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An adaptive anisotropic perfectly matched layer method for 3-D time harmonic electromagnetic scattering problems

Zhiming Chen · Tao Cui · Linbo Zhang

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Abstract We develop an anisotropic perfectly matched layer (PML) method for solving the time harmonic electromagnetic scattering problems in which the PML coordinate stretching is performed only in one direction outside a cuboid domain. The PML parameters such as the thickness of the layer and the absorbing medium property are determined through sharp *a posteriori* error estimates. Combined with the adaptive finite element method, the proposed adaptive anisotropic PML method provides a complete numerical strategy to solve the scattering problem in the framework of FEM which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the choice of the thickness of the PML layer. Numerical experiments are included to illustrate the competitive behavior of the proposed adaptive method.

Keywords Electromagnetic scattering · Perfectly matched layer · Anisotropic

1 Introduction

We propose and study an adaptive anisotropic perfectly matched layer (PML) method for solving the time harmonic electromagnetic scattering problem with the perfectly conducting boundary condition

Zhiming Chen
LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190, China.
E-mail: zmchen@lsec.cc.ac.cn

Tao Cui
LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190, China.
E-mail: tcui@lsec.cc.ac.cn

Linbo Zhang
LSEC, Institute of Computational Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190, China.
E-mail: zlb@lsec.cc.ac.cn

$$\nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} = 0 \quad \text{in } \mathbb{R}^3 \backslash \bar{D}, \tag{1.1}$$

$$\mathbf{n}_D \times \mathbf{E} = \mathbf{g} \quad \text{on } \Gamma_D, \tag{1.2}$$

$$|\mathbf{x}| \Big[(\nabla \times \mathbf{E}) \times \hat{\mathbf{x}} - \mathbf{i}k\mathbf{E} \Big] \to 0 \quad \text{as } |\mathbf{x}| \to \infty. \tag{1.3}$$

Here $D \subset \mathbb{R}^3$ is a bounded domain with Lipschitz polyhedral boundary Γ_D, $\mathbf{E}$ is the electric field, $\mathbf{g}$ is determined by the incoming wave, $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and $\mathbf{n}_D$ is the unit outer normal to Γ_D. We assume the wave number $k \in \mathbb{R}$ is a constant. We remark that the results in this paper can be easily extended to solve the scattering problems with other boundary conditions such as Neumann or the impedance boundary condition on Γ_D, or to solve the electromagnetic wave propagation through inhomogeneous media with a variable wave number $k^2(\mathbf{x})$ inside some bounded domain.

Since the work of Bérenger [5] which proposed a PML technique for solving the time dependent Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [36], Teixeira and Chew [34] for the reviews). Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML method is to surround the computational domain by a layer of finite thickness with specially designed model medium that absorbs all the waves that propagate from inside the computational domain.

The convergence of the PML method using circular PML layers is studied in Lassas and Somersalo [27], Hohage et al [24] for the acoustic scattering problems and in Bao and Wu [3], Bramble and Pasciak [7] for the electromagnetic scattering problems. It is proved in [27], [24], [7] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinity.

The adaptive PML method was first proposed in Chen and Wu [14] for a scattering problem by periodic structures (the grating problem). It is extended in Chen and Liu [12], Chen and Wu [15] for the acoustic scattering problem and in Chen and Chen [10] for electromagnetic scattering problems in which one uses the *a posteriori* error estimate to determine the PML parameters. Combined with the adaptive finite element method, the adaptive PML method provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer.

A posteriori error estimates are computable quantities in terms of the discrete solution and data that measure the actual discrete errors without the knowledge of exact solutions. The adaptive finite element method based on *a posteriori* error estimates provides a systematic way to achieve the optimal computational complexity by refining the mesh according to the local *a posteriori* error estimator on the elements. A posteriori error estimates for the Nédélec $H(\text{curl})$-conforming edge elements are obtained in Monk [29] for Maxwell scattering problems, in Beck et al [4] for eddy current problems, and in Chen et al [13] for Maxwell cavity problems. The restriction in [29], [4] that the domain should be convex or have smooth boundary in order to ensure the regularity of the functions in the Helmholtz decomposition is removed in [13] by using the Birman-Solomyak decomposition [6].

The main purpose of this paper is to propose an anisotropic PML method for the electromagnetic scattering problem (1.1)-(1.3) in which the PML layer is placed outside a cuboid domain. The main advantage of the anisotropic PML method as opposed to the circular PML method is that it provides greater flexibility and efficiency to solve problems involving anisotropic scatterers. One widely used anisotropic PML method in the literature is the uniaxial PML method. The convergence of the uniaxial PML method has been considered recently in Chen and Wu [15], Chen and Zheng [16], and Kim and Pasciak [26] for the 2D acoustic scattering problem. The stability of the uniaxial PML method in 3D is still an open problem due to the difficulty of the corner regions resulting from stretching the PML coordinate in three different directions. In our method, the PML coordinate stretching is performed only in one direction outside the cuboid domain. The stability of the PML problem is proved by extending the idea in [27], [7], [28] for circular or smooth PML layers. The convergence of our PML method is then proved by using the Stratton-Chu integral representation formula of the exterior Dirichlet problem for the time-harmonic Maxwell equation and the idea of the complex coordinate stretching. We also consider the finite element *a posteriori* error estimates and develop the adaptive anisotropic PML method. We also remark similar idea of defining PML layer outside a cuboid domain is also proposed in Trenev [35] for 2D Helmholtz equations and numerically tested.

The layout of the paper is as follows. In section 2 we construct our anisotropic PML formulation for (1.1)-(1.3) by following the method of complex coordinate stretching in Chew and Weedon [17]. In section 3 we prove the exponential decay of the PML extension based on the Stratton-Chu integral representation formula. In section 4 we show the stability of the PML problem in the PML layer. The results in Sections 3 and 4 are then used to prove the exponential convergence of the PML method in section 5. In section 6 we introduce the finite element approximation. In section 7 we derive the *a posteriori* error estimate which includes both the PML error and the finite element discretization error. Finally in section 8 we describe our adaptive algorithm and present two examples to show the competitive behavior of the adaptive method.

2 The PML equation

We first recall some notation. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain with boundary Γ whose unit outer normal is denoted by $\mathbf{n}$. The space

$$H(\mathrm{curl};\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla \times \mathbf{v} \in L^2(\Omega)^3\}$$

is a Hilbert space under the graph norm. The starting point to introduce the traces in $H(\mathrm{curl};\Omega)$ is the following Green formula

$$\int_\Omega (\nabla \times \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \nabla \times \mathbf{v})\, d\mathbf{x} = \langle \mathbf{n} \times \mathbf{u}, \mathbf{n} \times \mathbf{v} \times \mathbf{n} \rangle_\Gamma, \tag{2.1}$$

for any $\mathbf{u}, \mathbf{v} \in H^1(\Omega)^3$, where $\langle \cdot, \cdot \rangle_\Gamma$ is the duality pairing between $H^{-1/2}(\Gamma)^3$ and $H^{1/2}(\Gamma)^3$. Let $V_\pi(\Gamma) = \pi_\tau(H^{1/2}(\Gamma)^3)$, where for any $\mathbf{u} \in H^{1/2}(\Gamma)^3$, $\pi_\tau(\mathbf{u}) = \mathbf{n}\times\mathbf{u}\times\mathbf{n}$. We observe from (2.1) that for any $\mathbf{u} \in H(\mathrm{curl};\Omega)$, the tangential trace $\gamma_\tau \mathbf{u} = \mathbf{n} \times \mathbf{u}|_\Gamma$ can be defined as a continuous linear map on $V_\pi(\Gamma)$, that is, $\gamma_\tau \mathbf{u} \in V_\pi(\Gamma)'$. The mapping $\gamma_\tau : H(\mathrm{curl};\Omega) \to V_\pi(\Gamma)'$ is, however, not surjective. It is proved in Buffa et al [8] that the map γ_τ is a surjective mapping to the space

$$H^{-1/2}(\mathrm{Div};\Gamma) = \{\boldsymbol{\lambda} \in V_\pi(\Gamma)' : \mathrm{div}_\Gamma \boldsymbol{\lambda} \in H^{-1/2}(\Gamma)\},$$

which is a Hilbert space under the graph norm. It is known [8] that for $\mathbf{u} \in H(\mathrm{curl};\Omega)$, the surface divergence of $\mathbf{n} \times \mathbf{u}$ on Γ, $\mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{u}) = -\nabla \times \mathbf{u} \cdot \mathbf{n} \in H^{-1/2}(\Gamma)$. In the following we denote $Y(\Gamma) = H^{-1/2}(\mathrm{Div};\Gamma)$.

For any $\mathbf{v} \in H(\mathrm{curl};\Omega)$, we define the weighted norm

$$\|\mathbf{v}\|_{H(\mathrm{curl};\Omega)} = \left(d_\Omega^{-2}\|\,\mathbf{v}\,\|^2_{L^2(\Omega)} + \|\,\nabla \times \mathbf{v}\,\|^2_{L^2(\Omega)}\right)^{1/2}, \tag{2.2}$$

where d_Ω is the diameter of Ω. We use the weighted $H^{1/2}(\Gamma)$ norm,

$$\|\,v\,\|_{H^{1/2}(\Gamma)} = \left(d_\Omega^{-1}\|\,v\,\|^2_{L^2(\Gamma)} + |v|^2_{\frac{1}{2},\Gamma}\right)^{1/2}, \tag{2.3}$$

and the weighted $Y(\Gamma)$ norm

$$\|\,\boldsymbol{\mu}\,\|_{Y(\Gamma)} = \left(d_\Omega^{-2}\|\boldsymbol{\mu}\|^2_{V'_\pi(\Gamma)} + \|\mathrm{div}_\Gamma\boldsymbol{\mu}\|^2_{H^{-1/2}(\Gamma)}\right)^{1/2},$$

where

$$|v|^2_{\frac{1}{2},\Gamma} = \int_\Gamma\int_\Gamma \frac{|v(\mathbf{x}) - v(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|^3}\,ds(\mathbf{x})ds(\mathbf{x}').$$

Thus, for any $\mathbf{u} \in H(\mathrm{curl};\Omega)$, since $\mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{u}) = -\nabla \times \mathbf{u} \cdot \mathbf{n}$ on Γ, we have

$$\|\mathbf{n} \times \mathbf{u}\|_{Y(\Gamma)} = \left(d_\Omega^{-2}\|\mathbf{n} \times \mathbf{u}\|^2_{V'_\pi(\Gamma)} + \|\nabla \times \mathbf{u} \cdot \mathbf{n}\|^2_{H^{-1/2}(\Gamma)}\right)^{1/2}. \tag{2.4}$$

By the scaling argument and the trace theorem we know that there exist constants C_1, C_2 independent of d_Ω, such that for any $\boldsymbol{\lambda} \in Y(\Gamma)$,

$$C_1\|\,\boldsymbol{\lambda}\,\|_{Y(\Gamma)} \le \inf_{\substack{\gamma_\tau(\mathbf{u})|_\Gamma = \boldsymbol{\lambda} \\ \mathbf{u} \in H(\mathrm{curl};\Omega)}} \|\mathbf{u}\|_{H(\mathrm{curl};\Omega)} \le C_2\|\,\boldsymbol{\lambda}\,\|_{Y(\Gamma)}. \tag{2.5}$$

Let D be contained in the interior of the domain

$$B_1 = \{\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3 : |x_i| < L_i/2, \quad i = 1, 2, 3\}.$$

Let $\Gamma_1 = \partial B_1$ and $\mathbf{n}_1$ the unit outer normal to Γ_1. Given a tangential vector $\boldsymbol{\lambda}$ on Γ_1, the Calderon operator $G_e : Y(\Gamma_1) \to Y(\Gamma_1)$ is the Dirichlet-to-Neumann operator defined by

$$G_e(\boldsymbol{\lambda}) = \frac{1}{\mathbf{i}k}\mathbf{n}_1 \times (\nabla \times \mathbf{E}^s),$$

where $\mathbf{E}^s$ satisfies

$$\nabla \times \nabla \times \mathbf{E}^s - k^2\mathbf{E}^s = 0 \quad \text{in } \mathbb{R}^3\backslash\bar{B}_1, \tag{2.6}$$

$$\mathbf{n}_1 \times \mathbf{E}^s = \boldsymbol{\lambda} \quad \text{on } \Gamma_1, \tag{2.7}$$

$$|\mathbf{x}|\Big[(\nabla \times \mathbf{E}^s) \times \hat{\mathbf{x}} - \mathbf{i}k\mathbf{E}^s\Big] \to 0 \ \text{ as } |\mathbf{x}| \to \infty. \tag{2.8}$$

Let $a : H(\mathrm{curl};\Omega_1) \times H(\mathrm{curl};\Omega_1) \to \mathbb{C}$, where $\Omega_1 = B_1\backslash\bar{D}$, be the sesquilinear form

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega_1} (\nabla \times \mathbf{u} \cdot \nabla \times \bar{\mathbf{v}} - k^2\mathbf{u} \cdot \bar{\mathbf{v}})d\mathbf{x} + \mathbf{i}k\langle G_e(\mathbf{n}_1 \times \mathbf{u}), \mathbf{n}_1 \times \mathbf{v} \times \mathbf{n}_1\rangle_{\Gamma_1}.$$

The scattering problem (1.1)-(1.3) is equivalent to the following weak formulation: Given $\mathbf{g} \in Y(\Gamma_D)$, find $\mathbf{E} \in H(\mathrm{curl};\Omega_1)$ such that $\mathbf{n}_D \times \mathbf{E} = \mathbf{g}$ on Γ_D, and

$$a(\mathbf{E},\mathbf{v}) = 0, \qquad \forall \mathbf{v} \in H_D(\mathrm{curl};\Omega_1), \tag{2.9}$$

where $H_D(\mathrm{curl};\Omega_1) = \{\mathbf{v} \in H(\mathrm{curl};\Omega_1) : \mathbf{n} \times \mathbf{v} = 0 \ \text{ on } \Gamma_D\}$.

The existence of a unique solution of the variational problem (2.9) is known [20], [32], [30]. For the later analysis we need the inf-sup condition for the sesquilinear form $a(\cdot,\cdot)$.

Lemma 1 *There exists a constant $C > 0$ such that the following inf-sup condition holds*

$$\sup_{\mathbf{v}\in H_D(\mathrm{curl};\Omega_1)} \frac{|a(\mathbf{u},\mathbf{v})|}{\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_1)}} \ge C\|\mathbf{u}\|_{H(\mathrm{curl};\Omega_1)}, \quad \forall \mathbf{u} \in H_D(\mathrm{curl};\Omega_1). \tag{2.10}$$

Proof. For any $\mathbf{u} \in H_D(\mathrm{curl};\Omega_1)$, denote $\mathbf{u}^s$ the unique solution of (2.6)-(2.8) with $\boldsymbol{\lambda} = \mathbf{n}_1 \times \mathbf{u}$ on Γ_1. Let $\tilde{\mathbf{v}} \in H(\mathrm{curl};\mathbb{R}^3)$ be the extension of $\mathbf{v} \in H_D(\mathrm{curl};\Omega_1)$ satisfying $\|\tilde{\mathbf{v}}\|_{H(\mathrm{curl};\mathbb{R}^3)} \le C\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_1)}$. The existence of such extension for $H(\mathrm{curl})$ functions on Lipschitz domains is proved e.g. in Chen et al [11].

Let B_1 be included in the ball B_R, $R > 0$. Since $\mathbf{u}^s$ satisfies (2.6), by multiplying the equation by $\bar{\tilde{\mathbf{v}}}$ and integrating by parts over the domain $B_R\backslash\bar{B}_1$ we obtain

$$\begin{aligned}&\langle \mathbf{n}_1 \times \nabla \times \mathbf{u}^s, \mathbf{n}_1 \times \mathbf{v} \times \mathbf{n}_1\rangle_{\Gamma_1}\\ &= \int_{B_R\backslash\bar{\Omega}_1} (\nabla \times \mathbf{u}^s \cdot \nabla \times \bar{\tilde{\mathbf{v}}} - k^2\mathbf{u}^s \cdot \bar{\tilde{\mathbf{v}}})d\mathbf{x} + \langle \hat{\mathbf{x}} \times \nabla \times \mathbf{u}^s, \hat{\mathbf{x}} \times \tilde{\mathbf{v}} \times \hat{\mathbf{x}}\rangle_{\partial B_R}.\end{aligned}$$

Thus

$$a(\mathbf{u},\mathbf{v}) = \int_{B_R\backslash\bar{D}} (\nabla \times \mathbf{u} \cdot \nabla \times \bar{\tilde{\mathbf{v}}} - k^2\mathbf{u} \cdot \bar{\tilde{\mathbf{v}}})d\mathbf{x} + \mathbf{i}k\langle G_e(\hat{\mathbf{x}} \times \mathbf{u}), \hat{\mathbf{x}} \times \tilde{\mathbf{v}} \times \hat{\mathbf{x}}\rangle_{\partial B_R}.$$

The lemma now follows by using the inf-sup condition for the sesquilinear form based on the Dirichlet-to-Neumann mapping on the spherical boundary, cf. e.g. Monk [29, Lemma 10.9]. This completes the proof. $\square$

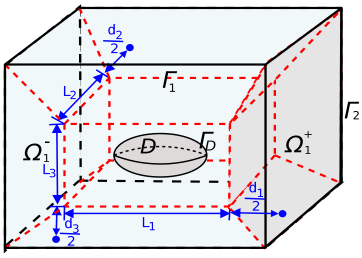

Fig. 2.1 Setting of the scattering problem with the PML layer.

Now we turn to the introduction of the absorbing PML layer. Let

$$B_2 = \{\mathbf{x} \in \mathbb{R}^3 : |x_i| < L_i/2 + d_i/2, \quad i = 1,2,3\}$$

be the domain which contains B_1. We assume that

$$\theta := 1 + \frac{d_1}{L_1} = 1 + \frac{d_2}{L_2} = 1 + \frac{d_3}{L_3}.$$

Then the diameter of B_2 is $d = \theta L$, where $L = (L_1^2 + L_2^2 + L_3^2)^{1/2}$. The domain $\Omega^{\mathrm{PML}} = B_2 \backslash \bar{B}_1$ is divided into six square frusta $\Omega_i^{\pm}, i = 1, 2, 3$, where

$$\Omega_i^+ = \{\mathbf{x} : x_j = rs_j, s_i = L_i/2, |s_j| \le L_j/2, j \ne i, j = 1, 2, 3, 1 < r < \theta\},$$
$$\Omega_i^- = \{\mathbf{x} : x_j = rs_j, s_i = -L_i/2, |s_j| \le L_j/2, j \ne i, j = 1, 2, 3, 1 < r < \theta\}.$$

Notice that $r = r(\mathbf{x}) = x_i/(\pm L_i/2)$ in $\Omega_i^{\pm}$. For $t \ge 0$, let $\alpha(t) = \eta(t) + \mathbf{i}\sigma(t)$ be the model medium property, where $\eta(t) = 1 + \zeta\sigma(t)$ with a constant $\zeta \ge 0$, and $\sigma(t) \ge 0$ for $t \ge 0$, $\sigma(t) = 0$ for $t \le 1$. The choice $\zeta > 0$, which is also used in the engineering literature [34], corresponds to introduce the additional damping for the evanescent waves propagating from B_1 in the PML region. We will show that this choice will enhance the elliptic coerciveness of the PML operator (see Lemma 8 and the remark after Lemma 8 below).

