Gamma = 0$. Here $\delta_{\bar{x}_j}$ is the Dirac measure at $\bar{x}_j$, $1 \le j \le N$. Denote by $K_j = K(\bar{x}_j)$, $\phi_j = -\dfrac{1}{2\pi K_j} \ln|x - \bar{x}_j|$ and $U = u - \sum_{j=1}^N \mathrm{q}_j \phi_j$. Note that the flow rates q_i for $i \in I_D$ in (1.5) are unknown. They are determined through the following additional conditions which are obtained by requiring $u|_{\Gamma_i} \approx \alpha_i$ as the approximation of the boundary conditions (1.2):

$$-\frac{\mathrm{q}_i}{2\pi K_i} \ln \delta + \sum_{j\ne i} \mathrm{q}_j \phi_j(\bar{x}_i) + U(\bar{x}_i) = \alpha_i, \qquad i \in I_D. \tag{1.6}$$

The first main result in this paper is the following error estimate between the solution u_δ of the original problem (1.4) and the solution u of (1.5)-(1.6)

$$\max_{x\in\bar{\Omega}_\delta} |u - u_\delta| \le C\delta |\ln\delta| \sum_{j=1}^N |\mathrm{q}_j|.$$

To the best of our knowledge, this error estimate is the first in the literature although the convergence of the solution u_δ to u in certain sense as $\delta \to 0$ is well-known (cf. e.g. [19]). The proof of the estimate depends on a sharp estimate between the well flow rate and the well bore pressure (Lemma 2.2) and an explicit dependence on p of the constant in the classical $W^{1,p}$ interior estimate for the second order elliptic equations whose proof, for the sake of completeness, is briefly recalled in the appendix of this paper.

Now we introduce an equivalent variational formulation for (1.5) which is suitable for our multiscale approximation. Let $\Omega_j, 1 \le j \le N$, be mutually disjoint subdomains inside Ω such that $\bar{B}_j \subset\subset \Omega_j$. Let G_j be the Green function associated with the domain Ω_j

$$-\mathrm{div}\,(K(x)\nabla G_j) = \delta_{\bar{x}_j} \quad \text{in } \Omega_j, \qquad G_j|_{\Sigma_j} = 0, \tag{1.7}$$

where $\Sigma_j = \partial\Omega_j$. Now for any $v \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} v(\bar{x}_j) &= \int_{\Omega_j} -\mathrm{div}\,(K(x)\nabla G_j) v dx \\ &= \int_{\Omega_j} K(x)\nabla G_j \cdot \nabla v dx - \int_{\Sigma_j} K\frac{\partial G_j}{\partial \nu} v ds. \end{aligned}$$

On the other hand, from (1.5) we know that

$$\sum_{j=1}^N \mathrm{q}_j v(\bar{x}_j) = \int_\Omega K(x)\nabla u \cdot \nabla v dx.$$

Let $G_j = 0$ for $x \in \Omega\backslash\bar{\Omega}_j$ and $\zeta = u - \sum_{j=1}^N \mathrm{q}_j G_j$, then we know that $\zeta \in H_0^1(\Omega)$ satisfies the following variational form

$$\int_\Omega K(x)\nabla\zeta \cdot \nabla v dx = -\sum_{j=1}^N \mathrm{q}_j \int_{\Sigma_j} K\frac{\partial G_j}{\partial\nu} v dx \quad \forall v \in C_0^\infty(\Omega). \tag{1.8}$$

Denote by $w_j = G_j + \dfrac{1}{2\pi K_j}\ln|x - \bar{x}_j|$ in Ω_j, then the complementary condition (1.6) to determine q_i for $i \in I_D$ becomes

$$-\frac{\mathrm{q}_i}{2\pi K_i}\ln\delta + \mathrm{q}_i w_i(\bar{x}_i) + \zeta(\bar{x}_i) = \alpha_i, \quad i \in I_D. \tag{1.9}$$

We note that for ζ the singularities of the original solution u are removed and we will use the over-sampling multiscale finite element method to discretize it on a coarse grid. The multiscale convergence of the method will be established for locally periodic coefficients, i.e. assuming $K(x) = K(x, x/\varepsilon)$, where $K(x,\cdot)$ is periodic with respect to the unit square Y. The proof depends on several new homogenization error estimates for Green functions which, in particular, improve some of the bounds in [2].

In practical computations, the local Green function G_j will be solved on a fine grid with the grid size much less than the characteristic size ε of the small structure of the medium property. It is well-known that the discrete Green function fails to converge in the vicinity of the singularity, i.e. the well location. In this paper we will introduce a new algorithm to remove this difficulty by extending the widely used Peaceman technique [24, 25] in the engineering literature. Our new algorithm improves the Peaceman method in that our method is convergent for any heterogeneous media and any regular finite element meshes.

The paper is organized as follows: In §2 we prove the estimate between the solution of the problem (1.4) and the solution of the problem (1.5)-(1.6). In §3 we introduce our multiscale algorithm based on the weak formulation (1.8) and present the convergence result of the algorithm. In §4 we establish some new bounds for the homogenization of Green functions which will be used in §5 to complete the multiscale error analysis. The approximation of the local Green functions will be considered in §6. We show some numerical results in §7 for both periodic and randomly generated coefficients to demonstrate the accuracy of the proposed method. Finally, we briefly discuss the proof of the interior $W^{1,p}$ estimate and the weak maximum principle in §8.

2. The mathematical modeling . Let Ω be a convex polygon in $\mathbf{R}^2$ with boundary Γ. Let $D \subseteq \Omega$ be a domain in $\mathbf{R}^2$ with Lipschitz boundary. For each integer $m \geq 0$ and $1 \leq p \leq +\infty$, we denote by $W^{m,p}(D)$ the standard Sobolev space of real functions having all their weak derivatives of order up to m in the Lebesgue space $L^p(D)$. The norm and the semi-norm of $W^{m,p}(D)$ will be denoted by $\|\cdot\|_{m,p,D}$ and $|\cdot|_{m,p,D}$, respectively. As usual, when $p = 2$, $W^{m,2}(D)$ is denoted by $H^m(D)$ with norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$.

Let $B(x_0, r) = \{x \in \mathbf{R}^2 : |x - x_0| < r\}$. Let $B_j = B(\bar{x}_j, \delta), 1 \leq j \leq N$, be mutually disjoint subdomains inside Ω that are occupied by the wells. Set $\Gamma_j = \partial B_j$ and $\Omega_\delta = \Omega \backslash (\cup_{j=1}^N \bar{B}_j)$. We consider the following problem

$$-\mathrm{div}\,(K(x)\nabla u_\delta) = 0 \quad \text{ in } \Omega_\delta, \tag{2.1}$$

$$u_\delta|_{\Gamma_j} = \text{const}\,, \qquad \int_{\Gamma_j} K\frac{\partial u_\delta}{\partial \nu} ds = \mathrm{q}_j \quad \text{ for } j \in I_M, \tag{2.2}$$

$$u_\delta|_{\Gamma_j} = \alpha_j \quad \text{ for } j \in I_D, \quad u_\delta|_\Gamma = 0. \tag{2.3}$$

Here I_D, I_M are index sets such that $I_D \cup I_M = \{1, 2, \cdots, N\}$, α_j for $j \in I_D$ and q_i for $i \in I_M$ are known constants. We imposed the following conditions on the coefficient $K(x)$:

(H1) $K \in C^{0,1}(\bar{\Omega})$, there exist constants $\lambda, \Lambda > 0$ such that $\lambda \leq K(x) \leq \lambda^{-1}$ and $|K(x) - K(y)| \leq \Lambda|x - y|$ for any $x, y \in \bar{\Omega}$. As we are interested in highly variable coefficients, without loss of generality, we assume $\Lambda \geq 1$.

As already indicated in §1, we will consider the following approximation of the problem (2.1)-(2.3): Find u and the constants $\{\mathrm{q}_i\}_{i \in I_D}$, such that

$$-\mathrm{div}\,(K(x)\nabla u) = \sum_{j=1}^N \mathrm{q}_j \delta_{\bar{x}_j} \quad \text{ in } \Omega, \tag{2.4}$$

$$-\frac{\mathrm{q}_i}{2\pi K_i}\ln\delta + \sum_{j \neq i} \mathrm{q}_j \phi_j(\bar{x}_i) + U(\bar{x}_i) = \alpha_i \quad \text{ for } i \in I_D, \tag{2.5}$$

$$u|_\Gamma = 0. \tag{2.6}$$

Here $\delta_{\bar{x}_j}$ is the Dirac measure at $\bar{x}_j$, $K_j = K(\bar{x}_j)$, $\phi_j(x) = -\dfrac{1}{2\pi K_j}\ln|x - \bar{x}_j|$, and $U = u - \sum_{j=1}^N \mathrm{q}_j\phi_j$. The relation (2.5) can be viewed as an approximation of the boundary condition $u|_{\Gamma_i} \approx \alpha_i$ for $i \in I_D$. The main purpose of this section is to derive a pointwise error estimate between the solutions u_δ and u. We start with the following lemma.

LEMMA 2.1. *There exists a constant $\gamma_0 \in (0, 1)$ such that for $\delta\Lambda \leq \gamma_0$, the problem* (2.4)-(2.6) *has a unique solution. More precisely, there exist unique constants*

$\{\mathrm{q}_i\}_{i\in I_D}$ *such that* $u=\sum_{j=1}^N \mathrm{q}_j\Theta_j$ *satisfies* (2.4)-(2.6)*, where* Θ_j *is the Green function satisfying*

$$-\mathrm{div}\,(K(x)\nabla\Theta_j)=\delta_{\bar{x}_j}\quad \text{in } \Omega,\quad \Theta_j|_\Gamma=0. \tag{2.7}$$

Proof. It is clear that $U=\sum_{j=1}^N \mathrm{q}_j\eta_j$ with $\eta_j=\Theta_j-\phi_j$. Substituting it into (2.5) we know that $\mathrm{q}_i, i\in I_D$, satisfy the following linear system of equations:

$$\Big[-\frac{1}{2\pi K_i}\ln\delta+\eta_i(\bar{x}_i)\Big]\mathrm{q}_i+\sum_{j\in I_D, j\neq i}\Theta_j(\bar{x}_i)\mathrm{q}_j=\alpha_i-\sum_{j\in I_M}\mathrm{q}_j\Theta_j(\bar{x}_i). \tag{2.8}$$

It is easy to see that $\eta_i\in H^1(\Omega)$ satisfies

$$-\mathrm{div}\,(K(x)\nabla\eta_i)=\mathrm{div}\,((K(x)-K_i)\nabla\phi_i)\quad \text{in } \Omega.$$

We now apply the weak maximum principle (cf. e.g. in [12, p.191] or [18, p.63]) in the subdomain $B_{1/\Lambda}=B(\bar{x}_i,1/\Lambda)$ to get

$$\begin{aligned}\max_{x\in B_{1/\Lambda}}(-\eta_i) &\le \max_{x\in\partial B_{1/\Lambda}}(-\eta_i)^+ + C|B_{1/\Lambda}|^{1/2}\|\,(K-K_j)\nabla\phi_j\,\|_{0,\infty,B_{1/\Lambda}}\\ &\le \max_{x\in\partial B_{1/\Lambda}}\phi_i + C(1/\Lambda)\cdot\Lambda\\ &= -\frac{1}{2\pi K_i}\ln\frac{1}{\Lambda}+C,\end{aligned}$$

where we have used the fact that $\Theta_i\ge 0$ on $\partial B_{1/\Lambda}$. Moreover, we have $0\le\Theta_j(\bar{x}_i)\le C$ for $j\neq i$. Thus for sufficiently small $\delta\Lambda\le\gamma_0<1$, the associated matrix for the linear system of equations (2.8) is strictly diagonally dominant

$$\begin{aligned}-\frac{1}{2\pi K_i}\ln\delta+\eta_i(\bar{x}_i)-\sum_{j\in I_D, j\neq i}\Theta_j(\bar{x}_i) &\ge -\frac{1}{2\pi K_i}\ln(\delta\Lambda)-C\\ &\ge -\frac{1}{4\pi K_i}\ln\gamma_0>0,\end{aligned} \tag{2.9}$$

for any $i\in I_D$. Therefore, the constants $\mathrm{q}_i, i\in I_D$, satisfying (2.8) are uniquely existent. This completes the proof. ◻

Now we turn to the error estimate. The key building block is the following basic lemma.

LEMMA 2.2. *For any* $j\in I_M$*, let* v *be the solution of the following problem*

$$-\mathrm{div}\,(K(x)\nabla v)=0\quad \text{in } \Omega_\delta, \tag{2.10}$$

$$v|_{\Gamma_i}=\mathrm{const}\,,\quad \int_{\Gamma_i}K\frac{\partial v}{\partial\nu}ds=0\quad \text{for } i\in I_M,\quad i\neq j, \tag{2.11}$$

$$v|_{\Gamma_j}=1,\quad v|_{\Gamma_i}=0\quad \text{for } i\in I_D,\quad v|_\Gamma=0. \tag{2.12}$$

Then there exists a constant C *depending only on* λ *such that*

$$2\pi\lambda\left(\ln\frac{D_j}{\delta}\right)^{-1}\le\int_{\Gamma_j}K\frac{\partial v}{\partial\nu}ds\le C\left(\ln\frac{d_j}{\delta}\right)^{-1}, \tag{2.13}$$

where $d_j=\frac{1}{2}\min(\min_{x\in\Gamma}|x-\bar{x}_j|,\min_{i\neq j}|\bar{x}_i-\bar{x}_j|)$ *and* $D_j=\max_{x\in\Gamma}|x-\bar{x}_j|$.

We observe that since $K \in C^{0,1}(\bar{\Omega})$ and Ω is a convex polygon in $\mathbf{R}^2$, the $W^{2,p}$ theory for elliptic equations implies that $v \in W^{2,p}(\Omega_\delta)$ for some $p > 2$ and thus $v \in C^1(\bar{\Omega}_\delta)$. Now Hopf's lemma and (2.11) implies that v cannot attain its maximum or minimum value on $\Gamma_i, i \in I_M, i \neq j$. Therefore, by maximum principle we have $0 \le v \le 1$ in Ω_δ. Again by Hopf's lemma we have $\partial v/\partial\nu > 0$ on Γ_j. The significance of this lemma lies in that it establishes the relation between the flux flow rate and the well bore pressure for each well B_j, $j \in I_M$.

Proof. We first prove the upper bound in (2.13). Consider the problem

$$\begin{aligned} &-\mathrm{div}\,(K(x)\nabla v_j) = 0 \quad \text{ in } B(\bar{x}_j, d_j)\backslash\bar{B}(\bar{x}_j,\delta) \\ &v_j|_{\Gamma_j} = 1, \quad v_j|_{\partial B(\bar{x}_j,d_j)} = 0, \end{aligned}$$

and let $\psi = v - v_j$. Then we know that $-\mathrm{div}\,(K(x)\nabla\psi) = 0$ in $B(\bar{x}_j, d_j)\backslash\bar{B}(\bar{x}_j,\delta)$, $\psi = 0$ on Γ_j and $\psi \ge 0$ on $\partial B(\bar{x}_j, d_j)$. By maximum principle we have $\psi \ge 0$ in $B(\bar{x}_j, d_j)\backslash\bar{B}(\bar{x}_j,\delta)$. This implies $\partial\psi/\partial\nu \le 0$ on Γ_j and thus

$$\int_{\Gamma_j} K\frac{\partial v}{\partial\nu}ds \le \int_{\Gamma_j} K\frac{\partial v_j}{\partial\nu}ds. \tag{2.14}$$

On the other hand, let θ_j be the Green function in $B(\bar{x}_j, d_j)$:

$$-\mathrm{div}\,(K(x)\nabla\theta_j) = \delta_{\bar{x}_j} \quad \text{ in } B(\bar{x}_j, d_j), \qquad \theta_j|_{\partial B(\bar{x}_j,d_j)} = 0.$$

We know that $\psi_j = v_j - \theta_j/(\min_{x\in\Gamma_j}\theta_j(x))$ satisfies the equation $-\mathrm{div}\,(K(x)\nabla\psi_j) = 0$ in $B(\bar{x}_j, d_j)\backslash\bar{B}(\bar{x}_j,\delta)$, $\psi_j = 0$ on $\partial B(\bar{x}_j, d_j)$ and $\psi \le 0$ on Γ_j. By maximum principle, $\psi_j \le 0$ in $B(\bar{x}_j, d_j)\backslash\bar{B}(\bar{x}_j,\delta)$ and thus $\partial\psi_j/\partial\nu \ge 0$ on $\partial B(\bar{x}_j, d_j)$. Therefore,

$$\int_{\Gamma_j} K\frac{\partial\psi_j}{\partial\nu}ds = -\int_{\partial B(\bar{x}_j,d_j)} K\frac{\partial\psi_j}{\partial\nu}ds \le 0,$$

which yields

$$\int_{\Gamma_j} K\frac{\partial v_j}{\partial\nu}ds \le \frac{1}{\min_{x\in\Gamma_j}\theta_j(x)}\int_{\Gamma_j} K\frac{\partial\theta_j}{\partial\nu}ds = \frac{1}{\min_{x\in\Gamma_j}\theta_j(x)}. \tag{2.15}$$

Let $\theta_j^\Delta = \dfrac{1}{2\pi}\ln\dfrac{d_j}{|x-\bar{x}_j|}$ be the Green function of the Laplacian $-\Delta$ in $B(\bar{x}_j, d_j)$. Then by the well-known property of the Green function (cf. e.g. [20, Theorem 7.1] or [14, Theorem 2.1]) we know that there exists a constant C depending only on λ in (H1) such that

$$\theta_j(x) \ge C\theta_j^\Delta(x) = C\ln\frac{d_j}{|x-\bar{x}_j|} \qquad \text{for } x \in B(\bar{x}_j, d_j).$$

Thus we obtain $\min_{x\in\Gamma_j}\theta_j(x) \ge C\ln(d_j/\delta)$ and this completes the proof of the upper bound by combining (2.14) with (2.15).

Now we turn to the proof of the lower bound in (2.13) by extending an argument in [19]. First we observe by multiplying (2.10) with v, integrating by parts and using the boundary conditions (2.11)-(2.12) that

$$\int_{\Omega_\delta} K(x)|\nabla v|^2 dx = \sum_{i=1}^N \int_{\Gamma_i} K\frac{\partial v}{\partial\nu}v ds = \int_{\Gamma_j} K\frac{\partial v}{\partial\nu}ds. \tag{2.16}$$

Introduce a polar coordinate system centered at $\bar{x}_j$. For any $x = (\rho, \theta) \in \Omega_\delta$, denote by L_x the half line started at $\bar{x}_j$ and passed through x. Clearly L_x can have only finite intersections with the wells. Assume that

$$L(\theta) = \Omega_\delta \cap L_x = \{y = (r, \theta) \in \Omega_\delta : r \in (r_0, r_1) \cup \cdots \cup (r_k, \delta)\},$$

where $y_0 = (r_0, \theta) \in \Gamma$, $y_{k+1} = (\delta, \theta) \in \Gamma_j$ and $y_m = (r_m, \theta)$, $1 \le m \le k$, on some Γ_i, $i \neq j$, $1 \le i \le N$. In most cases, $k = 0$ that means L_x has no intersections with the wells. Now since $v = 1$ on Γ_j, $v = \text{const}$ on Γ_i, for any $i \neq j$, and $v = 0$ on Γ, we have

$$\begin{aligned}
1 = -\int_{L(\theta)} \frac{\partial v}{\partial r} dr &\le \epsilon \int_{L(\theta)} r \Big|\frac{\partial v}{\partial \nu}\Big|^2 dr + \frac{1}{4\epsilon} \int_{L(\theta)} \frac{1}{r} dr \\
&\le \epsilon \int_{L(\theta)} r \Big|\frac{\partial v}{\partial \nu}\Big|^2 dr + \frac{1}{4\epsilon} \int_{\delta}^{r_0} \frac{1}{r} dr \\
&= \epsilon \int_{L(\theta)} r \Big|\frac{\partial v}{\partial \nu}\Big|^2 dr + \frac{1}{4\epsilon} \ln \frac{r_0}{\delta} \\
&\le \epsilon \int_{L(\theta)} r \Big|\frac{\partial v}{\partial \nu}\Big|^2 dr + \frac{1}{4\epsilon} \ln \frac{D_j}{\delta},
\end{aligned}$$

where $\epsilon > 0$ is a constant to be determined later. Note that

$$\int_{\Omega_\delta} |\nabla v|^2 dx = \int_0^{2\pi} d\theta \int_{L(\theta)} \Big(r\Big|\frac{\partial v}{\partial r}\Big|^2 + \frac{1}{r}\Big|\frac{\partial v}{\partial \theta}\Big|^2 \Big) dr \ge \int_0^{2\pi} \int_{L(\theta)} r \Big|\frac{\partial v}{\partial \nu}\Big|^2 dr d\theta.$$

Thus

$$2\pi \le \epsilon \int_{\Omega_\delta} |\nabla v|^2 dx + \frac{\pi}{2\epsilon} \ln \frac{D_j}{\delta} \le \frac{\epsilon}{\lambda} \int_{\Omega_\delta} K(x) |\nabla v|^2 dx + \frac{\pi}{2\epsilon} \ln \frac{D_j}{\delta},$$

which yields, after taking $\epsilon = \frac{1}{2}\ln(D_j/\delta) > 0$, that

$$\int_{\Omega_\delta} K(x) |\nabla v|^2 dx \ge 2\pi\lambda \Big(\ln \frac{D_j}{\delta} \Big)^{-1}.$$

This completes the proof by using (2.16). $\square$

The following theorem is the main result of this section.

