Upscaling of well singularities in the flow transport through heterogeneous porous media

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Abstract. We consider the steady flow transport through highly heterogeneous porous media driven by extraction wells. We develop a new upscaling technique which lumps the small-scale details of the medium property into a few representative macroscopic parameters on a coarse scale that preserve the large-scale behavior of the medium and are more appropriate for simulations. The method is based on the recently introduced over-sampling multiscale finite element method and the introduction of new base functions that locally resolve the well singularities. Numerical experiments are carried out for flow transport in both periodic and randomly generated log-normal permeabilities to demonstrate the efficiency and accuracy of the proposed method.

1. Introduction

The central difficulty in the modeling of subsurface flow and transport is the accounting for the spatial variability in the parameters used to characterize the relevant physical properties of the natural porous media. In realistic situations, the precise spatial distribution of the parameters required to characterize the problem is never available due to the lack of enough data. Thus sophisticated geological and geostatistical modeling tools are used in practice to generate highly detailed medium parameters based on some site-specific measurements and experience from other sites. There exists a vast literature on the upscaling or homogenization techniques that lump the small-scale details of the medium into a few representative macroscopic parameters on a coarse scale which preserve the large-scale behavior of the medium and are more appropriate for reservoir simulations. We refer to the book of Christakos [3] for more information on the random field modeling of the natural porous medium parameters and the recent review paper [15] on the existent upscaling techniques in the engineering literature.

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The study of steady flow through highly heterogeneous porous medium driven by extraction wells is of great importance in hydrology, petroleum reservoir engineering. It is observed in the engineering literature (cf. e.g. [5] and [12] and the references therein) that in the near-well region, many of the existing upscaling methods do not provide satisfactory results. The reason may be explained as the standard upscaling methods generally assume the pressure field is slowly varying, that is clearly not true in the vicinity of the flowing wells [5]. This fact may also be explained mathematically from the homogenization theory for the second order elliptic equations with periodic coefficients. In the homogenization theory, multiscale convergence is ensured under the assumption that the source should be at least in H^{-1} so that the solution is bounded in Sobolev space H^1. As we will see below, however, well singularities can be modeled as Dirac sources and thus in the near-well region, the solution behaves like Green function which is not uniformly bounded in H^1.

The recently introduced multiscale finite element method [8, 9] for solving elliptic equations with oscillating coefficients provides an effective way to capture the large scale structures of the solutions on a coarse mesh without resolving all the fine scale structures. The central idea of the method is to incorporate the local small scale information of the leading order differential operator into the finite element bases. It is through these multiscale bases and the finite element formulation that the effect of small scales on the large scales is correctly captured. In section 2 we will describe one engineering upscaling technique and discuss its relation with the multiscale finite element method.

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with Lipschitz boundary Γ. $B(x_0, r)$ will be denoted as the disk centered at x_0 with radius $r > 0$. Let $B_j = B(\bar{x}_j, \delta), 1 \le j \le N$, be mutually disjoint subdomains inside Ω that are occupied by the wells. Denote by $\Omega_\delta = \Omega \backslash (\cup_{j=1}^N \bar{B}_j)$. We consider the following single phase pressure equation which is formed by combining Darcy's law with the conservation of mass

$$-\mathrm{div}\,(K(x)\nabla u_\delta) = 0 \quad \text{ in } \Omega_\delta, \tag{1}$$

where u_δ is pressure, K is the permeability which is typically highly variable in space. We will impose homogeneous Dirichlet boundary condition $u_\delta|_\Gamma = 0$ on the exterior boundary. The other types of boundary conditions can be treated similarly without any essential difficulties. On the well boundary $\Gamma_j = \partial B_j$, two quantities are of particular importance in practical applications: the well bore pressure $u_\delta|_{\Gamma_j}$ and the well flow rate $\int_{\Gamma_j} K \frac{\partial u_\delta}{\partial \nu} ds$, where ν is the unit outer normal to $\partial\Omega_\delta$. The boundary condition to be imposed on Γ_j is either the Dirichlet boundary condition which fixes the well bore pressure α_j (assume to be constant)

$$u_\delta|_{\Gamma_j} = \alpha_j, \tag{2}$$

or the following mixed boundary condition which fixes the well flow rate q_j

$$u_\delta|_{\Gamma_j} = c_j = \text{const}\,, \qquad \int_{\Gamma_j} K \frac{\partial u_\delta}{\partial \nu} ds = \mathrm{q}_j. \tag{3}$$

The constant c_j in (3) is unknown.

The purpose of this paper is to develop a complete multiscale coarse grid algorithm for solving steady flow problem involving well singularities in heterogeneous porous medium based on the multiscale finite element method. The additional well singularities of the problem are resolved locally by adding finite element base functions. The final coarse grid model is based on a variational formulation which is different from the heuristic techniques in [5, 12]. This rigorous variational coarse grid model allows us to establish a complete multiscale error analysis for locally periodic medium properties based on the homogenization theory. We emphasize that as pointed out in [6], the spatial periodicity assumption does not a priori restrict the applicability of the results only to media which do exhibit such strict repetitive spatial ordering in the properties of interest. The numerical experiments carried out for random log-normal relative permeabilities demonstrate clearly the applicability of our method beyond the periodic structures.