Denote $\tilde{r}$ the complex stretching of r

$$\tilde{r}(\mathbf{x}) := \int_0^{r(\mathbf{x})} \alpha(t)dt = \int_0^{r(\mathbf{x})} \eta(t)dt + \mathbf{i}\int_0^{r(\mathbf{x})} \sigma(t)dt,$$

and define the complex coordinates $\tilde{x}_j = \tilde{r}(\mathbf{x})s_j$, $j = 1, 2, 3$, then we know that

$$\tilde{x}_j = \beta(r(\mathbf{x}))x_j, \text{ where } \beta(t) = \hat{\eta}(t) + \mathbf{i}\hat{\sigma}(t), \hat{\eta} = 1 + \zeta\hat{\sigma}, \hat{\sigma} = \frac{1}{t}\int_0^t \sigma(t)dt. \quad (2.11)$$

We know that $r(\mathbf{x})$ is continuous in Ω^{PML} and thus the complex coordinate stretching function $\tilde{x}_j$ is a continuous function in Ω^{PML}. We set $\tilde{\mathbf{x}} = \mathbf{x}$ for $\mathbf{x} \in \bar{B}_1$.

In this paper we make the following assumption on the medium property.

(H1) $\sigma = \hat{\sigma} = \sigma_0$ for $t \ge r_0 > 1$, where σ_0 is a constant, $\hat{\sigma}'(t) \ge 0$, for $t \ge 1$, and $\zeta \ge \sqrt{2} \max_{i,j=1,2,3} \dfrac{L_i}{L_j}$.

The requirement that the medium property $\sigma = \hat{\sigma}$ is constant for $t \ge r_0$ has been also used in [27] and [7]. To derive the PML equation, we first notice that by the Stratton-Chu integral representation formula, the solution $\mathbf{E}^s$ of the exterior Dirichlet problem (2.6)-(2.8) satisfies

$$\mathbf{E}^s = \Psi_{\mathrm{SL}}^k(\boldsymbol{\mu}) + \Psi_{\mathrm{DL}}^k(\boldsymbol{\lambda}) \quad \text{in } \mathbb{R}^3 \backslash \bar{B}_1, \quad (2.12)$$

where $\boldsymbol{\mu} = G_e(\boldsymbol{\lambda}) \in Y(\Gamma_1)$ is the Neumann trace of $\mathbf{E}^s$ on Γ_1, and $\Psi_{\mathrm{SL}}^k, \Psi_{\mathrm{DL}}^k$ are respectively the Maxwell single and double layer potential (cf. e.g. [9])

$$\Psi_{\mathrm{SL}}^k(\boldsymbol{\mu})(\mathbf{x}) = \mathbf{i}k\Psi_{\mathbf{A}}^k(\boldsymbol{\mu})(\mathbf{x}) + \mathbf{i}k^{-1}\nabla\left[\Psi_V^k(\mathrm{div}_{\Gamma_1}\boldsymbol{\mu})(\mathbf{x})\right], \quad \forall \mathbf{x} \in \mathbb{R}^3 \backslash \bar{B}_1, \quad (2.13)$$

$$\Psi_{\mathrm{DL}}^k(\boldsymbol{\lambda})(\mathbf{x}) = \nabla \times \left[\Psi_{\mathbf{A}}^k(\boldsymbol{\lambda})(\mathbf{x})\right], \quad \forall \mathbf{x} \in \mathbb{R}^3 \backslash \bar{B}_1. \quad (2.14)$$

Here Ψ_V^k and $\Psi_{\mathbf{A}}^k$ are the scalar and vector single layer potential for the Helmholtz kernel equation

$$\Psi_V^k(\phi)(\mathbf{x}) = \int_{\Gamma_1} \phi(\mathbf{y})G_k(\mathbf{x}, \mathbf{y})ds(\mathbf{y}), \quad \Psi_{\mathbf{A}}^k(\boldsymbol{\phi})(\mathbf{x}) = \int_{\Gamma_1} \boldsymbol{\phi}(\mathbf{y})G_k(\mathbf{x}, \mathbf{y})ds(\mathbf{y})$$

with $G_k(\mathbf{x},\mathbf{y})=\frac{e^{\mathbf{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$ being the fundamental solution of the 3D Helmholtz equation.

We follow the method of complex coordinate stretching [17] to introduce the PML equation. For any $z\in\mathbb{C}$, denote $z^{1/2}$ the analytic branch of $\sqrt{z}$ such that $\mathrm{Re}(z^{1/2})>0$ for any $z\in\mathbb{C}\backslash(-\infty,0]$. Let

$$\rho(\tilde{\mathbf{x}},\mathbf{y})=\left[(\tilde{x}_1-y_1)^2+(\tilde{x}_2-y_2)^2+(\tilde{x}_3-y_3)^2\right]^{1/2}$$

be the complex distance and define $G_k(\tilde{\mathbf{x}},\mathbf{y})=\frac{e^{\mathbf{i}k\rho(\tilde{\mathbf{x}},\mathbf{y})}}{4\pi\rho(\tilde{\mathbf{x}},\mathbf{y})}$. It is easy to see that $G_k(\tilde{\mathbf{x}},\mathbf{y})$ is smooth for $\mathbf{x}\in\mathbb{R}^3\backslash\bar{B}_1$ and $\mathbf{y}\in\bar{B}_1$. We define the modified scalar and vector single layer potential for the Helmholtz equation

$$\tilde{\Psi}_V^k(\phi)(\mathbf{x})=\int_{\Gamma_1}\phi(\mathbf{y})G_k(\tilde{\mathbf{x}},\mathbf{y})ds(\mathbf{y}),\qquad\forall\phi\in H^{-1/2}(\Gamma_1),$$
$$\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\phi})(\mathbf{x})=\int_{\Gamma_1}\boldsymbol{\phi}(\mathbf{y})G_k(\tilde{\mathbf{x}},\mathbf{y})ds(\mathbf{y}),\qquad\forall\boldsymbol{\phi}\in H^{-1/2}(\Gamma_1)^3,$$

and the modified single and double layer potential

$$\tilde{\Psi}_{\mathrm{SL}}^k(\boldsymbol{\mu})(\mathbf{x})=\mathbf{i}k\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\mu})(\mathbf{x})+\mathbf{i}k^{-1}\tilde{\nabla}\left[\tilde{\Psi}_V^k(\mathrm{div}_{\Gamma_1}\boldsymbol{\mu})(\mathbf{x})\right],$$
$$\tilde{\Psi}_{\mathrm{DL}}^k(\boldsymbol{\lambda})(\mathbf{x})=\tilde{\nabla}\times\left[\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})(\mathbf{x})\right].$$

Here $\tilde{\nabla}=(\partial/\partial\tilde{x}_1,\partial/\partial\tilde{x}_2,\partial/\partial\tilde{x}_3)^T$ is the gradient operator with respect to the stretched coordinates.

For any $\boldsymbol{\lambda}\in Y(\Gamma_1)$, let $\mathbb{E}(\boldsymbol{\lambda})(\mathbf{x})$ be the PML extension

$$\mathbb{E}(\boldsymbol{\lambda})(\mathbf{x})=\tilde{\Psi}_{\mathrm{SL}}^k(\boldsymbol{\mu})(\mathbf{x})+\tilde{\Psi}_{\mathrm{DL}}^k(\boldsymbol{\lambda})(\mathbf{x})\quad\text{ for }\mathbf{x}\in\mathbb{R}^3\backslash\bar{B}_1,\tag{2.15}$$

where $\boldsymbol{\mu}=G_e(\boldsymbol{\lambda})$. It is easy to see that $\mathbf{n}_1\times\mathbb{E}(\boldsymbol{\lambda})=\boldsymbol{\lambda}$ on Γ_1.

For the solution $\mathbf{E}$ of the scattering problem (2.9), let $\tilde{\mathbf{E}}=\mathbb{E}(\mathbf{n}_1\times\mathbf{E}|_{\Gamma_1})$ be the PML extension of $\mathbf{n}_1\times\mathbf{E}|_{\Gamma_1}$. Then $\mathbf{n}_1\times\tilde{\mathbf{E}}=\mathbf{n}_1\times\mathbf{E}|_{\Gamma_1}$ on Γ_1. It is obvious that $\tilde{\mathbf{E}}$ satisfies

$$\tilde{\nabla}\times\tilde{\nabla}\times\tilde{\mathbf{E}}-k^2\tilde{\mathbf{E}}=0\quad\text{ in }\mathbb{R}^3\backslash\bar{B}_1.$$

Let $\mathbf{F}:\Omega^{\mathrm{PML}}\to\mathbb{C}^3$ be defined by

$$F_j(\mathbf{x})=\beta(r(\mathbf{x}))x_j,\qquad j=1,2,3.$$

Then $\tilde{\mathbf{x}}=\mathbf{F}(\mathbf{x})$ and

$$\tilde{\nabla}\times=J^{-1}D\mathbf{F}\nabla\times D\mathbf{F}^T,\qquad J=\det(D\mathbf{F}),\quad D\mathbf{F}\text{ the Jacobian matrix.}\tag{2.16}$$

When $\mathbf{F}:\Omega^{\mathrm{PML}}\to\mathbb{R}^3$ is a real transform, (2.16) is known, cf. e.g. [30, P.78]. For the complex valued transform, the identity then follows from the principle of analytic continuation. By (2.16) we obtain easily the desired PML equation

$$\nabla\times A\nabla\times(B\tilde{\mathbf{E}})-k^2A^{-1}(B\tilde{\mathbf{E}})=0\quad\text{ in }\mathbb{R}^3\backslash\bar{B}_1,$$

where $A=J^{-1}D\mathbf{F}^TD\mathbf{F}$ and $B=D\mathbf{F}^T$.

The PML problem is then to find $\hat{\mathbf{E}}$, which approximates $\mathbf{E}$ in Ω_1 and $B\mathbf{E}$ in $\Omega^{\rm PML} = B_2 \backslash \bar{B}_1$, as the solution of the following system

$$\nabla \times A \nabla \times \hat{\mathbf{E}} - k^2 A^{-1} \hat{\mathbf{E}} = 0 \quad \text{in } \Omega_2 = B_2 \backslash \bar{D}, \tag{2.17}$$

$$\mathbf{n}_D \times \hat{\mathbf{E}} = \mathbf{g} \quad \text{on } \Gamma_D, \qquad \mathbf{n}_2 \times \hat{\mathbf{E}} = 0 \quad \text{on } \Gamma_2. \tag{2.18}$$

The well-posedness of the PML problem (2.17)-(2.18) and the convergence of its solution to the solution of the original problem (1.1)-(1.3) will be studied in section 5.

To conclude this section, for the sake of later reference, we write down the explicit formula for the matrix A in the domain $\Omega_1^{\pm}$. The formulas in the other domains are similar. We notice that $r(\mathbf{x}) = x_1/s_1, s_1 = \pm L_1/2$ depending $\mathbf{x} \in \Omega_1^{\pm}$. By $\mathbf{F}(\mathbf{x}) = \beta(r(\mathbf{x}))\mathbf{x}$ it is easy to check that

$$D\mathbf{F} = \beta I + (\alpha - \beta)\mathbf{s}\mathbf{t}^T = \begin{pmatrix} \alpha & 0 & 0 \\ \frac{(\alpha-\beta)s_2}{s_1} & \beta & 0 \\ \frac{(\alpha-\beta)s_3}{s_1} & 0 & \beta \end{pmatrix}, \quad J = \det(D\mathbf{F}) = \alpha\beta^2, \tag{2.19}$$

where $\mathbf{s} = (s_1, s_2, s_3)^T$, $\mathbf{t} = s_1^{-1}(1, 0, 0)^T$, and

$$A = J^{-1} D\mathbf{F}^T D\mathbf{F} = \begin{pmatrix} \frac{\alpha}{\beta^2} + \frac{(\alpha-\beta)^2}{\alpha\beta^2} \frac{s_2^2+s_3^2}{s_1^2} & \frac{\alpha-\beta}{\alpha\beta} \frac{s_2}{s_1} & \frac{\alpha-\beta}{\alpha\beta} \frac{s_3}{s_1} \\ \frac{\alpha-\beta}{\alpha\beta} \frac{s_2}{s_1} & \frac{1}{\alpha} & 0 \\ \frac{\alpha-\beta}{\alpha\beta} \frac{s_3}{s_1} & 0 & \frac{1}{\alpha} \end{pmatrix}. \tag{2.20}$$

From the property of elementary matrix we have $D\mathbf{F}^{-1} = \beta^{-1}(I + \frac{\beta-\alpha}{\alpha}\mathbf{s}\mathbf{t}^T)$. It is easy to see that $1 \le |\alpha|, |\beta| \le 1 + (1+\zeta)\sigma_{\max}$, $\|D\mathbf{F}\| \le C_0(1+\sigma_{\max})$, $\|D\mathbf{F}^{-1}\| \le C_0(1+\sigma_{\max})$, $\|A\| \le C_0^2(1+\sigma_{\max})^2$, and $\|A^{-1}\| \le C_0^2(1+\zeta)(1+\sigma_{\max})^3$, where $\sigma_{\max} = \max_{1 \le t \le r_0} |\sigma(t)|$ and $C_0 = (1+\zeta)(1+2L/\min(L_1, L_2, L_3))$ with $L = \sqrt{L_1^2 + L_2^2 + L_3^2}$.

3 Exponential decay of the PML extension

In this section we prove the exponential decay of the PML extension (2.15). We start with the following elementary lemma.

Lemma 2 *For any $z_i = a_i + \mathbf{i} b_i$ with $a_i, b_i \in \mathbb{R}$, $i = 1, 2, 3$, such that $a_1 b_1 + a_2 b_2 + a_3 b_3 \ge 0$ and $a_1^2 + a_2^2 + a_3^2 > 0$, we have*

$$\mathrm{Im}(z_1^2 + z_2^2 + z_3^2)^{1/2} \ge \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.$$

Proof. The proof extends the proof of Lemma 3.2 in [15]. For any $a, b \in \mathbb{R}$ we know that

$$\mathrm{Im}(a + \mathbf{i}b)^{1/2} = \mathrm{sgn}(b) \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}.$$

Here we used the convention that $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\mathrm{Re}(z^{1/2}) > 0$ for any $z \in \mathbb{C}\backslash(-\infty, 0]$. It is easy to check that $\mathrm{Im}(a+\mathbf{i}b)^{1/2}$ is a decreasing function in $a \in \mathbb{R}$. Let $z_1^2+z_2^2+z_3^2 = a+\mathbf{i}b$, then

$$a+\mathbf{i}b = \left(\sqrt{a_1^2+a_2^2+a_3^2} + \mathbf{i}\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2}}\right)^2 - \frac{(a_1b_2-a_2b_1)^2+(a_1b_3-a_3b_1)^2+(a_2b_3-a_3b_2)^2}{a_1^2+a_2^2+a_3^2}.$$

Let

$$a' = a + \frac{(a_2b_1-a_1b_2)^2+(a_1b_3-a_3b_1)^2+(a_2b_3-a_3b_2)^2}{a_1^2+a_2^2+a_3^2},$$

since $b = 2(a_1b_1+a_2b_2+a_3b_3) \geq 0$, we have

$$\mathrm{Im}(a'+\mathbf{i}b)^{1/2} = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}}.$$

On the other hand, since $a' \geq a$, we know that $\mathrm{Im}(a+\mathbf{i}b)^{1/2} \geq \mathrm{Im}(a'+\mathbf{i}b)^{1/2}$. This completes the proof. $\square$

In the reminder of the paper we need the following assumption which is rather mild in the practical applications as we are interested in the convergence of the PML method when θ sufficiently large.

(H2) $\theta > r_0$, $\bar{\sigma} = \displaystyle\int_1^\theta \sigma(t)dt \geq \sqrt{3/2}$.

Lemma 3 *Let (H1)-(H2) be satisfied. Then for any* $\mathbf{x} \in \Gamma_2$ *and* $\mathbf{y} \in \bar{B}_1$*,*

$$\mathrm{Im}\ \rho(\tilde{\mathbf{x}}, \mathbf{y}) \geq \gamma\,\bar{\sigma}, \qquad \gamma = \frac{L_{\min}}{2}\frac{\theta L_{\min}}{(1+\theta+\zeta\bar{\sigma})L} \geq \frac{L_{\min}}{2}\frac{L_{\min}}{(2+\zeta\sigma_{\max})L},$$

where $L_{\min} = \min(L_1, L_2, L_3)$.

Proof. Let $z_j = \tilde{x}_j - y_j = (\hat{\eta}(r(\mathbf{x}))x_j - y_j) + \mathbf{i}\hat{\sigma}(r(\mathbf{x}))x_j$. Since $r(\mathbf{x}) = \theta$, $|\mathbf{x}| \geq \theta L_{\min}/2$ for $\mathbf{x} \in \Gamma_2$ and $|\mathbf{y}| \leq L/2$ for $\mathbf{y} \in \bar{B}_1$, we have

$$|\hat{\eta}\mathbf{x}| - |\mathbf{y}| \geq \hat{\eta}(\theta)\theta L_{\min}/2 - L/2 = \theta L_{\min}/2 + \bar{\sigma}\zeta L_{\min}/2 - L/2 \geq \theta L_{\min}/2, \quad (3.1)$$

where we have used (H1)-(H2). This implies,

$$\sum_{j=1}^3 (\hat{\eta}(\theta)x_j - y_j)\cdot\hat{\sigma}(\theta)x_j \geq \hat{\sigma}(\theta)|\mathbf{x}|(|\hat{\eta}(\theta)\mathbf{x}| - |\mathbf{y}|) \geq \bar{\sigma}\theta L_{\min}^2/4.$$

On the other hand, since $|\mathbf{x}| \leq \theta L/2$ for $\mathbf{x} \in \Gamma_2$,

$$|\hat{\eta}\mathbf{x} - \mathbf{y}| \leq (1+\zeta\hat{\sigma}(\theta))\theta L/2 + L/2 = (1+\theta+\zeta\bar{\sigma})L/2.$$

The lemma now follows from Lemma 2. $\square$

In this paper we are interested in the convergence of the PML method when $d = \theta L$, the diameter of B_2, tends to infinite. The other PML parameters such as $r_0, \zeta, \sigma_{\max}$ are held fixed once they are chosen to satisfy the conditions imposed in (H1) and (H3) below. In the following we will use C to denote the generic constants that are independent of d but may depend on k, r_0, ζ, $\sigma_{\max}$, and $L_j, j = 1, 2, 3$.

