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A Source Transfer Domain Decomposition Method For Helmholtz Equations in Unbounded Domain Part II: Extensions

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Abstract. In this paper we extend the source transfer domain decomposition method (STDDM) introduced by the authors to solve the Helmholtz problems in two-layered media, the Helmholtz scattering problems with bounded scatterer, and Helmholtz problems in 3D unbounded domains. The STDDM is based on the decomposition of the domain into non-overlapping layers and the idea of source transfer which transfers the sources equivalently layer by layer so that the solution in the final layer can be solved using a PML method defined locally outside the last two layers. The details of STDDM is given for each extension. Numerical results are presented to demonstrate the efficiency of STDDM as a preconditioner for solving the discretization problem of the Helmholtz problems considered in the paper.

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Key words: Helmholtz equation, high frequency waves, PML, source transfer.

1. Introduction

The source transfer domain decomposition method (STDDM) is introduced by the authors in [11] to solve the following 2D Helmholtz problems:

$$\Delta u + k^2 u = f \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

$$r^{1/2}\left(\frac{\partial u}{\partial r} - \mathbf{i}ku\right) \to 0 \quad \text{as } r = |x| \to \infty, \tag{1.2}$$

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where $k > 0$ is the wave number and $f \in H^{-1}_{\rm comp}(\mathbb{R}^2)$, that is, $f \in H^{-1}(\mathbb{R}^2)$ and has compact support.

Helmholtz equation (1.1) appears in diverse scientific and engineering applications including acoustics, elasticity, and electromagnetics. It is well-known that the efficient algebraic solver for large wave number discrete Helmholtz equation resulting from finite difference or finite element discretization is challenging due to the huge number of degrees of freedom required and the highly indefinite nature of the discrete problem. There exist considerable efforts in the literature for finding efficient algorithms for solving discrete Helmholtz equations, see e.g. Benamou and Després [4], Gander et al [15], Brandt and Livshit [6], Elman et at [12], and the review articles Erlangga [14], Osei-Kuffuor and Saad [19] and the references therein. The STDDM is motivated by the recent work of Engquist and Ying [13] in which a sweeping preconditioner is constructed by an approximate LDL^t factorization which eliminates the unknowns layer by layer. The Schur complement matrix of the factorization is approximated by using a moving perfectly matched layer (PML) technique.

The purpose of this paper is to extend the STDDM to solve the Helmholtz problems in two-layered media, the Helmholtz scattering problems with bounded scatterer, and Helmholtz problems in 3D unbounded domains. Let $\Omega_i = \{x \in \mathbb{R}^2 : \zeta_i < x_2 < \zeta_{i+1}\}$, $i = 1, \cdots, N$, be the layers whose union covers the support of the source f. Let $\Omega_0 = \{x \in \mathbb{R}^2 : x_2 < \zeta_1\}$ and $\Omega_{N+1} = \{x \in \mathbb{R}^2 : x_2 > \zeta_{N+1}\}$. Let f_i be the restriction of f in Ω_i and vanish outside Ω_i. It is clear that

$$u(x) = -\int_{\mathbb{R}^2} f(y)G(x,y)dy = -\sum_{i=1}^{N}\int_{\Omega_i} f_i(y)G(x,y)dy,\ \ G(x,y) = \frac{\mathbf{i}}{4}H_0^{(1)}(k|x-y|).$$

Let $\bar{f}_1 = f_1$. The key idea of STDDM is to define a source transfer operator Ψ_{i+1} that transfers the source from Ω_i to Ω_{i+1} in the sense that

$$\int_{\Omega_i} \bar{f}_i(y)G(x,y)dy = \int_{\Omega_{i+1}} \Psi_{i+1}(\bar{f}_i)(y)G(x,y)dy, \quad \forall x \in \Omega_j, \ \ j > i+1. \tag{1.3}$$

Then for $\bar{f}_{i+1} = f_{i+1} + \Psi_{i+1}(\bar{f}_i)$ we have

$$u(x) = -\int_{\Omega_N} f_N(y)G(x,y)dy - \int_{\Omega_{N-1}} \bar{f}_{N-1}(y)G(x,y)dy, \quad \forall x \in \Omega_N. \tag{1.4}$$

The solution u in Ω_N only involves the sources in Ω_N and Ω_{N-1} and thus can be solved locally by using the PML method defined outside only two layers Ω_N and Ω_{N-1}. Once the solution u in Ω_N is known, the solution in the other layers can be computed successively by solving the half-space Helmholtz problem using the transferred sources. This heuristic idea is made rigorous in the setting of PML method in [11].