THEOREM 2.3. *Let the assumption (H1) be satisfied. Let u_δ and u be the solutions of* (2.1)-(2.3) *and* (2.4)-(2.6), *respectively. Then there exists a constant C independent of δ, q_j and Λ such that*

$$\max_{x \in \bar{\Omega}_\delta} |u - u_\delta| \le C\Lambda(1 + \ln \Lambda)^{1/2} \Big(\sum_{j=1}^N |\mathrm{q}_j| \Big) \delta |\ln \delta|. \tag{2.17}$$

The proof of this theorem is divided into several steps. Without loss of generality we may assume $I_M = \{1, 2, \cdots, N_0\}$ and $I_D = \{N_0 + 1, \cdots, N\}$. For any $j \in I_M$ we consider the following auxiliary problem

$$-\text{div}\,(K(x)\nabla u_j) = \sum_{i=1}^j \mathrm{q}_i \delta_{\bar{x}_i} \quad \text{in } \hat{\Omega}_j = \Omega_\delta \cup (\cup_{i=1}^j \bar{B}_j), \tag{2.18}$$

$$u_j|_{\Gamma_i} = \text{const}\,, \quad \int_{\Gamma_i} K \frac{\partial u_j}{\partial \nu} ds = \mathrm{q}_j \quad \text{for } i = j+1, \cdots, N_0, \tag{2.19}$$

$$u_j|_{\Gamma_i} = \alpha_i \quad \text{for } i \in I_D, \quad u_j|_\Gamma = 0. \tag{2.20}$$

Note that j wells in the original problem have been removed for the solution u_j. Let $u_0 = u_\delta$, then we have

$$\| u - u_\delta \|_{0,\infty,\Omega_\delta} \le \sum_{j=1}^{N_0} \| u_j - u_{j-1} \|_{0,\infty,\Omega_\delta} + \| u - u_{N_0} \|_{0,\infty,\Omega_\delta}. \tag{2.21}$$

So it remains to estimate $\| u_j - u_{j-1} \|_{0,\infty,\Omega_\delta}$ for $1 \le j \le N_0$ and $\| u_{N_0} - u \|_{0,\infty,\Omega_\delta}$. For any $j, 1 \le j \le N_0$, we have $u_j - u_{j-1} = \psi - \psi_\delta$ with

$$\begin{aligned}
&-\mathrm{div}\,(K(x)\nabla\psi) = \mathrm{q}_j \delta_{\bar{x}_j} \quad \text{in } \hat{\Omega}_j = \hat{\Omega}_{j-1} \cup \bar{B}_j, \\
&\psi|_{\Gamma_i} = \mathrm{const}\,, \quad \int_{\Gamma_i} K \frac{\partial \psi}{\partial \nu} ds = 0 \quad \text{for } i = j+1, \cdots, N_0, \\
&\psi|_{\Gamma_i} = 0 \quad \text{for } i \in I_D, \quad \psi|_\Gamma = 0.
\end{aligned}$$

and

$$\begin{aligned}
&-\mathrm{div}\,(K(x)\nabla\psi_\delta) = 0 \quad \text{in } \hat{\Omega}_{j-1}, \\
&\psi_\delta|_{\Gamma_j} = \mathrm{const}\,, \quad \int_{\Gamma_j} K \frac{\partial \psi_\delta}{\partial \nu} ds = \mathrm{q}_j, \\
&\psi_\delta|_{\Gamma_i} = \mathrm{const}\,, \quad \int_{\Gamma_i} K \frac{\partial \psi_\delta}{\partial \nu} ds = 0 \quad \text{for } i = j+1, \cdots, N_0, \\
&\psi_\delta|_{\Gamma_i} = 0 \quad \text{for } i \in I_D, \quad \psi_\delta|_\Gamma = 0.
\end{aligned}$$

Let $\phi_j = -\dfrac{1}{2\pi K_j} \ln|x - \bar{x}_j|$, $z_\delta = \psi_\delta - q_j\phi_j$ and $z = \psi - q_j\phi_j$. Then $v_\delta = z_\delta - z = \psi_\delta - \psi$ satisfies

$$-\mathrm{div}\,(K(x)\nabla v_\delta) = 0 \quad \text{in } \hat{\Omega}_{j-1}, \tag{2.22}$$

$$v_\delta|_{\Gamma_j} = \psi_\delta|_{\Gamma_j} + \frac{\mathrm{q}_j}{2\pi K_j} \ln\delta - z|_{\Gamma_j}, \quad \int_{\Gamma_j} K \frac{\partial v_\delta}{\partial \nu} ds = 0, \tag{2.23}$$

$$v_\delta|_{\Gamma_i} = \mathrm{const}\,, \quad \int_{\Gamma_i} K \frac{\partial v_\delta}{\partial \nu} ds = 0 \quad \text{for } i = j+1, \cdots, N_0, \tag{2.24}$$

$$v_\delta|_{\Gamma_i} = 0 \quad \text{for } i \in I_D, \quad v_\delta|_\Gamma = 0. \tag{2.25}$$

To estimate $\| v_\delta \|_{0,\infty,\Omega_\delta}$, we split v_δ into two parts: $v_\delta = v_1 + v_2$, where v_1 satisfies the problem

$$\begin{aligned}
&-\mathrm{div}\,(K(x)\nabla v_1) = 0 \quad \text{in } \hat{\Omega}_{j-1}, \\
&v_1|_{\Gamma_j} = z(\bar{x}_j) - z|_{\Gamma_j}, \\
&v_1|_{\Gamma_i} = \mathrm{const}\,, \quad \int_{\Gamma_i} K \frac{\partial v_1}{\partial \nu} ds = 0 \quad \text{for } i = j+1, \cdots, N_0, \\
&v_1|_{\Gamma_i} = 0 \quad \text{for } i \in I_D, \quad v_1|_\Gamma = 0,
\end{aligned}$$

and v_2 satisfies the problem

$$\begin{aligned}
&-\mathrm{div}\,(K(x)\nabla v_2) = 0 \quad \text{in } \hat{\Omega}_{j-1}, \\
&v_2|_{\Gamma_j} = \psi_\delta|_{\Gamma_j} + \frac{\mathrm{q}_j}{2\pi K_j} \ln\delta - z(\bar{x}_j) = \mathrm{const}\,, \quad \int_{\Gamma_j} K \frac{\partial v_2}{\partial \nu} ds = -\int_{\Gamma_j} K \frac{\partial v_1}{\partial \nu} ds, \\
&v_2|_{\Gamma_i} = \mathrm{const}\,, \quad \int_{\Gamma_i} K \frac{\partial v_2}{\partial \nu} ds = 0 \quad \text{for } i = j+1, \cdots, N_0, \\
&v_2|_{\Gamma_i} = 0 \quad \text{for } i \in I_D, \quad v_2|_\Gamma = 0.
\end{aligned}$$

Our program is the following. We first estimate $\max_{x\in\Gamma_j}|z(x)-z(\bar{x}_j)|$ which gives the estimate $\|v_1\|_{0,\infty,\hat{\Omega}_{j-1}}$ by maximum principle. We then use a comparison principle to give an estimate for the flow rate $\displaystyle\int_{\Gamma_j} K\frac{\partial v_1}{\partial\nu}dx$ which in turn gives the same estimate for the flow rate $\displaystyle\int_{\Gamma_j} K\frac{\partial v_2}{\partial\nu}dx$. Then Lemma 2.2 gives the estimate for $v_2|_{\Gamma_j}$ and thus an application of the maximum principle yields the desired bound for $\|v_2\|_{0,\infty,\hat{\Omega}_{j-1}}$.

To derive a bound for $\max_{x\in\Gamma_j}|z(x)-z(\bar{x}_j)|$, we first recall the $W^{1,p}$ interior estimate for elliptic equations which plays key role in the analysis in this paper.

LEMMA 2.4. ($W^{1,p}$ interior estimate) *Let $D\subset\mathbf{R}^2$ be a bounded domain with Lipschitz boundary. Consider the elliptic equation*

$$Lu := -\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = \frac{\partial f_i}{\partial x_i}\quad \text{in } D,$$

where the usual Einstein summation convention for repeated indices is used. Assume that $a_{ij}\in C^{0,1}(\bar{D})$, $f_i\in L^\infty(D)$ and there exist constants $\lambda,\Lambda>0$ such that $\lambda|\xi|^2\le a_{ij}\xi_i\xi_j\le\lambda^{-1}|\xi|^2$ for any $x\in\bar{D}$, $\xi\in\mathbf{R}^2$, and $|a_{ij}(x)-a_{ij}(y)|\le\Lambda|x-y|$ for any $x,y\in\bar{D}$, $i,j=1,2$. Let $d=\mathrm{diam}(D)$ and $Q\subset D$ be a square with edges parallel to the coordinate axes such that $\mathrm{dist}\,(Q,\partial D)\ge C_0 d$ for some constant $C_0>0$. Then for any $p>2$, there exists a constant C depending only on λ and C_0 such that

$$\|\nabla u\|_{0,p,Q}\le Cp|Q|^{\frac{1}{p}}\Big\{(d^{-1}+\Lambda)\|\nabla u\|_{0,D}+\sum_{i=1}^2\|f_i\|_{0,\infty,D}\Big\}.$$

$W^{1,p}$ interior estimates for elliptic equations are well-known in the literature. The importance in above estimate is the explicit dependent of the bound on p which is crucial in our analysis. the proof of this lemma will be briefly recalled in the appendix at the end of this paper.

LEMMA 2.5. *Under the assumption of Theorem 2.3 we have the following estimate*

$$\max_{x\in\Gamma_j}|z(x)-z(\bar{x}_j)|\le C\Lambda(1+\ln\Lambda)^{1/2}|q_j|\delta|\ln\delta|.$$

Proof. Recall that $d_j=\frac{1}{2}\min(\min_{x\in\Gamma}|x-\bar{x}_j|,\min_{i\neq j}|\bar{x}_i-\bar{x}_j|)$. By Morrey's embedding theorem (cf. e.g. [12, §7.8]) we know that

$$\max_{x\in\Gamma_j}|z(x)-z(\bar{x}_j)|\le C(2,p)\delta^\gamma\|\nabla z\|_{0,p,B(\bar{x}_j,\delta)}\le C(2,p)\delta^\gamma\|\nabla z\|_{0,p,B(\bar{x}_j,d_j/4)},$$

where $\gamma=1-2/p$, $p>2$, and $C(2,p)$ is uniformly bounded for $p\ge3$. Recall that $z=\psi-\mathrm{q}_j\phi_j$ satisfies

$$-\mathrm{div}\,(K(x)\nabla z)=\mathrm{q}_j\mathrm{div}\,((K-K_j)\nabla\phi_j)\quad \text{in } \hat{\Omega}_j, \tag{2.26}$$

$$z|_{\Gamma_i}=\psi|_{\Gamma_i}-\mathrm{q}_j\phi_j|_{\Gamma_i},\qquad \int_{\Gamma_i}K\frac{\partial z}{\partial\nu}ds=0\quad \text{for } i=j+1,\cdots,N_0, \tag{2.27}$$

$$z|_{\Gamma_i}=-\mathrm{q}_j\phi_j|_{\Gamma_i}\quad \text{for } i\in I_D,\qquad z|_\Gamma=-\mathrm{q}_j\phi_j|_\Gamma. \tag{2.28}$$

By $W^{1,p}$ interior estimate in Lemma 2.4 and (H1) we know that

$$
\begin{aligned}
& \| \nabla z \|_{0,p,B(\bar{x}_j,d_j/4)} \\
& \le Cpd_j^{\frac{2}{p}} \Big\{ (d_j^{-1} + \Lambda) \| \nabla z \|_{0,\hat{\Omega}_j} + |\mathrm{q}_j| \| (K - K_j) \nabla \phi_j \|_{0,\infty,\hat{\Omega}_j} \Big\} \\
& \le Cp \Big\{ \Lambda \| \nabla z \|_{0,\hat{\Omega}_j} + |\mathrm{q}_j| \Lambda \Big\}.
\end{aligned} \tag{2.29}
$$

Let $\omega_j \in C^\infty(\hat{\Omega}_j)$ be the cut-off function such that $\omega_j = 1$ for $|x - \bar{x}_j| \ge d_j$, $\omega_j = 0$ for $|x - \bar{x}_j| \le d_j/2$, and $|\nabla \omega_j| \le C d_j^{-1}$ in $\hat{\Omega}_j$. Then multiplying (2.26) by $z - \mathrm{q}_j \phi_j \omega_j$ and using standard energy estimate argument we obtain that

$$
\| \nabla z \|_{0,\hat{\Omega}_j} \le C |\mathrm{q}_j| \Big\{ \| (K - K_j) \nabla \phi_j \|_{0,\Omega} + \| \nabla (\phi_j \omega_j) \|_{0,\Omega} \Big).
$$

By (H1) we know that

$$
\begin{aligned}
& \| (K - K_j) \nabla \phi_j \|_{0,\Omega}^2 \\
&= \int_{B(\bar{x}_j, 1/\Lambda)} |(K - K_j) \nabla \phi_j|^2 dx + \int_{\Omega\{|x - \bar{x}_j| \ge 1/\Lambda\}} |(K - K_j) \nabla \phi_j|^2 dx \\
&\le C \Lambda^2 |B(\bar{x}_j, 1/\Lambda)| + C \int_{\Omega\{|x - \bar{x}_j| \ge 1/\Lambda\}} |x - \bar{x}_j|^{-2} dx \\
&\le C + C \int_0^{2\pi} d\theta \int_{1/\Lambda}^{D_j} r^{-1} dr \\
&\le C + C \ln \Lambda,
\end{aligned}
$$

where $D_j = \operatorname{dist}(\bar{x}_j, \Gamma)$ and $\Omega\{|x - \bar{x}_j| \ge 1/\Lambda\} = \{x \in \Omega : |x - \bar{x}_j| \ge 1/\Lambda\}$. Moreover, it is easy to see that $\| \nabla (\phi_j \omega_j) \|_{0,\Omega} \le C$. Thus we conclude that

$$
\| \nabla z \|_{0,\hat{\Omega}_j} \le C (1 + \ln \Lambda)^{1/2} |\mathrm{q}_j|.
$$

Thus we have from (2.26), (2.29) that

$$
\max_{x \in \Gamma_j} |z(x) - z(\bar{x}_j)| \le C \delta^{1 - \frac{2}{p}} p \Lambda (1 + \ln \Lambda)^{1/2} |\mathrm{q}_j|.
$$

This completes the proof by taking $p = \ln(1/\delta)$. □

LEMMA 2.6. *Under the assumption of Theorem 2.3 we have, for any $j \in I_M$,*

$$
\| u_j - u_{j-1} \|_{0,\infty,\hat{\Omega}_{j-1}} \le C \Lambda (1 + \ln \Lambda)^{1/2} |\mathrm{q}_j| \delta |\ln \delta|.
$$

Proof. Applying maximum principle, Hopf's lemma and Lemma 2.5 we know that

$$
\| v_1 \|_{0,\infty,\hat{\Omega}_{j-1}} \le \max_{x \in \Gamma_j} |z(x) - z(\bar{x}_j)| \le C \Lambda (1 + \ln \Lambda)^{1/2} |\mathrm{q}_j| \delta |\ln \delta| := \hat{C}.
$$

Denote by $V = \hat{C} v \pm v_1$, where v is defined in Lemma 2.2. Then we know that

$$
\begin{aligned}
& -\operatorname{div}(K(x) \nabla V) = 0 \quad \text{in } \hat{\Omega}_{j-1}, \\
& V|_{\Gamma_j} \ge 0, \quad V|_{\Gamma_i} = 0 \quad \text{for } i \in I_D, \quad V|_\Gamma = 0, \\
& V|_{\Gamma_i} = \text{const}, \quad \int_{\Gamma_i} K \frac{\partial V}{\partial \nu} ds = 0 \quad \text{for } i = j+1, \cdots, N_0.
\end{aligned}
$$

Again by maximum principle we have $V \geq 0$ in $\hat{\Omega}_{j-1}$, which implies

$$\int_{\Gamma_j} K \frac{\partial V}{\partial \nu} ds = -\Big\{ \sum_{i \in I_D} \int_{\Gamma_i} K \frac{\partial V}{\partial \nu} ds + \int_{\Gamma} K \frac{\partial V}{\partial \nu} ds \Big\} \geq 0.$$

Thus, by Lemma 2.2,

$$\Big| \int_{\Gamma_j} K \frac{\partial v_1}{\partial \nu} ds \Big| \leq \hat{C} \int_{\Gamma_j} K \frac{\partial v}{\partial \nu} ds \leq C\Lambda(1 + \ln \Lambda)^{1/2} |\mathrm{q}_j| \delta.$$

Now by (2.23) and the fact that $v_\delta = v_1 + v_2$ we deduce that

$$\Big| \int_{\Gamma_j} K \frac{\partial v_2}{\partial \nu} ds \Big| \leq C\Lambda(1 + \ln \Lambda)^{1/2} |\mathrm{q}_j| \delta.$$

Since $v_2|_{\Gamma_j} = \text{const}$, we may apply Lemma 2.2 to $v_2/(v_2|_{\Gamma_j})$ and obtain

$$\frac{1}{v_2|_{\Gamma_j}} \int_{\Gamma_j} K \frac{\partial v_2}{\partial \nu} ds \geq 2\pi\lambda \Big(\ln \frac{D_j}{\delta} \Big)^{-1}$$

which yields $|v_2|_{\Gamma_j}| \leq C\Lambda(1+\ln\Lambda)^{1/2}|\mathrm{q}_j|\delta|\ln\delta|$. Another application of the maximum principle implies

$$\| v_2 \|_{0,\infty,\hat{\Omega}_{j-1}} \leq C\Lambda(1 + \ln \Lambda)^{1/2} |\mathrm{q}_j| \delta |\ln \delta|.$$

This completes the proof since $u_j - u_{j-1} = v_\delta = v_1 + v_2$. ☐

Now we are in the position to complete the proof of Theorem 2.3.

Proof of Theorem 2.3. By (2.21) and Lemma 2.6 we are left to estimate $\| u_{N_0} - u \|_{0,\infty,\Omega_\delta}$. Let $\phi_j = -\dfrac{1}{2\pi K_j} \ln |x - \bar{x}_j|$, $U_{N_0} = u_{N_0} - \sum_{j=1}^N \mathrm{q}_j \phi_j$ and $U = u - \sum_{j=1}^N \mathrm{q}_j \phi_j$, then we have $e = u - u_{N_0} = U - U_{N_0}$ satisfies

$$\begin{aligned} &-\mathrm{div}\,(K(x)\nabla e) = 0 \quad \text{ in } \hat{\Omega}_{N_0} \\ &e|_{\Gamma_i} = u|_{\Gamma_i} - \alpha_i \quad \text{ for } i \in I_D, \qquad e|_{\Gamma} = 0. \end{aligned}$$

From (2.5) we have

$$\begin{aligned} u|_{\Gamma_i} - \alpha_i &= u|_{\Gamma_i} + \frac{\mathrm{q}_i}{2\pi K_i} \ln \delta - \sum_{j \neq i} \mathrm{q}_j \phi_j(\bar{x}_i) + U(\bar{x}_i) \\ &= \sum_{j \neq i} \mathrm{q}_j (\phi_j|_{\Gamma_i} - \phi_j(\bar{x}_i)) + U|_{\Gamma_i} - U(\bar{x}_i). \end{aligned}$$

Direct calculation implies

$$\max_{x \in \Gamma_i} |\phi_j(x) - \phi_j(\bar{x}_i)| \leq C\delta \| \nabla \phi_j \|_{0,\infty,B_j} \leq C\delta \quad \text{ for any } j \neq i.$$

The estimation for $U|_{\Gamma_i} - U(\bar{x}_i)$ can be carried out by using the similar argument in Lemma 2.5 to get

$$\max_{x \in \Gamma_i} |U(x) - U(\bar{x}_i)| \leq C\Lambda(1 + \ln \Lambda)^{1/2} \Big(\sum_{j=1}^N |\mathrm{q}_j| \Big) \delta |\ln \delta|.$$

Thus by maximum principle we get

$$\| u_{N_0} - u \|_{0,\infty,\hat{\Omega}_{N_0}} \le |u|_{\Gamma_i} - \alpha_i| \le C\Lambda(1+\ln\Lambda)^{1/2}\Big(\sum_{j=1}^{N} |\mathrm{q}_j|\Big)\delta|\ln\delta|.$$

This completes the proof. $\Box$

The following theorem shows that for $i \in I_D$, the solution q_i of (2.4)-(2.6) also provides good approximation to the well flow rate $\int_{\Gamma_i} K\frac{\partial u_\delta}{\partial \nu} ds$ of the problem (2.1)-(2.3).