Lemma 4 *Let (H1)-(H2) be satisfied. Then for any* $\mathbf{x} \in \Gamma_2, \mathbf{y} \in \bar{B}_1$,

(i) $|G_k(\tilde{\mathbf{x}}, \mathbf{y})| \le C d^{-1} e^{-\gamma k \bar{\sigma}}$;
(ii) $|\partial G_k(\tilde{\mathbf{x}}, \mathbf{y})/\partial y_j| \le Ckd^{-1} e^{-\gamma k \bar{\sigma}}, \quad j = 1, 2, 3$;
(iii) $|\partial G_k(\tilde{\mathbf{x}}, \mathbf{y})/\partial x_j| \le Ckd^{-1} e^{-\gamma k \bar{\sigma}}, \quad j = 1, 2, 3$;
(iv) $|\partial^2 G_k(\tilde{\mathbf{x}}, \mathbf{y})/\partial x_i \partial y_j| \le Ck^2 d^{-1} e^{-\gamma k \bar{\sigma}}, \quad i, j = 1, 2, 3.$

Proof. Note that when $|\hat{\eta}\mathbf{x} - \mathbf{y}| \ge L\bar{\sigma}$, since $|\hat{\sigma}\mathbf{x}| \le \bar{\sigma}L/2$ for $\mathbf{x} \in \Gamma_2$, we have

$$|\rho(\tilde{\mathbf{x}}, \mathbf{y})| \ge \left(|\hat{\eta}\mathbf{x} - \mathbf{y}|^2 - L^2\bar{\sigma}^2/4\right)^{1/2} \ge \frac{1}{2}|\hat{\eta}\mathbf{x} - \mathbf{y}|. \tag{3.2}$$

On the other hand, when $|\hat{\eta}\mathbf{x} - \mathbf{y}| \le L\bar{\sigma}$, by Lemma 3 we know that

$$|\rho(\tilde{\mathbf{x}}, \mathbf{y})| \ge \operatorname{Im} \rho(\tilde{\mathbf{x}}, \mathbf{y}) \ge \gamma\bar{\sigma} \ge \frac{1}{2}\gamma L^{-1}|\hat{\eta}\mathbf{x} - \mathbf{y}| \ge C|\hat{\eta}\mathbf{x} - \mathbf{y}|. \tag{3.3}$$

Thus by (3.1)

$$|\rho(\tilde{\mathbf{x}}, \mathbf{y})|^{-1} \le Cd^{-1}, \quad \forall \mathbf{x} \in \Gamma_2, \mathbf{y} \in \bar{B}_1, \tag{3.4}$$

which combines with Lemma 3 implies

$$|G_k(\tilde{\mathbf{x}}, \mathbf{y})| \le Cd^{-1} e^{-k \mathrm{Im} \rho(\tilde{\mathbf{x}}, \mathbf{y})} \le Cd^{-1} e^{-k\gamma\bar{\sigma}}, \quad \forall \mathbf{x} \in \Gamma_2, \mathbf{y} \in \bar{B}_1.$$

This shows (i). Next, notice that, $|\tilde{x}_j - y_j| \le |\hat{\eta}\mathbf{x} - \mathbf{y}| + \bar{\sigma}L/2$ for $\mathbf{x} \in \Gamma_2, \mathbf{y} \in \bar{B}_1$. Thus if $|\hat{\eta}\mathbf{x} - \mathbf{y}| \ge \bar{\sigma}L$, by (3.2),

$$\frac{|\tilde{x}_j - y_j|}{|\rho(\tilde{\mathbf{x}}, \mathbf{y})|} \le \frac{|\hat{\eta}\mathbf{x} - \mathbf{y}| + \bar{\sigma}L/2}{|\hat{\eta}\mathbf{x} - \mathbf{y}|/2} \le C, \quad \forall \mathbf{x} \in \Gamma_2, \mathbf{y} \in \bar{B}_1,$$

and if $|\hat{\eta}\mathbf{x} - \mathbf{y}| \le \bar{\sigma}L$, by Lemma 3,

$$\frac{|\tilde{x}_j - y_j|}{|\rho(\tilde{\mathbf{x}}, \mathbf{y})|} \le \frac{|\hat{\eta}\mathbf{x} - \mathbf{y}| + \bar{\sigma}L/2}{\mathrm{Im}\rho(\tilde{\mathbf{x}}, \mathbf{y})} \le C, \quad \forall \mathbf{x} \in \Gamma_2, \mathbf{y} \in \bar{B}_1.$$

Therefore

$$\frac{|\tilde{x}_j - y_j|}{|\rho(\tilde{\mathbf{x}}, \mathbf{y})|} \le C, \quad \forall \mathbf{x} \in \Gamma_2, \mathbf{y} \in \bar{B}_1,$$

which yields

$$\left|\frac{\partial \rho(\tilde{\mathbf{x}}, \mathbf{y})}{\partial y_j}\right| \le C, \quad \left|\frac{\partial \rho(\tilde{\mathbf{x}}, \mathbf{y})}{\partial x_j}\right| \le C \left|\frac{\partial F_j}{\partial x_j}\right| \le C, \quad \forall \mathbf{x} \in \Gamma_2, \mathbf{y} \in \bar{B}_1.$$

Moreover, by (3.4), $|\mathbf{i}k\rho^{-1} - \rho^{-2}| \le Cd^{-1} + Cd^{-2} \le Cd^{-1}$. Now (ii) and (iii) follows from the fact that

$$\frac{\partial G_k(\tilde{\mathbf{x}}, \mathbf{y})}{\partial x_j} = \frac{1}{4\pi}(\mathbf{i}k\rho^{-1} - \rho^{-2})\frac{\partial \rho}{\partial x_j} e^{\mathbf{i}k\rho}, \quad \frac{\partial G_k(\tilde{\mathbf{x}}, \mathbf{y})}{\partial y_j} = \frac{1}{4\pi}(\mathbf{i}k\rho^{-1} - \rho^{-2})\frac{\partial \rho}{\partial y_j} e^{\mathbf{i}k\rho}.$$

The estimate (iv) can be proved similarly by using the fact that

$$\left|\frac{\partial \rho(\tilde{\mathbf{x}}, \mathbf{y})^2}{\partial x_i \partial y_j}\right| = \left|-\frac{\partial F_i}{\partial x_i}\frac{\delta_{ij}}{\rho} - \frac{\partial F_i}{\partial x_i}\frac{\tilde{x}_i - y_i}{\rho^2} \cdot \frac{\partial \rho}{\partial y_j}\right| \le Cd^{-1}.$$

This completes the proof. $\square$

Now we are in the position to estimate the modified Maxwell single and double layer potentials $\tilde{\Psi}^k_{\mathrm{SL}}(\boldsymbol{\mu})$ and $\tilde{\Psi}^k_{\mathrm{DL}}(\boldsymbol{\lambda})$.

Lemma 5 *For any $\boldsymbol{\mu} \in Y(\Gamma_1)$, let*

$$\mathbf{v}(\mathbf{x}) = \mathbf{i}k\tilde{\Psi}^k_{\mathbf{A}}(\boldsymbol{\mu})(\mathbf{x}) + \mathbf{i}k^{-1}\tilde{\nabla}\left[\tilde{\Psi}^k_V(\mathrm{div}_{\Gamma_1}\boldsymbol{\mu})(\mathbf{x})\right],$$

be the modified Maxwell single layer potential. Then

$$\|\,\mathbf{n}_2 \times B\mathbf{v}\,\|_{Y(\Gamma_2)} \le C(1+kd)e^{-k\gamma\bar{\sigma}}\|\,\boldsymbol{\mu}\,\|_{Y(\Gamma_1)}, \tag{3.5}$$

$$\|\,\mathbf{n}_2 \times A\nabla \times B\mathbf{v}\,\|_{Y(\Gamma_2)} \le C(1+kd)e^{-k\gamma\bar{\sigma}}\|\,\boldsymbol{\mu}\,\|_{Y(\Gamma_1)}. \tag{3.6}$$

Proof. We only prove (3.5). The estimate (3.6) can be proved similarly. Denote

$$\mathbf{v}_1(\mathbf{x}) = \tilde{\Psi}^k_{\mathbf{A}}(\boldsymbol{\mu})(\mathbf{x}) = \int_{\Gamma_1} G_k(\tilde{\mathbf{x}}, \mathbf{y})\boldsymbol{\mu}(\mathbf{y})ds(\mathbf{y}).$$

For any $f \in L^\infty(\Gamma_2)$, it is easy to see that

$$\|f\|_{H^{-1/2}(\Gamma_2)} = \sup_{\phi\in H^{1/2}(\Gamma_2)} \frac{|\langle f, \phi\rangle_{\Gamma_2}|}{\|\phi\|_{H^{1/2}(\Gamma_2)}} \le d^{1/2}\|f\|_{L^2(\Gamma_2)} \le Cd^{3/2}\|f\|_{L^\infty(\Gamma_2)}.$$

Similarly, for any $\boldsymbol{\lambda} \in L^\infty(\Gamma_2)^3 \cap V'_\pi(\Gamma_2)$, $\|\boldsymbol{\lambda}\|_{V'_\pi(\Gamma_2)} \le Cd^{3/2}\|\boldsymbol{\lambda}\|_{L^\infty(\Gamma_2)}$. Thus, from (2.4)

$$\begin{aligned}
&\|\,\mathbf{n}_2 \times B\mathbf{v}_1\,\|_{Y(\Gamma_2)} \\
&\le Cd^{-1}\|\mathbf{n}_2 \times B\mathbf{v}_1\|_{V'_\pi(\Gamma_2)} + C\|\nabla \times B\mathbf{v}_1 \cdot \mathbf{n}_2\|_{H^{-1/2}(\Gamma_2)} \\
&\le Cd^{1/2}\|\mathbf{n}_2 \times B\mathbf{v}_1\|_{L^\infty(\Gamma_2)} + Cd^{3/2}\|\nabla \times B\mathbf{v}_1\|_{L^\infty(\Gamma_2)},
\end{aligned} \tag{3.7}$$

which yields, since $B = \alpha_0 I$ on Γ_2, where $\alpha_0 = \eta(r_0) + \mathbf{i}\sigma(r_0)$,

$$\|\,\mathbf{n}_2 \times B\mathbf{v}_1\,\|_{Y(\Gamma_2)} \le Cd^{1/2}(\|\mathbf{v}_1\|_{L^\infty(\Gamma_2)} + d\|\nabla\mathbf{v}_1\|_{L^\infty(\Gamma_2)}). \tag{3.8}$$

On the other hand,

$$\begin{aligned}
&\|\mathbf{v}_1\|_{L^\infty(\Gamma_2)} + d\|\nabla\mathbf{v}_1\|_{L^\infty(\Gamma_2)} \\
&\le \max_{\mathbf{x}\in\Gamma_2}\Big(\|G_k(\tilde{\mathbf{x}}, \cdot)\|_{H^{1/2}(\Gamma_1)} + d\|\nabla_{\mathbf{x}}G_k(\tilde{\mathbf{x}}, \cdot)\|_{H^{1/2}(\Gamma_1)}\Big)\|\boldsymbol{\mu}\|_{H^{-1/2}(\Gamma_1)}.
\end{aligned} \tag{3.9}$$

For any $\mathbf{x} \in \Gamma_2$, since for $\mathbf{y}, \mathbf{y}' \in \Gamma_1$,

$$|G_k(\tilde{\mathbf{x}}, \mathbf{y}) - G_k(\tilde{\mathbf{x}}, \mathbf{y}')| \le C\|\nabla_{\mathbf{y}}G_k(\tilde{\mathbf{x}}, \cdot)\|_{L^\infty(\Gamma_1)}|\mathbf{y} - \mathbf{y}'|,$$

we have

$$\|G_k(\tilde{\mathbf{x}}, \cdot)\|_{H^{1/2}(\Gamma_1)} \le CL^{1/2}\|G_k(\tilde{\mathbf{x}}, \cdot)\|_{L^\infty(\Gamma_1)} + CL^{3/2}\|\nabla_{\mathbf{y}}G_k(\tilde{\mathbf{x}}, \cdot)\|_{L^\infty(\Gamma_1)}.$$

This implies, by Lemma 4,

$$\|G_k(\tilde{\mathbf{x}}, \cdot)\|_{H^{1/2}(\Gamma_1)} \le CL^{1/2}d^{-1}(1+kL)e^{-k\gamma\bar{\sigma}}. \tag{3.10}$$

Similarly

$$\|\nabla_{\mathbf{x}}G_k(\tilde{\mathbf{x}}, \cdot)\|_{H^{1/2}(\Gamma_1)} \le CkL^{1/2}d^{-1}(1+kL)e^{-k\gamma\bar{\sigma}}. \tag{3.11}$$

Substituting (3.10)-(3.11) into (3.8) and(3.9) we obtain

$$\| \mathbf{n}_2 \times B\mathbf{v}_1 \|_{Y(\Gamma_2)} \le C(1+kd)e^{-k\gamma\bar{\sigma}} \| \boldsymbol{\mu} \|_{H^{-1/2}(\Gamma_1)}.$$

It remains to estimate $\mathbf{n}_2 \times B\mathbf{v}_2$ with

$$\mathbf{v}_2(\mathbf{x}) = \tilde{\nabla} \left[\tilde{\Psi}_V^k(\mathrm{div}_{\Gamma_1}\boldsymbol{\mu})(\mathbf{x}) \right] = B^{-1}\nabla \left[\tilde{\Psi}_V^k(\mathrm{div}_{\Gamma_1}\boldsymbol{\mu})(\mathbf{x}) \right].$$

By (3.7) we have

$$\begin{aligned} &\| \mathbf{n}_2 \times B\mathbf{v}_2 \|_{Y(\Gamma_2)} \\ \le\ & Cd^{1/2} \|\nabla \tilde{\Psi}_V^k(\mathrm{div}_{\Gamma_1}\boldsymbol{\mu})\|_{L^\infty(\Gamma_2)} \\ \le\ & Cd^{1/2} \max_{\mathbf{x}\in\Gamma_2} \|\nabla_{\mathbf{x}} G_k(\tilde{\mathbf{x}},\cdot)\|_{H^{1/2}(\Gamma_1)} \|\mathrm{div}_{\Gamma_1}\boldsymbol{\mu}\|_{H^{-1/2}(\Gamma_1)} \\ \le\ & Ck(1+kL)e^{-k\gamma\bar{\sigma}} \|\mathrm{div}_{\Gamma_1}\boldsymbol{\mu}\|_{H^{-1/2}(\Gamma_1)}, \end{aligned}$$

where we have used (3.11). In conclusion,

$$\begin{aligned} &\| \mathbf{n}_2 \times B\mathbf{v} \|_{Y(\Gamma_2)} \\ \le\ & k\| \mathbf{n}_2 \times B\mathbf{v}_1 \|_{Y(\Gamma_2)} + k^{-1} \| \mathbf{n}_2 \times B\mathbf{v}_2 \|_{Y(\Gamma_2)} \\ \le\ & Ck(1+kd)e^{-k\gamma\bar{\sigma}} \|\boldsymbol{\mu}\|_{H^{-1/2}(\Gamma_1)} + C(1+kL)e^{-k\gamma\bar{\sigma}} \|\mathrm{div}_{\Gamma_1}\boldsymbol{\mu}\|_{H^{-1/2}(\Gamma_1)} \\ \le\ & C(1+kd)e^{-k\gamma\bar{\sigma}} \| \boldsymbol{\mu} \|_{Y(\Gamma_1)}. \end{aligned}$$

This completes the proof. □

Lemma 6 *For any $\boldsymbol{\lambda} \in Y(\Gamma_1)$, let*

$$\mathbf{v}(x) = \tilde{\Psi}_{\mathrm{DL}}^k(\boldsymbol{\lambda}) = \tilde{\nabla} \times \left[\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})(\mathbf{x}) \right]$$

be the modified Maxwell double layer potential. Then

$$\| \mathbf{n}_2 \times B\mathbf{v} \|_{Y(\Gamma_2)} \le C(1+kd)e^{-k\gamma\bar{\sigma}} \| \boldsymbol{\lambda} \|_{Y(\Gamma_1)}, \tag{3.12}$$

$$\| \mathbf{n}_2 \times A\nabla \times B\mathbf{v} \|_{Y(\Gamma_2)} \le C(1+kd)e^{-k\gamma\bar{\sigma}} \| \boldsymbol{\lambda} \|_{Y(\Gamma_1)}. \tag{3.13}$$

Proof. We only show (3.12). (3.13) can be proved similarly. For any $\mathbf{x} \in \Gamma_2$, since

$$\mathbf{v}(\mathbf{x}) = JD\mathbf{F}\nabla \times \left[B\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})(\mathbf{x}) \right],$$

we have by (3.7)

$$\begin{aligned} &\| \mathbf{n}_2 \times B\mathbf{v} \|_{Y(\Gamma_2)} \\ \le\ & Cd^{1/2} \|\mathbf{n}_2 \times A\nabla \times B\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})\|_{L^\infty(\Gamma_2)} + Cd^{3/2} \|\nabla \times A\nabla \times B\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})\|_{L^\infty(\Gamma_2)} \\ =\ & Cd^{1/2} \|\mathbf{n}_2 \times A\nabla \times B\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})\|_{L^\infty(\Gamma_2)} + Ck^2 d^{3/2} \|A^{-1} B\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})\|_{L^\infty(\Gamma_2)}, \\ \le\ & Cd^{1/2} \left(\|\nabla \tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})\|_{L^\infty(\Gamma_2)} + k^2 d \|\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})\|_{L^\infty(\Gamma_2)} \right), \end{aligned}$$

where we have used the fact that $\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda})$ satisfies the PML equation

$$\nabla \times A\nabla \times B\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda}) - k^2 A^{-1} B\tilde{\Psi}_{\mathbf{A}}^k(\boldsymbol{\lambda}) = 0 \quad \text{in } \mathbb{R}^3 \backslash \bar{B}_1.$$

But

$$\|\nabla\tilde{\Psi}^k_{\mathbf{A}}(\boldsymbol{\lambda})\|_{L^\infty(\Gamma_2)} + k^2 d\|\tilde{\Psi}^k_{\mathbf{A}}(\boldsymbol{\lambda})\|_{L^\infty(\Gamma_2)}$$
$$\le \max_{\mathbf{x}\in\Gamma_2}\left(k^2 d\|G_k(\tilde{\mathbf{x}},\cdot)\|_{H^{1/2}(\Gamma_1)} + \|\nabla_{\mathbf{x}}G_k(\tilde{\mathbf{x}},\cdot)\|_{H^{1/2}(\Gamma_1)}\right)\|\boldsymbol{\lambda}\|_{H^{-1/2}(\Gamma_1)}.$$

Now by using (3.10)-(3.11) we get

$$\|\,\mathbf{n}_2\times B\mathbf{v}\,\|_{Y(\Gamma_2)} \le C(1+kd)e^{-k\gamma\bar{\sigma}}\|\boldsymbol{\lambda}\|_{H^{-1/2}(\Gamma_1)}.$$

This completes the proof. $\square$

4 The PML equation in the layer

We consider in this section the Dirichlet problem of the PML equation in the layer

$$\nabla\times A\nabla\times\mathbf{w} - k^2A^{-1}\mathbf{w} = 0 \quad \text{in } \Omega^{\mathrm{PML}}, \tag{4.1}$$
$$\mathbf{n}_1\times\mathbf{w} = 0 \text{ on } \Gamma_1, \quad \mathbf{n}_2\times\mathbf{w} = \mathbf{q} \text{ on } \Gamma_2, \tag{4.2}$$

where $\mathbf{q}\in Y(\Gamma_2)$. Introduce the following sesquilinear form

$$c(\mathbf{u},\mathbf{v}) = \int_{\Omega^{\mathrm{PML}}}(A\nabla\times\mathbf{u}\cdot\nabla\times\bar{\mathbf{v}} - k^2A^{-1}\mathbf{u}\cdot\bar{\mathbf{v}})d\mathbf{x}.$$

Then the weak formulation for (4.1)-(4.2) is: Given $\mathbf{q}\in Y(\Gamma_2)$, find $\mathbf{w}\in H(\mathrm{curl};\Omega^{\mathrm{PML}})$ such that $\mathbf{n}_1\times\mathbf{w}=0$ on Γ_1, $\mathbf{n}_2\times\mathbf{w}=\mathbf{q}$ on Γ_2, and

$$c(\mathbf{w},\mathbf{v}) = 0, \qquad \forall\ \mathbf{v}\in H_0(\mathrm{curl};\Omega^{\mathrm{PML}}). \tag{4.3}$$

We will extend the idea in [7] to show the well-posedness of the problem (4.3) for sufficiently large d. The first objective is to show that under the assumption (H1) the matrix A is coercive. We start with the following elementary lemma.