The layout of the paper is as follows. In section 2 we will review the basic ingredients of the PML method and STDDM method for constant wave number. In sections 3-5 we propose STDDM for solving Helmholtz problems in two-layered media, the Helmholtz scattering problems with bounded scatterer, and Helmholtz problems in 3D unbounded domains separately. The details of STDDM for each extension is provided and numerical examples are included to show the effective behavior of STDDM as a preconditioner.

2. Review of STDDM

In this section we first recall some basic facts of the PML method and then introduce the STDDM algorithm. Let the source f be supported inside $B_l = \{x \in \mathbb{R}^2 : \left|x_1\right| < l_1, \left|x_2\right| < l_2\}$. It is known that the solution of the Helmholtz equation (1.1) with the Sommerfeld radiation condition (1.2) has the following integral representation

$$u(x) = -\int_{\mathbb{R}^2} f(y)G(x,y)dy \;\text{ in } \mathbb{R}^2, \quad G(x,y) = \frac{\mathbf{i}}{4}H_0^{(1)}(k|x-y|). \tag{2.1}$$

Here $H_0^{(1)}(z)$, for $z \in \mathbb{C}$, is the first Hankel function of order zero and $G(x,y)$ is the fundamental solution of the Helmholtz equation of constant wave number k

$$\Delta u + k^2 u = -\delta_y(x) \;\text{ in } \mathbb{R}^2.$$

The integral in (2.1) is well-defined if f is a smooth function with compact support in $\mathbb{R}^2$. In the general case when $f \in H^{-1}_{\text{comp}}(\mathbb{R}^2)$, it should be understood in the sense of distribution.

The PML method is based on the complex coordinate stretching outside the domain B_l [7]. Let $\alpha_1(x_1) = 1 + \mathbf{i}\sigma_1(x_1), \alpha_2(x_2) = 1 + \mathbf{i}\sigma_2(x_2)$ be the model medium property, where σ_j is a bounded piecewise continuous function on $\mathbb{R}$ and satisfies, for $j = 1, 2$,

$$\sigma_j \geq 0,\; \sigma_j(t) = \sigma_j(-t),\; \sigma_j = 0 \text{ for } |t| \leq l_j, \text{ and } \sigma_j \geq \gamma_0 > 0 \text{ for } |t| \geq M, \tag{2.2}$$

where $M > \max(l_1, l_2)$ is a constant. Denote by $\tilde{x}(x) = (\tilde{x}(x_1), \tilde{x}(x_2))^T$ the complex coordinate, where

$$\tilde{x}(x_j) = \int_0^{x_j} \alpha_j(t)dt = x_j + \mathbf{i}\int_0^{x_j} \sigma_j(t)dt, \quad j = 1, 2. \tag{2.3}$$

Notice that $\tilde{x}_j(x_j)$ depends only on x_j and for this reason the method is called the uniaxial PML method. For any $z \in \mathbb{C}$, denote by $z^{1/2}$ the analytic branch of $\sqrt{z}$ such that $\mathrm{Re}\,(z^{1/2}) > 0$ for any $z \in \mathbb{C}\backslash[0, +\infty)$. We define the complex distance

$$\rho(\tilde{x}, \tilde{y}) = \left[(\tilde{x}(x_1) - \tilde{y}(y_1))^2 + (\tilde{x}(x_2) - \tilde{y}(y_2))^2\right]^{1/2}.$$