THEOREM 2.7. *Let* $\mathrm{q}_j^\delta = \mathrm{q}_j$ *for* $j \in I_M$, *and for* $i \in I_D$, $\mathrm{q}_i^\delta = \int_{\Gamma_i} K\frac{\partial u_\delta}{\partial \nu} ds$ *for* $i \in I_D$, *where* u_δ *is the solution of the problem* (2.1)-(2.3). *Then there exists a constant* C *independent of* δ, q_j *and* Λ *such that for* $\delta\Lambda \le \gamma_0$,

$$|\mathrm{q}_i - \mathrm{q}_i^\delta| \le C\Lambda(1+\ln\Lambda)^{1/2}\sum_{j=1}^{N}\Big(|\mathrm{q}_j| + |\mathrm{q}_j^\delta|\Big)\delta|\ln\delta| \qquad \forall i \in I_D. \tag{2.30}$$

Here the constant γ_0 *is defined in Lemma 2.1.*

Proof. We begin with introducing the auxiliary problem: Find $\tilde{u}_\delta$ such that

$$-\mathrm{div}\,(K(x)\nabla\tilde{u}_\delta) = \sum_{j=1}^{N} \mathrm{q}_j^\delta \delta_{\bar{x}_j} \quad \text{in } \Omega, \quad \tilde{u}_\delta|_\Gamma = 0.$$

By Lemma 2.6 we can show that

$$\max_{x\in\bar{\Omega}_\delta} |\tilde{u}_\delta - u_\delta| \le C\Lambda(1+\ln\Lambda)^{1/2}\Big(\sum_{j=1}^{N} |\mathrm{q}_j^\delta|\Big)\delta|\ln\delta|.$$

Hence, by Theorem 2.3,

$$|(u-\tilde{u}_\delta)|_{\Gamma_i}| \le C\Lambda(1+\ln\Lambda)^{1/2}\sum_{j=1}^{N}\Big(|\mathrm{q}_j^\delta| + |\mathrm{q}_j|\Big)\delta|\ln\delta|.$$

On the other hand, let Θ_j be the Green function in (2.7) and $\eta_j = \Theta_j - \phi_j$. Then, since $\mathrm{q}_j = \mathrm{q}_j^\delta$ for $j \in I_M$, we have

$$(u-\tilde{u}_\delta)|_{\Gamma_i} = \sum_{j\in I_D} (\mathrm{q}_j - \mathrm{q}_j^\delta)\Theta_j|_{\Gamma_i} \quad \forall i \in I_D.$$

Therefore, we have, for any $i \in I_D$,

$$\Big[-\frac{1}{2\pi K_i}\ln\delta + \eta_i(\bar{x}_i)\Big](\mathrm{q}_i - \mathrm{q}_i^\delta) + \sum_{j\in I_D, j\ne i}\Theta_j(\bar{x}_i)(\mathrm{q}_j - \mathrm{q}_j^\delta) = b_i, \tag{2.31}$$

with

$$b_i = (u-\tilde{u}_\delta)|_{\Gamma_i} + (\eta_i(\bar{x}_i) - \eta_i|_{\Gamma_i})(\mathrm{q}_i - \mathrm{q}_i^\delta) + \sum_{j\in I_D, j\ne i}(\Theta_j(\bar{x}_i) - \Theta_j|_{\Gamma_i})(\mathrm{q}_j - \mathrm{q}_j^\delta).$$

Similar to the argument in Lemma 2.5, we can show that

$$|\eta_i(\bar{x}_i) - \eta_i|_{\Gamma_i}| + |\Theta_j(\bar{x}_i) - \Theta_j|_{\Gamma_i}| \le C\Lambda(1+\ln\Lambda)^{1/2}\delta|\ln\delta|, \qquad \forall i \in I_D.$$

Thus

$$|b_i| \le C\Lambda(1+\ln\Lambda)^{1/2}\sum_{j=1}^{N}\Big(|\mathrm{q}_j| + |\mathrm{q}_j^\delta|\Big)\delta|\ln\delta| \qquad \forall i \in I_D.$$

This completes the proof upon using (2.9) which ensures the associated coefficient matrix in (2.31) is strictly diagonally dominant. ◻

3. The multiscale method. In this section we are going to introduce our multiscale method to solve the problem (2.4)-(2.6) which is a good approximation of the original problem (2.1)-(2.3) for small δ according to the analysis in last section. We make the following local periodicity assumption for the coefficient:

(H2) $K(x) = K_\varepsilon(x) = K(x, x/\varepsilon)$ for some small parameter $\varepsilon > 0$, we assume $K \in C^1(\bar{\Omega}; C_p^1(\mathbf{R}^2))$, where $C_p^1(\mathbf{R}^2)$ stands for the collection of all $C^1(\mathbf{R}^2)$ periodic functions with respect to the unit square Y.

As we have already pointed out in §1, the local periodicity assumption is not required in the definition of our method. This assumption allows us to use the homogenization theory to obtain the asymptotic structure of the solutions. The numerical experiments to be presented in §7 indicate that our method works fine for the well-accepted random log-normal permeability models in the engineering literature.

Our method is based on the over-sampling multiscale finite element method in [10] which reduces the so-called resonance error in the previous multiscale finite element method in [15, 16]. We first briefly recall the basic facts of the over-sampling multiscale finite element method. Let $\mathcal{M}_H$ be a regular and quasi-uniform triangulation of Ω and V_H the standard conforming linear finite element space over $\mathcal{M}_H$. For any $T \in \mathcal{M}_H$ with nodes $\{x_i^T\}_{i=1}^3$, let H_T denote the size of T, $P_1(T)$ the set of linear polynomials defined in T, and $\{\varphi_i^T\}_{i=1}^3$the basis of $P_1(T)$ satisfying $\varphi_i^T(x_j^T) = \delta_{ij}, i, j = 1, 2, 3$. For any $T \in \mathcal{M}_H$, we denote by $S = S(T)$ a macro-element which contains T and satisfies the following condition:

(H3) $H_S \le C_1 H_T$ and $\mathrm{dist}\,(\partial T, \partial S) \ge \delta_0 H_T$ for some positive constants C_1, δ_0 independent of H. The minimum angles of $S(T)$ is bounded below by some positive constant θ_0 independent of H.

Let $MS(S)$ be the multiscale finite element space spanned by $\psi_i^S, i = 1, 2, 3$, with $\psi_i^S \in H^1(S)$ being the solution of the problem

$$-\mathrm{div}\,(K_\varepsilon(x)\nabla\psi_i^S) = 0 \quad \text{in } S, \qquad \psi_i^S|_{\partial S} = \varphi_i^S. \tag{3.1}$$

Here $\{\varphi_i^S\}_{i=1}^3$ is the nodal basis of $P_1(S)$ such that $\varphi_i^S(x_j^S) = \delta_{ij}$, $i, j = 1, 2, 3$. The over-sampling multiscale finite element base functions over T is defined by

$$\bar{\psi}_i^T = c_{ij}^T \psi_j^S|_T \quad \text{in } T, \tag{3.2}$$

with the constants c_{ij}^T so chosen that

$$\varphi_i^T = c_{ij}^T \varphi_j^S|_T \quad \text{in } T. \tag{3.3}$$

The existence of the constants c_{ij}^T is guaranteed because $\{\varphi_j^S\}_{j=1}^3$ also forms the basis of $P_1(T)$. In practical applications, the base functions ψ_i^S of $MS(S)$ will be further approximated by solving (3.1) on a mesh of S with a mesh size resolving ε using linear finite element methods [10]. These base functions can be precomputed initially to generate the coarse grid operator which turns out to be only a small overhead of the overall computations. This is also a common practice in many upscaling methods [26].

Let $OMS(T) = \text{span}\{\bar{\psi}_i^T\}_{i=1}^3$ and $\Pi_T : OMS(T) \to P_1(T)$ the projection

$$\Pi_T\psi = c_i\varphi_i^T \quad \text{ if } \psi = c_i\bar{\psi}_i^T \in OMS(T).$$

Let $\bar{X}_H$ be the finite element space

$$\bar{X}_H = \{\psi_H : \psi_H|_T \in OMS(T), \ \forall T \in \mathcal{M}_H\}$$

and define $\Pi_H : \bar{X}_H \to \Pi_{T\in\mathcal{M}_H} P_1(T)$ through the relation $\Pi_H\psi_H|_T = \Pi_T\psi_H$ for any $T \in \mathcal{M}_H$, $\psi_H \in \bar{X}_H$. The over-sampling multiscale finite element space is then defined as

$$X_H = \{\psi_H \in \bar{X}_H : \Pi_H\psi_H \in V_H \subset H^1(\Omega)\}.$$

In general, $X_H \not\subset H^1(\Omega)$ and the requirement $\Pi_H\psi_H \in V_H$ is to impose certain continuity of the functions $\psi_H \in X_H$ across the inter-element boundaries.

For any $j = 1, 2, \cdots, N$, let Ω_j be a convex polygon inside Ω which consists of a union of elements in $\mathcal{M}_H$ such that $\bar{B}_j \subset\subset \Omega_j$. We assume $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ for any $i \neq j$. Let G_j^ε be the Green function

$$-\text{div}\,(K_\varepsilon(x)\nabla G_j^\varepsilon) = \delta_{\bar{x}_j} \quad \text{ in } \Omega_j, \quad G_j^\varepsilon|_{\Sigma_j} = 0, \tag{3.4}$$

where $\Sigma_j = \partial\Omega_j$. Let u_ε and $\{\mathrm{q}_i\}_{i\in I_D}$ be the solution of the problem (2.4)-(2.6). Then the argument in §1 yields that $\zeta_\varepsilon = u_\varepsilon - \sum_{j=1}^N \mathrm{q}_j G_j^\varepsilon \in H_0^1(\Omega)$ and $\{\mathrm{q}_i\}_{i\in I_D}$ satisfies

$$\int_\Omega K_\varepsilon(x)\nabla\zeta_\varepsilon \cdot \nabla\chi dx = -\sum_{j=1}^N \mathrm{q}_j \int_{\Sigma_j} K_\varepsilon \frac{\partial G_j^\varepsilon}{\partial\nu}\chi ds \quad \forall\chi \in H_0^1(\Omega), \tag{3.5}$$

$$-\frac{\mathrm{q}_i}{2\pi K_i}\ln\delta + \mathrm{q}_i w_i(\bar{x}_i) + \zeta_\varepsilon(\bar{x}_i) = \alpha_i \quad \forall i \in I_D. \tag{3.6}$$

Here and in the following G_j^ε is extended to be zero outside Ω_j, and $w_j = G_j^\varepsilon - \phi_j$ with $\phi_j = -\dfrac{1}{2\pi K_j}\ln|x - \bar{x}_j|$.

Let $X_H^0 = \{\psi_H \in X_H : \Pi_H\psi_H = 0 \ \text{ on } \Gamma\}$. Then we introduce the following discrete problem: Find $\zeta_H \in X_H^0$ and $\{\mathrm{q}_i^H\}_{i\in I_D}$ such that

$$\sum_{T\in\mathcal{M}_H}\int_T K_\varepsilon(x)\nabla\zeta_H \cdot \nabla\chi_H dx = -\sum_{j=1}^N \mathrm{q}_j^H \int_{\Sigma_j} K_\varepsilon \frac{\partial G_j^\varepsilon}{\partial\nu}\hat{\chi}_H ds \quad \forall\chi_H \in X_H^0, \tag{3.7}$$

$$-\frac{\mathrm{q}_i^H}{2\pi K_i}\ln\delta + \mathrm{q}_i^H w_i(\bar{x}_i) + \zeta_H(\bar{x}_i) = \alpha_i \quad \forall i \in I_D, \tag{3.8}$$

where $\hat{\chi}_H = \Pi_H\chi_H \in V_H^0 = V_H \cap H_0^1(\Omega)$, and we set $\mathrm{q}_j^H = \mathrm{q}_j$ for $j \in I_M$ to simplify the notation.

Now we are going to prove the existence of unique solutions of the discrete problem (3.7)-(3.8). For any $j, 1 \le j \le N$, let $\zeta_j^\varepsilon \in H_0^1(\Omega)$ be the solution of the problem

$$\int_\Omega K_\varepsilon(x)\nabla\zeta_j^\varepsilon \cdot \nabla\chi dx = -\int_{\Sigma_j} K_\varepsilon \frac{\partial G_j^\varepsilon}{\partial\nu}\chi ds \quad \forall\chi \in H_0^1(\Omega), \tag{3.9}$$

and $\zeta_j^H \in X_H^0$ the solution of the corresponding discrete problem

$$\sum_{T\in\mathcal{M}_H}\int_T K_\varepsilon(x)\nabla\zeta_j^H \cdot \nabla\chi_H dx = -\int_{\Sigma_j} K_\varepsilon \frac{\partial G_j^\varepsilon}{\partial\nu}\hat{\chi}_H ds \quad \forall\chi_H \in X_H^0. \tag{3.10}$$

The following theorem which will be proved in §5 is the basic building block of our convergence analysis. First we make the following assumption on the local domains Ω_j.

(H4) Let $H_j = \text{dist}\,(\bar{x}_j, \Sigma_j)$ and there exist constants $C_2, C_3 > 0$ such that $C_2 H_i \le H_j \le C_3 H_i$ for any $i, j = 1, 2, \cdots, N$.

This assumption requires that the local domains for computing local Green functions should be roughly of the same size. Our numerical experiments in §7 indicate that in practical computations, the size of the local domains need not be very large (two or three times of the size of the coarse mesh would be enough).

THEOREM 3.1. *Let the assumptions (H1)-(H4) be fulfilled. Then there exists a constant C independent of ε, δ, H and $H_j, 1 \le j \le N$, such that for any $i, j = 1, 2, \cdots, N$, the following error estimate is valid*

$$\max_{x\in\bar{B}_j}|\zeta_i^\varepsilon(x) - \zeta_i^H(\bar{x}_j)| \le C\Big\{H + \frac{H^2}{H_j^2}\Big|\ln\frac{H}{H_j}\Big|^2 + \frac{\varepsilon}{H} + \sqrt{\frac{\varepsilon}{H_j}}\Big\}|\ln H|.$$

Now we are in the position to prove the existence of unique solution of the discrete problem (3.7)-(3.8).

THEOREM 3.2. *Let the assumptions (H1)-(H4) be fulfilled. Then these exists a constant γ_1 such that if $\delta/\varepsilon \le \gamma_1$, the problem* (3.7)-(3.8) *has a unique solution. Moreover, the following error estimates are valid, for $i \in I_M$,*

$$\max_{x\in\Gamma_i}|\zeta_\varepsilon - \zeta_H(\bar{x}_i)| \le C \max_{1\le j\le N}\max_{x\in\Gamma_i}|\zeta_j^\varepsilon - \zeta_j^H(\bar{x}_i)|\sum_{j=1}^N|\mathrm{q}_j|, \tag{3.11}$$

and, for $i \in I_D$,

$$|\mathrm{q}_i - \mathrm{q}_i^H| \le C \max_{1\le j\le N}|\zeta_j^\varepsilon(\bar{x}_i) - \zeta_j^H(\bar{x}_i)|\sum_{j=1}^N|\mathrm{q}_j|. \tag{3.12}$$

Proof. By definition (3.10) we know that $\zeta_H = \sum_{j=1}^N \mathrm{q}_j^H\zeta_j^H$. Substituting it into (3.8) we know that $\{\mathrm{q}_i^H\}_{i\in I_D}$ satisfies the linear system of equations

$$\Big[-\frac{1}{2\pi K_i}\ln\delta + w_i(\bar{x}_i) + \zeta_i^H(\bar{x}_i)\Big]\mathrm{q}_i^H + \sum_{j\ne i, j\in I_D}\zeta_j^H(\bar{x}_i)\mathrm{q}_j^H = \alpha_i - \sum_{j\in I_M}\mathrm{q}_j\zeta_j^H(\bar{x}_i), \quad \forall i \in I_D \tag{3.13}$$

where we have used the definition $\mathrm{q}_j^H = \mathrm{q}_j$ for $j \in I_M$. Let $\mathcal{A} = (\alpha_{ij}) \in \mathbf{R}^{I_D \times I_D}$ be the associated coefficient matrix with elements

$$\alpha_{ii} = -\frac{1}{2\pi K_i} \ln \delta + w_i(\bar{x}_i) + \zeta_i^H(\bar{x}_i), \qquad \alpha_{ij} = \zeta_j^H(\bar{x}_i) \qquad \text{for } i \neq j.$$

Our goal is to prove $\mathcal{A}$ is strictly diagonally dominant. Let Θ_j^ε be the Green function defined in (2.7) and $\eta_j = \Theta_j^\varepsilon - \phi_j$. Then from the definition of ζ_i^ε in (3.9) we know that

$$\zeta_j^\varepsilon = \begin{cases} \Theta_j^\varepsilon & \text{outside } \Omega_j, \\ \Theta_j^\varepsilon - G_j^\varepsilon & \text{in } \Omega_j. \end{cases} \tag{3.14}$$

Thus, for $i \neq j$, we have

$$\begin{aligned} |\alpha_{ij}| = |\zeta_j^H(\bar{x}_i)| &\leq |\zeta_j^H(\bar{x}_i) - \zeta_j^\varepsilon(\bar{x}_i)| + |\zeta_j^\varepsilon(\bar{x}_i)| \\ &= |\zeta_j^H(\bar{x}_i) - \zeta_j^\varepsilon(\bar{x}_i)| + |\Theta_j^\varepsilon(\bar{x}_i)| \leq C, \end{aligned}$$

where we have used Theorem 3.1 and the well-known uniform estimate for Green function Θ_j^ε away from the singularity (cf. e.g. [20, 14]). On the other hand, since $w_i = G_i^\varepsilon - \phi_i$, we deduce from (2.9) and Theorem 3.1 that

$$\begin{aligned} \alpha_{ii} &= -\frac{1}{2\pi K_i} \ln \delta + w_i(\bar{x}_i) + \zeta_i^\varepsilon(\bar{x}_i) + \zeta_i^H(\bar{x}_i) - \zeta_i^\varepsilon(\bar{x}_i) \\ &= -\frac{1}{2\pi K_i} \ln \delta + \eta_i(\bar{x}_i) + \zeta_i^H(\bar{x}_i) - \zeta_i^\varepsilon(\bar{x}_i) \\ &\geq -\frac{1}{2\pi K_i} \ln \frac{\delta}{\varepsilon} - C. \end{aligned}$$

Therefore, for sufficiently small δ/ε, we have

$$\alpha_{ii} - \sum_{j \neq i, j \in I_D} |\alpha_{ij}| \geq C > 0. \tag{3.15}$$

This shows the coefficient matrix $\mathcal{A}$ is strictly diagonally dominant and thus (3.7)-(3.8) has a unique solution. To derive the error estimate, we subtract the equations (3.13) from (2.8), use (3.14) to get

$$\begin{aligned} &\alpha_{ii}(\mathrm{q}_i^H - \mathrm{q}_i) + \sum_{j \neq i, j \in I_D} \zeta_j^H(\bar{x}_j)(\mathrm{q}_j^H - \mathrm{q}_j) \\ &= \mathrm{q}_i(\zeta_i^\varepsilon(\bar{x}_i) - \zeta_i^H(\bar{x}_i)) + \sum_{j \neq i} \mathrm{q}_j(\zeta_j^\varepsilon(\bar{x}_i) - \zeta_j^H(\bar{x}_i)) \\ &= \sum_{j=1}^N \mathrm{q}_j(\zeta_j^\varepsilon(\bar{x}_i) - \zeta_j^H(\bar{x}_i)) \qquad \forall i \in I_D. \end{aligned}$$

Now by using (3.15) we deduce that

$$\max_{i \in I_D} |\mathrm{q}_i - \mathrm{q}_i^H| \leq C \max_{1 \leq j \leq N} |\zeta_j^\varepsilon(\bar{x}_i) - \zeta_j^H(\bar{x}_i)| \sum_{j=1}^N |\mathrm{q}_j|. \tag{3.16}$$

This proves (3.12). Now for any $i \in I_M$, we have

$$\begin{aligned}
&\max_{x\in\Gamma_i} |\zeta_\varepsilon(x) - \zeta_H(\bar{x}_i)| \\
&\le \max_{x\in\Gamma_i} \sum_{j=1}^N \mathrm{q}_j |\zeta_j^\varepsilon(x) - \zeta_j^H(\bar{x}_i)| + \sum_{j=1}^N |\mathrm{q}_j - \mathrm{q}_j^H||\zeta_j^H(\bar{x}_i)| \\
&\le \max_{1\le j\le N} \max_{x\in\Gamma_i} |\zeta_j^\varepsilon(x) - \zeta_j^H(\bar{x}_i)| \sum_{j=1}^N |\mathrm{q}_j| + \sum_{j\in I_D} |\mathrm{q}_j - \mathrm{q}_j^H||\zeta_j^H(\bar{x}_i)| \\
&\le C \max_{1\le j\le N} \max_{x\in\Gamma_i} |\zeta_j^\varepsilon(x) - \zeta_j^H(\bar{x}_i)| \sum_{j=1}^N |\mathrm{q}_j|.
\end{aligned}$$

This completes the proof. □

In practical applications, the most interesting quantities are the well bore pressure for the wells that prescribe the flow rate and the flow rate for the wells that prescribe the well bore pressure. The following theorem which is the main result of this paper provides the error estimate for these quantities. The proof of this theorem is a direct consequence of Theorems 3.1 and 3.2.