Lemma 7 *Let $C=(c_{ij})\in\mathbb{R}^{3\times3}$ be a symmetric matrix such that $c_{23}=c_{32}=0$ and $c_{22}=c_{33}$. Assume that $c_{11}+c_{22}>0$ and $c_{11}c_{22}\ge c_{12}^2+c_{13}^2$. Then the eigenvalues of C is bounded below by $\frac{c_{11}c_{22}-(c_{12}^2+c_{13}^2)}{c_{11}+c_{22}}$.*

Proof. It is easy to see that

$$\det(C-\lambda I) = (c_{22}-\lambda)(\lambda^2-(c_{11}+c_{22})\lambda + c_{11}c_{22}-(c_{12}^2+c_{13}^2)).$$

The eigenvalues of C are $\lambda_1=c_{22}$ and $\lambda_\pm=\frac{1}{2}(c_{11}+c_{22}\pm\sqrt{\Delta})$, where $\Delta=(c_{11}+c_{22})^2-4c_{11}c_{22}+4(c_{12}^2+c_{13}^2)\ge0$. It is clear that $\lambda_+\ge\lambda_-$ and

$$\lambda_- = \frac{1}{2}(c_{11}+c_{22}-\sqrt{\Delta}) = \frac{1}{2}\frac{4c_{11}c_{22}-4(c_{12}^2+c_{13}^2)}{c_{11}+c_{22}+\sqrt{\Delta}} \ge \frac{c_{11}c_{22}-(c_{12}^2+c_{13}^2)}{c_{11}+c_{22}},$$

where we have used the fact that $\Delta\le(c_{11}+c_{22})^2$ since $c_{11}c_{22}-(c_{12}^2+c_{13}^2)\ge0$. This completes the proof because

$$\lambda_1 = c_{22} \ge \frac{c_{11}c_{22}}{c_{11}+c_{22}} \ge \frac{c_{11}c_{22}-(c_{12}^2+c_{13}^2)}{c_{11}+c_{22}}. \qquad \square$$

Lemma 8 *Let (H1) be satisfied. Then*

$$\mathrm{Re}(A(\mathbf{x})\xi\cdot\bar{\xi}) \geq \frac{1}{(1+\zeta^2)(1+|\alpha|)|\alpha|^2|\beta|^2}\xi\cdot\bar{\xi}, \qquad \forall\xi\in\mathbb{C}^3, \mathbf{x}\in\Omega^{\mathrm{PML}}.$$

Proof. We only prove the lemma for $\mathbf{x}\in\Omega_1^{\pm}$. The other cases are similar. By (2.20), write $A(\mathbf{x})=(a_{ij}(\mathbf{x}))$ we know that for any $\mathbf{x}\in\Omega_1^{\pm}$,

$$\mathrm{Re}(A(\mathbf{x})\xi\cdot\bar{\xi}) = \sum_{i,j=1}^{3}\mathrm{Re}(a_{ij}(\mathbf{x})\xi_i\bar{\xi}_j) \geq \left[\min_{j=1,2,3}\lambda_j(\mathbf{x})\right](\xi\cdot\bar{\xi}),$$

where $\lambda_j(\mathbf{x})$, $j=1,2,3$, are the eigenvalues of the symmetric matrix $\mathrm{Re}\,A(\mathbf{x})$.

We will use Lemma 7 to prove the lemma. First it is obvious that $\mathrm{Re}(a_{22})>0$. Next by direct calculation we have

$$\begin{aligned}\mathrm{Re}\left[(\alpha-\beta)^2\bar{\alpha}\bar{\beta}^2\right] &= (\sigma-\hat{\sigma})^2\left[(\zeta^2-1)(\eta\hat{\eta}^2-\eta\hat{\sigma}^2-2\sigma\hat{\sigma}\hat{\eta})+2(\sigma\hat{\eta}^2-\sigma\hat{\sigma}^2+2\hat{\sigma}\hat{\eta}\eta)\right]\\ &\geq (\sigma-\hat{\sigma})^2(\zeta^2-1)(\eta\hat{\eta}^2-\eta\hat{\sigma}^2-2\sigma\hat{\sigma}\hat{\eta}),\end{aligned}$$

where we have used $\hat{\eta}^2\geq\hat{\sigma}^2$. It is easy to show that $\eta\hat{\eta}^2-\eta\hat{\sigma}^2-2\sigma\hat{\sigma}\hat{\eta}\geq-\zeta\sigma\hat{\sigma}^2$ since $\zeta^2\geq 2$ by (H1). Thus

$$\mathrm{Re}\frac{(\alpha-\beta)^2}{\alpha\beta^2} \geq -(\sigma-\hat{\sigma})^2(\zeta^2-1)\frac{\zeta\sigma\hat{\sigma}^2}{|\alpha|^2|\beta|^4} \geq -\frac{\zeta\sigma\hat{\sigma}^2}{|\beta|^4}.$$

On the other hand, it is easy to check that

$$\mathrm{Re}\frac{\alpha}{\beta^2} = \frac{\eta(\hat{\eta}^2-\hat{\sigma}^2)+2\sigma\hat{\sigma}\hat{\eta}}{|\beta|^4} \geq \frac{\eta\hat{\eta}^2+\sigma\hat{\sigma}\hat{\eta}}{|\beta|^4}, \tag{4.4}$$

where we have used $\sigma\hat{\eta}\geq\eta\hat{\sigma}$ from the definition of η and $\hat{\eta}$. Therefore, since $|s_2|\leq L_2/2, |s_3|\leq L_3/2$ and $|s_1|=L_1/2$, we obtain by using (H1) that

$$\mathrm{Re}(a_{11}) \geq \frac{\eta\hat{\eta}^2}{|\beta|^4}-\frac{\zeta\sigma\hat{\sigma}^2}{|\beta|^4}\cdot 2\max_{i,j}\frac{L_i^2}{L_j^2} \geq \frac{\eta\hat{\eta}^2}{|\beta|^4}-\frac{\zeta^3\sigma\hat{\sigma}^2}{|\beta|^4} \geq \frac{\eta\hat{\eta}^2}{|\beta|^4}-\frac{\eta\hat{\eta}^2}{|\beta|^4}=0.$$

This show that $\mathrm{Re}(a_{11})+\mathrm{Re}(a_{22})>0$.

To proceed we notice that by (2.20)

$$\begin{aligned}&\mathrm{Re}(a_{11})\mathrm{Re}(a_{22})-(\mathrm{Re}(a_{12})^2+\mathrm{Re}(a_{13})^2)\\ &=\mathrm{Re}\frac{\alpha}{\beta^2}\cdot\frac{1}{\alpha}+\left[\mathrm{Re}\frac{(\alpha-\beta)^2}{\alpha\beta^2}\cdot\frac{1}{\alpha}-\left(\mathrm{Re}\frac{\alpha-\beta}{\alpha\beta}\right)^2\right]\frac{s_2^2+s_3^2}{s_1^2}\\ &=\mathrm{Re}\frac{\alpha}{\beta^2}\cdot\frac{1}{\alpha}+\left[\mathrm{Re}\frac{\alpha}{\beta^2}\cdot\mathrm{Re}\frac{1}{\alpha}-\left(\mathrm{Re}\frac{1}{\beta}\right)^2\right]\frac{s_2^2+s_3^2}{s_1^2}\\ &=\mathrm{Re}\frac{\alpha}{\beta^2}\cdot\frac{1}{\alpha}-\frac{(\sigma-\hat{\sigma})^2}{|\alpha|^2|\beta|^4}\frac{s_2^2+s_3^2}{s_1^2}.\end{aligned}$$

By (4.4) and (H1) we know that

$$\begin{aligned}
&\mathrm{Re}(a_{11})\mathrm{Re}(a_{22}) - (\mathrm{Re}(a_{12})^2 + \mathrm{Re}(a_{13})^2) \\
&\geq \frac{\eta^2\hat{\eta}^2 + \sigma\hat{\sigma}\eta\hat{\eta}}{|\alpha|^2|\beta|^4} - \frac{(\sigma-\hat{\sigma})^2}{|\alpha|^2|\beta|^4} \cdot 2\max_{i,j}\frac{L_i^2}{L_j^2} \\
&\geq \frac{\hat{\eta}^2 + \hat{\sigma}^2}{|\alpha|^2|\beta|^4} + \frac{\sigma^2}{|\alpha|^2|\beta|^4}\left(\zeta^2 - 2\max_{i,j}\frac{L_i^2}{L_j^2}\right) \\
&\geq \frac{1}{|\alpha|^2|\beta|^2},
\end{aligned} \tag{4.5}$$

where we have used the fact that $\eta^2\hat{\eta}^2 \geq \hat{\eta}^2 + \zeta^2\sigma^2$ and $(\sigma-\hat{\sigma})^2 \leq \sigma^2$. This completes the proof by Lemma 8 by the fact that $\mathrm{Re}(a_{11}) + \mathrm{Re}(a_{22}) \leq (1+\zeta^2)(1+|\alpha|)$. $\square$

We remark that if $\zeta = 0$, then $\eta = \hat{\eta} = 1$ and (4.5) becomes

$$\mathrm{Re}(a_{11})\mathrm{Re}(a_{22}) - (\mathrm{Re}(a_{12})^2 + \mathrm{Re}(a_{13})^2) \geq \frac{1+\sigma\hat{\sigma}}{|\alpha|^2|\beta|^4} - \frac{(\sigma-\hat{\sigma})^2}{|\alpha|^2|\beta|^4} \cdot 2\max_{i,j}\frac{L_i^2}{L_j^2}.$$

Thus, in order to guarantee the ellipticity, we require $(\sigma-\hat{\sigma})^2 \cdot 2\max_{i,j}\frac{L_i^2}{L_j^2} < 1$. Since $\sigma - \hat{\sigma} = t\hat{\sigma}'$, if we take $\hat{\sigma}' = c_0(r_0-t)^2(t-1)^2$ for $1 \leq t \leq r_0$ as suggested in [7] and used in our numerical experiments, then c_0 should be taken very small. On the other hand, there is no such restriction for the choice of ζ in (H1) by Lemma 8.

Lemma 9 *Let (H1) be satisfied and fix some $r_1 > r_0$. Then any solution of the problem* (4.1)-(4.2) *satisfies*

$$\|\nabla \times \mathbf{w}\|_{L^2(\Omega_{r_0})} \leq C\|\mathbf{w}\|_{L^2(\Omega_{r_1})},$$

where $\Omega_{r_i} = \{\mathbf{x} \in \Omega^{\mathrm{PML}} : |x_j| \leq r_i L_j/2, j = 1,2,3\}$, $i = 0, 1$.

Proof. The argument is standard. Let $\chi \in C^\infty(\Omega^{\mathrm{PML}})$ be the cut-off function such that $0 \leq \chi \leq 1$, $\chi = 1$ in Ω_{r_0}, $\chi = 0$ in $\Omega^{\mathrm{PML}}\backslash\bar{\Omega}_{r_1}$, and $|\nabla\chi| \leq C/[(r_1-r_0)L_{\min}/2] \leq C$. By multiplying (4.1) by $\chi^2\bar{\mathbf{w}} \in H_0(\mathrm{curl};\Omega^{\mathrm{PML}})$, we obtain

$$\int_{\Omega^{\mathrm{PML}}}\left(A\nabla\times\mathbf{w}\cdot\nabla\times(\chi^2\bar{\mathbf{w}}) - k^2A^{-1}\mathbf{w}\cdot\chi^2\bar{\mathbf{w}}\right)d\mathbf{x} = 0.$$

Since $\nabla\times(\chi^2\bar{\mathbf{w}}) = \chi\nabla\times(\chi\bar{\mathbf{w}}) + \nabla\chi\times(\chi\bar{\mathbf{w}})$, we have

$$\begin{aligned}
&\int_{\Omega^{\mathrm{PML}}} A\nabla\times\mathbf{w}\cdot\nabla\times(\chi^2\bar{\mathbf{w}})d\mathbf{x} \\
&= \int_{\Omega^{\mathrm{PML}}}\left(A\chi\nabla\times\mathbf{w}\cdot\nabla\times(\chi\bar{\mathbf{w}}) + A\nabla\times\mathbf{w}\cdot\nabla\chi\times(\chi\bar{\mathbf{w}})\right)d\mathbf{x} \\
&= \int_{\Omega^{\mathrm{PML}}}\left(A\nabla\times\chi\mathbf{w}\cdot\nabla\times(\chi\bar{\mathbf{w}}) - A(\nabla\chi\times\mathbf{w})\cdot\nabla\times(\chi\bar{\mathbf{w}})\right)d\mathbf{x} \\
&\quad + \int_{\Omega^{\mathrm{PML}}}\left(A\nabla\times\chi\mathbf{w}\cdot\nabla\chi\times\bar{\mathbf{w}} - A\nabla\chi\times\mathbf{w}\cdot\nabla\chi\times\bar{\mathbf{w}}\right)d\mathbf{x}.
\end{aligned}$$

On the other hand, since $\|A\| \leq C$ and $\|A^{-1}\| \leq C$, by using Lemma 8 and standard argument we obtain that

$$\int_{\Omega^{\mathrm{PML}}}|\nabla\times(\chi\mathbf{w})|^2d\mathbf{x} \leq C\|\mathbf{w}\|^2_{L^2(\Omega_{r_1})}.$$

This completes the proof. $\square$

Lemma 10 *Let (H1)-(H2) be satisfied. Then any solution of the problem* (4.1)-(4.2) *satisfies the following estimate*

$$\|\,\mathbf{n}_2 \times A\nabla \times \mathbf{w}\,\|_{Y(\Gamma_2)} \le C(1+kd)\|\,\mathbf{q}\,\|_{Y(\Gamma_2)} + C\|\mathbf{w}\|_{L^2(\Omega_{r_1})}.$$

Proof. Denote by $\mathcal{D} = \Omega^{\rm PML}\backslash\bar{\Omega}_{r_0}$. Let $\mathbf{U} \in H(\mathrm{curl};\mathcal{D})$ such that $\mathbf{n}_2 \times \mathbf{U} = \mathbf{q}$ on Γ_2 and $\mathbf{n} \times \mathbf{U} = 0$ on Γ_{r_0}. Multiplying (4.1) by $\bar{\mathbf{w}} - \bar{\mathbf{U}}$ and integrating by parts over $\mathcal{D}$ we obtain

$$\begin{aligned}
&\int_{\mathcal{D}} (A\nabla \times \mathbf{w} \cdot \nabla \times \bar{\mathbf{w}} - k^2 A^{-1}\mathbf{w} \cdot \bar{\mathbf{w}})d\mathbf{x} \\
&= \int_{\mathcal{D}} (A\nabla \times \mathbf{w} \cdot \nabla \times \bar{\mathbf{U}} - k^2 A^{-1}\mathbf{w} \cdot \bar{\mathbf{U}})d\mathbf{x} + \langle \mathbf{n} \times A\nabla \times \mathbf{w}, \mathbf{w}\rangle_{\Gamma_{r_0}}.
\end{aligned}$$

Since $A = \alpha_0^{-1} I$ in $\mathcal{D}$, where $\alpha_0 = \alpha(r_0)$, we have by taking the imaginary part of the equation and using the standard argument that

$$\begin{aligned}
&\int_{\mathcal{D}} \left(\frac{\sigma_0}{|\alpha_0|^2}|\nabla \times \mathbf{w}|^2 + k^2\sigma_0|\mathbf{w}|^2\right) d\mathbf{x} \\
&\le C(\|\nabla \times \mathbf{U}\|^2_{L^2(\mathcal{D})} + k^2\|\mathbf{U}\|^2_{L^2(\mathcal{D})}) + |\langle \mathbf{n} \times A\nabla \times \mathbf{w}, \mathbf{w}\rangle_{\Gamma_{r_0}}| \\
&\le C(1+kd)^2\|\mathbf{U}\|^2_{H(\mathrm{curl};\mathcal{D})} + |\langle \mathbf{n} \times A\nabla \times \mathbf{w}, \mathbf{w}\rangle_{\Gamma_{r_0}}|.
\end{aligned}$$

The estimate holds for any $\mathbf{U} \in H(\mathrm{curl};\mathcal{D})$ such that $\mathbf{n}_2 \times \mathbf{U} = \mathbf{q}$ on Γ_2 and $\mathbf{n} \times \mathbf{U} = 0$ on Γ_{r_0}. By (2.5) we get

$$\int_{\mathcal{D}} |\nabla \times \mathbf{w}|^2 + k^2|\mathbf{w}|^2 \le C(1+kd)^2\|\,\mathbf{q}\,\|^2_{Y(\Gamma_2)} + |\langle \mathbf{n} \times A\nabla \times \mathbf{w}, \mathbf{w}\rangle_{\Gamma_{r_0}}|. \tag{4.6}$$

To estimate the second term, we multiply the equation (4.1) by $\bar{\mathbf{w}}$ and integrate by parts over Ω_{r_0} to get

$$\int_{\Omega_{r_0}} (A\nabla \times \mathbf{w} \cdot \nabla \times \bar{\mathbf{w}} - k^2 A^{-1}\mathbf{w} \cdot \bar{\mathbf{w}})d\mathbf{x} + \langle \mathbf{n} \times A\nabla \times \mathbf{w}, \mathbf{w}\rangle_{\partial\Omega_{r_0}} = 0,$$

which implies, since $\mathbf{n} \times \mathbf{w} = 0$ on Γ_1,

$$\begin{aligned}
|\langle \mathbf{n} \times A\nabla \times \mathbf{w}, \mathbf{w}\rangle_{\Gamma_{r_0}}| &\le C(\|\nabla \times \mathbf{w}\|^2_{L^2(\Omega_{r_0})} + k^2\|\mathbf{w}\|^2_{L^2(\Omega_{r_0})}) \\
&\le C\|\mathbf{w}\|^2_{L^2(\Omega_{r_1})},
\end{aligned}$$

where we have used Lemma 9. Substitute the estimate to (4.6) we have

$$\int_{\mathcal{D}} |\nabla \times \mathbf{w}|^2 + k^2|\mathbf{w}|^2 \le C(1+kd)^2\|\,\mathbf{q}\,\|^2_{Y(\Gamma_2)} + C\|\mathbf{w}\|^2_{L^2(\Omega_{r_1})}. \tag{4.7}$$

Again by (2.5) and the equation (4.1) we have then

$$\begin{aligned}
\|\,\mathbf{n}_2 \times A\nabla \times \mathbf{w}\,\|_{Y(\Gamma_2)} &\le C\|A\nabla \times \mathbf{w}\|_{H(\mathrm{curl};\mathcal{D})} \\
&= C(d^{-2}\|A\nabla \times \mathbf{w}\|^2_{L^2(\mathcal{D})} + \|k^2 A^{-1}\mathbf{w}\|^2_{L^2(\mathcal{D})})^{1/2}.
\end{aligned}$$

This competes the proof by using (4.7). □

In the following we need the following assumption on the medium property.