Now we define

$$\tilde{u}(x) = u(\tilde{x}) = -\int_{\mathbb{R}^2} f(y)G(\tilde{x}, \tilde{y})dy, \quad \forall x \in \mathbb{R}^2. \tag{2.4}$$

Since f is supported inside B_l we know that $\tilde{y}(y) = y$ and $\tilde{u}$ is well-defined in $H^1_{\rm loc}(\mathbb{R}^2)$ and decays exponentially as $|x| \to \infty$. Obviously $\tilde{u} = u$ in B_l and $\tilde{u}$ satisfies $\tilde{\Delta}\tilde{u} + k^2\tilde{u} = f$ in $\mathbb{R}^2$, where $\tilde{\Delta}$ is the Laplacian with respect to the stretched coordinate $\tilde{x}$. This yields by the chain rule that $\tilde{u}$ satisfies the PML equation

$$J^{-1}\nabla\cdot(A\nabla\tilde{u}) + k^2\tilde{u} = f \quad \text{in } \mathbb{R}^2, \tag{2.5}$$

where $A(x) = \mathrm{diag}\left(\frac{\alpha_2(x_2)}{\alpha_1(x_1)}, \frac{\alpha_1(x_1)}{\alpha_2(x_2)}\right)$ is a diagonal matrix and $J(x) = \alpha_1(x_1)\alpha_2(x_2)$.

Let $B_L = (-l_1 - d_1, l_1 + d_1) \times (-l_2 - d_2, l_2 + d_2)$, $d_1, d_2 > 0$, be the rectangle including B_l. The PML problem whose solution approximates $\tilde{u}$ in (2.4) in the bounded domain B_L is defined by

$$\nabla\cdot(A\nabla\hat{u}) + k^2 J\hat{u} = f \quad \text{in } B_L, \tag{2.6}$$

$$\hat{u} = 0 \quad \text{on } \partial B_L. \tag{2.7}$$

Here (2.6) is obtained by multiplying (2.5) by J and noticing that $Jf = f$ since f is supported in B_l where $J = 1$.

Now we introduce the STDDM method for the truncated PML equation (2.6)-(2.7). We start by introducing some notation. For any $a, b \in R$, we denote $\Omega(a,b) = \{x \in \mathbb{R}^2 : a < x_2 < b\}$. We also use the notation $\Omega(-\infty, b)$ for the half-space $\{x \in \mathbb{R}^2 : x_2 < b\}$. Let $-l_2 = \zeta_1 < \zeta_2 < \cdots < \zeta_{N+1} = l_2$, $N > 1$, be a division of the interval $(-l_2, l_2)$. For simplicity, we assume $\zeta_i = \zeta_1 + (i-1)\Delta\zeta$, where $\Delta\zeta = 2l_2/N$. The general case of non-equally spaced division can be considered similarly. Let $\Omega_i = \Omega(\zeta_i, \zeta_{i+1})$ and $\Gamma_i = \{x \in \mathbb{R}^2 : x_2 = \zeta_i\}$.

We define $f_i = f|_{\Omega_i}$ in Ω_i and $f_i = 0$ in $\mathbb{R}^2\backslash\bar{\Omega}_i$. Denote by $\beta_i = \beta_i(x_2)$ a smooth function defined in Ω_i such that

$$\beta_i = 1, \beta_i' = 0 \ \text{ on } \Gamma_i, \quad \beta_i = \beta_i' = 0 \ \text{ on } \Gamma_{i+1}, \quad |\beta_i'| \le C(\Delta\zeta)^{-1} \ \text{ in } (\zeta_i, \zeta_{i+1}). \tag{2.8}$$

We will also use the PML complex coordinate stretching outside the domain $(-l_1, l_1) \times (\zeta_i, \zeta_{i+2})$ as $\tilde{x}_i(x) = (\tilde{x}_i(x_1), \tilde{x}_i(x_2))^T$, where $\tilde{x}_i(x_1) = \tilde{x}(x_1)$ and

$$\tilde{x}_i(x_2) = \begin{cases} x_2 + \mathbf{i}\int_{\zeta_{i+2}}^{x_2} \sigma_2(t + \zeta_N - \zeta_{i+2})dt & \text{if } x_2 > \zeta_{i+2}, \\ x_2 & \text{if } \zeta_i \le x_2 \le \zeta_{i+2}, \\ x_2 + \mathbf{i}\int_{\zeta_i}^{x_2} \sigma_2(t - \zeta_i + \zeta_1)dt & \text{if } x_2 < \zeta_i. \end{cases} \tag{2.9}$$