THEOREM 3.3. *Let the assumptions (H1)-(H4) be fulfilled. Then there exists a constant C independent of ε, δ, H and H_j such that, for $j \in I_M$,*

$$\max_{x\in\Gamma_j} |\zeta_\varepsilon - \zeta_H(\bar{x}_j)| \le C\Big\{H + \frac{H^2}{H_j^2}\Big|\ln\frac{H}{H_j}\Big|^2 + \frac{\varepsilon}{H} + \sqrt{\frac{\varepsilon}{H_j}}\Big\}|\ln H| \sum_{i=1}^N |\mathrm{q}_i|,$$

and, for $j \in I_D$,

$$|\mathrm{q}_j - \mathrm{q}_j^H| \le C\Big\{H + \frac{H^2}{H_j^2}\Big|\ln\frac{H}{H_j}\Big|^2 + \frac{\varepsilon}{H} + \sqrt{\frac{\varepsilon}{H_j}}\Big\}|\ln H| \sum_{i=1}^N |\mathrm{q}_i|.$$

4. Homogenization results for Green functions. In this section we prove several homogenization results for Green functions which will be used in the subsequent analysis.

Let $D \subseteq \Omega$ be a bounded domain with Lipschitz boundary. Given $f \in L^2(D)$, we consider the following problem with rapidly oscillating coefficients: find $w_\varepsilon \in H_0^1(D)$ such that

$$\int_D a\Big(x, \frac{x}{\varepsilon}\Big)\nabla w_\varepsilon \cdot \nabla\varphi\, dx = \int_D f\varphi\, dx \quad \forall\varphi \in H_0^1(D). \tag{4.1}$$

Here $a(x, x/\varepsilon) = (a_{ij}(x, x/\varepsilon))$ is a symmetric matrix which satisfies the uniform ellipticity condition

$$\mu|\xi|^2 \le a_{ij}(x,y)\xi_i\xi_j \le \mu^{-1}|\xi|^2, \qquad \forall\xi \in \mathbf{R}^2, x \in \bar{D}, y \in \mathbf{R}^2, \tag{4.2}$$

for some constant $\mu > 0$. Furthermore, we assume that $a_{ij} \in C^1(\bar{D}; C_p^1(\mathbf{R}^2))$, where $C_p^1(\mathbf{R}^2)$ stands for the collection of all periodic $C^1(\mathbf{R}^2)$ functions with respect to the unit square Y.

Let $w_0 \in H_0^1(D)$ be the unique solution of the homogenized problem

$$\int_D a^*(x)\nabla w_0 \cdot \nabla\varphi\, dx = \int_D f\varphi\, dx \quad \forall\varphi \in H_0^1(D), \tag{4.3}$$

where $a^*(x) = (a^*_{ij}(x))$ with

$$a^*_{ij}(x) = \frac{1}{|Y|}\int_Y a_{ik}(x,y)\Big(\delta_{kj} - \frac{\partial \chi^j}{\partial y_k}(x,y)\Big)dy, \tag{4.4}$$

and $\chi^j(x,y)$ is the periodic solution of the cell problem

$$-\frac{\partial}{\partial y_i}\Big(a_{ik}(x,y)\frac{\partial \chi^j}{\partial y_k}(x,y)\Big) = \frac{\partial}{\partial y_i}a_{ij}(x,y) \ \text{ in } Y, \quad \int_Y \chi^j(x,y)dy = 0. \tag{4.5}$$

Here δ_{kj} is the Kronecker delta, i.e. $\delta_{kj} = 1$ for $k = j$, and $\delta_{kj} = 0$ for $k \neq j$.

Set

$$w_1^\varepsilon(x) = w_0(x) - \varepsilon\chi^k\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k},$$

then by simple calculations we get

$$\begin{aligned}
a_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial w_1^\varepsilon}{\partial x_j} &= a^*_{ij}(x)\frac{\partial w_0}{\partial x_j} - G_i^k\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k}\\
&\quad - \varepsilon a_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial \chi^k}{\partial x_j}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k}\\
&\quad - \varepsilon a_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\chi^k\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial^2 w_0}{\partial x_j\partial x_k},
\end{aligned} \tag{4.6}$$

where

$$G_i^k(x,y) = a^*_{ik}(x) - a_{ij}(x,y)\Big(\delta_{kj} - \frac{\partial \chi^k}{\partial y_j}(x,y)\Big). \tag{4.7}$$

Due to (4.4) and (4.5) we know that $\int_Y G_i^k(x,y)\,dy = 0$ and $\partial G_i^k(x,y)/\partial y_i = 0$. Thus there exists a skew-symmetric matrix $\alpha^k(x,y) = (\alpha^k_{ij}(x,y))$ such that [17, p.6]

$$G_i^k(x,y) = \frac{\partial}{\partial y_j}\Big(\alpha^k_{ij}(x,y)\Big), \quad \int_Y \alpha^k_{ij}(x,y)dy = 0.$$

With this notation, we can rewrite

$$G_i^k\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k} = \varepsilon\frac{\partial}{\partial x_j}\Big(\alpha^k_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k}\Big) - \varepsilon\alpha^k_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial^2 w_0}{\partial x_j\partial x_k} - \varepsilon\frac{\partial \alpha^k_{ij}}{\partial x_j}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k}.$$

Therefore, noting that the first term on the right hand side of the above formula is divergence free, we deduce from (4.1) and (4.3) that

$$\int_D a\Big(x,\frac{x}{\varepsilon}\Big)\nabla(w_\varepsilon - w_1^\varepsilon)\cdot\nabla\varphi dx = \varepsilon\int_D f_i\frac{\partial\varphi}{\partial x_i}dx \qquad \forall\varphi\in H_0^1(D) \tag{4.8}$$

where $f_i = f_i(w_0)$ is defined by

$$\begin{aligned}
f_i(w_0) = a_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial \chi^k}{\partial x_j}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k} + a_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\chi^k\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial^2 w_0}{\partial x_j\partial x_k}\\
-\alpha^k_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial^2 w_0}{\partial x_j\partial x_k} - \frac{\partial \alpha^k_{ij}}{\partial x_j}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k}.
\end{aligned} \tag{4.9}$$

Let $\theta_\varepsilon = \theta_\varepsilon(w_0) \in H^1(D)$ be the boundary corrector that satisfies

$$-\text{div}\left(a\Big(x, \frac{x}{\varepsilon}\Big)\nabla\theta_\varepsilon\right) = 0 \quad \text{in } D, \quad \theta_\varepsilon|_{\partial D} = \chi^k\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k}. \tag{4.10}$$

The following theorem is known (cf. e.g. [17, 22, 4]).

THEOREM 4.1. *Assume that $w_0 \in H^2(D) \cap W^{1,\infty}(D)$. Then there exists a constant C independent of ε, the domain D, and the function f such that*

$$\| \nabla w_\varepsilon - \nabla(w_1^\varepsilon + \varepsilon\theta_\varepsilon) \|_{0,D} \le C\varepsilon(| w_0 |_{2,D} + | w_0 |_{1,D}).$$

Moreover, the boundary corrector θ_ε satisfies the estimate

$$\| \varepsilon\nabla\theta_\varepsilon \|_{0,D} \le C\varepsilon(| w_0 |_{2,D} + | w_0 |_{1,D}) + C\sqrt{\varepsilon|\partial D|}\, | w_0 |_{1,\infty,D},$$

where $|\partial D|$ stands for the length of the boundary ∂D.

The following maximum norm error estimate which improves the result in [2] is of independent interest.

THEOREM 4.2. *Let D be a convex polygon in $\mathbf{R}^2$. Then there exists a constant $p_0 > 2$ and a constant C depending only on the ellipticity constant μ in (4.2) and p_0 such that, for any $p \in (2, p_0]$,*

$$\| w_\varepsilon - w_0 \|_{0,\infty,D} \le C\varepsilon\frac{2p}{p-2}|D|^{\frac{1}{2}-\frac{1}{p}}\| f \|_{0,p,D}, \qquad \forall f \in L^p(D).$$

Proof. Let $\psi_\varepsilon = w_\varepsilon - (w_1^\varepsilon + \varepsilon\theta_\varepsilon)$. Then from (4.8) and (4.10) we know that ψ_ε satisfies

$$-\text{div}\left(a\Big(x, \frac{x}{\varepsilon}\Big)\nabla\psi_\varepsilon\right) = -\varepsilon\sum_{i=1}^{2}\frac{\partial f_i}{\partial x_i} \quad \text{in } D, \quad \psi_\varepsilon|_{\partial D} = 0.$$

It is easy to see from (4.9) that $\| f_i \|_{0,p,D} \le C(|w_0|_{2,p,D} + |w_0|_{1,p,D})$. By the classical weak maximum principle (for a proof see §8.2 of this paper) we have

$$\begin{aligned}\| \psi_\varepsilon \|_{0,\infty,D} &\le C\frac{2p}{p-2}|D|^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_{i=1}^{2}\| \varepsilon f_i \|_{0,p,D}^2\Big)^{1/2}\\ &\le C\varepsilon\frac{2p}{p-2}|D|^{\frac{1}{2}-\frac{1}{p}}(|w_0|_{2,p,D} + |w_0|_{1,p,D}), \qquad \forall p > 2.\end{aligned}$$

Here the constant C depends only on μ, and in particular, independent of p. Since D is a convex polygon, elliptic regularity theory implies that there exists a constant $p_0 > 2$ such that

$$\| w_0 \|_{2,p_0,D} \le C(p_0, \text{diam}(D))\| f \|_{0,p_0,D}.$$

Moreover, we have

$$\| w_0 \|_{2,D} \le C(2, \text{diam}(D))\| f \|_{0,D}.$$

Thus, by the Marcinkiewicz interpolation theorem, we obtain, for any $p \in (2, p_0]$, that

$$\begin{aligned}\| w_0 \|_{2,p,D} &\le C(p_0, \text{diam}(D))^{1-\rho}C(2, \text{diam}(D))^{\rho}\| f \|_{0,p,D}\\ &\le C(\text{diam}(D))\| f \|_{0,p,D},\end{aligned} \tag{4.11}$$

where ρ satisfies $1/p = (1-\rho)/2 + \rho/p_0$. The constant $C(\mathrm{diam}(D))$ depends on the upper bound of $\mathrm{diam}(D)$ which is bounded by assumption. Therefore, we have

$$\| \psi_\varepsilon \|_{0,\infty,D} \le C\varepsilon \ \frac{2p}{p-2} \ |D|^{\frac{1}{2}-\frac{1}{p}} \| f \|_{0,p,D}.$$

Note that $\displaystyle\int_D \nabla w_0 dx = 0$ since $w_0 \in H_0^1(D)$, by the embedding estimate (see e.g. [12, Lemmas 7.16 and 7.12]), we know that

$$\| \nabla w_0 \|_{0,\infty,D} \le C \ \frac{2p}{p-2} \ |D|^{\frac{1}{2}-\frac{1}{p}} |w_0|_{2,p,D} \le C \ \frac{2p}{p-2} \ |D|^{\frac{1}{2}-\frac{1}{p}} \| f \|_{0,p,D},$$

where we have used (4.11) . This completes the proof by noticing that $\psi_\varepsilon = w_\varepsilon - w_0 - \varepsilon\chi^k \dfrac{\partial w_0}{\partial x_k} - \varepsilon\theta_\varepsilon$, and $\| \theta_\varepsilon \|_{0,\infty,D} \le \| \chi^k \dfrac{\partial w_0}{\partial x_k} \|_{0,\infty,\partial D} \le C \| \nabla w_0 \|_{0,\infty,D}$ by the maximum principle. □

Now let $x_0 \in D$ and $\mathrm{dist}\,(x_0, \partial D) = d > 0$. We consider the Green function G_ε that satisfies

$$-\mathrm{div}\left(a\left(x, \frac{x}{\varepsilon}\right)\nabla G_\varepsilon\right) = \delta_{x_0} \quad \text{in } D, \quad G_\varepsilon|_{\partial D} = 0. \tag{4.12}$$

Let G_0 be the homogenized Green function

$$-\mathrm{div}\,(a^*(x)\nabla G_0) = \delta_{x_0} \quad \text{in } D, \quad G_0|_{\partial D} = 0. \tag{4.13}$$

A direct consequence of Theorem 4.2 is the following important error estimate for $G_\varepsilon - G_0$.

THEOREM 4.3. *Let D be a convex polygon in $\mathbf{R}^2$. Then, for any $q \in [1,2)$, the following error estimate is valid*

$$\| G_\varepsilon - G_0 \|_{0,q,D} \le C\varepsilon \ \frac{2q}{2-q} \ |D|^{\frac{1}{q}-\frac{1}{2}}, \tag{4.14}$$

where the constant C is independent of ε and q.

Furthermore, assume that $\mathrm{diam}(D) \le Cd$, *then there exists a constant C independent of ε such that*

$$\|G_\varepsilon - G_0\|_{L^2(D\setminus\bar{B}(x_0,d/2))} \le C\varepsilon\left(1 + \ln\frac{d}{\varepsilon}\right). \tag{4.15}$$

Proof. By the definition of Green functions we have

$$(w_\varepsilon - w_0)(x_0) = \int_D (G_\varepsilon(y) - G_0(y))f(y)dy \quad \forall f \in C_0^\infty(D).$$

By Theorem 4.2 we obtain that there exists a constant $p_0 > 2$ such that, for any $p \in (2, p_0]$,

$$\begin{aligned} \left| \int_D (G_\varepsilon - G_0) f dx \right| &\le C\varepsilon \ \frac{2p}{p-2} \ |D|^{\frac{1}{2}-\frac{1}{p}} \| f \|_{0,p,D} \\ &= C\varepsilon \frac{2q}{2-q} |D|^{\frac{1}{q}-\frac{1}{2}} \| f \|_{0,p,D} \quad \forall f \in C_0^\infty(D), \end{aligned} \tag{4.16}$$

where $p^{-1}+q^{-1}=1$. Now standard compactness argument implies the above estimate is also valid for $f \in L^p(D)$. This proves (4.14) when $q \in [p_0/(p_0-1), 2)$.

Now let $q_0 = p_0/(p_0-1)$. For any $q \in [1, q_0)$, by Hölder inequality and (4.14) for the case when $q = q_0$, we have

$$
\begin{aligned}
\| G_\varepsilon - G_0 \|_{0,q,D} \le \| G_\varepsilon - G_0 \|_{0,q_0,D} |D|^{\frac{1}{q}-\frac{1}{q_0}} &\le C\varepsilon \frac{2q_0}{2-q_0} |D|^{\frac{1}{q}-\frac{1}{2}} \\
&\le C\varepsilon \frac{2q}{2-q} |D|^{\frac{1}{q}-\frac{1}{2}},
\end{aligned}
$$

where we have used the fact that $2q/(2-q) \ge 2$ for $q \ge 1$. This proves (4.14) for all $q \in [1, 2)$.

To show (4.15), we first note that by well-known properties of Green functions (cf. e.g. [14]) we have $|G_\varepsilon| + |G_0| \le C(1 + |\ln|x-x_0||)$ for any $x \in D$ and some constant C independent of ε. Thus we have $|G_\varepsilon| + |G_0| \le C(1 + |\ln d|) \le C$ for $x \in D \backslash \bar{B}(x_0, d/2)$. By taking $f = (G_\varepsilon - G_0)\omega$ in (4.16) with $\omega \in C^\infty(D)$ the cut-off function satisfying $0 \le \omega \le 1$, $\omega = 0$ in $B(x_0, d/4)$, and $\omega = 1$ outside $B(x_0, d/2)$, we obtain, for any $p \in [p_0, 2)$,

$$
\begin{aligned}
\int_D \omega |G_\varepsilon - G_0|^2 dx &\le C\varepsilon \frac{2p}{p-2} |D|^{\frac{1}{2}-\frac{1}{p}} \|\omega(G_\varepsilon - G_0)\|_{0,D}^{\frac{2}{p}} \|\omega(G_\varepsilon - G_0)\|_{0,\infty,D}^{\frac{p-2}{p}} \\
&\le C\varepsilon \frac{2p}{p-2} |D|^{\frac{1}{2}-\frac{1}{p}} \|\omega(G_\varepsilon - G_0)\|_{0,D}^{\frac{2}{p}}.
\end{aligned}
$$

Thus, since $\frac{p}{2p-2} < 1$ due to $p > 2$, we have

$$
\begin{aligned}
\|G_\varepsilon - G_0\|_{0,D\backslash B(x_0,d/2)} &\le \left(C\varepsilon \frac{2p}{p-2} \right)^{\frac{p}{2p-2}} d^{\frac{p-2}{2p-2}} \\
&\le C\varepsilon \frac{2p}{p-2} \left(\frac{d}{\varepsilon} \right)^{\frac{p-2}{2p-2}}.
\end{aligned}
$$

This completes the proof of (4.15) upon taking $p = \min(p_0, 2 + 1/\ln(d/\varepsilon))$. $\square$

Now we define $G_1^\varepsilon = G_0(x) - \varepsilon \chi^k \left(x, \frac{x}{\varepsilon} \right) \frac{\partial G_0}{\partial x_k}$. Our next goal is to derive an energy error estimate for $G_\varepsilon - G_1^\varepsilon$ away from the singularity. First we recall the following results for the Green function G_0.

LEMMA 4.4. *Assume that* $\operatorname{diam}(D) \le Cd$. *Then there exists a constant* C *depending on* a^* *and the upper bound of the size of the domain* D *such that*

(i) $|G_0(x)| \le C$, $|\nabla G_0(x)| \le Cd^{-1}$ *for any* $x \in D \backslash \bar{B}(x_0, d/2)$*;*

(ii) $\| D^2 G_0 \|_{0,D\backslash \bar{B}(x_0,d/2)} \le Cd^{-1}$.

Proof. First we recall the following well-known properties of Green function (cf. e.g. [14, Theorem 3.3]), where $\delta(x) = \operatorname{dist}(x, \partial D)$,

$$
|G_0(x)| \le C\delta(x)|x-x_0|^{-1}, \qquad |\nabla G_0(x)| \le C|x-x_0|^{-1} \qquad \forall x \in D.
$$

This proves (i) by noticing that

$$
\max_{x \in D\backslash \bar{B}(x_0,d/2)} |G_0(x)| \le \max_{|x-x_0|=d/2} |G_0(x)|
$$

as a consequence of maximum principle. The argument to prove (ii) is standard by considering the equation satisfied by ωG_0 in D with appropriate cut-off function ω. Here we omit the details. $\square$

Now we are in the position to give the second error estimate for the Green functions.

THEOREM 4.5. *Assume that* $\mathrm{diam}(D) \leq Cd$. *Then there exists a constant* C *independent of* ε *such that*

$$\| \nabla(G_\varepsilon - G_1^\varepsilon) \|_{0,D\setminus\bar{B}(x_0,3d/4)} \leq C\sqrt{\varepsilon/d}.$$

Proof. Let $\theta_\varepsilon(G_0)$ be the boundary corrector defined as in (4.10) with w_0 being replaced by G_0. Then it is easy to see that $\psi_\varepsilon = G_\varepsilon - G_1^\varepsilon - \varepsilon\theta_\varepsilon(G_0)$ satisfies

$$-\mathrm{div}\left(a\left(x, \frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon\right) = -\varepsilon\sum_{i=1}^{2}\frac{\partial \bar{f}_i}{\partial x_i} \quad \text{in } D\setminus\{x_0\}, \quad \psi_\varepsilon|_{\partial D} = 0, \tag{4.17}$$

where $\bar{f}_i = f_i(G_0)$ is defined in (4.9).