(H3) $r_0 \max_{1\le t\le r_0} |\hat{\sigma}'(t)| \le \dfrac{1}{2(1+\zeta^2)^{3/2}}$.

The following theorem is the main result of this section.

Theorem 1 *Let (H1)-(H3) be satisfied. The problem* (4.1)-(4.2) *has a unique solution for sufficiently large d. Moreover, there exists a constant $C>0$ independent of d such that*

$$\| \mathbf{n}_1 \times (\nabla \times \mathbf{w}) \|_{Y(\Gamma_1)} \le C \| \mathbf{q} \|_{Y(\Gamma_2)}. \tag{4.8}$$

We will use the duality argument to prove the theorem. We first recall the following lemma formulated in [7, Theorem 3.2] (see also Girault and Raviart [22, Theorem 2.1]). The lemma can be viewed as a variant of the Fredholm alternative.

Lemma 11 *Let $A_0(\cdot,\cdot)$, $I(\cdot,\cdot)$ be bounded sesquilinear forms on a complex Hilbert space V with norm $\|\cdot\|_V$. Let W be another Hilbert space with V compactly embedded in W. Suppose that $|I(v,v)| \le C_1\|v\|_V\|v\|_W$ for all $v\in V$ and $\|v\|_V^2 \le C_2|A_0(v,v)|$ for all $v \in V$. Set $A = A_0 + I$ and assume that the only $u\in V$ satisfying $A(u,v)=0$ for all $v\in V$ is $u=0$. Then, there exists $C_3>0$ such that for all $u\in V$,*

$$\|u\|_V \le C_3 \sup_{v\in V} \frac{|A(u,v)|}{\|v\|_V}.$$

The proof of the following lemma will be given in the appendix of the paper.

Lemma 12 *Let (H1)-(H3) be satisfied. Then for any $\mathbf{U}\in L^2(\mathbb{R}^3)^3$ supported in Ω_{r_1}, there exists a function $\mathbf{v}$ in $H(\mathrm{curl};\mathbb{R}^3)$ such that*

$$\nabla\times A\nabla\times\mathbf{v} - k^2A^{-1}\mathbf{v} = A^{-1}\mathbf{U} \quad \text{in } \mathbb{R}^3. \tag{4.9}$$

Moreover, we have the estimate $\|\mathbf{v}\|_{H(\mathrm{curl};\mathbb{R}^3)} \le C\|\mathbf{U}\|_{L^2(\Omega_{r_1})}$.

Lemma 13 *Let (H1)-(H3) be satisfied and $\theta > r_1$. Then there exists a function $\mathbf{u}$ in $H(\mathrm{curl};\mathbb{R}^3\backslash\bar{B}_1)$ such that*

$$\nabla\times A\nabla\times\mathbf{u} - k^2A^{-1}\mathbf{u} = A^{-1}\mathbf{U} \quad \text{in } \mathbb{R}^3\backslash\bar{B}_1, \tag{4.10}$$

$$\mathbf{n}_1\times\mathbf{u} = 0 \quad \text{on } \Gamma_1. \tag{4.11}$$

Moreover, the following estimate holds

$$\| \mathbf{n}_2\times\mathbf{u} \|_{Y(\Gamma_2)} + \| \mathbf{n}_2\times A\nabla\times\mathbf{u} \|_{Y(\Gamma_2)} \le C(1+kd)e^{-k\sigma_0(\theta-r_1)L_{\min}/2}\|\mathbf{U}\|_{L^2(\Omega_{r_1})}.$$

Proof. We first construct the function $\mathbf{u}$ that satisfies (4.10)-(4.11). Let $\mathbf{v}$ be the function defined in Lemma 12 and $\mathbf{u}_1 = \mathbb{E}(\mathbf{n}_1\times\mathbf{v}|_{\Gamma_1})$ the PML extension given in (2.15). Then

$$\nabla\times A\nabla\times(B\mathbf{u}_1) - k^2A^{-1}(B\mathbf{u}_1) = 0 \quad \text{in } \mathbb{R}^2\backslash\bar{B}_1,$$
$$\mathbf{n}_1\times\mathbf{u}_1 = \mathbf{n}_1\times\mathbf{v} \quad \text{on } \Gamma_1.$$

Moreover, by the argument in Lemmas 5-6 we know that

$$\|\mathbf{n}_{r_1}\times B\mathbf{u}_1\|_{Y(\Gamma_{r_1})} + \|\mathbf{n}_{r_1}\times A\nabla\times B\mathbf{u}_1\|_{Y(\Gamma_{r_1})} \le C(1+kd)\|\mathbf{n}_1\times\mathbf{v}\|_{Y(\Gamma_1)}. \tag{4.12}$$

It is clear that $\mathbf{u} = \mathbf{v} - B\mathbf{u}_1$ satisfies (4.10)-(4.11). It remains to show that $\mathbf{u}$ satisfies the desired estimate.

Since $A = \alpha_0^{-1} I$ outside Ω_{r_1}, we know that $\mathbf{u}$ is the solution of a time-harmonic Maxwell scattering problem with the complex wave number $\tilde{k} = k\alpha_0 = k(\eta_0 + \mathbf{i}\sigma_0)$, $\eta_0, \sigma_0 > 0$. By the Stratton-Chu integral representation we have, for $\mathbf{x} \in \mathbb{R}^3 \backslash \bar{\Omega}_{r_1}$,

$$\mathbf{u}(\mathbf{x}) = \Psi_{\mathrm{SL}}^{\tilde{k}}(\boldsymbol{\mu})(\mathbf{x}) + \Psi_{\mathrm{DL}}^{\tilde{k}}(\boldsymbol{\lambda})(\mathbf{x}),$$

where $\boldsymbol{\lambda} = \mathbf{n}_{r_1} \times \mathbf{u}$ on Γ_{r_1}, $\boldsymbol{\mu} = \frac{1}{\mathbf{i}\tilde{k}} \mathbf{n}_{r_1} \times \nabla \times \mathbf{u}$ on Γ_{r_1}, and

$$\begin{aligned}
&\Psi_{\mathrm{SL}}^{\tilde{k}}(\boldsymbol{\mu})(\mathbf{x}) = \mathbf{i}\tilde{k}\Psi_{\mathbf{A}}^{\tilde{k}}(\boldsymbol{\mu})(\mathbf{x}) + \mathbf{i}\tilde{k}^{-1}\nabla\left[\Psi_V^{\tilde{k}}(\mathrm{div}_{\Gamma_{r_1}}\boldsymbol{\mu})(\mathbf{x})\right] \qquad \forall \mathbf{x} \in \mathbb{R}^3 \backslash \bar{\Omega}_{r_1}, \\
&\Psi_{\mathrm{DL}}^{\tilde{k}}(\boldsymbol{\lambda})(\mathbf{x}) = \nabla \times \left[\Psi_{\mathbf{A}}^{\tilde{k}}(\boldsymbol{\lambda})(\mathbf{x})\right] \qquad \forall \mathbf{x} \in \mathbb{R}^3 \backslash \bar{\Omega}_{r_1}
\end{aligned}$$

with the vector and scalar single layer potentials

$$\Psi_{\mathbf{A}}^{\tilde{k}}(\boldsymbol{\lambda})(\mathbf{x}) = \int_{\Gamma_{r_1}} G_{\tilde{k}}(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})d\mathbf{x}, \quad \Psi_V^{\tilde{k}}(\phi)(\mathbf{x}) = \int_{\Gamma_{r_1}} G_{\tilde{k}}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})d\mathbf{x}.$$

Recall that $G_{\tilde{k}}(\mathbf{x},\mathbf{y}) = \frac{e^{\mathbf{i}\tilde{k}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$. For any $\mathbf{x} \in \Gamma_2, \mathbf{y} \in \Gamma_{r_1}$, $|\mathbf{x}-\mathbf{y}| \geq (\theta - r_1)L_{\min}/2$. Thus $|G_{\tilde{k}}(\mathbf{x},\mathbf{y})| \leq Cd^{-1}e^{-k\sigma_0(\theta-r_1)L_{\min}/2}$ for $\mathbf{x} \in \Gamma_2, \mathbf{y} \in \Gamma_{r_1}$. Similarly, we have

$$\begin{aligned}
&|\nabla_{\mathbf{x}} G_{\tilde{k}}(\mathbf{x},\mathbf{y})| + |\nabla_{\mathbf{y}} G_{\tilde{k}}(\mathbf{x},\mathbf{y})| \leq Ckd^{-1}e^{-k\sigma_0(\theta-r_1)L_{\min}/2}, \quad \forall \mathbf{x} \in \Gamma_2, \mathbf{y} \in \Gamma_{r_1} \\
&|\nabla_{\mathbf{x}}\nabla_{\mathbf{y}} G_{\tilde{k}}(\mathbf{x},\mathbf{y})| \leq Ck^2 d^{-1}e^{-k\sigma_0(\theta-r_1)L_{\min}/2}, \quad \forall \mathbf{x} \in \Gamma_2, \mathbf{y} \in \Gamma_{r_1}.
\end{aligned}$$

By the similar argument in Lemma 5 and Lemma 6, we can obtain

$$\begin{aligned}
&\|\, \mathbf{n}_2 \times \mathbf{u} \,\|_{Y(\Gamma_2)} + \|\, \mathbf{n}_2 \times A\nabla \times \mathbf{u} \,\|_{Y(\Gamma_2)} \\
&\leq C(1+kd)e^{-k\sigma_0(\theta-r_1)L_{\min}/2}(\|\, \boldsymbol{\lambda} \,\|_{Y(\Gamma_{r_1})} + \|\, \boldsymbol{\mu} \,\|_{Y(\Gamma_{r_1})}).
\end{aligned}$$

This completes the proof since by (4.12) and Lemma 12

$$\|\, \boldsymbol{\lambda} \,\|_{Y(\Gamma_{r_1})} + \|\, \boldsymbol{\mu} \,\|_{Y(\Gamma_{r_1})} \leq C\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_{r_1})} \leq C\|\mathbf{U}\|_{L^2(\Omega_{r_1})}. \qquad \square$$

Now we are in the position to prove Theorem 1.

Proof of Theorem 1. Multiply the equation (4.1) by $\mathbf{u}$, integrate by parts over Ω^{PML}, and use (4.10), we have

$$\int_{\Omega^{\mathrm{PML}}} A^{-1}\mathbf{U} \cdot \mathbf{w} d\mathbf{x} + \langle \mathbf{n} \times A\nabla \times \mathbf{w}, \bar{\mathbf{u}} \rangle_{\partial\Omega^{\mathrm{PML}}} + \langle \mathbf{n} \times A\nabla \times \mathbf{u}, \bar{\mathbf{w}} \rangle_{\partial\Omega^{\mathrm{PML}}} = 0.$$

This yields, by $\mathbf{n}_1 \times \mathbf{u} = 0, \mathbf{n}_1 \times \mathbf{w} = 0$ on Γ_1,

$$\begin{aligned}
&\left|\int_{\Omega^{\mathrm{PML}}} A^{-1}\mathbf{U} \cdot \mathbf{w} d\mathbf{x}\right| \\
&\leq |\langle \mathbf{n}_2 \times A\nabla \times \mathbf{w}, \mathbf{n}_2 \times \bar{\mathbf{u}} \times \mathbf{n}_2 \rangle_{\Gamma_2}| + \langle \mathbf{n}_2 \times A\nabla \times \mathbf{u}, \bar{\mathbf{q}} \times \mathbf{n}_2 \rangle_{\Gamma_2}| \\
&\leq \|\, \mathbf{n}_2 \times A\nabla \times \mathbf{w} \,\|_{Y(\Gamma_2)} \|\, \mathbf{n}_2 \times \mathbf{u} \,\|_{Y(\Gamma_2)} + \|\, \mathbf{n}_2 \times A\nabla \times \mathbf{u} \,\|_{Y(\Gamma_2)} \|\, \mathbf{q} \,\|_{Y(\Gamma_2)}.
\end{aligned}$$

By Lemma 10 and Lemma 13 we know that

$$
\begin{aligned}
&\|\,\mathbf{n}_2\times A\nabla\times\mathbf{w}\,\|_{Y(\Gamma_2)}\|\,\mathbf{n}_2\times\mathbf{u}\,\|_{Y(\Gamma_2)}\\
\le{}& C\left[(1+kd)\|\mathbf{q}\|_{Y(\Gamma_2)}+\|\mathbf{w}\|_{L^2(\Omega_{r_1})}\right]\cdot C(1+kd)e^{-k\sigma_0(\theta-r_1)L_{\min}/2}\|\mathbf{U}\|_{L^2(\Omega_{r_1})}\\
\le{}& C\left[\|\mathbf{q}\|_{Y(\Gamma_2)}+(1+kd)e^{-k\sigma_0(\theta-r_1)L_{\min}/2}\|\mathbf{w}\|_{L^2(\Omega_{r_1})}\right]\|\mathbf{U}\|_{L^2(\Omega_{r_1})}, \qquad (4.13)
\end{aligned}
$$

where we have used the fact that $(1+kd)^2e^{-k\sigma_0(\theta-r_1)L_{\min}/2}\le C$ for sufficiently large d. By Lemma 13 and the fact that $(1+kd)e^{-k\sigma_0(\theta-r_1)L_{\min}/2}\le C$ for sufficiently large d we have

$$\|\,\mathbf{n}_2\times A\nabla\times\mathbf{u}\,\|_{Y(\Gamma_2)}\le C\|\mathbf{U}\|_{L^2(\Omega_{r_1})}. \qquad (4.14)$$

Thus combining (4.12)-(4.14) and taking $\mathbf{U}=\chi_1 A\bar{\mathbf{w}}$, where χ_1 is the characteristic function of Ω_{r_1}, we obtain

$$\|\mathbf{w}\|_{L^2(\Omega_{r_1})}\le C(1+kd)e^{-k\sigma_0(\theta-r_1)L_{\min}/2}\|\mathbf{w}\|_{L^2(\Omega_{r_1})}+C\|\,\mathbf{q}\,\|_{Y(\Gamma_2)}. \qquad (4.15)$$

To prove the uniqueness of the problem (4.1)-(4.2), we set $\mathbf{q}=0$. Then it is easy to see from (4.15) that $\mathbf{w}=0$ in Ω_{r_1} for sufficiently large d. The uniqueness of the solution then follows by the principle of unique continuation. The existence of the solution then follows from Lemma 11 and the uniqueness (see [7, Theorem 5.1] for a similar argument).

To show the desired estimate (4.15), we first note that it follows from (4.15) that for sufficiently large d

$$\|\mathbf{w}\|_{L^2(\Omega_{r_1})}\le C\|\,\mathbf{q}\,\|_{Y(\Gamma_2)}.$$

Now by using the trace inequality (2.5) and Lemma 9

$$\|\,\mathbf{n}_1\times\nabla\times\mathbf{w}\,\|_{Y(\Gamma_1)}\le C(\|\nabla\times\mathbf{w}\|_{L^2(\Omega_{r_0})}+L^{-2}\|\mathbf{w}\|_{L^2(\Omega_{r_0})})\le C\|\mathbf{w}\|_{L^2(\Omega_{r_1})}.$$

This completes the proof. $\square$

5 The convergence of the PML method

We first reformulate (2.17)-(2.18) in the bounded domain Ω_1 by imposing the boundary condition

$$\mathbf{n}_1\times(\nabla\times\hat{\mathbf{E}})|_{\Gamma_1}=\hat{G}_e(\mathbf{n}_1\times\hat{\mathbf{E}}|_{\Gamma_1}),$$

where the approximate Calderon operator $\hat{G}_e: Y(\Gamma_1)\to Y(\Gamma_1)$ is defined as

$$\hat{G}_e(\boldsymbol{\lambda}):=\frac{1}{\mathbf{i}k}\,\mathbf{n}_1\times(\nabla\times\mathbf{u}), \qquad (5.1)$$

with $\mathbf{u}$ satisfying

$$
\begin{aligned}
&\nabla\times A(\nabla\times\mathbf{u})-k^2A^{-1}\mathbf{u}=0 \quad \text{in } \Omega^{\rm PML}, && (5.2)\\
&\mathbf{n}_1\times\mathbf{u}=\boldsymbol{\lambda} \text{ on } \Gamma_1, \qquad \mathbf{n}_2\times\mathbf{u}=0 \text{ on } \Gamma_2. && (5.3)
\end{aligned}
$$

By Theorem 1 we know that $\hat{G}_e$ is well-defined for sufficiently large d. Based on the operator $\hat{G}_e$, let $\hat{a} : H(\mathrm{curl}; \Omega_1) \times H(\mathrm{curl}; \Omega_1) \to \mathbb{C}$ be the sesquilinear form

$$\hat{a}(\hat{\mathbf{E}}, \mathbf{v}) = \int_{\Omega_1} (\nabla \times \hat{\mathbf{E}} \cdot \nabla \times \bar{\mathbf{v}} - k^2 \hat{\mathbf{E}} \cdot \bar{\mathbf{v}}) d\mathbf{x} + \mathbf{i}k \langle \hat{G}_e(\mathbf{n}_1 \times \hat{\mathbf{E}}), \mathbf{n}_1 \times \mathbf{v} \times \mathbf{n}_1 \rangle_{\Gamma_1}.$$

Then the weak formulation of (2.17)-(2.18) on the bounded domain Ω_1 is: Given $\mathbf{g} \in Y(\Gamma_D)$, find $\hat{\mathbf{E}} \in H(\mathrm{curl}; \Omega_1)$ such that $\mathbf{n}_D \times \hat{\mathbf{E}} = \mathbf{g}$ on Γ_D, and

$$\hat{a}(\hat{\mathbf{E}}, \mathbf{v}) = 0, \qquad \forall \mathbf{v} \in H_D(\mathrm{curl}; \Omega_1). \tag{5.4}$$

Lemma 14 *Let (H1)-(H3) be satisfied. Then, for sufficiently large d, we have*

$$\| (\hat{G}_e - G_e)(\boldsymbol{\lambda}) \|_{Y(\Gamma_1)} \le C(1 + kd) e^{-k\gamma\bar{\sigma}} \| \boldsymbol{\lambda} \|_{Y(\Gamma_1)},$$

for any $\boldsymbol{\lambda} \in Y(\Gamma_1)$.