We denote $A_i(x) = \mathrm{diag}\left(\dfrac{\tilde{x}_i(x_2)'}{\tilde{x}_i(x_1)'}, \dfrac{\tilde{x}_i(x_1)'}{\tilde{x}_i(x_2)'}\right)$ and $J_i(x) = \tilde{x}_i(x_1)'\tilde{x}_i(x_2)'$.

The STDDM consists of two algorithms, the source transfer algorithm and the wave expansion algorithm.

Algorithm 2.1. (SOURCE TRANSFER)
1° *Let* $\hat{f}_1 = f_1$ *in* Ω_1;
2° *For* $i = 1, \cdots, N-2$, *compute* $\hat{f}_{i+1} = f_{i+1} + \hat{\Psi}_{i+1}(\hat{f}_i)$ *in* $\Omega_{i+1} \cap B_L$, *where*

$$\hat{\Psi}_{i+1}(\hat{f}_i) = \begin{cases} J_i^{-1}\nabla\cdot(A_i\nabla(\beta_{i+1}\hat{u}_i)) + k^2(\beta_{i+1}\hat{u}_i) & \textit{in } \Omega_{i+1} \cap B_L, \\ 0 & \textit{elsewhere}, \end{cases}$$

and $\hat{u}_i$ solves the problem

$$
\begin{aligned}
J_i^{-1}\nabla\cdot(A_i\nabla\hat{u}_i)+k^2\hat{u}_i &= -\hat{f}_i \quad \text{in } \Omega_i^{\mathrm{PML}}, && (2.10)\\
\hat{u}_i &= 0 \quad \text{on } \partial\Omega_i^{\mathrm{PML}}, && (2.11)
\end{aligned}
$$

where $\Omega_i^{\mathrm{PML}}=(-l_1-d_1,l_1+d_1)\times(\zeta_i-d_2,\zeta_{i+2}+d_2)$.

Algorithm 2.2. (WAVE EXPANSION)
1° *Solve $\hat{v}_N$ such that*

$$
\begin{aligned}
J_{N-1}^{-1}\nabla\cdot(A_{N-1}\nabla\hat{v}_N)+k^2\hat{v}_N &= f_N+\hat{f}_{N-1} \quad \text{in } \Omega_{N-1}^{\mathrm{PML}}, && (2.12)\\
\hat{v}_N &= 0 \quad \text{on } \partial\Omega_{N-1}^{\mathrm{PML}}. && (2.13)
\end{aligned}
$$

2° *For $i=N-1,\cdots,2$, find $\hat{v}_i$ such that*

$$
\begin{aligned}
J_{i-1}^{-1}\nabla\cdot(A_{i-1}\nabla\hat{v}_i)+k^2\hat{v}_i &= f_i+\hat{f}_{i-1} \quad \text{in } D_i^{\mathrm{PML}}, && (2.14)\\
\hat{v}_i &= \hat{v}_{i+1} \quad \text{on } \partial D_i^{\mathrm{PML}}\cap\Gamma_{i+1}, && (2.15)\\
\hat{v}_i &= 0 \quad \text{on } \partial D_i^{\mathrm{PML}}\backslash\Gamma_{i+1}, && (2.16)
\end{aligned}
$$

where $D_i^{\mathrm{PML}}=(-l_1-d_1,l_1+d_1)\times(\zeta_{i-1}-d_2,\zeta_{i+1})$.

We make the following assumption on the medium property which is rather mild in practical applications.