Since the size of the wells δ is negligible in practical situations, the first approximation to be made is to replace (1)-(3) by the following problem

$$-\mathrm{div}\,(K(x)\nabla u) = \sum_{j=1}^{N} \mathrm{q}_j \delta_{\bar{x}_j} \quad \text{ in } \Omega \tag{4}$$

with the boundary condition $u|_\Gamma = 0$. Here $\delta_{\bar{x}_j}$ is the Dirac measure at $\bar{x}_j$, $1 \le j \le N$. Denote by $K_j = K(\bar{x}_j)$, $\phi_j = -\dfrac{1}{2\pi K_j} \ln|x-\bar{x}_j|$ and $U = u - \sum_{j=1}^N \mathrm{q}_j\phi_j$. Note that the flow rates q_i for $i \in I_D$ in (4) are unknown. They are determined through the following additional conditions which are obtained by requiring $u|_{\Gamma_i} \approx \alpha_i$ as the approximation of the boundary conditions (2):

$$-\frac{\mathrm{q}_i}{2\pi K_i} \ln\delta + \sum_{j\ne i} \mathrm{q}_j\phi_j(\bar{x}_i) + U(\bar{x}_i) = \alpha_i, \qquad i \in I_D. \tag{5}$$

We derive the following error estimate between the solution u_δ of the original problem (1)-(3) and the solution u of (4)-(5)

$$\max_{x\in\bar{\Omega}_\delta} |u-u_\delta| \le C\delta|\ln\delta| \sum_{j=1}^{N} |\mathrm{q}_j|.$$

To the best of our knowledge, this error estimate is the first in the literature although the convergence of the solution u_δ to u in certain sense as $\delta \to 0$ is well-known (cf. e.g. [10]). Instead of presenting the full proof of the estimate which can be found in [2], we will present in section 3 the key ideas of the proof in a simplified situation of single well and constant permeability. In this case, the proof is more transparent and the estimate is sharper (without the logarithmic factors).

To introduce our multiscale algorithm, we first introduce an equivalent variational formulation for (4) which is suitable for our multiscale approximation. Let $\Omega_j, 1 \le j \le N$, be mutually disjoint subdomains inside Ω such that $\bar{B}_j \subset\subset \Omega_j$.

Let G_j be the Green function associated with the domain Ω_j

$$-\mathrm{div}\,(K(x)\nabla G_j) = \delta_{\bar{x}_j} \quad \text{in } \Omega_j, \quad G_j|_{\Sigma_j} = 0, \tag{6}$$

where $\Sigma_j = \partial\Omega_j$. Now for any $v \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} v(\bar{x}_j) &= \int_{\Omega_j} -\mathrm{div}\,(K(x)\nabla G_j) v dx \\ &= \int_{\Omega_j} K(x)\nabla G_j \nabla v dx - \int_{\Sigma_j} K \frac{\partial G_j}{\partial \nu} v ds. \end{aligned}$$

On the other hand, from (4) we know that

$$\sum_{j=1}^{N} \mathrm{q}_j v(\bar{x}_j) = \int_\Omega K(x)\nabla u \nabla v dx.$$

Let $G_j = 0$ for $x \in \Omega \backslash \bar{\Omega}_j$ and $\zeta = u - \sum_{j=1}^{N} \mathrm{q}_j G_j$, then we know that $\zeta \in H_0^1(\Omega)$ satisfies the following variational form

$$\int_\Omega K(x)\nabla\zeta\nabla v dx = -\sum_{j=1}^{N} \mathrm{q}_j \int_{\Sigma_j} K\frac{\partial G_j}{\partial \nu} v dx \quad \forall v \in C_0^\infty(\Omega). \tag{7}$$

Denote by $w_j = G_j + \dfrac{1}{2\pi K_j} \ln|x - \bar{x}_j|$ in Ω_j, then the complementary condition (5) to determine q_i for $i \in I_D$ becomes

$$-\frac{\mathrm{q}_i}{2\pi K_i} \ln\delta + \mathrm{q}_i w_i(\bar{x}_i) + \zeta(\bar{x}_i) = \alpha_i, \quad i \in I_D. \tag{8}$$

We note that for ζ the singularities of the original solution u are removed and we will use the over-sampling multiscale finite element method to discretize it on a coarse grid.

In section 5, we will discuss the implementation of our algorithm and show some numerical results for randomly generated coefficients to demonstrate the accuracy of the method.

2. Upscaling of the permeability

The purpose of this section is to show that one of the well-known engineering upscaling techniques (see e.g. [11]) is equivalent to the multiscale finite element method proposed in [7, 8]. We remark that multiscale finite element method is shown to be convergent under the condition that the permeability is locally periodic $K_\varepsilon(x) = K(x, x/\varepsilon)$, where $K(x,\cdot)$ is periodic with respect to the second variable. As a consequence, the convergence of the engineering approach described in this section is guaranteed.