Proof. For any $\boldsymbol{\lambda} \in Y(\Gamma_1)$, let $\mathbb{E}(\boldsymbol{\lambda})$ be the PML extension defined in (2.15). It is easy to see that $\frac{1}{\mathbf{i}k} \mathbf{n}_1 \times \mathbb{E}(\boldsymbol{\lambda}) = G_e(\boldsymbol{\lambda})$ on Γ_1. Now by (5.2)-(5.3), we know that $(G_e - \hat{G}_e)(\boldsymbol{\lambda}) = \frac{1}{\mathbf{i}k} \mathbf{n}_1 \times (\nabla \times \mathbf{v})$, where $\mathbf{v}$ satisfies

$$\nabla \times A(\nabla \times \mathbf{v}) - k^2 A^{-1} \mathbf{v} = 0 \ \text{ in } \Omega^{\mathrm{PML}},$$
$$\mathbf{n}_1 \times \mathbf{v} = 0 \text{ on } \Gamma_1, \quad \mathbf{n}_2 \times \mathbf{v} = \mathbf{n}_2 \times B\mathbb{E}(\boldsymbol{\lambda}) \text{ on } \Gamma_2.$$

By Theorem 1, Lemma 5 and Lemma 6, we have

$$\| \mathbf{n}_1 \times (\nabla \times B\mathbf{v}) \|_{Y(\Gamma_1)} \le C \| \mathbf{n}_2 \times B\mathbb{E}(\boldsymbol{\lambda}) \|_{Y(\Gamma_2)} \le C(1 + kd) e^{-k\gamma\bar{\sigma}} \| \boldsymbol{\lambda} \|_{Y(\Gamma_1)}.$$

This completes the proof. □

The following theorem is the main result of this section.

Theorem 2 *Let (H1)-(H3) be satisfied. Then for sufficiently large $d > 0$, the PML problem* (2.17)-(2.18) *has a unique solution $\hat{\mathbf{E}} \in H(\mathrm{curl}; \Omega_2)$. Moreover, we have the following estimate*

$$\|\mathbf{E} - \hat{\mathbf{E}}\|_{H(\mathrm{curl}; \Omega_1)} \le C(1 + kd) e^{-k\gamma\bar{\sigma}} \| \mathbf{n}_1 \times \mathbf{E} \|_{Y(\Gamma_1)}. \tag{5.5}$$

Proof. First by (2.9) and (5.4) we have, for any $\mathbf{v} \in \mathbf{H}_D(\mathrm{curl}; \Omega_1)$,

$$\begin{aligned} \hat{a}(\mathbf{E} - \hat{\mathbf{E}}, \mathbf{v}) &= \hat{a}(\mathbf{E}, \mathbf{v}) - a(\mathbf{E}, \mathbf{v}) \\ &= \mathbf{i}k \langle (\hat{G}_e - G_e)(\mathbf{n}_1 \times \mathbf{E}), \mathbf{n}_1 \times \mathbf{v} \times \mathbf{n}_1 \rangle_{\Gamma_1}. \end{aligned} \tag{5.6}$$

By Lemma 1, Lemma 6 and Lemma 14 we know that for sufficiently large d,

$$\begin{aligned} \sup_{\mathbf{v} \in H_D(\mathrm{curl}; \Omega_1)} \frac{|\hat{a}(\mathbf{E}, \mathbf{v})|}{\|\mathbf{v}\|_{H_D(\mathrm{curl}; \Omega_1)}} &\ge C \|\mathbf{E}\|_{H_D(\mathrm{curl}; \Omega_1)} - C(1 + kd) e^{-k\gamma\bar{\sigma}} \|\mathbf{n}_1 \times \mathbf{E}\|_{Y(\Gamma_1)} \\ &\ge C \|\mathbf{E}\|_{H_D(\mathrm{curl}; \Omega_1)}. \end{aligned}$$

This shows that the PML problem (2.17)-(2.18) has a unique solution. The desired estimate then follows from (5.6), the above inf-sup condition, and Lemma 14. This completes the proof. □

6 Finite element approximation

We start by introducing the weak formulation of the PML problem (2.17)-(2.18). Let

$$b(\mathbf{u}, \mathbf{v}) = \int_{\Omega_2} (A\nabla \times \mathbf{u} \cdot \nabla \times \bar{\mathbf{v}} - k^2 A^{-1}\mathbf{u} \cdot \bar{\mathbf{v}}) d\mathbf{x}. \tag{6.1}$$

Then the weak formulation of (2.17)-(2.18) is: Given $\mathbf{g} \in Y(\Gamma_D)$, find $\hat{\mathbf{E}} \in \mathbf{H}(\text{curl}, \Omega_2)$, such that $\mathbf{n}_D \times \hat{\mathbf{E}} = \mathbf{g}$ on Γ_D, $\mathbf{n}_2 \times \hat{\mathbf{E}} = 0$ on Γ_2, and

$$b(\hat{\mathbf{E}}, \mathbf{v}) = 0, \qquad \forall \mathbf{v} \in H_0(\text{curl}; \Omega_2). \tag{6.2}$$

Let $\mathcal{M}_h$ be a regular partition of the domain Ω_2 whose elements may have curved boundaries on Γ_D. We will use the lowest order Nédélec edge element [31] for which the finite element space $\mathbf{U}_h$ over $\mathcal{M}_h$ is defined by

$$\mathbf{U}_h = \{\mathbf{u} \in H(\text{curl}; \Omega_2) : \mathbf{u}|_K = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x}, \forall \mathbf{a}_K, \mathbf{b}_K \in \mathbb{R}^3, \forall K \in \mathcal{M}_h\}.$$

Degrees of freedom of functions $\mathbf{u} \in \mathbf{U}_h$ on every $K \in \mathcal{M}_h$ are $\int_{e_i} \mathbf{u} \cdot \mathbf{dl}, i = 1, \ldots, 6$, where $e_1, \ldots, e_6$ are the six edges of K. Denote by $\overset{\circ}{\mathbf{U}}_h = \mathbf{U}_h \cap H_0(\text{curl}; \Omega_2)$. The finite element approximation to (6.2) reads as follows: Find $\mathbf{E}_h \subset \mathbf{U}_h$ such that $\mathbf{n} \times \mathbf{E}_h = \mathbf{g}_h$ on Γ_D, $\mathbf{n} \times \mathbf{E}_h = 0$ on Γ_2, and

$$b(\mathbf{E}_h, \mathbf{v}_h) = 0, \qquad \forall \mathbf{v}_h \in \overset{\circ}{\mathbf{U}}_h. \tag{6.3}$$

Here $\mathbf{g}_h$ is some edge element approximation of $\mathbf{g}$ on Γ_D. The existence and uniqueness of the discrete problem (6.3) is a difficult problem due to the non-coerciveness of the sesquilinear form $b : H(\text{curl}; \Omega_2) \times H(\text{curl}; \Omega_2) \to \mathbb{C}$. By extending the argument in [30, Section 7.2] for the Maxwell cavity problem, the unique existence of (6.3) for a sufficiently small mesh size $h < h^*$ can be proved by using the unique existence of the continuous problem (6.2). In this paper we are interested in *a posteriori* error estimates and the associated adaptive algorithm. Thus in the following, we simply assume the discrete problem (6.3) has a unique solution $\mathbf{E}_h$.

For any $K \in \mathcal{M}_h$, we denote by h_K its diameter. Let $\mathcal{F}_h$ be the set of all faces of the mesh $\mathcal{M}_h$ that do not lie on Γ_D and Γ_2. For any $F \in \mathcal{F}_h$, h_F stands for its diameter. For any interior face F which is a common face of K_1 and K_2 in $\mathcal{M}_h$, we define the following jump residuals across F

$$[\mathbf{n} \times (A\nabla \times \mathbf{E}_h)] = \mathbf{n}_F \times (A\nabla \times (\mathbf{E}_h|_{K_1} - \mathbf{E}_h|_{K_2})),$$
$$[k^2 A^{-1}\mathbf{E}_h \cdot \mathbf{n}] = k^2 A^{-1}(\mathbf{E}_h|_{K_1} - \mathbf{E}_h|_{K_2}) \cdot \mathbf{n}_F,$$

using the convention that the unit norm vector $\mathbf{n}_F$ to F points from K_2 to K_1. The local error indicator η_K for any $K \in \mathcal{M}_h$ is defined as

$$\begin{aligned}\eta_K^2 = \quad & h_K^2 \| k^2 A^{-1}\mathbf{E}_h - \nabla \times (A\nabla \times \mathbf{E}_h) \|^2_{\mathbf{L}^2(K)} \\ & + h_K^2 \| \operatorname{div}(k^2 A^{-1}\mathbf{E}_h) \|^2_{L^2(K)} \\ & + h_K \| [\mathbf{n} \times (A\nabla \times \mathbf{E}_h)] \|^2_{L^2(\partial K)} + h_K \| [k^2 A^{-1}\mathbf{E}_h \cdot \mathbf{n}] \|^2_{L^2(\partial K)}.\end{aligned}$$

The following theorem is the main result of this paper.

Theorem 3 *Let (H1)-(H3) be satisfied. Then for sufficiently large d, there exists a constant C depending on the minimum angle of the mesh $\mathcal{M}_h$ but independent of d such that the following a posteriori error estimate is valid*

$$\|\mathbf{E}-\mathbf{E}_h\|_{H(\mathrm{curl};\Omega_1)} \le C\|\,\mathbf{g}-\mathbf{g}_h\,\|_{Y(\Gamma_D)} + C\Big(\sum_{K\in\mathcal{M}_h}\eta_K^2\Big)^{1/2}$$
$$+C(1+kd)e^{-k\gamma\bar{\sigma}}\|\,\mathbf{n}_1\times\mathbf{E}_h\,\|_{Y(\Gamma_1)}.$$

The proof of this theorem will be given in Section 7. One of the key ingredients of the *a posteriori* error analysis is the Birman-Solomyak decomposition theorem in Lipschitz domains [6], [21], [13]. More precisely, the following result whose proof can be found in [21], [13] will be used.

Lemma 15 *For any $\mathbf{v}\in H_0(\mathrm{curl},\Omega_2)$, there exists a $\mathbf{v}_s\in H_0(\mathrm{curl},\Omega_2)\cap H^1(\Omega_2)^3$ and a $\varphi\in H_0^1(\Omega_2)$ such that $\mathbf{v}=\mathbf{v}_s+\nabla\varphi$ in Ω_2, and*

$$\|\,\mathbf{v}_s\,\|_{H^1(\Omega_2)}+\|\,\varphi\,\|_{H^1(\Omega_2)}\le C\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_2)}.$$

Let V_h be the standard H^1-conforming linear finite element space over $\mathcal{M}_h$ and $\mathring{V}_h=H_0^1(\Omega_2)\cap V_h$. In Section 7, we will use the Clément operator $r_h: H_0^1(\Omega_2)\to\mathring{V}_h$ in [18] and the Beck-Hiptmair-Hoppe-Wohlmuth interpolation operator $\pi_h: H^1(\Omega_2)^3\cap H_0(\mathrm{curl};\Omega_2)\to\mathring{\mathbf{U}}_h$ in [4] which satisfy the following estimates

$$\|\,\varphi-r_h\varphi\,\|_{L^2(K)}\le Ch_K\|\,\nabla\varphi\,\|_{L^2(\tilde{K})}, \tag{6.4}$$
$$\|\,\varphi-r_h\varphi\,\|_{L^2(F)}\le Ch_F^{1/2}\|\,\nabla\varphi\,\|_{L^2(\tilde{F})}, \tag{6.5}$$
$$\|\,\mathbf{v}-\pi_h\mathbf{v}\,\|_{L^2(K)}\le Ch_K\|\,\nabla\mathbf{v}\,\|_{L^2(\tilde{K})}, \tag{6.6}$$
$$\|\,\mathbf{v}-\pi_h\mathbf{v}\,\|_{L^2(F)}\le Ch_F^{1/2}\|\,\nabla\mathbf{v}\,\|_{L^2(\tilde{F})}, \tag{6.7}$$

where $\tilde{\mathcal{A}}$ is the union of elements in $\mathcal{M}_h$ with non-empty intersection with $\mathcal{A}$, $\mathcal{A}=K\in\mathcal{M}_h$ or $F\in\mathcal{F}_h$.

7 A posteriori error analysis

In this section, we prove the *a posteriori* error estimates in Theorem 3. To begin with, let $\mathbf{u}\in H(\mathrm{curl};\Omega_1)$ such that $\mathbf{n}_D\times\mathbf{u}=\mathbf{g}-\mathbf{g}_h$ on Γ_D, then $\mathbf{E}-\mathbf{E}_h-\mathbf{u}\in H_D(\mathrm{curl};\Omega_1)$. Thus by (2.10) we have

$$\|\mathbf{E}-\mathbf{E}_h-\mathbf{u}\|_{H(\mathrm{curl};\Omega_1)}\le C\sup_{\mathbf{v}\in H_D(\mathrm{curl};\Omega_1)}\frac{|a(\mathbf{E}-\mathbf{E}_h-\mathbf{u},\mathbf{v})|}{\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_1)}}.$$

Since $|a(\mathbf{u},\mathbf{v})|\le C\|\mathbf{u}\|_{H(\mathrm{curl};\Omega_1)}\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_1)}$, we obtain

$$\|\mathbf{E}-\mathbf{E}_h\|_{H(\mathrm{curl};\Omega_1)}\le C\|\mathbf{u}\|_{H(\mathrm{curl};\Omega_1)}+C\sup_{\mathbf{v}\in H_D(\mathrm{curl};\Omega_1)}\frac{|a(\mathbf{E}-\mathbf{E}_h,\mathbf{v})|}{\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_1)}}.$$

The above estimate is valid for any $\mathbf{u} \in H(\mathrm{curl};\Omega_1)$ such that $\mathbf{n} \times \mathbf{u} = \mathbf{g} - \mathbf{g}_h$ on Γ_D, we get by the trace theorem

$$\|\mathbf{E} - \mathbf{E}_h\|_{H(\mathrm{curl};\Omega_1)} \leq C\|\,\mathbf{g} - \mathbf{g}_h\,\|_{Y(\Gamma_D)} + C \sup_{\mathbf{v}\in H_D(\mathrm{curl};\Omega_1)} \frac{|a(\mathbf{E} - \mathbf{E}_h, \mathbf{v})|}{\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_1)}}. \tag{7.1}$$

For any $\mathbf{v} \in H_D(\mathrm{curl};\Omega_1)$, we extend $\mathbf{v}$ to Ω^{PML}, denoted by $\mathfrak{E}(\mathbf{v})$, such that $\mathbf{w} = \mathfrak{E}(\mathbf{v})|_{\Omega^{\mathrm{PML}}}$ satisfies

$$\nabla \times A\nabla \times \mathbf{w} - k^2 A^{-1}\mathbf{w} = 0 \quad \text{in } \Omega^{\mathrm{PML}}, \tag{7.2}$$

$$\mathbf{n}_1 \times \mathbf{w} = \mathbf{n}_1 \times \mathbf{v} \quad \text{on } \Gamma_1, \qquad \mathbf{n}_2 \times \mathbf{w} = 0 \quad \text{on } \Gamma_2. \tag{7.3}$$

We know from Theorem 1 that $\mathfrak{E}(\mathbf{v})$ is well-defined. Moreover, by (5.1)

$$\hat{G}_e(\mathbf{n}_1 \times \mathbf{v}) = \frac{1}{\mathbf{i}k}\mathbf{n}_1 \times \nabla \times \mathfrak{E}(\mathbf{v}).$$

Lemma 16 (Error representational formula) *For any* $\mathbf{v} \in H(\mathrm{curl};\Omega_1)$*, let* $\tilde{\mathbf{v}}$ *in* $H(\mathrm{curl};\Omega_2)$ *be its extension defined by*

$$\tilde{\mathbf{v}} = \overline{\mathfrak{E}(\bar{\mathbf{v}})}.$$

Then for any $\mathbf{v}_h \in \overset{\circ}{\mathbf{U}}_h$*, we have*

$$a(\mathbf{E} - \mathbf{E}_h, \mathbf{v}) = -b(\mathbf{E}_h, \tilde{\mathbf{v}} - \mathbf{v}_h) + \mathbf{i}k\langle(\hat{G}_e - G_e)(\mathbf{n}_1 \times \mathbf{E}_h), \mathbf{n}_1 \times \mathbf{v} \times \mathbf{n}_1\rangle_{\Gamma_1}.$$

Proof. By (2.9) and the definition of the sesquilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we have

$$\begin{aligned}
&a(\mathbf{E} - \mathbf{E}_h, \mathbf{v}) \\
&= -\int_{\Omega_1} \left(\nabla \times \mathbf{E}_h \cdot \nabla \times \bar{\mathbf{v}} - k^2 \mathbf{E}_h \cdot \bar{\mathbf{v}}\right) d\mathbf{x} \\
&\quad -\mathbf{i}k\langle G_e(\mathbf{n}_1 \times \mathbf{E}_h), \mathbf{n}_1 \times \mathbf{v} \times \mathbf{n}_1\rangle_{\Gamma_1} \\
&= -b(\mathbf{E}_h, \tilde{\mathbf{v}}) + \int_{\Omega^{\mathrm{PML}}} \left(A\nabla \times \mathbf{E}_h \cdot \nabla \times \bar{\tilde{\mathbf{v}}} - k^2 A^{-1}\mathbf{E}_h \cdot \bar{\tilde{\mathbf{v}}}\right) d\mathbf{x} \\
&\quad -\mathbf{i}k\langle G_e(\mathbf{n}_1 \times \mathbf{E}_h), \mathbf{n}_1 \times \mathbf{v} \times \mathbf{n}_1\rangle_{\Gamma_1}.
\end{aligned}$$

But $\bar{\tilde{\mathbf{v}}} = \mathfrak{E}(\bar{\mathbf{v}})$ satisfies

$$\nabla \times A\nabla \times \bar{\tilde{\mathbf{v}}} - k^2 A^{-1}\bar{\tilde{\mathbf{v}}} = 0 \quad \text{in } \Omega^{\mathrm{PML}},$$

we have

$$\begin{aligned}
&\int_{\Omega^{\mathrm{PML}}} \left(A\nabla \times \mathbf{E}_h \cdot \nabla \times \bar{\tilde{\mathbf{v}}} - k^2 A^{-1}\mathbf{E}_h \cdot \bar{\tilde{\mathbf{v}}}\right) d\mathbf{x} \\
&= \langle \mathbf{n} \times \mathbf{E}_h, \mathbf{n} \times A\nabla \times \tilde{\mathbf{v}} \times \mathbf{n}\rangle_{\Gamma_1\cup\Gamma_2}.
\end{aligned}$$

Since $\mathbf{n} \times \mathbf{E}_h = 0$ on Γ_2 and $\mathbf{n} = -\mathbf{n}_1$ on Γ_1 for the domain Ω^{PML}, we then get

$$\begin{aligned}
&\int_{\Omega^{\mathrm{PML}}} \left(A\nabla \times \mathbf{E}_h \cdot \nabla \times \bar{\tilde{\mathbf{v}}} - k^2 A^{-1}\mathbf{E}_h \cdot \bar{\tilde{\mathbf{v}}}\right) d\mathbf{x} \\
&= -\langle \mathbf{n}_1 \times \mathbf{E}_h, \mathbf{n}_1 \times A\nabla \times \tilde{\mathbf{v}} \times \mathbf{n}_1\rangle_{\Gamma_1} \\
&= -\langle \mathbf{n}_1 \times \mathfrak{E}(\mathbf{E}_h), \mathbf{n}_1 \times A\nabla \times \tilde{\mathbf{v}} \times \mathbf{n}_1\rangle_{\Gamma_1},
\end{aligned}$$

where $\mathfrak{E}(\mathbf{E}_h)$ is the extension of $\mathbf{E}_h$ in $\Omega^{\rm PML}$ by (7.2)-(7.3). Now integrating by parts twice and using the equation (7.2) we obtain

$$\begin{aligned}
&\int_{\Omega^{\rm PML}}\left(A\nabla\times\mathbf{E}_h\cdot\nabla\times\bar{\tilde{\mathbf{v}}}-k^2A^{-1}\mathbf{E}_h\cdot\bar{\tilde{\mathbf{v}}}\right)d\mathbf{x}\\
&=\int_{\Omega^{\rm PML}}\left(A\nabla\times\mathfrak{E}(\mathbf{E}_h)\cdot\nabla\times\bar{\tilde{\mathbf{v}}}-k^2A^{-1}\mathfrak{E}(\mathbf{E}_h)\cdot\bar{\tilde{\mathbf{v}}}\right)d\mathbf{x}\\
&=-\langle\mathbf{n}_1\times\bar{\tilde{\mathbf{v}}},\mathbf{n}_1\times\nabla\times\overline{\mathfrak{E}(\mathbf{E}_h)}\times\mathbf{n}_1\rangle_{\Gamma_1}\\
&=\mathbf{i}k\langle\hat{G}_e(\mathbf{n}_1\times\mathbf{E}_h),\mathbf{n}_1\times\tilde{\mathbf{v}}\times\mathbf{n}_1\rangle_{\Gamma_1}.
\end{aligned}$$

This completes the proof because $\tilde{\mathbf{v}}=\mathbf{v}$ on Γ_1. $\square$

Now we are in the position to prove the main result of this paper.