(H1) $l_1\le l_2$, $d_1=2d_2$, and

$$\int_{l_1}^{l_1+d_2}\sigma_1(t)dt=\int_{l_2}^{l_2+d_2}\sigma_2(t)dt=\bar{\sigma},\quad \int_{l_1+d_2}^{l_1+d_1}\sigma_1(t)dt\ge\bar{\sigma}.$$

The following theorem proved in [11] provides the theoretical justification of the above STDDM method.

Theorem 2.1. *Let (H1) be satisfied. Let $\hat{v}=\hat{v}_N$ in $\Omega(\zeta_N,+\infty)\cap B_L$, $\hat{v}=\hat{v}_i$ in $\Omega_i\cap B_L$ for all $i=3,\cdots,N-1$, and $\hat{v}=\hat{v}_2$ in $\Omega(-\infty,\zeta_2)\cap B_L$. We have*

$$\|\hat{u}-\hat{v}\|_{H^1(B_L)}\le Ce^{-\frac{1}{2}k\gamma\bar{\sigma}}\|f\|_{H^1(B_l)'}$$

where $\gamma=\frac{d_2}{\sqrt{d_2^2+(2l_2+d_1+d_2)^2}}$.

Let T be the solution operator of (2.6)-(2.7) defined by $T(f)=\hat{u}$ and $\hat{T}$ be the output operator of the the STDDM defined by $\hat{T}(f)=\hat{v}$. Then Theorem 2.1 indicates that $\hat{T}$ is a good approximation of T if the PML parameters are chosen such that $e^{-\frac{1}{2}k\gamma\bar{\sigma}}$ is sufficiently small. In this case the discretization of $\hat{T}$ will be a good preconditioner of the corresponding discretization of T. This idea has been confirmed in [11] for Helmholtz problems with constant wave number in 2D unbounded domains. In the following we will extend the STDDM to solve the Helmholtz problems in two-layered media, the Helmholtz scattering problems with bounded scatterer, and Helmholtz problems in 3D unbounded domains.

3. Helmholtz problems in two-layered media

We consider in this section the Helmholtz problems in two-layered media in 2D. The method can be extended to 3D cases when combining the SDTTM method to be discussed in Section 5. Suppose the wave number k in (1.1) is defined by:

$$k(x) = \begin{cases} k_1, & x_2 < 0; \\ k_2, & x_2 \geq 0. \end{cases}$$

Based on the proof in [11], we can see that there is no need to just transfer the source from bottom to up. In fact we can also transfer source from bottom and from up to the middle simultaneously. This is particularly important for the case of two-layered media. The most important idea is that the times of transferring source N should be as few as possible.

Let the interface between k_1 and k_2 is included Ω_M and $N \geq 5$. We propose the following source transfer algorithm for the Helmholtz problems in two-layered media.

Algorithm 3.1. (SOURCE TRANSFER FOR THE TWO-LAYERED PROBLEM)
1° *Let* $\hat{f}_1 = f_1$ *in* Ω_1 *and* $\hat{f}_N = f_N$ *in* Ω_N*;*
2° *For* $i = 1, \cdots, M-2$*, compute* $\hat{f}_{i+1} = f_{i+1} + \hat{\Psi}_{i+1}(\hat{f}_i)$ *in* $\Omega_{i+1} \cap B_L$*, where*

$$\hat{\Psi}_{i+1}(\hat{f}_i) = \begin{cases} J_i^{-1} \nabla \cdot (A_i \nabla(\beta_{i+1} \hat{u}_i)) + k^2(\beta_{i+1} \hat{u}_i) & \text{in } \Omega_{i+1} \cap B_L, \\ 0 & \text{elsewhere,} \end{cases}$$

and $\hat{u}_i$ *solves the problem*

$$J_i^{-1} \nabla \cdot (A_i \nabla \hat{u}_i) + k^2 \hat{u}_i = -\hat{f}_i \quad \text{in } \hat{\Omega}_i^{\rm PML}, \tag{3.1}$$