Note that $\| \bar{f}_i \|_{0,D\setminus\bar{B}(x_0,d/2)} \leq Cd^{-1}$ by Lemma 4.4. Let $\omega \in C^\infty(D)$ be the cut-off function such that $\omega = 1$ in $D\setminus\bar{B}(x_0,3d/4)$, $\omega = 0$ in $B(x_0,d/2)$, $|\nabla\omega| \leq Cd^{-1}$ in D. By multiplying (4.17) with $\omega\psi_\varepsilon$ and using (4.15), we can obtain $\| \nabla\psi_\varepsilon \|_{0,D\setminus\bar{B}(x_0,3d/4)} \leq C(\varepsilon/d)(1 + |\ln\varepsilon/d|)$. Moreover, by Theorem 4.1 and Lemma 4.4 we have

$$\begin{aligned}
\| \varepsilon\nabla\theta_\varepsilon(G_0) \|_{0,D} &= \| \varepsilon\nabla\theta_\varepsilon(\omega G_0) \|_{0,D} \\
&\leq C\varepsilon(|\omega G_0|_{2,D} + |\,\omega G_0\,|_{1,D}) + C\sqrt{\varepsilon d}\ |\omega G_0|_{1,\infty,D} \\
&\leq C\varepsilon/d + C\sqrt{\varepsilon/d}.
\end{aligned}$$

This completes the proof since $\| \nabla(G_\varepsilon - G_1^\varepsilon) \|_{0,D\setminus\bar{B}(x_0,3d/4)} \leq \| \nabla\psi_\varepsilon \|_{0,D\setminus\bar{B}(x_0,3d/4)} + \| \varepsilon\nabla\theta_\varepsilon(G_0) \|_{0,D}$. ∎

5. Multiscale convergence analysis. In this section we prove the basic error estimate in Theorem 3.1. For any $j, 1 \leq j \leq N$, we introduce the homogenized Green function G_j^* as follows

$$-\mathrm{div}\,(K^*(x)\nabla G_j^*) = \delta_{\bar{x}_j} \quad \text{in } \Omega_j, \quad G_j^*|_{\Sigma_j} = 0, \tag{5.1}$$

where $K^*(x)$ is the homogenized coefficient defined in (4.4) with $a_{ij}(x,x/\varepsilon) = K(x,x/\varepsilon)\delta_{ij}$, where δ_{ij} is the Kronecker delta. Let $\zeta_j^* \in H_0^1(\Omega)$ be the solution of the homogenized problem of (3.9)

$$\int_\Omega K^*(x)\nabla\zeta_j^* \cdot \nabla\chi dx = -\int_{\Sigma_j} K^*\frac{\partial G_j^*}{\partial\nu}\chi ds \quad \forall\chi \in H_0^1(\Omega). \tag{5.2}$$

Finally, we denote by $R_H : H_0^1(\Omega) \to V_H^0$ the standard Ritz projection operator

$$\int_\Omega K^*(x)\nabla(R_H\varphi - \varphi) \cdot \nabla v_H dx = 0 \quad \forall v_H \in V_H^0,\ \forall\varphi \in H_0^1(\Omega). \tag{5.3}$$

Then by the triangle inequality we have the following decomposition

$$\begin{aligned}
\max_{x\in\bar{B}_j}|\zeta_i^\varepsilon(x) - \zeta_i^H(\bar{x}_j)| \leq \max_{x\in\bar{B}_j}|\zeta_i^\varepsilon(x) - \zeta_i^*(x)| &+ \max_{x\in\bar{B}_j}|\zeta_i^*(x) - (R_H\zeta_i^*)(\bar{x}_j)| \\
&+ |(R_H\zeta_i^*)(\bar{x}_j) - \zeta_i^H(\bar{x}_j)|.
\end{aligned} \tag{5.4}$$

LEMMA 5.1. *Recall that* $H_j = \mathrm{dist}\,(\bar{x}_j, \Sigma_j)$. *There exists a constant* C *independent of* H_j *and* $p > 2$ *such that*

(i) $\| \nabla\zeta_j^* \|_{0,\infty,B(\bar{x}_j,3H_j/4)} \le CH_j^{-1}|\ln H_j|$,

(ii) $|\zeta_j^*|_{2,p,B(\bar{x}_j,3H_j/4)} \le CpH_j^{\frac{2}{p}-2}|\ln H_j|, \quad \forall p > 2.$

Proof. First, let Θ_j^* be the Green function of the domain Ω

$$-\mathrm{div}\,(K^*(x)\nabla\Theta_j^*) = \delta_{\bar{x}_j} \quad \text{in } \Omega, \quad \Theta_j^*|_\Gamma = 0. \tag{5.5}$$

Then we have $\zeta_j^* = \Theta^* - G_j^*$ by the definition (5.1) and (5.2). From (5.2) we also know that

$$\int_{\Omega_j} K^*(x)\nabla\zeta_j^* \cdot \nabla\chi dx = 0 \quad \forall \chi \in H_0^1(\Omega_j).$$

By the maximum principle, we have $\| \zeta_j^* \|_{0,\infty,\Omega_j} \le \| \zeta_j^* \|_{0,\infty,\Sigma_j} = \| \Theta_j^* \|_{0,\infty,\Sigma_j} \le C|\ln H_j|$ by the standard estimate for Green functions. Now interior gradient estimate [14] implies

$$\| \nabla\zeta_j^* \|_{0,\infty,B(\bar{x}_j,3H_j/4)} \le CH_j^{-1}\| \zeta_j^* \|_{0,\infty,\Omega_j} \le CH_j^{-1}|\ln H_j|.$$

This proves (i). Next, denote by $\psi_k = \partial\zeta_j^*/\partial x_k$, $k = 1, 2$. We differentiate the equation satisfied by ζ_j^* in Ω_j and know that

$$-\mathrm{div}\,(K^*(x)\nabla\psi_k) = \mathrm{div}\,\Big(\frac{\partial K^*}{\partial x_k}\nabla\zeta_j^*\Big) \quad \text{in } \Omega_j.$$

By the interior $W^{1,p}$ estimate in Lemma 2.4 we have

$$\begin{aligned}
&\| \nabla\psi_k \|_{0,p,B(\bar{x}_j,H_j/2)} \\
&\le CpH_j^{\frac{2}{p}}\Big\{(H_j^{-1}+1)\| \nabla\psi_k \|_{0,B(\bar{x}_j,3H_j/4)} + \| (\partial K^*/\partial x_k)\nabla\zeta_j^* \|_{0,\infty,B(\bar{x}_j,3H_j/4)}\Big\} \\
&\le CpH_j^{\frac{2}{p}}\Big\{CH_j^{-1}\cdot H_j^{-2}\| \zeta_j^* \|_{0,\Omega_j} + CH_j^{-1}|\ln H_j|\Big\} \\
&\le CpH_j^{\frac{2}{p}-2}|\ln H_j|,
\end{aligned}$$

where we have used (i) and $\| \zeta_j^* \|_{0,\infty,\Omega_j} \le C|\ln H_j|$. This proves (ii). ◻

Now we are ready to estimate the first term in (5.4).

PROPOSITION 5.2. *Under the assumption of Theorem 3.1, we have*

$$\max_{x\in\bar{B}_j} |\zeta_i^\varepsilon(x) - \zeta_i^*(x)| \le C\frac{\varepsilon}{H_j}|\ln H_j|, \quad \forall 1 \le i, j \le N.$$

Proof. In the proof we will suppress the subscript i and thus write $\zeta^\varepsilon = \zeta_i^\varepsilon, \zeta^* = \zeta_i^*$. Denote by $\zeta_1^\varepsilon = \zeta^* - \varepsilon\chi^k(x, x/\varepsilon)\dfrac{\partial\zeta^*}{\partial x_k}$. Then similar to (4.8) we know that

$$\int_{\Omega_j} K\Big(x, \frac{x}{\varepsilon}\Big)\nabla(\zeta^\varepsilon - \zeta_1^\varepsilon)\cdot\nabla\chi dx = \varepsilon\int_{\Omega_j}\sum_{k=1}^2 \tilde{f}_k\frac{\partial\chi}{\partial x_k}dx \quad \forall \chi \in H_0^1(\Omega_j),$$

where $\tilde{f}_k = f_k(\zeta^*)$ is defined in (4.9).

By Lemma 5.1 we know that $\| \tilde{f}_k \|_{0,s,B(\bar{x}_j,H_j/2)} \le CsH_j^{\frac{2}{s}-2}|\ln H_j|$ for any $s > 2$. Now by the interior maximum principle [12, Theorem 8.17] we have

$$\begin{aligned}
\max_{x\in\bar{B}_j}|\zeta^\varepsilon - \zeta_1^\varepsilon| &\le C\Big\{H_j^{-\frac{2}{q}}\| \zeta^\varepsilon - \zeta_1^\varepsilon \|_{0,q,B(\bar{x}_j,H_j/4)} + \sum_{k=1}^2 \varepsilon H_j^{1-\frac{2}{s}}\| \tilde{f}_k \|_{0,s,B(\bar{x}_j,H_j/4)}\Big\} \\
&\le CH_j^{-\frac{2}{q}}\| \zeta^\varepsilon - \zeta_1^\varepsilon \|_{0,q,B(\bar{x}_j,H_j/4)} + Cs\frac{\varepsilon}{H_j}|\ln H_j|, \quad \forall q > 1, s > 2.
\end{aligned}$$

On the other hand, by Lemma 5.1,

$$\begin{aligned}\| \zeta^\varepsilon - \zeta_1^\varepsilon \|_{0,q,B(\bar{x}_j,H_j/4)} &\le \| \zeta^\varepsilon - \zeta^* \|_{0,q,B(\bar{x}_j,H_j/4)} + C\varepsilon \| \nabla \zeta^* \|_{0,q,B(\bar{x}_j,H_j/4)} \\ &\le \| \zeta^\varepsilon - \zeta^* \|_{0,q,B(\bar{x}_j,H_j/4)} + C\frac{\varepsilon}{H_j} H_j^{\frac{2}{q}} |\ln H_j|, \quad \forall q > 1.\end{aligned}$$

Therefore, by taking $s = 3$, we have

$$\max_{x \in \bar{B}_j} |\zeta_i^\varepsilon - \zeta_i^*| \le C H_j^{-\frac{2}{q}} \| \zeta_i^\varepsilon - \zeta_i^* \|_{0,q,B(\bar{x}_j,H_j/4)} + C\frac{\varepsilon}{H_j} |\ln H_j|, \quad \forall 1 < q < 2.$$

Since $\zeta_i^\varepsilon = \Theta_i^\varepsilon - G_i^\varepsilon$ by (3.14) and $\zeta_i^* = \Theta_i^* - G_i^*$, we deduce by using Theorem 4.3 that

$$\begin{aligned}\| \zeta_i^\varepsilon - \zeta_i^* \|_{0,q,B(\bar{x}_j,H_j/4)} &\le \| \Theta_i^\varepsilon - \Theta_i^* \|_{0,q,B(\bar{x}_j,H_j/4)} + \| G_i^\varepsilon - G_i^* \|_{0,q,B(\bar{x}_j,H_j/4)} \\ &\le C\varepsilon \ \frac{2q}{2-q} (|\Omega|^{\frac{1}{q}-\frac{1}{2}} + |\Omega_j|^{\frac{1}{q}-\frac{1}{2}}) \\ &\le C\varepsilon \ \frac{2q}{2-q}.\end{aligned}$$

Therefore, we have

$$\max_{x \in \bar{B}_j} |\zeta_i^\varepsilon - \zeta_i^*| \le C\varepsilon \ \frac{2q}{2-q} \ H_j^{-\frac{2}{q}} + C\frac{\varepsilon}{H_j} |\ln H_j|, \quad \forall 1 \le q < 2.$$

By taking $q = 2\ln(1/H_j)/(1 + \ln(1/H_j)) < 2$, we finally obtain

$$\max_{x \in \bar{B}_j} |\zeta_i^\varepsilon - \zeta_i^*| \le C\frac{\varepsilon}{H_j} |\ln H_j|.$$

This completes the proof. $\square$

PROPOSITION 5.3. *Under the assumption of Theorem 3.1 we have*

$$\max_{x \in \bar{B}_j} |\zeta_i^*(x) - (R_H \zeta_i^*)(\bar{x}_j)| \le C\Big\{ \frac{\delta}{H_j} + \frac{H^2}{H_j^2} \Big| \ln \frac{H}{H_j} \Big|^2 \Big\} |\ln H_j|.$$

Proof. At first, by using Lemma 5.1, we obtain

$$\max_{x \in \bar{B}_j} |\zeta_i^*(x) - \zeta_i^*(\bar{x}_j)| \le \delta \| \nabla \zeta_i^* \|_{0,\infty,\bar{B}_j} \le \frac{\delta}{H_j} |\ln H_j|. \tag{5.6}$$

To estimate $\zeta_i^*(\bar{x}_j) - (R_H \zeta_i^*)(\bar{x}_j)$ we use the classical interior L^∞ error estimates for finite element methods in [27] and get

$$\begin{aligned}&|\zeta_i^*(\bar{x}_j) - (R_H \zeta_i^*)(\bar{x}_j)| \\ &\le C\Big| \ln \frac{H}{H_j} \Big| \inf_{\chi_H \in V_H} \| \zeta_i^* - \chi_H \|_{0,\infty,B(\bar{x}_j,H_j/2)} + C H_j^{-1} \| \zeta_i^* - (R_H \zeta_i^*) \|_{0,B(\bar{x}_j,H_j/2)} \\ &\le C H^{2-\frac{2}{p}} \Big| \ln \frac{H}{H_j} \Big| |\zeta_i^*|_{2,p,B(\bar{x}_j,H_j/2)} + C H_j^{-1} \| \zeta_i^* - (R_H \zeta_i^*) \|_{0,\Omega_i} \\ &\le Cp \Big(\frac{H}{H_j} \Big)^{2-\frac{2}{p}} \Big| \ln \frac{H}{H_j} \Big| |\ln H_j| + C H_j^{-1} \| \zeta_i^* - (R_H \zeta_i^*) \|_{0,\Omega_j} \quad \forall p > 2,\end{aligned} \tag{5.7}$$

where have used the standard finite element interpolation estimates and Lemma 5.1. Since $\zeta_i^* = \Theta_i^*$ in $\Omega\backslash\bar\Omega_i$, where Θ_i^* is the Green function defined in (5.5), we know that $\zeta_i^* \in H^2(\Omega\backslash\Omega_i)$ and $|\,\zeta_i^*\,|_{2,\Omega\backslash\bar\Omega_i} \le |\,\Theta_i^*\,|_{2,\Omega\backslash\bar\Omega_i} \le CH_i^{-1}$. On the other hand, from (5.4) we know that $\zeta_i^* \in H^1(\Omega_i)$ satisfies $\zeta_i^*|_{\Sigma_i} = (\Theta_i^* - G_i^*)|_{\Sigma_i} = \Theta_i^*|_{\Sigma_i}$ and

$$\int_{\Omega_i} K^*(x)\nabla\zeta_i^*\cdot\nabla\chi dx = 0 \qquad \forall\chi\in H_0^1(\Omega_i).$$

Thus, let $\omega_i \in C^\infty(\Omega_i)$ be the cut-off function such that $\omega_i = 1$ in $\Omega_i\backslash\bar B(\bar x_i, 3H_i/4)$, $\omega_i = 0$ in $B(\bar x_i, H_i/2)$ and $|\nabla\omega_i| \le CH_i^{-1}$ in Ω_i, we know that

$$|\,\zeta_i^*\,|_{2,\Omega_i} \le C|\,\omega_i\Theta_i^*\,|_{2,\Omega_i} \le CH_i^{-1}|\ln H_i|.$$

Therefore, standard finite element error analysis implies that

$$\begin{aligned}\|\,\zeta_i^* - R_H\zeta_i^*\,\|_{0,\Omega} &\le CH|\,\zeta_i^* - R_H\zeta_i^*\,|_{1,\Omega}\\ &\le CH|\,\zeta_i^* - I_H\zeta_i^*\,|_{1,\Omega_i} + CH|\,\zeta_i^* - I_H\zeta_i^*\,|_{1,\Omega\backslash\bar\Omega_i}\\ &\le CH^2H_i^{-1}|\ln H_i|,\end{aligned}$$

where $I_H : C(\bar\Omega) \to V_H$ is the standard linear finite element nodal interpolant. Substituting above estimates into (5.7) and using (H4), we get

$$|\zeta_i^*(\bar x_j) - (R_H\zeta_i^*)(\bar x_j)| \le Cp\Big(\frac{H}{H_j}\Big)^{2-\frac{2}{p}}\Big|\ln\frac{H}{H_j}\Big||\ln H_j| + C\Big(\frac{H}{H_j}\Big)^2|\ln H_j|, \qquad \forall p > 2.$$

This completes the proof by taking $p = \ln(H_j/H)$. □

Now we turn to the estimate of the third term $|(R_H\zeta_i^*)(\bar x_j) - \zeta_i^H(\bar x_j)|$ in (5.4). Denote by $w_H = (R_H\zeta_i^*) - \hat\zeta_i^H = R_H\zeta_i^* - \Pi_H\zeta_i^H \in V_H^0$. Then

$$|(R_H\zeta_i^*)(\bar x_j) - \zeta_i^H(\bar x_j)| = |w_H(\bar x_j)| + |\zeta_i^H(\bar x_j) - \hat\zeta_i^H(\bar x_j)|. \tag{5.8}$$

We first bound $w_H(\bar x_j)$. By the inverse estimate, we have

$$|w_H(\bar x_j)| \le \|\,w_H\,\|_{0,\infty,\Omega} \le C|\ln H|^{1/2}\|\,\nabla w_H\,\|_{0,\Omega}. \tag{5.9}$$

On the other hand, by (3.10), (5.2) and (5.3) we have that

$$\int_\Omega K^*(x)\nabla w_H\cdot\nabla\hat\chi_H dx = F_1(\hat\chi_H) + F_2(\hat\chi_H) \qquad \forall\chi_H \in X_H^0, \tag{5.10}$$

where $\hat\chi_H = \Pi_H\chi_H \in V_H^0$, and

$$\begin{aligned}F_1(\hat\chi_H) &= \int_{\Sigma_i} K_\varepsilon\frac{\partial G_i^\varepsilon}{\partial\nu}\hat\chi_H dx - \int_{\Sigma_i} K^*\frac{\partial G_i^*}{\partial\nu}\hat\chi_H dx,\\ F_2(\hat\chi_H) &= \int_\Omega K_\varepsilon(x)\cdot\nabla\zeta_i^H\nabla\chi_H dx - \int_\Omega K^*(x)\nabla\hat\zeta_i^H\cdot\nabla\hat\chi_H dx.\end{aligned}$$

To estimate the terms F_1 and F_2, we first recall the following result whose proof can be found in [10, Lemma 3.2].

Lemma 5.4. *Let $N(x,y)$ be periodic in y with respect to the unit square Y in $\mathbf{R}^2$ such that $\int_Y N(x,y)dy = 0$ for any $x\in D$. Moreover, assume that $|N(x,y)| + |\nabla_x N(x,y)| \le C$ for any $x\in D, y\in\mathbf{R}^2$. Then, for any $\xi\in H^1(D)\cap L^\infty(D)$, we have*

$$\Big|\int_D \xi(x)N\Big(x,\frac{x}{\varepsilon}\Big)dx\Big| \le C\varepsilon|D|^{1/2}\|\,\xi\,\|_{1,D} + C\varepsilon|\partial D|\|\,\xi\,\|_{0,\infty,D}.$$

LEMMA 5.5. *Under the assumption of Theorem 3.1 we have*

$$|F_1(\hat{\chi}_H)| \le C\Big(\frac{\varepsilon}{H_i}|\ln H|\Big)^{1/2}\| \nabla\hat{\chi}_H \|_{0,\Omega} \quad \forall \hat{\chi}_H \in V_H^0.$$

Proof. Let $\omega \in C^\infty(\Omega_i)$ be the cut-off function such that $\omega = 1$ in $\Omega_i\backslash \bar{B}_i(\bar{x}_i, 3H_i/4)$, $\omega = 0$ in $B(\bar{x}_i, H_i/2)$ and $|\nabla\omega| \le CH_i^{-1}, |D^2\omega| \le CH_i^{-2}$ in Ω_i. Then by integration by parts we get

$$F_1(\hat{\chi}_H) = \int_{\Omega_i}(K_\varepsilon(x)\nabla G_i^\varepsilon - K^*(x)\nabla G_i^*)\cdot\nabla(\omega\hat{\chi}_H)dx.$$

In the rest of the proof we will suppress the subscript i and write $G^\varepsilon = G_i^\varepsilon, G^* = G_i^*$, and $G_1^\varepsilon = G^* - \varepsilon\chi^k(x, x/\varepsilon)\dfrac{\partial G^*}{\partial x_k}$. Recall that we have written $a_{lj}(x, x/\varepsilon) = K_\varepsilon(x)\delta_{lj}$ and $a_{lj}^*(x) = K_{lj}^*(x)$. Then by (4.6) we know that

$$\begin{aligned}
F_1(\hat{\chi}_H) &= \int_{\Omega_i}\Big(a_{lj}\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial G^\varepsilon}{\partial x_j} - a_{lj}^*(x)\frac{\partial G^*}{\partial x_j}\Big)\frac{\partial(\omega\hat{\chi}_H)}{\partial x_l}dx\\
&= \int_{\Omega_i} a_{lj}\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial(G^\varepsilon - G_1^\varepsilon)}{\partial x_l}\frac{\partial(\omega\hat{\chi}_H)}{\partial x_j}dx\\
&- \int_{\Omega_i} G_l^k\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial G^*}{\partial x_k}\frac{\partial(\omega\hat{\chi}_H)}{\partial x_l}dx\\
&- \varepsilon\int_{\Omega_i}\hat{f}_l\frac{\partial(\omega\hat{\chi}_H)}{\partial x_l}dx := \mathrm{I} + \mathrm{II} + \mathrm{III}\,,
\end{aligned} \tag{5.11}$$

where $G_i^k(x, y)$ is defined in (4.7) which satisfies $\int_Y G_l^k(x, y)dy = 0$, and $\hat{f}_l$ is defined by

$$\hat{f}_l(G^*) = a_{lj}\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial\chi^k}{\partial x_j}\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial G^*}{\partial x_k} + a_{lj}\Big(x, \frac{x}{\varepsilon}\Big)\chi^k\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial^2 G^*}{\partial x_j\partial x_k}.$$

By Theorem 4.5, we know that

$$\begin{aligned}
\mathrm{I} &\le C\sqrt{\varepsilon/H_i}\| \nabla(\omega\hat{\chi}_H) \|_{0,\Omega_i} \le C\sqrt{\varepsilon/H_i}(\| \hat{\chi}_H \|_{0,\infty,\Omega_i} + \| \nabla\hat{\chi}_H \|_{0,\Omega_i})\\
&\le C\sqrt{\varepsilon/H_i}|\ln H|^{1/2}\| \nabla\hat{\chi}_H \|_{0,\Omega}.
\end{aligned}$$

where we have used finite element inverse estimate.