Let $\mathcal{M}_H$ be a finite element mesh of Ω with the mesh size H much larger than the ε, the characteristic length representing the small scale variability of the media. Usually, ε is equal to the correlation length in the statistical random field modeling of the media. Let W_H be the standard conforming linear finite element space over $\mathcal{M}_H$ and $W_H^0 = W_H \cap H_0^1(\Omega)$. In the engineering literature, the problem

$$-\mathrm{div}\,(K_\varepsilon(x)\nabla u_\varepsilon) = f \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \Gamma \tag{9}$$

is approximated by the homogenized or upscaled problem: Find $u_H^* \in W_H^0$ such that

$$\int_\Omega K^*(x)\nabla u_H^* \nabla v_H dx = \int_\Omega f v_H dx \quad \forall v_H \in W_H^0 \tag{10}$$

with K^* being piecewise constant on the coarse mesh $\mathcal{M}_H$. The so-called effective permeability matrix K^* on each $T \in \mathcal{M}_H$ is defined as follows. For any $G \in \mathbf{R}^2$, let θ_ε be the solution of the problem

$$-\mathrm{div}\,(K_\varepsilon \nabla\theta_\varepsilon) = 0 \text{ in } T, \quad \theta_\varepsilon|_{\partial T} = G \cdot x.$$

Simple integration by parts implies that

$$\overline{\nabla\theta_\varepsilon} = \frac{1}{|T|}\int_T \nabla\theta_\varepsilon dx = \frac{1}{|T|}\int_{\partial T} Gx \cdot \nu ds = \frac{1}{|T|}\int_T \nabla(Gx)dx = G. \tag{11}$$

On the other hand, since $Q = \overline{K_\varepsilon\nabla\theta_\varepsilon} = \frac{1}{|T|}\int_T K_\varepsilon\nabla\theta_\varepsilon dx$ is linear in G, there exists a matrix K^* such that

$$Q = K^* \cdot G \quad \Longleftrightarrow \quad -\overline{K_\varepsilon\nabla\theta_\varepsilon} = -K^*\overline{\nabla\theta_\varepsilon}. \tag{12}$$

The multiscale finite element method introduced in [7, 8] is based on multiscale finite element base functions. For any $T \in \mathcal{M}_H$, let ψ_i, $i = 1, 2, 3$, be the linear nodal bases. Define ϕ_i^T, $i = 1, 2, 3$, as the solution of the local problem

$$-\mathrm{div}\,(K_\varepsilon\nabla\phi_i^T) = 0 \ \text{ in } T, \quad \phi_i^T|_{\partial T} = \psi_i.$$

Denote by $V(T) = \mathrm{span}\,\{\phi_i^T, i = 1, 2, 3\}$ and V_H the finite element space

$$V_H = \{v_H \in H^1(\Omega) : v_H|_T \in V(T)\}.$$

Then the multiscale finite element approximation to (9) is: Find $u_H \in V_H^0 = V_H \cap H_0^1(\Omega)$ such that

$$\int_\Omega K_\varepsilon \nabla u_H \nabla v_H dx = \int_\Omega f v_H dx \quad \forall v_H \in V_H^0. \tag{13}$$

We have the following proposition concerning the relationship of these two upscaling methods.

Proposition 2.1. *Let A^* and A be the stiffness matrices corresponding to the discrete problems* (10) *and* (13)*, respectively. Then we have $A^* = A$.*

Proof. It is clear that we only need to prove the element stiffness matrices are the same, that is, for any $T \in \mathcal{M}_T$ and $i, j = 1, 2, 3$,

$$\int_T K^*\nabla\psi_i\nabla\psi_j dx = \int_T K_\varepsilon\nabla\phi_i^T\nabla\phi_j^T dx.$$

To do that, we note that by (11), we have $\overline{\nabla\phi_i^T} = \nabla\psi_i$. Since $\nabla\psi_i, \nabla\psi_j$ are constant over T, we get by (12) that

$$\int_T K^*\nabla\psi_i\nabla\psi_j dx = |T|K^*\overline{\nabla\phi_i^T}\nabla\psi_j = \int_T K_\varepsilon\nabla\phi_i^T\nabla\psi_j dx,$$

which, after integration by parts and using the definition of the multiscale base functions, yields

$$\begin{aligned}\int_T K^*\nabla\psi_i\nabla\psi_j dx = \int_{\partial T}(K_\varepsilon\nabla\phi_i^T\cdot\nu)\psi_j ds &= \int_{\partial T}(K_\varepsilon\nabla\phi_i^T\cdot\nu)\phi_j^T ds\\ &= \int_T K_\varepsilon\nabla\phi_i^T\nabla\phi_j^T dx.\end{aligned}$$

This completes the proof. □

3. The mathematical modeling

Let Ω be a convex polygon in $\mathbf{R}^2$ with boundary Γ. Let $B(x_0,r) = \{x \in \mathbf{R}^2 : |x-x_0| < r\}$. Let $B_j = B(\bar{x}_j,\delta), 1 \le j \le N$, be mutually disjoint subdomains inside Ω that are occupied by the wells. Set $\Gamma_j = \partial B_j$ and $\Omega_\delta = \Omega\backslash(\cup_{j=1}^N \bar{B}_j)$. We consider the following problem

$$-\mathrm{div}\,(K(x)\nabla u_\delta) = 0 \quad \text{in } \Omega_\delta, \tag{14}$$

$$u_\delta|_{\Gamma_j} = \text{const}\,, \quad \int_{\Gamma_j} K\frac{\partial u_\delta}{\partial\nu}ds = \mathrm{q}_j \quad \text{for } j \in I_M, \tag{15}$$

$$u_\delta|_{\Gamma_j} = \alpha_j \quad \text{for } j \in I_D, \quad u_\delta|_\Gamma = 0. \tag{16}$$

Here I_D, I_M are index sets such that $I_D \cup I_M = \{1, 2, \cdots, N\}$, α_j for $j \in I_D$ and q_i for $i \in I_M$ are known constants. We imposed the following conditions on the coefficient $K(x)$:

(H1) $K \in C^{0,1}(\bar{\Omega})$, there exist constants $\lambda, \Lambda > 0$ such that $\lambda \le K(x) \le \lambda^{-1}$ and $|K(x) - K(y)| \le \Lambda|x-y|$ for any $x, y \in \bar{\Omega}$. As we are interested in highly variable coefficients, without lost of generality, we assume $\Lambda \ge 1$.