$$\hat{u}_i = 0 \quad \text{on } \partial \hat{\Omega}_i^{\rm PML}, \tag{3.2}$$

where $\hat{\Omega}_i^{\rm PML} = (-l_1 - d_1, l_1 + d_1) \times (\zeta_i - d_2, \zeta_{i+2})$.
3° *For* $i = N, \cdots, M+2$*, compute* $\hat{f}_{i-1} = f_{i-1} + \hat{\Psi}_{i-1}(\hat{f}_i)$ *in* $\Omega_{i-1} \cap B_L$*, where*

$$\hat{\Psi}_{i-1}(\hat{f}_i) = \begin{cases} J_{i-1}^{-1} \nabla \cdot (A_{i-1} \nabla(\beta_{i-1} \hat{u}_i)) + k^2(\beta_{i-1} \hat{u}_i) & \text{in } \Omega_{i-1} \cap B_L, \\ 0 & \text{elsewhere,} \end{cases}$$

and $\hat{u}_i$ *solves the problem*

$$J_{i-1}^{-1} \nabla \cdot (A_{i-1} \nabla \hat{u}_i) + k^2 \hat{u}_i = -\hat{f}_i \quad \text{in } \hat{\Omega}_i^{\rm PML}, \tag{3.3}$$

$$\hat{u}_i = 0 \quad \text{on } \partial \hat{\Omega}_i^{\rm PML}, \tag{3.4}$$

where $\hat{\Omega}_i^{\rm PML} = (-l_1 - d_1, l_1 + d_1) \times (\zeta_{i-1}, \zeta_{i+1} + d_2)$.

Clearly the step 2° and 3° can be done in parallel. After the completion of Algorithm 3.1, the source f_i, $i = 1, \cdots, M-2$, is equivalently transferred to the source $\hat{\Psi}_{M-1}(\bar{f}_{M-2})$ in Ω_{M-1} and the source f_i, $i = N, \cdots, M+2$, is equivalently transferred to the source

$\hat{\Psi}_{M+1}(\bar{f}_{M+2})$ in Ω_{M+1}. Now we define the PML complex coordinate stretching outside the domain $(-l_1,l_1)\times(\zeta_{M-1},\zeta_{M+2})$ as $\tilde{x}_M(x)=(\tilde{x}_M(x_1),\tilde{x}_M(x_2))^T$, where $\tilde{x}_M(x_1)=\tilde{x}(x_1)$ and

$$\tilde{x}_M(x_2)=\begin{cases} x_2+\mathbf{i}\int_{\zeta_{M+2}}^{x_2}\sigma_2(t+\zeta_N-\zeta_{M+2})dt & \text{if } x_2>\zeta_{M+2},\\ x_2 & \text{if } \zeta_{M-1}\le x_2\le\zeta_{M+2},\\ x_2+\mathbf{i}\int_{\zeta_{M-1}}^{x_2}\sigma_2(t-\zeta_{M-1}+\zeta_1)dt & \text{if } x_2<\zeta_{M-1}.\end{cases} \tag{3.5}$$

We denote $A_M(x)=\mathrm{diag}\left(\dfrac{\tilde{x}_M(x_2)'}{\tilde{x}_M(x_1)'},\dfrac{\tilde{x}_M(x_1)'}{\tilde{x}_M(x_2)'}\right)$ and $J_M(x)=\tilde{x}_M(x_1)'\tilde{x}_M(x_2)'$. From this definition, we have the following wave expansion algorithm for Helmholtz problems in two-layered media.

Algorithm 3.2. (Wave expansion for the two-layered problem)
1° *Solve $\hat{v}_M$ such that*

$$\begin{aligned} J_M^{-1}\nabla\cdot(A_M\nabla\hat{v}_M)+k^2\hat{v}_M &= f_M+\hat{f}_{M-1}+\hat{f}_{M+1} \quad \text{in} \quad \hat{\Omega}_M^{\rm PML}, &(3.6)\\ \hat{v}_M &= 0 \qquad \text{on} \ \ \partial\hat{\Omega}_M^{\rm PML}, &(3.7)\end{aligned}$$