By Lemma 4.4, we have, for any $T \in \mathcal{M}_H$, $T \subset \bar{\Omega}_i$, that

$$\begin{aligned}
&\Big|\frac{\partial G^*}{\partial x_k}\frac{\partial(\omega\hat{\chi}_H)}{\partial x_l}\Big|_{1,T}\\
&\le |\, G^* \,|_{2,T\backslash\bar{B}(\bar{x}_i,H_i/2)}|\, \omega\hat{\chi}_H \,|_{1,\infty,T} + |\, G^* \,|_{1,\infty,T\backslash B(\bar{x}_i,H_i/2)}|\, \omega\hat{\chi}_H \,|_{2,T}\\
&\le |\, G^* \,|_{2,T\backslash\bar{B}(\bar{x}_i,H_i/2)}(|\, \omega \,|_{1,\infty,T}\| \hat{\chi}_H \|_{0,\infty,T} + |\, \hat{\chi}_H \,|_{1,\infty,T})\\
&+ |\, G^* \,|_{1,\infty,T\backslash\bar{B}(\bar{x}_i,H_i/2)}(|\omega|_{2,\infty,T}\| \hat{\chi}_H \|_{0,T} + |\, \omega \,|_{1,\infty,T}|\, \hat{\chi}_H \,|_{1,T})\\
&\le C|\, G^* \,|_{2,T\backslash\bar{B}(\bar{x}_i,H_i/2)}(H_i^{-1}\| \hat{\chi}_H \|_{0,\infty,T} + H^{-1}|\, \hat{\chi}_H \,|_{1,T})\\
&+ CH_i^{-1}(H_i^{-2}H\| \hat{\chi}_H \|_{0,\infty,T} + H_i^{-1}|\, \hat{\chi} \,|_{1,T})\\
&\le C(|\, G^* \,|_{2,T\backslash\bar{B}(\bar{x}_i,H_i/2)} + H_i^{-2}H)(H_i^{-1}\| \hat{\chi}_H \|_{0,\infty,T} + H^{-1}|\, \hat{\chi}_H \,|_{1,T}).
\end{aligned}$$

Similarly, we have

$$\Big\| \frac{\partial G^*}{\partial x_k} \frac{\partial (\omega \hat{\chi}_H)}{\partial x_i} \Big\|_{0,\infty,T} \le C H_i^{-1} (H_i^{-1} \| \hat{\chi}_H \|_{0,\infty,T} + H^{-1} | \hat{\chi}_H |_{1,T})$$

Since the number of elements in Ω_i is bounded by H_i^2/H^2, we use Lemma 4.4 to obtain that

$$\Big\{ \sum_{T \in \mathcal{M}_H, T \subset \Omega_i} (| G^* |_{2,T \setminus \bar{B}(\bar{x}_i, H_i/2)} + H_i^{-2} H)^2 \Big\}^{1/2} \le C H_i^{-1}.$$

Moreover, by the finite element inverse estimate, we have

$$\begin{aligned}
&\Big\{ \sum_{T \in \mathcal{M}_H, T \subset \Omega_i} (H_i^{-1} \| \hat{\chi}_H \|_{0,\infty,T} + H^{-1} | \hat{\chi}_H |_{1,T})^2 \Big\}^{1/2} \\
&\le C H^{-1} (\| \hat{\chi}_H \|_{0,\infty,\Omega_i} + | \hat{\chi}_H |_{1,\Omega_i}) \\
&\le C H^{-1} | \ln H |^{1/2} | \hat{\chi}_H |_{1,\Omega}.
\end{aligned}$$

Thus, we can use Lemma 5.4 to estimate the second term in (5.11) to get

$$\begin{aligned}
\mathrm{II} &= - \sum_{T \in \mathcal{M}_H, T \subset \Omega_i} \int_T G_l^k \Big(x, \frac{x}{\varepsilon} \Big) \frac{\partial G^*}{\partial x_k} \frac{\partial (\omega \hat{\chi}_H)}{\partial x_l} dx \\
&\le C \varepsilon H \sum_{T \in \mathcal{M}_H, T \subset \Omega_i} \Big\{ \Big\| \frac{\partial G^*}{\partial x_k} \frac{\partial (\omega \hat{\chi}_H)}{\partial x_l} \Big\|_{1,T} + \Big\| \frac{\partial G^*}{\partial x_k} \frac{\partial (\omega \hat{\chi}_H)}{\partial x_l} \Big\|_{0,\infty,T} \Big\} \\
&\le C \frac{\varepsilon}{H_i} \, | \ln H |^{1/2} \| \nabla \hat{\chi}_H \|_{0,\Omega_i}.
\end{aligned}$$

Finally, by Lemma 4.4 we obtain

$$\begin{aligned}
\mathrm{III} &\le C \varepsilon (| G^* |_{1,\Omega_i \setminus B(\bar{x}_i, H_i/2)} + | G^* |_{2,\Omega_i \setminus B(\bar{x}_i, H_i/2)}) | \omega \hat{\chi}_H |_{1,\Omega_i} \\
&\le C \frac{\varepsilon}{H_i} | \ln H |^{1/2} \| \nabla \hat{\chi}_H \|_{0,\Omega}.
\end{aligned}$$

This completes the proof upon substituting above estimates into (5.11). ☐

It remains to estimate $F_2(\hat{\chi}_H)$. For any $\chi_H \in X_H$, $\hat{\chi}_H = \Pi_H \chi_H \in V_H$, we denote by $\chi_H^1 = \hat{\chi}_H - \varepsilon \chi^k(x, x/\varepsilon) \dfrac{\partial \hat{\chi}_H}{\partial x_k}$, and $\theta_\varepsilon(\hat{\chi}_H)$ the associated boundary corrector that satisfies, for any $T \in \mathcal{M}_H$, $S = S(T)$ the over-sampling element,

$$-\mathrm{div}\, (K_\varepsilon(x) \nabla \theta_\varepsilon(\hat{\chi}_H)) = 0 \quad \text{in } S, \qquad \theta_\varepsilon(\hat{\chi}_H)|_{\partial S} = \chi^k \Big(x, \frac{x}{\varepsilon} \Big) \frac{\partial \hat{\chi}_H}{\partial x_k} \quad \text{on } \partial S.$$

By the interior gradient estimate in [1], (H3), the maximum principle and the finite element inverse estimate, we have that

$$\begin{aligned}
| \theta_\varepsilon(\hat{\chi}_H) |_{1,\infty,T} \le C H^{-1} \| \theta_\varepsilon(\hat{\chi}_H) \|_{0,\infty,S} &\le C H^{-1} \| \nabla \hat{\chi}_H \|_{0,\infty,T} \\
&\le C H^{-2} \| \nabla \hat{\chi}_H \|_{0,T}. \qquad (5.12)
\end{aligned}$$

Since $\hat{\chi}_H$ is linear in S, by using (4.6) and the similar argument leading to (4.8), we have that $\psi_H = \chi_H - \chi_H^1 + \varepsilon \theta_\varepsilon(\hat{\chi}_H) \in H_0^1(S)$ satisfies

$$\int_S K_\varepsilon \nabla \psi_H \cdot \nabla \varphi dx = \varepsilon \int_S \tilde{f}_i \frac{\partial \varphi}{\partial x_i} dx + \int_S (K^* - \bar{K}_S^*) \nabla \hat{\chi}_H \cdot \nabla \varphi dx, \qquad (5.13)$$

for any $\varphi \in H_0^1(S)$, where $\bar{K}_S^* = \dfrac{1}{|S|}\displaystyle\int_S K^* dx$, and $\tilde{f}_i = f_i(\hat{\chi}_H)$ is defined in (4.9) which satisfies $\| \tilde{f}_i \|_{0,\infty,S} \le C\| \nabla\hat{\chi}_H \|_{0,\infty,S} \le CH^{-1}\| \nabla\hat{\chi}_H \|_{0,T}$. By taking $\varphi = \psi_H$ in (5.13) we obtain easily that

$$
\begin{aligned}
&\| \nabla(\chi_H - \hat{\chi}_H) \|_{0,T} \\
&\le \varepsilon |T|^{1/2} \| \nabla\theta_\varepsilon(\hat{\chi}_H) \|_{0,\infty,T} + C\varepsilon \| \tilde{f}_i \|_{0,S} + C\| (K^* - \bar{K}_S^*)\nabla\hat{\chi}_H \|_{0,S} \\
&\le C\varepsilon H \cdot H^{-2} \| \nabla\hat{\chi}_H \|_{0,T} + C(\varepsilon + H)\| \nabla\hat{\chi}_H \|_{0,S} \\
&\le C\Big(\frac{\varepsilon}{H} + H\Big)\| \nabla\hat{\chi}_H \|_{0,T}.
\end{aligned} \tag{5.14}
$$

Moreover, by the weak maximum principle [18, p.64], we deduce that

$$
\begin{aligned}
&\| \chi_H - \hat{\chi}_H \|_{0,\infty,T} \\
&\le \varepsilon \| \theta_\varepsilon(\hat{\chi}_H) \|_{0,\infty,T} + \| \psi_H \|_{0,\infty,T} \\
&\le \varepsilon \| \theta_\varepsilon(\hat{\chi}_H) \|_{0,\infty,T} + C|S|^{1/2}(\| \varepsilon\tilde{f}_i \|_{0,\infty,S} + \| (K^* - \bar{K}_S^*)\nabla\hat{\chi}_H \|_{0,\infty,S}) \\
&\le C\varepsilon \| \nabla\hat{\chi}_H \|_{0,\infty,T} + C(\varepsilon + H)\| \nabla\hat{\chi}_H \|_{0,S} \\
&\le C\Big(\frac{\varepsilon}{H} + H\Big)\| \nabla\hat{\chi}_H \|_{0,T}.
\end{aligned} \tag{5.15}
$$

The following lemma can be proved by the method in [4, Lemma 4.10].

Lemma 5.6. *Under the assumption (H1)-(H4) there exist positive constants γ_2 and C independent of H, ε such that if $H_T \le \gamma_2, \varepsilon/H_T \le \gamma_2$ for all $T \in \mathcal{M}_H$, the following stability estimates are valid*

$$
\| \nabla\hat{\chi}_H \|_{0,T} \le C\| \nabla\chi_H \|_{0,T}, \qquad \| \nabla\chi_H \|_{0,T} \le C\| \nabla\hat{\chi}_H \|_{0,T}
$$

for any $\chi_H \in X_H, \hat{\chi}_H = \Pi_H\chi_H \in V_H$.

Lemma 5.7. *Under the assumption of Theorem 3.1 we have*

$$
|F_2(\hat{\chi}_H)| \le C\Big(\frac{\varepsilon}{H} + H\Big)|\ln H|^{1/2}\| \nabla\hat{\chi}_H \|_{0,\Omega}
$$

Proof. For any $T \in \mathcal{M}_H$, we denote by $\zeta_{i,1}^H = \hat{\zeta}_i^H - \varepsilon\chi(x, x/\varepsilon)\dfrac{\partial\hat{\zeta}_i^H}{\partial x_k}$ and get

$$
\begin{aligned}
&\int_T K_\varepsilon(x)\nabla\zeta_i^H \cdot \nabla\chi_H dx - \int_T K^*(x)\nabla\hat{\zeta}_i^H \cdot \nabla\hat{\chi}_H dx \\
&= \int_T K_\varepsilon(x)\nabla(\zeta_i^H - \zeta_{i,1}^H)\cdot\nabla\chi_H dx + \int_T K_\varepsilon(x)\nabla\zeta_{i,1}^H \cdot \nabla(\chi_H - \chi_H^1)dx \\
&\quad + \int_T (K_\varepsilon(x)\nabla\zeta_{i,1}^H - K^*(x)\nabla\hat{\zeta}_i^H)\cdot\nabla\chi_H^1 dx + \int_T K^*\nabla\hat{\zeta}_i^H \cdot \nabla(\chi_H^1 - \hat{\chi}_H)dx \\
&:= \mathrm{IV} + \cdots \mathrm{VII}.
\end{aligned}
$$

By (5.14) and Lemma 5.6 we know that

$$
\begin{aligned}
|\,\mathrm{IV}\,| &\le C\Big(\frac{\varepsilon}{H} + H\Big)\| \nabla\hat{\zeta}_i^H \|_{0,T}\| \nabla\chi_H \|_{0,T} \le C\Big(\frac{\varepsilon}{H} + H\Big)\| \nabla\hat{\zeta}_i^H \|_{0,T}\| \nabla\hat{\chi}_H \|_{0,T}, \\
|\,\mathrm{V}\,| &\le C\Big(\frac{\varepsilon}{H} + H\Big)\| \nabla\zeta_{i,1}^H \|_{0,T}\| \nabla\hat{\chi}_H \|_{0,T} \le C\Big(\frac{\varepsilon}{H} + H\Big)\| \nabla\hat{\zeta}_i^H \|_{0,T}\| \nabla\hat{\chi}_H \|_{0,T}.
\end{aligned}
$$

Using (4.6), Lemma 5.4 and the argument in the proof of Lemma 5.5 we can get

$$|\,\mathrm{VI}\,| \le C\frac{\varepsilon}{H}\|\,\nabla\hat{\zeta}_i^H\,\|_{0,T}\|\,\nabla\hat{\chi}_H\,\|_{0,T}.$$

Finally, since $\hat{\zeta}_i^H$ is linear in T, we have

$$\begin{aligned}\mathrm{VII} &= -\varepsilon\int_T K_{lm}^*(x)\frac{\partial\hat{\zeta}_i^H}{\partial x_m}\frac{\partial}{\partial x_l}\Big(\chi^k\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial\hat{\chi}_H}{\partial x_k}\Big)dx\\ &= -\varepsilon\int_T K_{lm}^*(x)\frac{\partial\hat{\zeta}_i^H}{\partial x_m}\frac{\partial\chi^k}{\partial x_l}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial\hat{\chi}_H}{\partial x_k}dx - \int_T K_{lm}^*\frac{\partial\hat{\zeta}_i^H}{\partial x_m}\frac{\partial\chi^k}{\partial y_l}\Big(x,\frac{x}{\varepsilon}\Big)\frac{\partial\hat{\chi}_H}{\partial x_k}dx\\ &\le C\frac{\varepsilon}{H}\|\,\nabla\hat{\zeta}_i^H\,\|_{0,T}\|\,\nabla\hat{\chi}_H\,\|_{0,T},\end{aligned}$$

where we have used Lemma 5.4 to estimate the second term. Collecting these estimates together, we have

$$|F_2(\hat{\chi}_H)| \le C\Big(\frac{\varepsilon}{H}+H\Big)\|\,\nabla\hat{\zeta}_i^H\,\|_{0,\Omega}\|\,\nabla\hat{\chi}_H\,\|_{0,\Omega}.$$

Now let $\omega \in C^\infty(\Omega_i)$ be the cut-off function such that $\omega = 1$ in $\Omega_i\backslash\bar{B}(\bar{x}_i, 3H_i/4)$, $\omega = 0$ in $B(\bar{x}_i, H_i/2)$ and $|\nabla\omega| \le CH_i^{-1}$ in Ω_i, we deduce from (3.10) and Green formula that

$$\begin{aligned}\sum_{T\in\mathcal{M}_H}\int_T K_\varepsilon(x)\nabla\zeta_i^H\cdot\nabla\chi_H dx &= -\int_{\Sigma_i} K_\varepsilon\frac{\partial G_i^\varepsilon}{\partial\nu}\hat{\chi}_H ds\\ &= -\int_{\Omega_i} K_\varepsilon\nabla G_i^\varepsilon\cdot\nabla(\omega\hat{\chi}_H)dx\\ &\le C\|\,\nabla G_i^\varepsilon\,\|_{0,\Omega_i\backslash\bar{B}(\bar{x}_i,H_i/2)}\|\,\nabla(\omega\hat{\chi}_H)\,\|_{0,\Omega_i}\\ &\le C\|\,\nabla(\omega\hat{\chi}_H)\,\|_{0,\Omega_i}\qquad\forall\chi_H\in X_H,\end{aligned}$$

where we have used Theorem 4.5 and Lemma 4.4 to deduce $\|\,\nabla G_i^\varepsilon\,\|_{0,\Omega_i\backslash\bar{B}(\bar{x}_i,H_i/2)} \le C$. Now using the inverse estimate $\|\,\hat{\chi}_H\,\|_{0,\infty,\Omega} \le C|\ln H|^{1/2}\|\,\nabla\hat{\chi}_H\,\|_{0,\Omega}$, we deduce by taking $\chi_H = \zeta_i^H$ in above inequality to get $(\sum_{T\in\mathcal{M}_H}\|\,\nabla\zeta_i^H\,\|_{0,T}^2)^{1/2} \le C|\ln H|^{1/2}$. Thus $\|\,\nabla\hat{\zeta}_i^H\,\|_{0,\Omega} \le C|\ln H|^{1/2}$ by Lemma 5.6. This completes the proof. □

PROPOSITION 5.8. *Under the assumption of Theorem 3.1, we have*

$$|(R_H\zeta_i^*)(\bar{x}_j) - \zeta_i^H(\bar{x}_j)| \le C\Big(\frac{\varepsilon}{H}+H+\sqrt{\frac{\varepsilon}{H_i}}\Big)|\ln H|.$$

Proof. By (5.9)-(5.10), Lemmas 5.5 and 5.7 we deduce that $|w_H(\bar{x}_j)| \le C(H + \varepsilon/H + \sqrt{\varepsilon/H_i})|\ln H|$. It remains to estimate the term $\zeta_i^H(\bar{x}_j) - \hat{\zeta}_i^H(\bar{x}_j)$ in (5.8). By (5.15) and the estimate $\|\,\nabla\hat{\zeta}_i^H\,\|_{0,\Omega} \le C|\ln H|^{1/2}$ proved in Lemma 5.7, we have that, for any $T \in \mathcal{M}_H$ such that $\bar{x}_j \in T$,

$$|\zeta_i^H(\bar{x}_j) - \hat{\zeta}_i^H(\bar{x}_j)| \le \Big(\frac{\varepsilon}{H}+H\Big)\|\,\nabla\hat{\zeta}^H\,\|_{0,T} \le C\Big(\frac{\varepsilon}{H}+H\Big)|\ln H|^{1/2}.$$

This completes the proof. □

Proof of Theorem 3.1. This is now a direct consequence of (5.4) and Propositions 5.2, 5.3 and 5.8. □

6. Approximation of Green functions. In this section we consider the approximation of Green functions by finite element methods. Let D be a convex polygon in $\mathbf{R}^2$ and G the Green function defined by

$$-\mathrm{div}\,(K(x)\nabla G) = \delta_{x_0} \quad \text{in } D, \qquad G|_{\partial D} = 0. \tag{6.1}$$

Here $x_0 \in D$ and we assume K satisfies the assumption (H1) in §2. Let $\mathcal{M}_h$ be a triangulation of the domain such that x_0 is one of the nodes. Denote by V_h the standard conforming linear finite element space over $\mathcal{M}_h$ and $V_h^0 = V_h \cap H_0^1(\Omega)$. We introduce the discrete Green function $G_h \in V_h^0$ as follows:

$$\int_D K(x)\nabla G_h \cdot \nabla v_h dx = v_h(x_0) \qquad \forall v_h \in V_h^0. \tag{6.2}$$

Remember that we are interested in the values of G on the well boundary $\Gamma_\delta = \{x : |x - x_0| = \delta\}$ with $\delta \ll h$. It is well-known that discrete Green function G_h fails to give good approximation in the vicinity of the singularity.

One way to overcome this difficulty is to remove the singularity by considering the function $w = G - \phi$, where $\phi = -\dfrac{1}{2\pi K_0}\ln|x - x_0|$, $K_0 = K(x_0)$. It is easy to see that $w \in H^1(\Omega)$ satisfies $w = -\phi$ on ∂D and

$$\int_D K(x)\nabla w \cdot \nabla v dx = \int_D (K - K_0)\nabla\phi \cdot \nabla v dx \qquad \forall v \in H_0^1(D). \tag{6.3}$$

We denote by $w_h \in V_h$ its finite element approximation: $w_h = -I_h\phi$ on ∂D and

$$\int_D K(x)\nabla w_h \cdot \nabla v_h dx = \int_D (K - K_0)\nabla\phi \cdot \nabla v_h dx \qquad \forall v_h \in V_h^0. \tag{6.4}$$

Thus the value of the Green function G on Γ_δ may be approximated by $-\frac{1}{2\pi K_0}\ln\delta + w_h(x_0)$.

Now we are going to develop a new approach which extend the well-known Peaceman model in [24, 25] to any finite element triangulation and general heterogeneous coefficients. The method is based on the notation of effective radius $\bar{r}$ with which $G|_{\Gamma_\delta}$ is approximated by

$$\bar{\alpha}_h = -\frac{1}{2\pi K_0}\ln\frac{\delta}{\bar{r}} + G_h(x_0). \tag{6.5}$$

Thus once one knows the effective radius, one only need to solve the discrete Green function (6.2) which is simpler than to solve (6.4). In Peaceman [24, 25], the effective radius is given by $\bar{r} = 0.14(\Delta x^2 + \Delta y^2)^{1/2}$ for constant coefficient differential opertor and finite difference discretization with mesh spacing $\Delta x, \Delta y$ in x and y directions. The argument to derive the formula is heuristic. In this paper, we propose to define the effective radius through the relation

$$\frac{1}{2\pi K_0}\ln\bar{r} = w_h(x_0) - G_h(x_0). \tag{6.6}$$

Thus $\bar{\alpha}_h = -\dfrac{1}{2\pi K_0}\ln\delta + w_h(x_0)$. Unfortunately, the definition (6.6) depends on $w_h(x_0)$. The nice thing is that $w_h(x_0)$ can be computed using the information of G_h as we are now going to describe.