As already indicated in §1, we will consider the following approximation of the problem (14)-(16): Find u and the constants $\{\mathrm{q}_i\}_{i\in I_D}$, such that

$$-\mathrm{div}\,(K(x)\nabla u) = \sum_{j=1}^N \mathrm{q}_j\delta_{\bar{x}_j} \quad \text{in } \Omega, \tag{17}$$

$$-\frac{\mathrm{q}_i}{2\pi K_i}\ln\delta + \sum_{j\ne i}\mathrm{q}_j\phi_j(\bar{x}_i) + U(\bar{x}_i) = \alpha_i \quad \text{for } i \in I_D, \tag{18}$$

$$u|_\Gamma = 0. \tag{19}$$

Here $\delta_{\bar{x}_j}$ is the Dirac measure at $\bar{x}_j$, $K_j = K(\bar{x}_j)$, $\phi_j(x) = -\dfrac{1}{2\pi K_j}\ln|x-\bar{x}_j|$, and $U = u - \sum_{j=1}^N \mathrm{q}_j\phi_j$. The relation (18) can be viewed as an approximation of the boundary condition $u|_{\Gamma_i} \approx \alpha_i$ for $i \in I_D$.

The following error estimate is derived in [2].

Theorem 3.1. *Let the assumption (H1) be satisfied. Let u_δ and u be the solutions of* (14)-(16) *and* (17)-(19)*, respectively. Then there exists a constant $\gamma_0 \in (0,1)$ such that for $\delta\Lambda \le \gamma_0$, the problem* (17)-(19) *has a unique solution. Moreover, there exists a constant C independent of δ, q_j and Λ such that*

$$\max_{x\in\bar{\Omega}_\delta}|u-u_\delta| \le C\Lambda(1+\ln\Lambda)^{1/2}\Big(\sum_{j=1}^N|\mathrm{q}_j|\Big)\delta|\ln\delta|,$$

and, for any $i \in I_D$,

$$|\mathrm{q}_i - \mathrm{q}_i^\delta| \le C\Lambda(1+\ln\Lambda)^{1/2}\Big(\sum_{j=1}^N|\mathrm{q}_j|\Big)\delta|\ln\delta|,$$

where

$$\mathrm{q}_i^\delta = \int_{\Gamma_i} K_\varepsilon\frac{\partial u_\delta}{\partial\nu}ds \quad \textit{for } i \in I_D.$$

The proof of this theorem is rather complicated. It depends on a sharp estimate between the well flow rate and the well bore pressure and an explicit dependence on p of the constant in the classical $W^{1,p}$ interior estimate for the second order elliptic equations. Instead of presenting the full proof of the estimate which can be found in [2], we will discuss in the following a simplified situation of the single well flow in media with constant permeability.

The starting point of the proof is the following basic lemma.

Lemma 3.2. *Let $\Omega_\delta = \Omega\backslash\bar{B}(x_0,\delta)$, $\Gamma_\delta = \partial B(x_0,\delta)$ and $\Gamma = \partial\Omega$. Consider the problem:*

$$-\Delta v = 0 \;\text{ in } \Omega_\delta, \quad v = 1 \;\text{ on } \Gamma_\delta, \quad v = 0 \;\text{ on } \Gamma.$$

Then

$$\frac{2\pi}{\ln D_0/\delta} \le \int_{\Gamma_\delta}\frac{\partial v}{\partial\nu}ds \le \frac{2\pi}{\ln d_0/\delta},$$

where $d_0 = \min_{x\in\Gamma}|x-x_0|$ and $D_0 = \max_{x\in\Gamma}|x-x_0|$.

Proof. Let $r = |x-x_0|$. Consider the following barrier functions

$$v_\delta^+ = 1 - \frac{\ln(r/\delta)}{\ln(D_0/\delta)}, v_\delta^- = 1 - \frac{\ln(r/\delta)}{\ln(d_0/\delta)}.$$

Let $w_\delta^\pm = v_\delta^\pm - v$. Then it is clear that $-\Delta w_\delta^\pm = 0$ in Ω_δ, $w_\delta^\pm = 0$ on Γ_δ, and $w_\delta^+ \ge 0, w_\delta^- \le 0$ on Γ. Thus by maximum principle, we get $w_\delta^+ \ge 0$, $w_\delta^- \le 0$ in

Ω_δ. This yields $v_\delta^- \le v \le v_\delta^+$ in Ω_δ. By the definition of $v_\delta^\pm$, we obtain

$$-\frac{\ln r - \ln\delta}{r-\delta}\cdot\frac{1}{\ln(d_0/\delta)} \le \frac{v-1}{r-\delta} \le -\frac{\ln r - \ln\delta}{r-\delta}\cdot\frac{1}{\ln(D_0/\delta)} \quad \text{in } \Omega_\delta.$$

By letting $r\to\delta^+$, we get the desired estimate. □

A direct consequence of the basic lemma is the following corollary.