Proof of Theorem 3. Our starting point is (7.1). To estimate the second term in (7.1), for any $\mathbf{v}\in H(\mathrm{curl};\Omega_1)$ such that $\mathbf{n}_D\times\mathbf{v}=0$ on Γ_D, we denote $\tilde{\mathbf{v}}=\overline{\mathfrak{E}(\bar{\mathbf{v}})}$ its extension to $\Omega^{\rm PML}$. Thus $\tilde{\mathbf{v}}\in H_0(\mathrm{curl};\Omega_2)$. By Lemma 15, there exists $\mathbf{v}_s\in H_0(\mathrm{curl};\Omega_2)\cap H^1(\Omega_2)^3$ and $\varphi\in H_0^1(\Omega_2)$ such that $\tilde{\mathbf{v}}=\mathbf{v}_s+\nabla\varphi$, and

$$\|\mathbf{v}_s\|_{H^1(\Omega_2)}+\|\varphi\|_{H^1(\Omega_2)}\le C\|\tilde{\mathbf{v}}\|_{H(\mathrm{curl};\Omega_2)}.$$

By Theorem 1, $\|\tilde{\mathbf{v}}\|_{H(\mathrm{curl};\Omega^{\rm PML})}\le C\|\mathbf{n}_1\times\mathbf{v}\|_{H^{-1/2}(\mathrm{Div};\Gamma_1)}$. Thus by the trace theorem in $H(\mathrm{curl};\Omega_1)$, we have

$$\|\mathbf{v}_s\|_{H^1(\Omega_2)}+\|\varphi\|_{H^1(\Omega_2)}\le C\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_1)}. \tag{7.4}$$

Let

$$\mathbf{v}_h=\nabla r_h\varphi+\pi_h\mathbf{v}_s,$$

where $r_h:H_0^1(\Omega_2)\to\overset{\circ}{V}_h$ and $\pi_h:H^1(\Omega_2)^3\cap H_0(\mathrm{curl};\Omega_2)\to\overset{\circ}{\mathbf{U}}_h$ are the interpolation operators defined at the end of Section 6. By the error representation formula in Lemma 16, we have

$$\begin{aligned}
&a(\mathbf{E}-\mathbf{E}_h,\mathbf{v})\\
&=-b(\mathbf{E}_h,\mathbf{v}_s+\nabla\varphi-(\pi_h\mathbf{v}_s+\nabla r_h\varphi))+\mathbf{i}k\langle(\hat{G}_e-G_e)(\hat{\mathbf{x}}\times\mathbf{E}_h),(\mathbf{n}_1\times\mathbf{v})\times\mathbf{n}_1\rangle_{\Gamma_1}\\
&=-\int_{\Omega_2}\left(A\nabla\times\mathbf{E}_h\cdot\nabla\times(\bar{\mathbf{v}}_s-\pi_h\bar{\mathbf{v}}_s)-k^2A^{-1}\mathbf{E}_h\cdot(\bar{\mathbf{v}}_s-\pi_h\bar{\mathbf{v}}_s)\right)d\mathbf{x}\\
&\quad+\int_{\Omega_2}k^2A^{-1}\mathbf{E}_h\cdot\nabla(\bar{\varphi}-r_h\bar{\varphi})d\mathbf{x}\\
&\quad+\mathbf{i}k\langle(\hat{G}_e-G_e)(\mathbf{n}_1\times\mathbf{E}_h),(\mathbf{n}_1\times\mathbf{v})\times\mathbf{n}_1\rangle_{\Gamma_1}\\
&:=\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

By using integration by parts, the estimates (6.4)-(6.7), and standard argument in the *a posteriori* error analysis, we obtain

$$\begin{aligned}
|\mathrm{I}+\mathrm{II}|&\le C\Big(\sum_{K\in\mathcal{M}_h}\eta_K^2\Big)^{1/2}(\|\varphi\|_{H^1(\Omega_2)}+\|\mathbf{v}_s\|_{H^1(\Omega_2)})\\
&\le C\Big(\sum_{K\in\mathcal{M}_h}\eta_K^2\Big)^{1/2}\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_1)},
\end{aligned}$$

where we have used (7.4) in the last inequality. By Lemma 6 and trace inequality for $H(\mathrm{curl};\Omega_1)$, we have

$$|\mathrm{III}| \le C(1+kd)e^{-k\gamma\bar{\sigma}}\|\,\mathbf{n}_1\times\mathbf{E}_h\,\|_{Y(\Gamma_1)}\|\mathbf{v}\|_{H(\mathrm{curl};\Omega_1)}.$$

This completes the proof by (7.1). $\square$

8 Numerical examples

In this section we report two numerical examples to illustrate the performance of the adaptive anisotropic PML method. The implementation of the adaptive finite element method is based on the parallel adaptive finite element package PHG [33], [37] which is based on the unstructured mesh and MPI. The computations are performed on the cluster LSSC-III in the State Key Laboratory of Scientific and Engineering Computing of Chinese Academy of Sciences.

First we choose L_1, L_2, L_3 such that $D\subset B_1$. We take $\zeta=\sqrt{2}\max_{i,j}\frac{L_i}{L_j}$ in $\eta=1+\zeta\sigma$ and choose the medium property σ such that $\hat{\sigma}'=c_0(r_0-r)^2(r-1)^2$ for $1\le r\le r_0$. Then we choose r_0, c_0 and θ such that the exponentially decaying factor:

$$\omega = e^{-k\gamma\bar{\sigma}} \le 10^{-8}, \tag{8.1}$$

which makes the PML error negligible compared with the finite element discretization errors. Once the PML region and the medium property are fixed, we use the standard finite element adaptive strategy to modify the mesh according to the *a posteriori* error estimate.

The adaptive finite element algorithm is based on the *a posteriori* error estimate in Theorem 3. With the local error estimator η_K in Theorem 3 we define the global *a posteriori* error estimate

$$\mathcal{E} := \left(\sum_{K\in\mathcal{M}_h}\eta_K^2\right)^{1/2}.$$

Now we describe the adaptive algorithm used in this paper.

ALGORITHM. Given a tolerance tol >0 and the initial mesh $\mathcal{M}_0$. Set $\mathcal{M}_h=\mathcal{M}_0$.

1. Solve the discrete problem (6.3) on $\mathcal{M}_0$.
2. Compute the local error estimator η_K on each $K\in\mathcal{M}_0$, the global error estimate $\mathcal{E}$.
3. While $\mathcal{E} >$ tol do
   - Refine the elements in $\hat{\mathcal{M}}_h\subset\mathcal{M}_h$, where $\hat{\mathcal{M}}_h$ is the minimum subset of $\mathcal{M}_h$ such that

     $$\left(\sum_{K\in\hat{\mathcal{M}}_h}\eta_K^2\right)^{1/2} \ge \frac{1}{2}\mathcal{E}.$$

   - Solve the discrete problem (6.3) on $\mathcal{M}_h$.
   - Compute the local error estimator η_K on each $K\in\mathcal{M}_h$, the global error estimate $\mathcal{E}$.

end while.

This discrete algebraic system is solved by the MUMPS (MUltifrontal Massively Parallel Sparse direct Solver) [1], [2].

Example 1. Let the scatterer $D = [-0.5, 0.5] \times [-1, 1] \times [-1.5, 1.5]$, $L_1 = 2, L_2 = 3, L_3 = 4$ and $k = 4\pi$. We consider the scattering problem whose exact solution is known as

$$\mathbf{E} = \mathbf{M}_1^0(|\mathbf{x}|, \hat{\mathbf{x}}) = \nabla \times \{\mathbf{x} h_1^{(1)}(|\mathbf{x}|) Y_1^0(\hat{\mathbf{x}})\},$$

where $h_1^{(1)}(|\mathbf{x}|)$ is the spherical Hankel function of the first kind and order one, $Y_1^0(\hat{\mathbf{x}})$ is the zeroth spherical harmonics of order one. In this example we are interested in the accuracy of our adaptive PML method and the influence of different choices of the thickness of the PML layer to the performance of the adaptive PML method. For this purpose we choose different thickness of the layer $d_1 = 4, d_2 = 6, d_3 = 8$, $\theta = 3$, $r_0 = 2$, $c_0 = 13$ or $d_1 = 6, d_2 = 9, d_3 = 12$, $\theta = 4$, $r_0 = 2$, $c_0 = 10$.

Figures 8.2 shows the log$N$-log$||\mathbf{E} - \mathbf{E}_h||_{H(\text{curl})}$ and log$N$-log$\mathcal{E}$ curves with different choices of θ, where N is the number of the degrees of freedom. It indicates clearly that the meshes and the associated numerical compexity are quasi-optimal: $||\mathbf{E} - \mathbf{E}_h||_{H(\text{curl})} \approx CN^{-\frac{1}{3}}$ and $\mathcal{E} \approx CN^{-\frac{1}{3}}$ are valid asymptotically. This figure also shows the total computational costs are insensitive to the choice of the thickness of the PML layer using the adaptive PML method.

Fig 8.3 shows the far fields in the direction (1, 0, 0) when $\theta = 3$.

Example 2. Let the scatterer be the screen $\Sigma = [-0.5, 0.5] \times [-0.5, 0.5] \times \{0\}$. We set the incident wave $E^i = (e^{\mathbf{i}kx_3}, 0, 0)^T$. Let $k = 2\pi$ and take $L_1 = L_2 = 2$, $L_3 = 1$, $d_1 = d_2 = 4$, $d_3 = 2$, $r_0 = 2$ and $c_0 = 13$.

Figure 8.4 indicates that the meshes and the associated numberical compexity are quasi-optimal: $\mathcal{E} \approx CN^{-\frac{1}{3}}$ is valid asymptotically. The adaptive mesh on the $x_3 = 0$ is plotted in Figure 8.5 with 1370291 elements (3237584 DOFs). We observe the mesh is much refined around the scatterer.

Figures 8.6 shows the modulus of the far fields on the $x_1 - x_2$ plan for the different choices of the incident waves. We observe the far fields converge rather fast in our adaptive mesh refinement steps.

9 Appendix: Proof of Lemma 12

We prove the lemma by a constructive argument. For any $z \in \bar{\mathbb{C}}^{++} = \{z : \text{Re}(z) \geq 0, \text{Im}(z) \geq 0\}$ and $\mathbf{x} \in \mathbb{R}^3$, we let $\tilde{\mathbf{x}}_z = \mathbf{F}_z(\mathbf{x}) = \beta_z(r(\mathbf{x}))\mathbf{x}$, where $\beta_z(r(\mathbf{x})) = 1 + z\hat{\sigma}(r(\mathbf{x}))$. Let $\gamma(z)$ be the multivalued analytic function satisfying $\gamma(z)^2 = z$ defined on the Riemann surface corresponding to $\sqrt{z}$. We define the stretched complex distance

$$d(\tilde{\mathbf{x}}_z, \tilde{\mathbf{y}}_z) = \gamma\left[(\tilde{x}_z^1 - \tilde{y}_z^1)^2 + (\tilde{x}_z^2 - \tilde{y}_z^2)^2 + (\tilde{x}_z^3 - \tilde{y}_z^3)^2\right], \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3,$$

where $\tilde{\mathbf{x}}_z = (\tilde{x}_z^1, \tilde{x}_z^2, \tilde{x}_z^3)^T$, $\tilde{\mathbf{y}}_z = (\tilde{y}_z^1, \tilde{y}_z^2, \tilde{y}_z^3)^T$. We require that for $z \in \mathbb{R}$, $d(\tilde{\mathbf{x}}_z, \tilde{\mathbf{y}}_z)$ is on the top sheet of the Riemann surface in which $\text{Re}\gamma(z) \geq 0$. By the argument in the proof of [28, Theorem 2.8] we know that $J_z(\mathbf{y})G_k(\tilde{\mathbf{x}}_z, \tilde{\mathbf{y}}_z)$, where $J_z(\mathbf{y}) = \det(D\mathbf{F}_z(\mathbf{y}))$

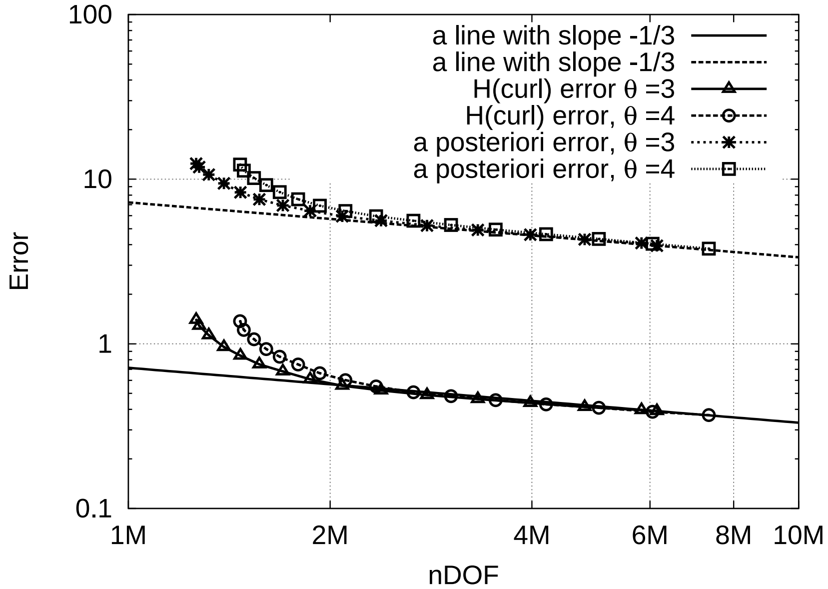

Fig. 8.2 The quasi-optimality of the adaptive mesh refinemensts of the error $||E - E_h||_{H(\text{curl};(\Omega_1))}$ and the *a posteriori* error estimate for Example 1 ($\theta = 3, 4$).

and $G_k(\tilde{\mathbf{x}}_z, \tilde{\mathbf{y}}_z) = \frac{e^{\mathbf{i}kd(\tilde{\mathbf{x}}_z, \tilde{\mathbf{y}}_z)}}{4\pi d(\tilde{\mathbf{x}}_z, \tilde{\mathbf{y}}_z)}$, is the fundamental solution of the stretched Helmholtz equation

$$(\tilde{\Delta}_z + k^2)G_k(\tilde{\mathbf{x}}_z, \tilde{\mathbf{y}}_z) = -\delta(\mathbf{x} - \mathbf{y}), \tag{9.1}$$

where $\tilde{\Delta}_z = J_z^{-1}\text{div}(J_z D\mathbf{F}_z^{-1} D\mathbf{F}_z^{-T}\nabla)$. When $z = \zeta + \mathbf{i}$, we write $\mathbf{F}_{\zeta+\mathbf{i}}(\mathbf{x}) = \mathbf{F}(\mathbf{x})$, $\tilde{\mathbf{x}}_{\zeta+\mathbf{i}} = \tilde{\mathbf{x}}$, and $\tilde{\Delta}_{\zeta+\mathbf{i}} = \tilde{\Delta}$, to be in conform with the notation in section 2.

Lemma 17 *Let (H1)-(H3) be satisfied. We have*

$$|d(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})| \geq C|\mathbf{x} - \mathbf{y}|, \quad -\text{Im}[d(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})] \leq C, \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3. \tag{9.2}$$

Proof. By definition we have $\tilde{x}_i - \tilde{y}_i = a_i + \mathbf{i}b_i$, where

$$a_i = x_i - y_i + \zeta(\hat{\sigma}(r(\mathbf{x}))x_i - \hat{\sigma}(r(\mathbf{y}))y_i), \quad b_i = \hat{\sigma}(r(\mathbf{x}))x_i - \hat{\sigma}(r(\mathbf{y}))y_i.$$

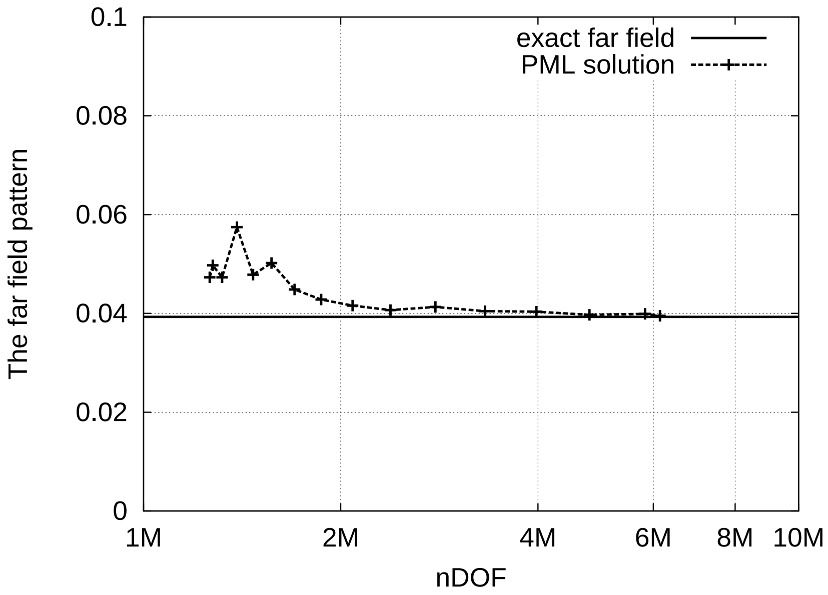

Fig. 8.3 The module of the far fields in the direction (1, 0, 0) for Example 1 ($\theta = 3$).