Let $\psi_h \in V_h$ be the function whose nodal values are given by

$$\psi_h(x_k) = \begin{cases} \phi(x_k) & \text{if } x_k \in \partial D, \\ 0 & \text{otherwise}, \end{cases}$$

then we have $w_h - \psi_h \in V_h^0$ and

$$\begin{aligned} w_h(x_0) = (w_h - \psi_h)(x_0) &= \int_D K(x)\nabla G_h \cdot \nabla(w_h - \psi_h)dx \\ &= \int_D (K - K_0)\nabla\phi \cdot \nabla G_h dx - \int_D K(x)\nabla G_h \cdot \nabla\psi_h dx, \end{aligned}$$

where we have used (6.2) and (6.4). Thus the effective radius $\bar{r}$ can be computed by only using the information of the discrete Green function G_h.

The following theorem establishes the convergence of the method.

THEOREM 6.1. *Let $d = \operatorname{dist}(x_0, \partial D)$ and $\delta \le h$. Then under the assumption (H1) there exists a constant C independent of h and d but may depend on λ, Λ and the upper bound of the size of the domain such that*

$$\max_{x\in\Gamma_\delta} |G(x) - \bar{\alpha}_h| \le C\frac{h}{d}\Big|\ln\frac{h}{d}\Big|.$$

Proof. First we notice that

$$\begin{aligned} \max_{x\in\Gamma_\delta} |G(x) - \bar{\alpha}_h| &= \max_{x\in\Gamma_\delta} |w(x) - w_h(x_0)| \\ &\le \max_{x\in\Gamma_\delta} |w(x) - w(x_0)| + |w(x_0) - w_h(x_0)|. \end{aligned} \tag{6.7}$$

By using the interior $W^{1,p}$ estimate in Lemma 2.4 we can prove from (6.3) that

$$\| \nabla w \|_{0,p,B(x_0,d/4)} \le Cpd^{\frac{2}{p}-1} \quad \forall p > 2. \tag{6.8}$$

Thus by using Morrey's embedding theorem as in the proof of Lemma 2.5 we have

$$\max_{x\in\Gamma_\delta} |w(x) - w(x_0)| \le C\frac{\delta}{d}\Big|\ln\frac{\delta}{d}\Big|. \tag{6.9}$$

On the other hand, we use the interior L^∞ error estimates for finite element methods in [27] to get

$$\begin{aligned} |w(x_0) - w_h(x_0)| \le\ & C\Big|\ln\frac{d}{h}\Big| \inf_{v_h\in V_h} \| w - v_h \|_{0,\infty,B(x_0,d/4)} \\ &+ Cd^{-1}\| w - w_h \|_{0,B(x_0,d/4)}. \end{aligned} \tag{6.10}$$

By standard finite element interpolation estimate and (6.8) we have

$$\inf_{v_h\in V_h} \| w - v_h \|_{0,\infty,B(x_0,d/4)} \le Ch^{1-\frac{2}{p}}\| \nabla w \|_{0,p,B(x_0,d/4)} \le Cp(h/d)^{1-\frac{2}{p}} \quad \forall p > 2.$$

By taking $p = \ln d/h$ we know that

$$\inf_{v_h\in V_h} \| w - v_h \|_{0,\infty,B(x_0,d/4} \le C\frac{h}{d}\Big|\ln\frac{h}{d}\Big|. \tag{6.11}$$

Finally, by standard duality argument we have

$$\| w - w_h \|_{0,B(x_0,d/4)} \le Ch \| \nabla (w - w_h) \|_{0,D} \le Ch \| \nabla w \|_{0,D} \le Ch. \qquad (6.12)$$

This completes the proof by combining (6.9)-(6.12) with (6.7). □

The next theorem shows that G_h provides good approximation of G near the boundary ∂D.

THEOREM 6.2. *There exists a constant C independent of h and d such that*

$$\| \nabla (G - G_h) \|_{0,D\setminus \bar{B}(x_0,d/2)} \le Ch/d.$$

Proof. We first note that $G - G_h$ satisfies

$$\int_D K(x)\nabla(G - G_h)\cdot \nabla v_h dx = 0 \quad \forall v_h \in V_h^0, \ \ \operatorname{supp} v_h \subset D\setminus\{x_0\}.$$

Thus we can use the similar argument in deriving finite element interior estimate in [27, §3] or [29, p.408] to get

$$\begin{aligned}
&\| \nabla (G - G_h) \|_{0,D\setminus \bar{B}(x_0,d/2)} \\
\le &\, C \inf_{v_h \in V_h} \left(\| \nabla (G - v_h) \|_{0,D\setminus \bar{B}(x_0,d/2)} + d^{-\frac{2}{q}} \| G - v_h \|_{0,q,D\setminus \bar{B}(x_0,d/2)} \right) \\
&+ C d^{-\frac{2}{q}} \| G - G_h \|_{0,q,D\setminus \bar{B}(x_0,d/2)}, \qquad \forall q \in (1,2).
\end{aligned}$$

By Lemma 4.4 we know that $| G |_{2,D\setminus \bar{B}(x_0,d/2)} \le C d^{-1}$. Thus we obtain

$$\begin{aligned}
&\| \nabla (G - G_h) \|_{0,D\setminus \bar{B}(x_0,d/2)} \\
\le &\, Ch/d + C d^{-\frac{2}{q}} \| G - G_h \|_{0,q,D\setminus \bar{B}(x_0,d/2)}, \qquad \forall q \in (1,2). \qquad (6.13)
\end{aligned}$$

Since D is a convex polygon, there exists a constant $p_0 > 2$ depending on the maximum angles of D such that for any $g \in L^{p_0}(D)$, the dual problem

$$-\mathrm{div}\,(K(x)\nabla\psi) = g \quad \text{ in } D, \qquad \psi|_{\partial D} = 0$$

has the regularity estimate $\| \psi \|_{2,p_0,D} \le C \| g \|_{2,p_0,D}$. Denote by $\psi_h \in V_h^0$ the Ritz project of ψ, that is $\psi_h \in V_h^0$ satisfies

$$\int_D K(x)\nabla(\psi - \psi_h)\cdot\nabla v_h dx = 0 \qquad \forall v_h \in V_h^0.$$

Then by (6.2) we get, for any $g \in C_0^\infty(D)$,

$$\begin{aligned}
(\psi - \psi_h)(x_0) &= \int_D G(x)g(x)dx - \int_D K(x)\nabla G_h \cdot \nabla \psi_h dx \\
&= \int_D G(x)g(x)dx - \int_D K(x)\nabla G_h \cdot \nabla \psi dx \\
&= \int_D (G - G_h) g(x) dx. \qquad (6.14)
\end{aligned}$$

By interior finite element error estimate we have that

$$|\psi(x_0) - \psi_h(x_0)| \le C h^{2-\frac{2}{p_0}} \| \psi \|_{2,p_0,D} \le C h^{2-\frac{2}{p_0}} \| g \|_{0,p_0,D}.$$

By taking $q = p_0/(p_0 - 1)$ we deduce from (6.14) that $\| G - G_h \|_{0,q,D} \le C h^{2-\frac{2}{p_0}} = C h^{\frac{2}{q}}$, which, together with (6.13), completes the proof as $2/q > 1$. □

7. Numerical results. We start this section by summarizing the algorithm proposed in this paper for solving the problem (2.4)-(2.6) which is a good approximation of the original problem (2.1)-(2.3) when the size of the wells is negligible.

Algorithm 7.1. Given the well bore pressure α_i for $i \in I_D$ and the well flow rate q_j for $j \in I_M$. The following procedure finds the approximate well bore pressure α_j^H for $j \in I_M$, the approximate well flow rate q_i^H for $i \in I_D$, and the approximate pressure $u_H = \zeta_H + \sum_{j=1}^N \mathrm{q}_j^H G_j^h$, where $\mathrm{q}_j^H = \mathrm{q}_j$ for $j \in I_M$.

- For $j = 1, \cdots, N$, compute the discrete Green function G_j^h on each subdomain Ω_j, i.e. $G_j^h \in V_h^0(\Omega_j)$ such that

$$\int_{\Omega_j} K(x)\nabla G_j^h \cdot \nabla v_h dx = v_h(\bar{x}_j) \qquad \forall v_h \in V_h^0(\Omega_j). \tag{7.1}$$

  Here $V_h^0(\Omega_j) = V_h(\Omega_j) \cap H_0^1(\Omega_j)$, and $V_h(\Omega_j)$ is the standard conforming linear finite element space over the mesh $\mathcal{M}_h(\Omega_j)$ of Ω_j with the mesh size h resolving the scale of the permeability field $K(x)$. Compute the associate effective radius $\bar{r}_j$ from G_j^h according to the discussion in §6. Thus the approximate value of the Green function G_j on Γ_j is

$$\alpha_j^h = G_j^h(\bar{x}_j) - \frac{1}{2\pi K_j} \ln \frac{\delta}{\bar{r}_j}, \qquad K_j = K(\bar{x}_j).$$

- Use the over-sampling multiscale finite element method to find $\zeta_H \in X_H^0$ and $\{\mathrm{q}_i^H\}_{i \in I_D}$ such that

$$\sum_{T \in \mathcal{M}_H} \int_T K(x)\nabla\zeta_H \cdot \nabla\chi_H dx = -\sum_{j=1}^N \mathrm{q}_j^H \int_{\Sigma_j} K \frac{\partial G_j^h}{\partial \nu} \hat{\chi}_H ds \qquad \forall \chi_H \in X_H^0,$$
$$\mathrm{q}_i^H \alpha_i^h + \zeta_H(\bar{x}_i) = \alpha_i \qquad \text{for } i \in I_D.$$

- Compute the approximate well bore pressure α_j^H for $j \in I_M$ through the relation

$$\alpha_j^H = \zeta_H(\bar{x}_j) + \mathrm{q}_j \alpha_j^h.$$

Now we present two numerical examples carried out for periodic and random log-normal permeabilities to demonstrate the accuracy and efficiency of our method. More extensive numerical experiments will be reported elsewhere. The computations are performed on the unit square $\Omega = (0,1) \times (0,1)$. We consider two wells $B_j = B(\bar{x}_j, \delta)$ with $x_1 = (\frac{1}{4}, \frac{3}{4})$, $x_2 = (\frac{3}{4}, \frac{1}{4})$ and the radius $\delta = 10^{-5}$. We impose mixed boundary condition (1.3) that fixes the well flow rate q_j for both wells. Thus we have $I_D = \emptyset$ and the interested quantities are well bore pressure (WBP) on each well. In the computations we always take $\mathrm{q}_1 = -1$ and $\mathrm{q}_2 = 1$ which corresponds to the situation that the well B_1 is an extraction well and B_2 is an injection well.

The domain Ω_j is taken as the square centered at $\bar{x}_j$ with the length of the sides being $2H_j$ so that $H_j = \mathrm{dist}\,(\bar{x}_j, \partial\Omega_j)$. The discrete Green function G_j^h in (7.1) is computed on a uniform finite element mesh of size $h = H_j/512$.

A uniform $L \times L$ finite element mesh of Ω is constructed by first dividing the domain Ω into $L \times L$ subrectangles and then connecting the lower-left and the upper-right vertices of each subrectangle. The finite elements of the mesh is divided into two groups: the lower-right and upper-left elements. For each triangle T, an over-sampling element $S(T)$ is created according to whether T is a lower-right element or an upper-left element as shown in Figure 7.1. Here we assume that the permeability field is known outside the domain Ω when the over-sampling element $S(T)$ is extended outside Ω. This assumption is not a restriction for practical applications because the permeability is generated by geostatistical method based on the statistical characteristics which are usually known outside the interested domain.

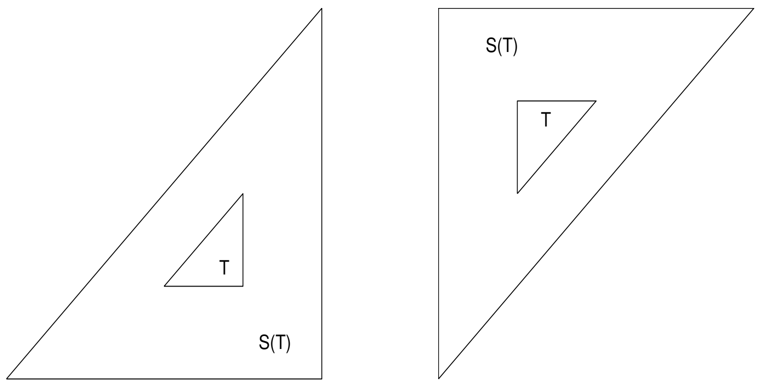

FIG. 7.1. *The element T and its over-sampling element $S(T)$: lower-right elements (left) and upper-left elements (right). The length of the horizontal and vertical edges of $S(T)$ is four times of the corresponding length of the edges of T.*

For each triangle T, the over-sampling multiscale base functions on the over-sampling element $S(T)$ are computed on a uniform triangular mesh which divides the horizontal and vertical edges of $S(T)$ into $4M$ subintervals, respectively. So the mesh size for solving the base functions is $1/LM$ in each space direction.

All the resultant linear algebraic system of equations is solved by the multigrid method with matrix dependent prolongation [30].

Example 7.1. In this example we assume the coefficient is

$$K_\varepsilon(x) = \frac{1}{(2 + 1.5\sin\frac{2\pi x}{\varepsilon})(2 + 1.5\sin\frac{2\pi y}{\varepsilon})}.$$

The exact solution of the problem is unknown and so we compare the coarse grid solution obtained by Algorithm 7.1 with the well resolved solution on a uniform 2048×2048 mesh using the method introduced in §6 which can be easily extended to multi-well problems. This well resolved solution will be referred as "exact" solution in the following. For example, when $\varepsilon = 1/64 = 0.015625$, the "exact" well bore pressures are $\alpha_1 = -5.3884973$ in the first well and $\alpha_2 = 5.3884973$ in the second well.

The homogenized coefficient matrix K^* is known $K^* = \frac{1}{\sqrt{7}}\mathrm{I}$, where I is the 2×2 identity matrix [16]. The standard homogenization method would be to approximate the original problem by

$$-\mathrm{div}\,(K^*(x)\nabla u^*) = \sum_{j=1}^{2} \mathrm{q}_j \delta_{\bar{x}_j} \quad \text{in } \Omega, \qquad u^*|_\Gamma = 0.$$

L	Well No.	WBP	Relative error
32	1	-4.3624648	0.19041
64	1	-4.3626761	0.19037
128	1	-4.3627285	0.19036
32	2	4.3624648	0.19041
64	2	4.3626761	0.19037
128	2	4.3627285	0.19036

TABLE 7.1
Results of the standard homogenization method, $\varepsilon = 1/64$.

Table 7.1 shows the well bore pressures computed by solving above homogenized problem using the Peaceman method over uniform meshes. It clearly indicates that the standard homogenization method fails to provide good approximations in the vicinity of well singularities. This shows the importance to develop new upscaling methods for flow transport problems involving well singularities.

Recall from Theorem 3.3 that neglecting the logarithm factors, the convergence rate of Algorithm 7.1 is $O(H + H^2/H_j^2 + \varepsilon/H + \sqrt{\varepsilon/H_j})$. In Table 7.2, we test the convergence of our method when ε/H is fixed to be 0.5. We see that the solution still converges as ε decreases. This shows that the constant in the ε/h term in the error estimate is small. We also observe that the convergence rate is better than $\sqrt{\varepsilon}$ when the resonant error term ε/H is not dominant. In Table 7.3, we show the errors for a fixed $\varepsilon = 1/64$. We observe that the errors increase as H decreases, indicating the existence of the error term ε/H in the error estimate. The resonant error is the strongest when $\varepsilon = H$. We remark that previous studies [15, 10] indicate that the cell resonance error is more visible for periodic coefficients, especially when ε/H is a rational number. But the cell resonance error is generically small for random coefficients. Table 7.4 shows the influence of the size H_j of local domain Ω_j to the overall convergence of the method. It indicates that in practical computations, the size of the local domain Ω_j need not be very large (two or three times of the size of the coarse mesh would be enough). Remember that smaller local domains Ω_j reduce the cost of computing the discrete Green functions. However, we emphasize that the discrete Green functions need only to be computed once which is a just small overhead of the overall computations.

Example 7.2. In this example, we test the performance of our method for a random log-normal permeability field. We generate the random field $K(x)$ on a uniform 1024×1024 mesh by using moving ellipse average technique [7] with the variance of the logarithm of the permeability $\sigma^2 = 1$, and the correlation lengths $l_x = l_y = 0.01$ in two space direction. Figure 7.2 shows a realization of the random permeability field. The "exact" well bore pressures are computed by solving the problem (2.4)-(2.6) on the fine 1024×1024 mesh are $\alpha_1 = -1.7383199$ and $\alpha_2 = 3.0999852$.

Table 7.5 shows the results computed by using the Algorithm 7.1 on the coarse 64×64 mesh with the size of the local domains $H_1 = H_2 = 2H$. The mesh size for

L	ε	Well No.	WBP	Relative error	Rate
8	1/16	1	-5.6634649	0.54608E-02	
16	1/32	1	-5.5312112	0.22704E-02	1.2662
32	1/64	1	-5.3842752	0.78354E-03	1.5349
8	1/32	2	5.6615759	0.57925E-02	
16	1/32	2	5.5281669	0.28195E-02	1.0387
32	1/64	2	5.3802340	0.15335E-02	0.8786

Table 7.2
Results of the Algorithm 7.1, $H_j = 1/8$, $M = 64$.

L	M	Well No.	WBP	Relative error	Rate
8	256	1	-5.3683706	0.37351E-02	
16	128	1	-5.3828001	0.10573E-02	1.8207
32	64	1	-5.3842752	0.78354E-03	0.4323
64	32	1	-5.3838442	0.86352E-03	-0.1402
8	256	2	5.3713129	0.31891E-02	
16	128	2	5.3825320	0.11070E-02	1.5256
32	64	2	5.3802340	0.15335E-02	-0.4702
64	32	2	5.3739254	0.27043E-02	-0.8184

Table 7.3
Results of the Algorithm 7.1, $\varepsilon = 1/64$, $H_j = 1/8$.

Well No.	H_j	WBP	Relative error	Rate
1	H	-5.3133941	0.139377E-01	
1	2H	-5.3519482	0.678280E-02	1.0390
1	4H	-5.3700536	0.342279E-02	0.9867
2	H	5.3123314	0.141349E-01	
2	2H	5.3497078	0.719858E-02	0.9735
2	4H	5.3653456	0.429650E-02	0.7445

Table 7.4
Influence of the size of the local domain Ω_j, $\varepsilon = 1/64, L = 64, M = 32$.

Well No.	WBP	Relative error
1	-1.7102777	0.1613E-01
2	3.0960784	0.1260E-02

TABLE 7.5

Results of the Algorithm 7.1 in the case of random log-normal field with $\sigma^2 = 1$ and $l_x = l_y = 0.01$.

solving the over-sampling base functions is $1/1024$. It clearly shows that the WBP by our method on the coarse 64×64 mesh is a very good approximation of the "exact" WBP on the fine 1024×1024 mesh. In Figure 7.3 we show the contour plots of the pressure field computed by Algorithm 7.1 on the 64×64. As a comparison, we show in Figure 7.4 the contour plots of the averaged "exact" pressure which is obtained by averaging the "exact" pressure field on the 1024×1024 mesh to the coarse 64×64 mesh. We observe from these figures and Table 7.5 that our method provides good approximation of the well bore pressure as well as the large scale structure of the "exact" solutions for random log-normal permeability field. A careful theoretical analysis of the upscaling method proposed in this paper for random permeability fields is an interesting open problem.