Corollary 3.3. *Consider the problem:*

$$-\Delta\zeta = 0 \quad \textit{in } \Omega_\delta, \quad \zeta|_\Gamma = 0, \quad \zeta|_{\Gamma_\delta} = \beta_\delta, \quad \int_{\Gamma_\delta}\frac{\partial\zeta}{\partial\nu}ds = \mathrm{q} > 0.$$

Then we have

$$\frac{\mathrm{q}}{2\pi}\ln\Big(\frac{d_0}{\delta}\Big) \le \beta_\delta \le \frac{\mathrm{q}}{2\pi}\ln\Big(\frac{D_0}{\delta}\Big).$$

Now we consider the problem with single well and constant permeability

$$-\Delta u_\delta = 0 \ \text{ in } \Omega_\delta, \quad u_\delta|_\Gamma = 0, \quad u_\delta|_{\Gamma_\delta} = \alpha_\delta, \quad \int_{\Gamma_\delta}\frac{\partial u_\delta}{\partial\nu}ds = \mathrm{q}, \tag{20}$$

and its limit problem

$$-\Delta u = \mathrm{q}\delta_{x_0} \ \text{ in } \Omega_\delta, \quad u|_\Gamma = 0. \tag{21}$$

Let $u = -\frac{\mathrm{q}}{2\pi}\ln|x-x_0| + w$ and $v_\delta = u_\delta - u$. Then we have

$$\begin{aligned} &-\Delta v_\delta = 0 \ \text{ in } \Omega_\delta, \quad v_\delta|_\Gamma = 0 \\ &v_\delta|_{\Gamma_\delta} = \alpha_\delta + \frac{\mathrm{q}}{2\pi}\ln\delta - w|_{\Gamma_\delta}, \quad \int_{\Gamma_\delta}\frac{\partial v_\delta}{\partial\nu}ds = 0. \end{aligned}$$

The problem now is reduced to estimate $\| v_\delta \|_{0,\infty,\Omega_\delta}$. Toward that goal, we first split v_δ into two parts: $v_\delta = v_1 + v_2$, where v_1 and v_2 satisfy

$$-\Delta v_1 = 0 \ \text{ in } \Omega_\delta, \quad v_1|_\Gamma = 0, \quad v_1 = w(x_0) - w \ \text{ on } \Gamma_\delta.$$

and

$$\begin{aligned} &-\Delta v_2 = 0 \ \text{ in } \Omega_\delta, \quad v_2|_\Gamma = 0, \\ &v_2|_{\Gamma_\delta} = \alpha_\delta + \frac{\mathrm{q}}{2\pi}\ln\delta - w(x_0) =: \alpha_\delta', \quad \int_{\Gamma_\delta}\frac{\partial v_2}{\partial\nu}ds = -\int_{\Gamma_\delta}\frac{\partial v_1}{\partial\nu}ds. \end{aligned}$$

Since w is smooth around x_0, it is easy to see that $|w - w(x_0)| \le C_0\delta$ on Γ_δ, which, in turn, implies that $|v_1| \le C_0\delta$ in Ω_δ.

Let $\theta = C_0\delta v \pm v_1$, then $-\Delta\theta = 0$ in Ω_δ, $\theta = 0$ on Γ and $\theta \ge 0$ on Γ_δ. Thus $\theta \ge 0$ in Ω_δ and $\frac{\partial\theta}{\partial\nu} \le 0$ on Γ. Therefore

$$\begin{aligned} \Big|\int_{\Gamma_\delta}\frac{\partial v_1}{\partial\nu}ds\Big| &= \Big|\int_\Gamma\frac{\partial v_1}{\partial\nu}ds\Big| \le C_0\delta\Big|\int_\Gamma\frac{\partial v}{\partial\nu}ds\Big| \\ &= C_0\delta\Big|\int_{\Gamma_\delta}\frac{\partial v}{\partial\nu}ds\Big| \le \frac{2\pi C_0\delta}{\ln(d_0/\delta)}. \end{aligned}$$

Now by the third equality in the definition of v_2, we know that

$$\Big|\int_{\Gamma_\delta}\frac{\partial v_2}{\partial\nu}ds\Big|\le\frac{2\pi C_0\delta}{\ln(d_0/\delta)},$$

which implies by using Corollary 3.3 that

$$|\alpha'_\delta|\le\frac{1}{2\pi}\frac{2\pi C_0\delta}{\ln(d_0/\delta)}\ln\Big(\frac{D_0}{\delta}\Big)\le C\delta$$

A final application of maximum principle gives $|v_2|\le C\delta$. This proves the following theorem.

Theorem 3.4. *Let u_δ and u be the solution of* (20) *and* (21)*, respectively. Then there exists a constant C independent of δ such that the following estimate is valid*

$$\max_{x\in\bar\Omega_\delta}|u_\delta-u|\le C\delta.$$

4. The multiscale method

In this section we are going to introduce our multiscale method to solve the problem (17)-(19) which is a good approximation of the original problem (14)-(16) for small δ according to the analysis in last section. We make the following local periodicity assumption for the coefficient:

(H2) $K(x)=K_\varepsilon(x)=K(x,x/\varepsilon)$ for some small parameter $\varepsilon>0$, we assume $K\in C^1(\bar\Omega;C^1_p(\mathbf{R}^2))$, where $C^1_p(\mathbf{R}^2)$ stands for the collection of all $C^1(\mathbf{R}^2)$ periodic functions with respect to the unit square Y.