Simple calculation shows that

$$
\begin{aligned}
|d(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})|^4 &= \left[\sum_{i=1}^{3}(a_i^2 - b_i^2)\right]^2 + 4\left[\sum_{i=1}^{3} a_i b_i\right]^2 \\
&= \sum_{i=1}^{3}(a_i^2 + b_i^2)^2 + 2(a_1a_2 + b_1b_2)^2 + 2(a_1a_3 + b_1b_3)^2 + 2(a_2a_3 + b_2b_3)^2 \\
&\quad - 2(a_1b_2 - a_2b_1)^2 - 2(a_1b_3 - a_3b_1)^2 - 2(a_2b_3 - a_3b_2)^2.
\end{aligned}
$$

It is easy to see by Young's inequality that

$$
a_i^2 + b_i^2 = (x_i - y_i)^2 + (1 + \zeta^2)b_i^2 + 2\zeta(x_i - y_i)b_i \geq \frac{1}{1 + \zeta^2}|x_i - y_i|^2.
$$

On the other hand, for $\mathbf{a} = (a_1, a_2, a_3)^T, \mathbf{b} = (b_1, b_2, b_3)^T$, we have

$$
(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2 = |\mathbf{a} \times \mathbf{b}|^2 = |(\mathbf{x} - \mathbf{y}) \times \mathbf{b}|^2.
$$

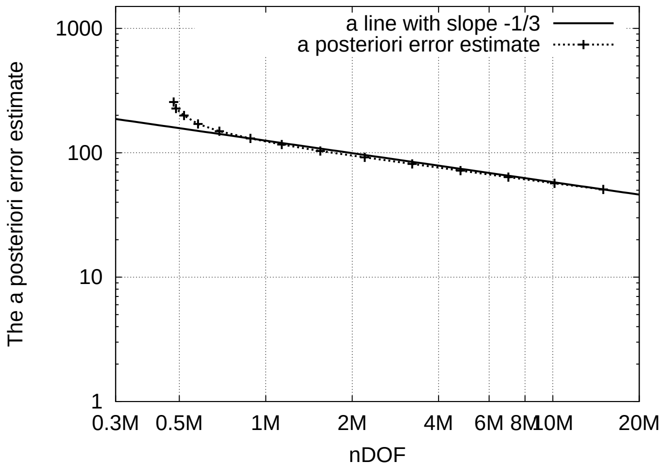

Fig. 8.4 The quasi-optimality of the adaptive mesh refinemensts of the *a posteriori* error estimate for Example 2.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \backslash \bar{\Omega}_{r_0}$ we have $\hat{\sigma}(r(\mathbf{x})) = \hat{\sigma}(r(\mathbf{y})) = \sigma_0$ and consequently $|(\mathbf{x}-\mathbf{y}) \times \mathbf{b}| = 0$. If one of $\mathbf{x}, \mathbf{y}$ is in Ω_{r_0}, without loss of generality, we may assume $\mathbf{y} \in \Omega_{r_0}$, we have

$$\begin{aligned} |(\mathbf{x}-\mathbf{y}) \times \mathbf{b}| &= |(\mathbf{x}-\mathbf{y}) \times (\hat{\sigma}(r(\mathbf{x}))\mathbf{x} - \hat{\sigma}(r(\mathbf{y}))\mathbf{y})| \\ &= |(\mathbf{x}-\mathbf{y}) \times \mathbf{y}(\hat{\sigma}(r(\mathbf{x})) - \hat{\sigma}(r(\mathbf{y})))| \\ &\leq |\mathbf{x}-\mathbf{y}| \cdot r_0 L/2 \cdot \max_{1 \leq t \leq r_0} |\hat{\sigma}'(t)| \|\nabla r\|_{L^\infty(R^3)} |\mathbf{x}-\mathbf{y}|. \end{aligned}$$

From the definition we know that $\|\nabla r\|_{L^\infty(R^3)} \leq \max_{i=1,2,3}(L_i/2)^{-1}$. Now by (H1) we have

$$\begin{aligned} |(\mathbf{x}-\mathbf{y}) \times \mathbf{b}| &\leq r_0 \max_{i=1,2,3}(L/L_i) \max_{1 \leq t \leq r_0} |\hat{\sigma}'(t)| |\mathbf{x}-\mathbf{y}|^2 \\ &\leq r_0 (1+\zeta^2)^{1/2} \max_{1 \leq t \leq r_0} |\hat{\sigma}'(t)| |\mathbf{x}-\mathbf{y}|^2. \end{aligned}$$

Thus by the assumption (H3) we obtain

$$|d(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})|^4 \geq \frac{1}{(1+\zeta^2)^2} |\mathbf{x}-\mathbf{y}|^4 - 2|(\mathbf{x}-\mathbf{y}) \times \mathbf{b}|^2 \geq \frac{1}{2(1+\zeta^2)^2} |\mathbf{x}-\mathbf{y}|^4.$$

This shows the first inequality in (9.2). To show the second estimate in (9.2). We first notice that if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \backslash \bar{\Omega}_{r_0}$, $\mathrm{Im}\left(\sum_{i=1}^3 (\tilde{x}_i - \tilde{y}_i)^2\right) = 2\sum_{i=1}^3 a_i b_i \geq 0$. Thus $\mathrm{Im}[d(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})] \geq 0$. For $\mathbf{y} \in \Omega_{r_0}$, if $|\mathbf{x}| \geq r_0 L$, then $\hat{\sigma}(r(\mathbf{x})) = \sigma_0, \hat{\sigma}(r(\mathbf{y})) \leq \sigma_0, |\mathbf{y}| \leq r_0 L/2$, and thus

$$\sum_{i=1}^3 a_i b_i \geq (\mathbf{x}-\mathbf{y}) \cdot (\hat{\sigma}(r(\mathbf{x}))\mathbf{x} - \hat{\sigma}(r(\mathbf{y}))\mathbf{y}) \geq \sigma_0 |\mathbf{x}|(|\mathbf{x}| - 2|\mathbf{y}|) \geq 0.$$

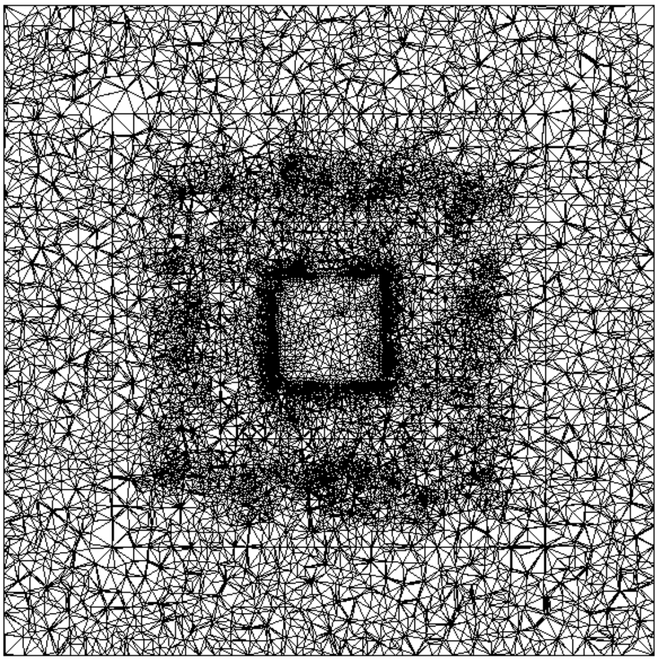

Fig. 8.5 The adaptive mesh on the $x_3 = 0$ plane with 1370291 elements (3237584 DOFs) for Example 2.

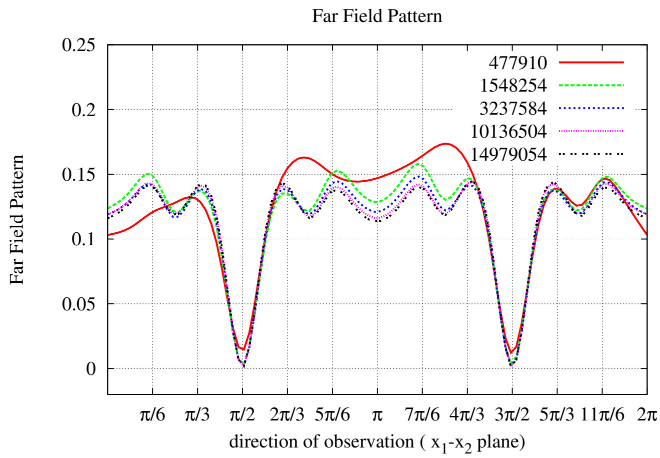

Fig. 8.6 The module of the far fields on the $x_1 - x_2$ plane for Example 2 when $E^i = (e^{\mathbf{i}2\pi x_3}, 0, 0)^T$, $d_1 = d_2 = 4$, $d_3 = 2$.

This implies $\mathrm{Im}[d(\tilde{\mathbf{x}},\tilde{\mathbf{y}})] \ge 0$. For the remaining case of $\mathbf{y} \in \Omega_{r_0}$, $|\mathbf{x}| \le r_0 L$, we obviously have $-\mathrm{Im}[d(\tilde{\mathbf{x}},\tilde{\mathbf{y}})] \le |d(\tilde{\mathbf{x}},\tilde{\mathbf{y}})| \le C$. This completes the proof. □

Now we are in the position to complete the proof Lemma 12.

Proof of Lemma 12. Let $\tilde{H}_0^1(\mathbb{R}^3)$ denote the completion of $C_0^\infty(\mathbb{R}^3)$ in the norm $\|\nabla v\|_{L^2(\mathbb{R}^3)}$. By Lemma 8

$$\mathrm{Re}(A^{-1}(\mathbf{x})\xi,\bar{\xi}) = \mathrm{Re}(A(A^{-1}\xi),\overline{A^{-1}\xi}) \ge C|A^{-1}\xi|^2 \ge C|\xi|^2, \qquad \forall \xi \in \mathbb{C}^3, \mathbf{x} \in \mathbb{R}^3.$$

Thus for any $\mathbf{U} \in L^2(\mathbb{R}^3)^3$ supported in Ω_{r_1}, there exists a function $\phi \in \tilde{H}_0^1(\mathbb{R}^3)$ such that

$$(A^{-1}\nabla\phi,\nabla v)_{\mathbb{R}^3} = (A^{-1}\mathbf{U},\nabla v)_{\mathbb{R}^3}, \qquad \forall v \in \tilde{H}_0^1(\mathbb{R}^3). \tag{9.3}$$

Let $\tilde{\mathbf{U}} = \mathbf{U} - \nabla\phi$, then $\nabla\cdot(A^{-1}\tilde{\mathbf{U}}) = 0$ in $\mathbb{R}^3$ and $\|\tilde{\mathbf{U}}\|_{L^2(\mathbb{R}^3)} \le C\|\mathbf{U}\|_{L^2(\Omega_{r_1})}$. Now we define

$$\mathbf{v}_1(\mathbf{x}) = \int_{\mathbb{R}^3} G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})J(\mathbf{y})B^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y}, \qquad B = D\mathbf{F}^T. \tag{9.4}$$

Since $J(\mathbf{y})G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ is the fundamental solution of the stretched Helmholtz equation, we know that

$$(\tilde{\Delta} + k^2)\mathbf{v}_1 = -B^{-1}\tilde{\mathbf{U}}. \tag{9.5}$$

Moreover, since $\tilde{\nabla}_{\tilde{\mathbf{x}}}G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) = -\tilde{\nabla}_{\tilde{\mathbf{y}}}G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$, we have

$$\begin{aligned}
\tilde{\nabla}\cdot\mathbf{v}_1(\mathbf{x}) &= -\int_{\mathbb{R}^3} \tilde{\nabla}_{\tilde{\mathbf{y}}}G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\cdot J(\mathbf{y})B^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y} \\
&= -\int_{\mathbb{R}^3} D\mathbf{F}^{-T}(\mathbf{y})\nabla_{\mathbf{y}}G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\cdot J(\mathbf{y})B^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y} \\
&= -\int_{\mathbb{R}^3} \nabla_{\mathbf{y}}G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\cdot A^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y}.
\end{aligned}$$

Thus $\tilde{\nabla}\cdot\mathbf{v}_1 = 0$ because $\nabla\cdot(A^{-1}\tilde{\mathbf{U}}) = 0$ and $G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ decays exponentially as $|\mathbf{y}| \to \infty$ for fixed $\mathbf{x}$. Now by the well-known identity $-\tilde{\Delta} = \tilde{\nabla}\times\tilde{\nabla} - \tilde{\nabla}\cdot\tilde{\nabla}$, we obtain from (9.5) that

$$\tilde{\nabla}\times\tilde{\nabla}\times\mathbf{v}_1 - k^2\mathbf{v}_1 = B^{-1}\tilde{\mathbf{U}},$$

which by (2.16) is equivalent to

$$\nabla\times A\nabla\times(B\mathbf{v}_1) - k^2A^{-1}(B\mathbf{v}_1) = A^{-1}\tilde{\mathbf{U}}.$$

This shows that $\mathbf{v} = B\mathbf{v}_1 - \nabla\phi$ satisfies the equation (4.9).

Now we estimate $\|\mathbf{v}\|_{H(\mathrm{curl};\mathbb{R}^3)}$. By (9.3) we have $\|\nabla\phi\|_{L^2(\mathbb{R}^3)} \le C\|\mathbf{U}\|_{L^2(\Omega_{r_1})}$, which yields

$$\|\mathbf{v}\|_{H(\mathrm{curl},\mathbb{R}^3)} \le \|B\mathbf{v}_1\|_{H(\mathrm{curl},\mathbb{R}^3)} + C\|\mathbf{U}\|_{L^2(\Omega_{r_1})} \le C\|\mathbf{v}_1\|_{H^1(\mathbb{R}^3)} + C\|\mathbf{U}\|_{L^2(\Omega_{r_1})}.$$

It is clear that

$$\begin{aligned}
\|\mathbf{v}_1\|_{H^1(\mathbb{R}^3\setminus\bar{\Omega}_{r_1})} \le{} & \left\|\int_{\Omega_{r_0}} G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})J(\mathbf{y})B^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y}\right\|_{H^1(\mathbb{R}^3\setminus\bar{\Omega}_{r_1})} \\
&+ \left\|\int_{\mathbb{R}^3\setminus\bar{\Omega}_{r_0}} G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})J(\mathbf{y})B^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y}\right\|_{H^1(\mathbb{R}^3\setminus\bar{\Omega}_{r_1})}
\end{aligned}$$

Since $G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ decays exponentially as $|\mathbf{x}|\to\infty$ for $\mathbf{y}\in\Omega_{r_0}$, we have

$$\left\|\int_{\Omega_{r_0}} G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})J(\mathbf{y})B^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y}\right\|_{H^1(\mathbb{R}^3\backslash\bar{\Omega}_{r_1})} \le C\|\tilde{\mathbf{U}}\|_{L^2(\Omega_{r_0})}.$$

Notice that for $\mathbf{x}\in\mathbb{R}^3\backslash\bar{\Omega}_{r_1}$ and $\mathbf{y}\in\mathbb{R}^3\backslash\bar{\Omega}_{r_0}$, $\hat{\sigma}(r(\mathbf{x}))=\hat{\sigma}(r(\mathbf{y}))=\sigma_0$, by Lemma 2 we have $\mathrm{Im}[d(\tilde{\mathbf{x}},\tilde{\mathbf{y}})]\ge\sigma_0|\mathbf{x}-\mathbf{y}|$, and consequently

$$\begin{aligned}&\left\|\int_{\mathbb{R}^3\backslash\bar{\Omega}_{r_0}} G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})J(\mathbf{y})B^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y}\right\|_{H^1(\mathbb{R}^3\backslash\bar{\Omega}_{r_1})}\\ &\le C\left\|\int_{\mathbb{R}^3\backslash\bar{\Omega}_{r_0}}\frac{e^{-k\sigma_0|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}|\tilde{\mathbf{U}}(\mathbf{y})|d\mathbf{y}\right\|_{H^1(\mathbb{R}^3\backslash\bar{\Omega}_{r_1})}.\end{aligned}$$

Denote by $h_1(\mathbf{x},\mathbf{y})=e^{-k\sigma_0|\mathbf{x}-\mathbf{y}|}(|\mathbf{x}-\mathbf{y}|^{-1}+|\mathbf{x}-\mathbf{y}|^{-2})$. By Cauchy-Schwarz inequality we have

$$\begin{aligned}\left\|\int_{\mathbb{R}^3}\frac{e^{-k\sigma_0|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}|\tilde{\mathbf{U}}(\mathbf{y})|d\mathbf{y}\right\|_{H^1(\mathbb{R}^3)} &\le C\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}h_1(\mathbf{x},\mathbf{y})|\tilde{\mathbf{U}}(\mathbf{y})|d\mathbf{y}\right|^2 d\mathbf{x}\\ &\le C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}h_1(\mathbf{x},\mathbf{y})|\tilde{\mathbf{U}}(\mathbf{y})|^2d\mathbf{y}d\mathbf{x}\cdot\int_{\mathbb{R}^3}h_1(\mathbf{x},\mathbf{y})d\mathbf{y}\\ &\le C\|\tilde{\mathbf{U}}\|_{L^2(\mathbb{R}^3)}.\end{aligned}\tag{9.6}$$

Thus we have $\|\mathbf{v}_1\|_{H^1(\mathbb{R}^3\backslash\bar{\Omega}_{r_1})}\le C\|\tilde{\mathbf{U}}\|_{L^2(\mathbb{R}^3)}$. To estimate $\|\mathbf{v}_1\|_{H^1(\Omega_{r_1})}$, we split the integration in (9.4) in two domains Ω_{2r_1} and $\mathbb{R}^3\backslash\bar{\Omega}_{2r_1}$. Since $G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ decays exponentially as $|\mathbf{y}|\to\infty$ for $\mathbf{x}\in\Omega_{r_1}$, we have

$$\left\|\int_{\mathbb{R}^3\backslash\bar{\Omega}_{2r_1}} G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})J(\mathbf{y})B^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y}\right\|_{H^1(\Omega_{r_1})}\le C\|\tilde{\mathbf{U}}\|_{L^2(\mathbb{R}^3)}.$$

For the integral in Ω_{2r_1}, we first note that since $r(\mathbf{x})$ is Lipschitz continuous, $|\tilde{x}_i-\tilde{y}_i|\le C|\mathbf{x}-\mathbf{y}|$. Thus the first estimate in (9.2) implies that $|\tilde{x}_i-\tilde{y}_i|/|d(\tilde{\mathbf{x}},\tilde{\mathbf{y}})|\le C$. By the second estimate in (9.2) we have $|e^{\mathbf{i}kd(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}|\le C$. Thus $|\partial G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})/\partial x_i|\le Ch_2(\mathbf{x},\mathbf{y})$ for any $\mathbf{x},\mathbf{y}\in\mathbb{R}^3$, where $h_2(\mathbf{x},\mathbf{y})=|\mathbf{x}-\mathbf{y}|^{-1}+|\mathbf{x}-\mathbf{y}|^{-2}$. Now it is easy to see that

$$\begin{aligned}\left\|\int_{\Omega_{2r_1}} G_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}})J(\mathbf{y})B^{-1}(\mathbf{y})\tilde{\mathbf{U}}(\mathbf{y})d\mathbf{y}\right\|_{H^1(\Omega_{r_1})} &\le C\left\|\int_{\Omega_{2r_1}}h_2(\mathbf{x},\mathbf{y})|\tilde{\mathbf{U}}(\mathbf{y})|d\mathbf{y}\right\|_{L^2(\Omega_{2r_1})}\\ &\le C\|\tilde{\mathbf{U}}\|_{L^2(\Omega_{2r_1})},\end{aligned}$$

where we have used the similar argument in (9.6) in the last inequality. This shows $\|\mathbf{v}_1\|_{H^1(\Omega_{r_1})}\le C\|\tilde{\mathbf{U}}\|_{L^2(\mathbb{R}^3)}\le C\|\mathbf{U}\|_{L^2(\Omega_{r_1})}$ and completes the proof. □

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