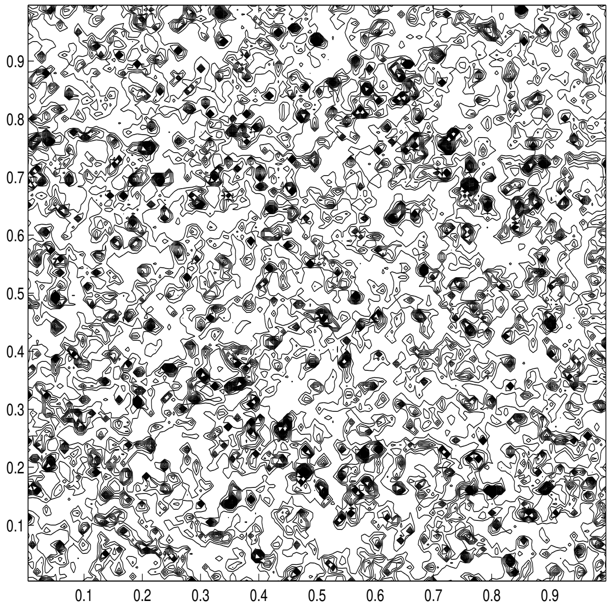

FIG. 7.2. *The random log-normal permeability field $K(x)$. The ratio of maximum to minimum is 4.15443E+03.*

8. Appendices. In this section we prove some of the technical results used in the paper.

8.1. Appendix 1: Interior $W^{1,p}$ estimate. In this subsection we briefly recall the proof of the $W^{1,p}$ interior estimate in Lemma 2.4. The main purpose here is to

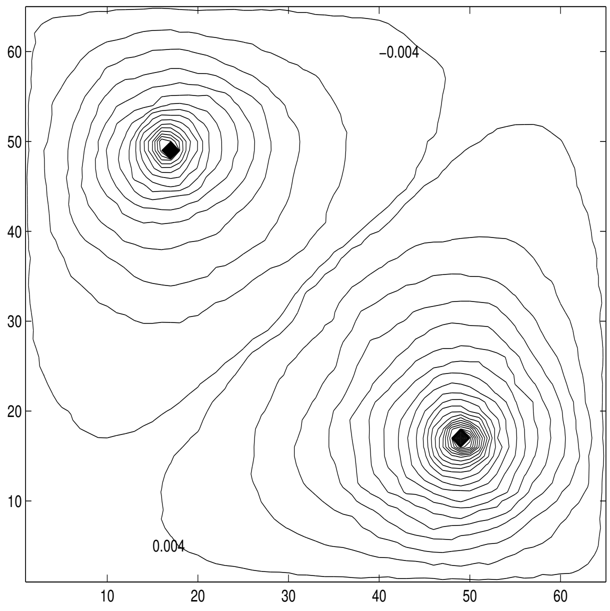

FIG. 7.3. *Contour plots of the pressure field on 64×64 mesh by using Algorithm 7.1.*

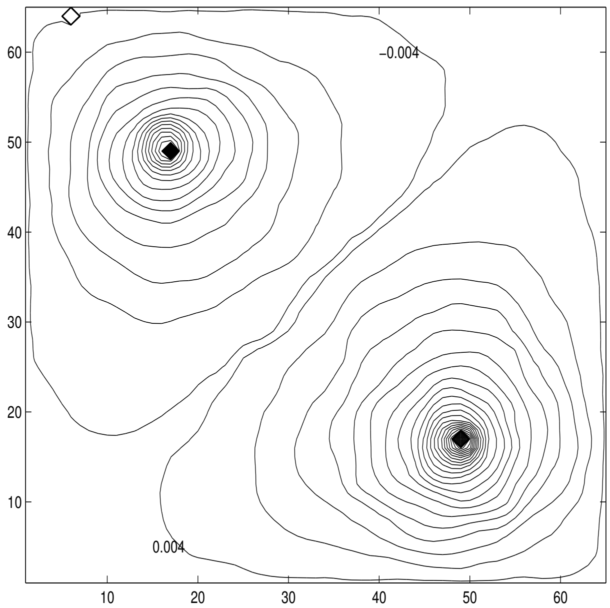

FIG. 7.4. *Contour plots of the averaged "exact" pressure field on the 64×64 mesh*

trace down the explicit dependence of the estimate on p which plays key role in the analysis of this paper. We follow the proof presented in Troianello [28].

The starting point is the Campanato space $L^{C}_{2,2}(D)$ which is a subset of the functions u in $L^2(D)$ such that

$$[u]_{2,2,D} := \sup_{x\in D, 0<\rho<+\infty} \left\{ \frac{1}{\rho^2} \int_{D(x,\rho)} |u(y) - u_{D(x,\rho)}|^2 dy \right\}^{1/2} < +\infty,$$

where $D(x,\rho) = B(x,\rho) \cap D$ and $u_{D(x,\rho)} = \frac{1}{|D(x,\rho)|} \int_{D(x,\rho)} u(y)dy$. $L^C_{2,2}(D)$ is a Banach space with the norm

$$\| u \|_{L^C_{2,2}(D)} = \Big(\| u \|^2_{0,D} + [u]^2_{2,2,D} \Big)^{1/2}.$$

It is easy to prove [28, p.29] that

$$[u]^2_{2,2,D} \leq \frac{1}{r^2} \| u \|^2_{0,D} + \sup_{x\in D, 0<\rho\leq r} \frac{1}{\rho^2} \int_{D(x,\rho)} |u(y) - u_{D(x,\rho)}|^2 dy. \tag{8.1}$$

Let Q_0 be a square in $\mathbf{R}^2$ with the edges parallel to the coordinate axes. Let $v \in L^1(Q_0)$. For any square $Q \subset Q_0$ with edges parallel to those of Q_0, we denote by

$$|v|_{*,Q_0} := \sup_{Q\subset Q_0} \frac{1}{|Q|} \int_Q |v - v_Q| dx.$$

It is easy to see that $|v|_{*,Q_0} \leq [v]_{2,2,Q_0}$. The following deep result due to John-Nirenberg will be used.

LEMMA 8.1. *Let $v \in L^C_{2,2}(Q_0)$. Then there exists two constants C_1, C_2 such that for any $Q \subset Q_0$,*

$$\text{meas}\,\{x \in Q : |v - v_Q| \geq t\} \leq C_1 |Q| e^{-\frac{C_2}{|v|_{*,Q_0}} t}.$$

Now for any $p \geq 1$ and $Q \subset Q_0$, we have

$$\begin{aligned} \int_Q |v - v_Q|^p &= p \int_0^\infty t^{p-1} \text{meas}\,\{x \in Q : |v - v_Q| > t\} dt \\ &\leq C_1 p |Q| \int_0^\infty t^{p-1} e^{-\frac{C_2}{|v|_{*,Q_0}} t} dt \\ &= p C_1 \Big(\frac{|v|_{*,Q_0}}{C_2} \Big)^p |Q| \int_0^\infty t^{p-1} e^{-t} dt. \end{aligned}$$

Denote by $\zeta(p-1) = \int_0^\infty t^{p-1} e^{-t} dt$. It is easy to see that $\zeta(p) = p\zeta(p-1)$. Thus $\zeta(p) \leq Cp^p$ for some constant C independent of p. Therefore, we have

$$\| v - v_Q \|_{0,p,Q} \leq Cp |Q|^{\frac{1}{p}} |v|_{*,Q_0} \leq Cp |Q|^{\frac{1}{p}} [v]_{2,2,Q_0} \quad \forall v \in L^C_{2,2}(Q_0). \tag{8.2}$$

LEMMA 8.2. ([28, Lemma 1.18]) *Assume that $\psi : (0, d_0) \to \mathbf{R}^+$ satisfies*

$$\psi(\rho) \leq K_1 \frac{\rho^2}{r^2} \psi(r) + \frac{r^2}{\rho^2} K_2, \qquad K_1 > 1, \quad 0 < \rho < r < d_0. \tag{8.3}$$

Then we have $\psi(\rho) \leq K_1 \psi(r) + (K_1^3/(K_1 - 1)) K_2$ for any $0 < \rho < r < d_0$.

Proof. Let $l \geq 0$ be the integer such that $1 \leq K_1^l \leq \frac{r}{\rho} \leq K_1^{l+1}$. Then repeatedly using (8.3) we have

$$\psi(\rho) \leq \frac{1}{K_1^l}\psi(\rho K_1^l) + K_1^2 K_2\Big(1 + \frac{1}{K_1} + \cdots + \frac{1}{K_1^{l-1}}\Big).$$

Moreover,

$$\psi(\rho K_1^l) \leq K_1\Big(\frac{\rho K_1^l}{r}\Big)^2\psi(r) + \Big(\frac{r}{K_1^l}\Big)^2 K_2 \leq K_1\psi(r) + K_1^2 K_2.$$

Thus

$$\psi(\rho) \leq K_1^{1-l}\psi(r) + K_1^2 K_2\Big(1 + \frac{1}{K_1} + \cdots + \frac{1}{K_1^l}\Big) \leq K_1\psi(r) + \frac{K_1^2 K_2}{1 - K_1^{-1}}.$$

This completes the proof. □

Now we are ready to complete the proof of Lemma 2.4.

Proof of Lemma 2.4. The starting point is the interior $L^C_{2,2}(D)$ estimate in [28, Lemma 3.2]

$$\begin{aligned}\| \nabla u - (\nabla u)_{B(x,\rho)} \|^2_{0,B(x,\rho)} \leq C(\lambda)\Big\{ &\frac{\rho^4}{r^4}\| \nabla u - (\nabla u)_{B(x,r)} \|^2_{0,B(x,r)} \\ &+ \Lambda^2 r^2 \| \nabla u \|^2_{0,B(x,r)} + \sum_{i=1}^2 \| f_i - (f_i)_{B(x,r)} \|^2_{0,B(x,r)}\Big\}\end{aligned}$$

for any $x \in D$, $0 < \rho < r$ and $B(x,r) \subset D$. Denote by $\psi(\rho) = \rho^{-2}\| \nabla u - (\nabla u)_{B(x,\rho)} \|^2_{0,B(x,\rho)}$. Then we have

$$\psi(\rho) \leq C(\lambda)\frac{\rho^2}{r^2}\psi(r) + \frac{r^2}{\rho^2}C(\lambda)\Big[\Lambda^2\| \nabla u \|^2_{0,B(x,r)} + \sum_{i=1}^2 [f_i]^2_{2,2,B(x,r)}\Big],$$

which implies by Lemma 8.2 that

$$\psi(\rho) \leq C(\lambda)\psi(r) + C(\lambda)(\Lambda^2\| \nabla u \|^2_{0,B(x,r)} + [f]^2_{2,2,B(x,r)}).$$

By taking $r = \mathrm{dist}\,(Q, \partial D)/2 =: d_1$ and using (8.1), we deduce that

$$\begin{aligned}[\nabla u]^2_{2,2,Q} &\leq d_1^{-2}\| \nabla u \|^2_{0,D} + C(\lambda)(\psi(d_1) + \Lambda^2\| \nabla u \|^2_{0,D} + \sum_{i=1}^2 [f_2]^2_{2,2,D}) \\ &\leq C(\lambda)\{(d_1^{-2} + \Lambda^2)\| \nabla u \|^2_{0,D} + \sum_{i=1}^2 [f_i]^2_{2,2,D}\}.\end{aligned}$$

On the other hand, it is easy to see that $[f_i]_{2,2,D} \leq C\| f_i \|_{0,\infty,D}$. Thus

$$[\nabla u]^2_{2,2,Q} \leq C(\lambda)\Big\{(d_1^{-2} + \Lambda^2)\| \nabla u \|^2_{0,D} + \sum_{i=1}^2 \|f_i\|^2_{0,\infty,D}\Big\}.$$

This completes the proof upon using (8.2). □

8.2. Appendix 2: The Weak Maximum Principle. In this subsection we prove the weak maximum principle which has been used in §4. The main purpose here is to trace down the explicit dependence of the constant on $p > 2$ in the estimate.

THEOREM 8.3. *Let $D \subset \mathbf{R}^2$ be a domain with Lipschitz boundary. Let $a_{ij} \in L^\infty(D)$ satisfy*

$$\mu|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \quad \textit{for } \xi \in \mathbf{R}^2 \quad \textit{a.e. in } D,$$

where $\mu > 0$. Let $f_1, f_2 \in L^p(D)$ for $p > 2$ and let

$$u \in H_0^1(D): \quad \int_D a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}dx = \int_D f_i \frac{\partial \varphi}{\partial x_i}dx \quad \forall \varphi \in H_0^1(D). \tag{8.4}$$

Then there exists a constant C independent of p and D such that

$$\| u \|_{0,\infty,D} \le C\frac{2p}{p-2}|D|^{\frac{1}{2}-\frac{1}{p}}\Big(\sum_{i=1}^2 \| f_i \|_{0,p,D}^2\Big)^{1/2}.$$

Proof. The proof follows mainly the idea in [18, Theorem B.2]. For any $k > 0$ define

$$\zeta = (\mathrm{sgn}u)\max(|u|-k, 0) = \begin{cases} u-k & \text{for } u \ge k \\ 0 & \text{for } |u| \le k \\ u+k & \text{for } u \le -k. \end{cases}$$

Clearly $\partial\zeta/\partial x_i = \partial u/\partial x_i$ on the set $A(k) = \{x \in D : |u(x)| \ge k\}$ whereas $\partial\zeta/\partial x_i = 0$ elsewhere. Thus by taking $\varphi = \zeta \in H_0^1(D)$ in (8.4) we obtain easily that

$$\| \nabla\zeta \|_{0,D} \le \frac{1}{\mu}\Big(\sum_{i=1}^2 \| f_i \|_{0,A(k)}^2\Big)^{1/2}.$$

Then by Hölder inequality, we get, for any $p > 2$,

$$\| \nabla\zeta \|_{0,D} \le \frac{1}{\mu}\Big(\sum_{i=1}^2 \| f_i \|_{0,p,D}^2\Big)^{1/2}|A(k)|^{\frac{1}{2}-\frac{1}{p}}, \tag{8.5}$$

where $|A(k)|$ stands for the Lebesgue measure of $A(k)$. Note that $L^\infty(D)$ is not embedded into $H^1(D)$ because of $D \subset \mathbf{R}^2$. We resort to a different argument from that of in [18, Theorem B.2].

Since $\zeta \in H_0^1(D)$, by the embedding estimate in [12, p.156] we know that

$$\| \zeta \|_{0,\frac{2s}{2-s},D} \le \frac{2s}{2-s}\| \nabla\zeta \|_{0,s,D} \quad \forall s \in [1, 2).$$

Thus, by Hölder inequality

$$\| \zeta \|_{0,\frac{2s}{2-s},D} \le \frac{2s}{2-s}|D|^{\frac{1}{s}-\frac{1}{2}}\| \nabla\zeta \|_{0,D} \quad \forall s \in [1, 2),$$

which, together with (8.5), implies

$$\| \zeta \|_{0,\frac{2s}{2-s},D} \le \frac{1}{\mu}\frac{2s}{2-s}|D|^{\frac{1}{s}-\frac{1}{2}}\Big(\sum_{i=1}^2 \| f_i \|_{0,p,D}^2\Big)^{1/2}|A(k)|^{\frac{1}{2}-\frac{1}{p}}.$$

The rest of the proof is similar to that of [18, Theorem B.2]. For any $h > k > 0$, we have $A(h) \subset A(k)$ and

$$(h-k)|A(h)|^{\frac{2-s}{2s}} \le \| \, |u| - k \, \|_{0,\frac{2s}{2-s},A(h)} \le \| \, |u| - k \, \|_{0,\frac{2s}{2-s},A(k)} = \| \zeta \|_{0,\frac{2s}{2-s},D}.$$

Consequently

$$|A(h)| \le \frac{1}{(h-k)^\alpha} \Big\{ \frac{1}{\mu} \frac{2s}{2-s} |D|^{\frac{1}{s}-\frac{1}{2}} \Big(\sum_{i=1}^{2} \| f_i \|_{0,p,D}^2 \Big)^{1/2} \Big\}^\alpha |A(k)|^\beta,$$

where $\alpha = 2s/(2-s) > 0$ and $\beta = \Big(\frac{1}{2} - \frac{1}{p}\Big)\alpha$. Note that $\beta > 1$ whenever $s \in (p/(p-1), 2)$. Then by applying Lemma B.1 in [18] we know that

$$|u(x)| \le \rho \qquad \text{a.e. in } D$$

with

$$\rho = \frac{1}{\mu} \frac{2s}{2-s} |D|^{\frac{1}{s}-\frac{1}{2}} \Big(\sum_{i=1}^{2} \| f_i \|_{0,p,D}^2 \Big)^{1/2} |D|^{\frac{\beta-1}{\alpha}} 2^{\frac{\beta}{\beta-1}}.$$

The key observation is that $\frac{1}{s} - \frac{1}{2} + \frac{\beta - 1}{\alpha} = \frac{\beta}{\alpha} = \frac{1}{2} - \frac{1}{p}$. Thus

$$\begin{aligned} \rho &= \frac{1}{\mu} \frac{2s}{2-s} |D|^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_{i=1}^{2} \| f_i \|_{0,p,D}^2 \Big)^{1/2} 2^{\frac{\beta}{\beta-1}} \\ &= \frac{1}{\mu} \frac{2p}{p-2} \Big(\sum_{i=1}^{2} \| f_i \|_{0,p,D}^2 \Big)^{1/2} \beta 2^{\frac{\beta}{\beta-1}} \qquad \forall p > 2, \beta > 1. \end{aligned}$$

This completes the proof upon taking $\beta = 2$, for example. □

Acknowledgment. We wish to thank Thomas Y. Hou for many inspiring discussions on the multiscale finite element methods and the upscaling of well singularities. We also wish to thank Dr. Yalchin R. Efendiev for providing us the code of generating the random log-normal permeability field.

REFERENCES

[1] M. Avellaneda and F.-H. Lin, *Compactness methods in the theory of homogenization*, Comm. Pure Appl. Math. **40** (1987), 803–847.

[2] M. Avellaneda and F.-H. Lin, *L^p bounds on singular integrals in homogenization*, Comm. Pure Appl. Math. **44** (1991), 897-910.

[3] A. Bensoussan, J.L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures,* North-Holland, Amsterdam, 1978.

[4] Z. Chen and T.Y. Hou, *A mixed multiscale finite element method for elliptic problem with oscillating coefficients,* Math. Comp. (to appear).

[5] G. Christakos, *Random Field Models in Earth Sciences,* Academic Press, San Diego, 1992.

[6] C. Conca, S. Natesan and M. Vanninathan, *Numerical solution of elliptic partial differential equations by Bloch waves method,* in XVII CEDYA: Congress on Differential Equations and Applications/VII: Congress on Applied Mathematics (Spanish) (Salamanca 2001), Dep. Mat. Apl., Univ. Salamanca, 2001, 63–83.

[7] L.J. Durlofsky, *Numerical calculation of equivalent grid block permeability tensors for heterogeneous porous media,* Water Resources Research **27** (1991), 699–708.

[8] L.J. Durlofsky, W.J. Milliken and A. Bernath, *Scale up in the Near-Well Region,* SPE 51940, Proceedings of the 15th SPE Symposium on Reservoir Simulation, Houston, February 14-17, 1999.

[9] B.B. Dykaar and P.K. Kitanidis, *Transmissivity of a heterogeneous formation*, Water Resource Research **29** (1993), 985-1001.

[10] Y.R. Efendiev, T.Y. Hou, and X.H. Wu, *The convergence of non-conforming multiscale finite element methods,* SIAM J. Numer. Anal. **37** (2000), 888–910.

[11] H.H. Gerke, U. Hornung, Y. Kelanemer, M. Slodička and S. Schumacher, *Optimal Control of Soil Venting: Mathematical Modelling and Applications,* Birkhäuser Verlag, Basel, 1999.

[12] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, Berlin, 1983.

[13] P. Grisvard, *Elliptic Problems on Non-smooth Domains,* Pitman, Boston, 1985.

[14] M. Grüter and K.-O. Widman, *The Green function for uniformly elliptic equations,* Manuscripta Math. **37** 1982, 303-342.

[15] T.Y. Hou and X.H. Wu, *A multiscale finite element method for elliptic problems in composite materials and porous media,* J. Comput. Phys. **134** (1997), 169–189.

[16] T.Y. Hou, X.H. Wu, and Z. Cai, *Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients,* Math. Comp. **68** (1999), 913–943.

[17] V.V. Jikov, S.M. Kozlov, and O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals,* Springer, Berlin, 1994.

[18] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications,* Academic Press, New York, 1980.

[19] D.Q. Li and S.X. Chen, *On asymptotic behaviors of the equi-valued surface boundary value problem for the second order self-adjoint elliptic equations,* Fudan Journal (Natural Sciences), 6-14, December 1978. (in Chinese)

[20] W. Littman, G. Stampacchia and H.F. Weinberger, *Regular points for elliptic equations with discontinueous coefficients,* Ann. Scuola Norm Sup Pisa, **17** (1963), 43-77.

[21] O. Mascarenhas and L.J. Durlofsky, *Scale up in the vicinity of horizontal wells,* Proceedings of the 20th Annual International Energy Agency Workshop and Symposium, Paris, September 22-24, 1999.

[22] S. Moskow and M. Vogelius, *First order corrections to the homogenized eigenvalues of a periodic composite medium. A convergence proof,* Proc. Royal Soc. Edinburgh A **127** (1997), 1263–1299.

[23] A. Matache, I. Babuška and C. Schwab, *Generalized p-fem in homogenization,* Numer. Math. **86**, 319–375.

[24] D.W. Peaceman, *Interpretation of well-block pressures in numerical reservoir simulations,* SPEJ, 183-194, June 1978.

[25] D.W. Peaceman, *Interpretation of well-block pressures in numerical reservoir simulations with non-square grid blocks and anisotropic permeability,* SPEJ, 531-543, June 1983.

[26] Ph. Renard and G. de Marsily, *Calculating equivalent permeability: A review,* Advances in Water Resources, **20** (1997), 253-278.

[27] A.H. Schatz and L.B. Wahlbin, *Interior maximum norm estimates for finite element methods,* Math. Comp. **31** (1977), 414-442.

[28] G.M. Troianiello, *Elliptic Differential Equations and Obstacle Problems,* Plenum Press, New York, 1987.

[29] L.B. Wahlbin, *Local behavior in Finite Element methods,* in Handbook of Numerical Analysis, Volume II, edited by P.G. Ciarlet and J.L. Lions, Elsevier, Paris, 1991, 355-521.

[30] P.M. De Zeeuw, *Matrix-dependent prolongation and restrictions in a blackbox multigrid solver,* J. Comput. Appl. Math. **33** (1990), 1–27.