Our method is based on the over-sampling multiscale finite element method in [7] which reduces the so-called resonance error in the previous multiscale finite element method in [8, 9]. We first briefly recall the basic facts of the over-sampling multiscale finite element method. Let $\mathcal{M}_H$ be a regular and quasi-uniform triangulation of Ω and V_H the standard conforming linear finite element space over $\mathcal{M}_H$. For any $T\in\mathcal{M}_H$ with nodes $\{x_i^T\}_{i=1}^3$, let H_T denote the size of T, $P_1(T)$ the set of linear polynomials defined in T, and $\{\varphi_i^T\}_{i=1}^3$the basis of $P_1(T)$ satisfying $\varphi_i^T(x_j^T)=\delta_{ij},i,j=1,2,3$. For any $T\in\mathcal{M}_H$, we denote by $S=S(T)$ a macro-element which contains T and satisfies the following condition:

(H3) $H_S\le C_1H_T$ and $\operatorname{dist}(\partial T,\partial S)\ge\delta_0H_T$ for some positive constants C_1, δ_0 independent of H. The minimum angles of $S(T)$ is bounded below by some positive constant θ_0 independent of H.

Let $MS(S)$ be the multiscale finite element space spanned by $\psi_i^S, i=1,2,3$, with $\psi_i^S\in H^1(S)$ being the solution of the problem

$$-\operatorname{div}(K_\varepsilon(x)\nabla\psi_i^S)=0\quad\text{in } S,\qquad \psi_i^S|_{\partial S}=\varphi_i^S.$$

Here $\{\varphi_i^S\}_{i=1}^3$ is the nodal basis of $P_1(S)$ such that $\varphi_i^S(x_j^S)=\delta_{ij}, i,j=1,2,3$. The over-sampling multiscale finite element base functions over T is defined by

$$\bar\psi_i^T=c_{ij}^T\psi_j^S|_T\quad\text{in } T,$$

with the constants c_{ij}^T so chosen that

$$\varphi_i^T = c_{ij}^T \varphi_j^S|_T \quad \text{ in } T.$$

The existence of the constants c_{ij}^T is guaranteed because $\{\varphi_j^S\}_{j=1}^3$ also forms the basis of $P_1(T)$. In practical applications, the base functions ψ_i^S of $MS(S)$ will be further approximated by solving (22) on a mesh of S with a mesh size resolving ε using linear finite element methods [7]. These base functions can be precomputed initially to generate the coarse grid operator which turns out to be only a small overhead of the overall computations. This is also a common practice in many upscaling methods [15].

Let $OMS(T) = \text{span}\{\bar{\psi}_i^T\}_{i=1}^3$ and $\Pi_T : OMS(T) \to P_1(T)$ the projection

$$\Pi_T\psi = c_i\varphi_i^T \quad \text{ if } \psi = c_i\bar{\psi}_i^T \in OMS(T).$$

Let $\bar{X}_H$ be the finite element space

$$\bar{X}_H = \{\psi_H : \psi_H|_T \in OMS(T), \ \forall T \in \mathcal{M}_H\}$$

and define $\Pi_H : \bar{X}_H \to \Pi_{T\in\mathcal{M}_H} P_1(T)$ through the relation $\Pi_H\psi_H|_T = \Pi_T\psi_H$ for any $T \in \mathcal{M}_H$, $\psi_H \in \bar{X}_H$. The over-sampling multiscale finite element space is then defined as

$$X_H = \{\psi_H \in \bar{X}_H : \Pi_H\psi_H \in V_H \subset H^1(\Omega)\}.$$

In general, $X_H \not\subset H^1(\Omega)$ and the requirement $\Pi_H\psi_H \in V_H$ is to impose certain continuity of the functions $\psi_H \in X_H$ across the interelement boundaries.

For any $j = 1, 2, \cdots, N$, let Ω_j be a convex polygon inside Ω which consists of a union of elements in $\mathcal{M}_H$ such that $\bar{B}_j \subset\subset \Omega_j$. We assume $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ for any $i \neq j$. Let G_j^ε be the Green function

$$-\text{div}\,(K_\varepsilon(x)\nabla G_j^\varepsilon) = \delta_{\bar{x}_j} \quad \text{ in } \Omega_j, \quad G_j^\varepsilon|_{\Sigma_j} = 0,$$

where $\Sigma_j = \partial\Omega_j$. Let u_ε and $\{\mathrm{q}_i\}_{i\in I_D}$ be the solution of the problem (17)-(19). Then the argument in §1 yields that $\zeta_\varepsilon = u_\varepsilon - \sum_{j=1}^N \mathrm{q}_j G_j^\varepsilon \in H_0^1(\Omega)$ and $\{\mathrm{q}_i\}_{i\in I_D}$ satisfies

$$\int_\Omega K_\varepsilon(x)\nabla\zeta_\varepsilon\nabla\chi dx = -\sum_{j=1}^N \mathrm{q}_j \int_{\Sigma_j} K_\varepsilon \frac{\partial G_j^\varepsilon}{\partial\nu}\chi ds \quad \forall\chi \in H_0^1(\Omega),$$

$$-\frac{\mathrm{q}_i}{2\pi K_i}\ln\delta + \mathrm{q}_i w_i(\bar{x}_i) + \zeta_\varepsilon(\bar{x}_i) = \alpha_i \quad \forall i \in I_D.$$

Here and in the following G_j^ε is extended to be zero outside Ω_j, and $w_j = G_j^\varepsilon - \phi_j$ with $\phi_j = -\dfrac{1}{2\pi K_j}\ln|x - \bar{x}_j|$.

Let $X_H^0 = \{\psi_H \in X_H : \Pi_H \psi_H = 0 \ \text{ on } \Gamma\}$. Then we introduce the following discrete problem: Find $\zeta_H \in X_H^0$ and $\{\mathrm{q}_i^H\}_{i\in I_D}$ such that

$$\sum_{T\in\mathcal{M}_H}\int_T K_\varepsilon(x)\nabla\zeta_H\nabla\chi_H dx = -\sum_{j=1}^N \mathrm{q}_j^H \int_{\Sigma_j} K_\varepsilon \frac{\partial G_j^\varepsilon}{\partial\nu}\hat\chi_H ds \qquad \forall \chi_H \in X_H^0,$$
$$-\frac{\mathrm{q}_i^H}{2\pi K_i}\ln\delta + \mathrm{q}_i^H w_i(\bar x_i) + \zeta_H(\bar x_i) = \alpha_i \qquad \forall i\in I_D,$$

where $\hat\chi_H = \Pi_H\chi_H \in V_H^0 = V_H\cap H_0^1(\Omega)$, and we set $\mathrm{q}_j^H = \mathrm{q}_j$ for $j\in I_M$ to simplify the notation.

We make the following assumption on the local domains Ω_j.

(H4) Let $H_j = \operatorname{dist}(\bar x_j, \Sigma_j)$ and there exist constants $C_2, C_3 > 0$ such that $C_2 H_i \le H_j \le C_3 H_i$ for any $i,j = 1,2,\cdots,N$.

In practical applications, the most interesting quantities are the well bore pressure for the wells that prescribe the flow rate and the flow rate for the wells that prescribe the well bore pressure. The following theorem establishes the error estimate for these quantities.

Theorem 4.1. *Let the assumptions (H1)-(H4) be fulfilled. Then there exists a constant C independent of ε,δ,H and H_j such that, for $j\in I_M$,*

$$\max_{x\in\Gamma_j}|\zeta_\varepsilon - \zeta_H(\bar x_j)| \le C\Big\{H + \frac{H^2}{H_j^2}\Big|\ln\frac{H}{H_j}\Big|^2 + \frac{\varepsilon}{H} + \sqrt{\frac{\varepsilon}{H_j}}\Big\}|\ln H|\sum_{i=1}^N|\mathrm{q}_i|,$$

and, for $j\in I_D$,

$$|\mathrm{q}_j - \mathrm{q}_j^H| \le C\Big\{H + \frac{H^2}{H_j^2}\Big|\ln\frac{H}{H_j}\Big|^2 + \frac{\varepsilon}{H} + \sqrt{\frac{\varepsilon}{H_j}}\Big\}|\ln H|\sum_{i=1}^N|\mathrm{q}_i|.$$

The proof of this theorem can be found in [2], in which, some sharp bounds on the homogenization of Green functions are derived.

5. Numerical results

We start this section by summarizing the algorithm proposed in this paper for solving the problem (17)-(19) which is a good approximation of the original problem (14)-(16) when the size of the wells is negligible.

Algorithm 1. Given the well bore pressure α_i for $i\in I_D$ and the well flow rate q_j for $j\in I_M$. The following procedure finds the approximate well bore pressure α_j^H for $j\in I_M$, the approximate well flow rate q_i^H for $i\in I_D$, and the approximate pressure $u_H = \zeta_H + \sum_{j=1}^N \mathrm{q}_j^H G_j^h$, where $\mathrm{q}_j^H = \mathrm{q}_j$ for $j\in I_M$.

- For $j = 1, \cdots, N$, compute the discrete Green function G_j^h on each subdomain Ω_j, i.e. $G_j^h \in V_h^0(\Omega_j)$ such that

$$\int_{\Omega_j} K(x)\nabla G_j^h \nabla v_h dx = v_h(\bar{x}_j) \quad \forall v_h \in V_h^0(\Omega_j). \tag{22}$$

Here $V_h^0(\Omega_j) = V_h(\Omega_j) \cap H_0^1(\Omega_j)$, and $V_h(\Omega_j)$ is the standard conforming linear finite element space over the mesh $\mathcal{M}_h(\Omega_j)$ of Ω_j with the mesh size h resolving the scale of the permeability field $K(x)$. Compute the associate effective radius $\bar{r}_j$ from G_j^h according to the following relation

$$\frac{1}{2\pi K_j}\ln \bar{r}_j = \int_{\Omega_j}(K - K_j)\nabla\phi\nabla G_j^h dx - \int_{\Omega_j} K\nabla G_j^h \nabla\psi_h, \tag{23}$$

where $\phi = -\frac{1}{2\pi K_j}\ln|x - x_j|$ and $\psi_h \in V_h(\Omega_j)$ whose nodal values are defined by

$$\psi_h(x_k) = \begin{cases} \phi(x_k) & \text{if } x_k \in \partial\Omega_j, \\ 0 & \text{otherwise,} \end{cases}$$

Thus the approximate value of the Green function G_j on Γ_j is

$$\alpha_j^h = G_j^h(\bar{x}_j) - \frac{1}{2\pi K_j}\ln\frac{\delta}{\bar{r}_j}, \qquad K_j = K(\bar{x}_j).$$

- Use the over-sampling multiscale finite element method to find $\zeta_H \in X_H^0$ and $\{\mathrm{q}_i^H\}_{i\in I_D}$ such that

$$\sum_{T\in\mathcal{M}_H}\int_T K(x)\nabla\zeta_H\nabla\chi_H dx = -\sum_{j=1}^N \mathrm{q}_j^H \int_{\Sigma_j} K\frac{\partial G_j^h}{\partial\nu}\hat{\chi}_H ds \quad \forall\chi_H \in X_H^0,$$

$$\mathrm{q}_i^H\alpha_i^h + \zeta_H(\bar{x}_i) = \alpha_i \quad \text{for } i \in I_D.$$

- Compute the approximate well bore pressure α_j^H for $j \in I_M$ through the relation

$$\alpha_j^H = \zeta_H(\bar{x}_j) + \mathrm{q}_j\alpha_j^h.$$

The formula (23) defining the effective radius extends the well-known Peaceman method [13, 14] in the engineering literature for five point finite difference discretization and constant permeability to the general case and is convergent for any finite element meshes and any heterogeneous porous media, as proved in [2].

Now we report one numerical example carried out for random log-normal permeability to demonstrate the accuracy and efficiency of our method. More extensive numerical experiments can be found in [2]. The computations are performed on the unit square $\Omega = (0,1)\times(0,1)$. We consider two wells $B_j = B(\bar{x}_j, \delta)$ with $x_1 = (\frac{1}{4}, \frac{3}{4})$, $x_2 = (\frac{3}{4}, \frac{1}{4})$. We impose mixed boundary condition (3) that fixes the well flow rate q_j for both wells. Thus we have $I_D = \emptyset$ and the interested quantities are well bore pressure (WBP) on each well. In the computations we take $\mathrm{q}_1 = -1$

Well No.	Radius	'Exact' WBP	WBP	Relative error
1	1.E-05	-1.7278769	-1.7013823	0.1533E-01
1	1.E-04	-1.3540093	-1.3275152	0.1957E-01
1	1.E-03	-0.9801418	-0.9536476	0.2703E-01
2	1.E-05	3.7894742	3.8199386	0.7773E-02
2	1.E-04	2.8264341	2.8558985	0.1042E-01
2	1.E-03	1.8623940	1.8918584	0.1582E-01

TABLE 1. Results of the Algorithm 1 in the case of random log-normal field with $\sigma^2 = 1.69$ and $l_x = l_y = 0.01$.

and $q_2 = 1$ which corresponds to the situation that the well B_1 is an extraction well and B_2 is an injection well.

The domain Ω_j is taken as the square centered at $\bar{x}_j$ with the length of the sides being $2H_j$ so that $H_j = \text{dist}\,(\bar{x}_j, \partial\Omega_j)$. The discrete Green function G_j^h in (22) is computed on a uniform finite element mesh of size $h = H_j/512$.

We generate the random field $K(x)$ on a uniform 1024×1024 mesh by using moving ellipse average technique [4] with the variance of the logarithm of the permeability $\sigma = 1.3$, and the correlation lengths $l_x = l_y = 0.01$ in two space direction. Figure 1 shows a realization of the random permeability field. The "exact" well bore pressures are computed by on the fine 1024×1024 mesh are $\alpha_1 = -1.7278769, \alpha_2 = 3.7894742$.

Table 1 shows the results computed by using the Algorithm 1 on the coarse 64×64 mesh with the size of the local domains $H_1 = H_2 = H$. The mesh size for solving the over-sampling base functions is $1/1024$. It clearly shows that the WBP by our method on the coarse 64×64 mesh is a very good approximation of the "exact" WBP on the fine 1024×1024 mesh.

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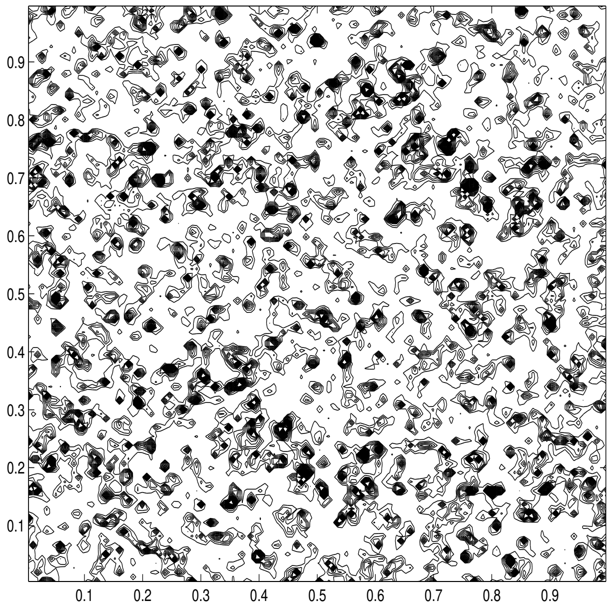

FIGURE 1. The random log-normal permeability field $K(x)$, The ratio of maximum to minimum is 5.0590E+04

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