where $\hat{\Omega}_M^{\rm PML}=(-l_1-d_1,l_1+d_1)\times(\zeta_{M-1}-d_2,\zeta_{M+2}+d_2)$.
2° *For $i=M-1,\cdots,2$, find $\hat{v}_i$ such that*

$$\begin{aligned} J_{i-1}^{-1}\nabla\cdot(A_{i-1}\nabla\hat{v}_i)+k^2\hat{v}_i &= f_i+\hat{f}_{i-1} \quad \text{in } \hat{D}_i^{\rm PML}, &(3.8)\\ \hat{v}_i &= \hat{v}_{i+1} \qquad \text{on } \partial\hat{D}_i^{\rm PML}\cap\Gamma_{i+1}, &(3.9)\\ \hat{v}_i &= 0 \qquad \text{on } \partial\hat{D}_i^{\rm PML}\backslash\Gamma_{i+1}, &(3.10)\end{aligned}$$

where $\hat{D}_i^{\rm PML}=(-l_1-d_1,l_1+d_1)\times(\zeta_{i-1}-d_2,\zeta_{i+1})$.
3° *For $i=M+1,\cdots,N-1$, find $\hat{v}_i$ such that*

$$\begin{aligned} J_i^{-1}\nabla\cdot(A_i\nabla\hat{v}_i)+k^2\hat{v}_i &= f_i+\hat{f}_{i+1} \quad \text{in } \hat{D}_i^{\rm PML}, &(3.11)\\ \hat{v}_i &= \hat{v}_{i-1} \qquad \text{on } \partial\hat{D}_i^{\rm PML}\cap\Gamma_i, &(3.12)\\ \hat{v}_i &= 0 \qquad \text{on } \partial\hat{D}_i^{\rm PML}\backslash\Gamma_i, &(3.13)\end{aligned}$$

where $\hat{D}_i^{\rm PML}=(-l_1-d_1,l_1+d_1)\times(\zeta_i,\zeta_{i+2}+d_2)$.

After we obtain $\hat{v}_i$, $i=2,\dots,N-1$, let $\hat{v}=\hat{v}_M$ in $\Omega(\zeta_{M-1},\zeta_{M+1})\cap B_L$, $\hat{v}=\hat{v}_i$ in $\Omega_i\cap B_L$ for all $i=3,\cdots,M-2$, $\hat{v}=\hat{v}_2$ in $\Omega(-\infty,\zeta_2)\cap B_L$, $\hat{v}=\hat{v}_i$ in $\Omega_i\cap B_L$ for all $i=M+2,\cdots,N-2$ and $\hat{v}=\hat{v}_{N-1}$ in $\Omega(\zeta_{N-1},+\infty)\cap B_L$. A similar result to Theorem 2.1 for the case two-layered media can be proved

$$\|\hat{u}-\hat{v}\|_{H^1(B_L)}\le Ce^{-\frac{1}{2}k_{\min}\gamma\bar{\sigma}}\|f\|_{H^1(B_l)'},$$

where $k_{\min}=\min(k_1,k_2)$ and $\gamma=\frac{d_2}{\sqrt{d_2^2+(2l_2+d_1+d_2)^2}}$. Here we omit the details.

Now we report a numerical example to show the performance of our method. In the remainder of this paper we will always use our STDDM as the preconditioner of GMRES method. The linear system of equations in each subdomain is solved by MUMPS [1, 2]. The computations are all carried out in MATLAB on Dell Precision T5500 with Intel(R) Xeon(R)CPU 2.67GHz and 72GB memory. We denote N_{iter} the number of iterations of the preconditioned GMRES method and T_{solve} the overall solution time in seconds.

Example 3.1. Let $L := l_1 + d_1 = l_2 + d_2 = 0.5$ and choose the thickness of the PML layer $d_1 = 2\lambda$ and $d_2 = \lambda$. We take the medium property

$$\sigma_j(t) = \tilde{\sigma}_j \left(\frac{|t| - l_j}{d_2} \right)^m, \; m \geq 1 \text{ integer}, \; \tilde{\sigma}_j > 0 \text{ is a constant}, \; j = 1, 2,$$

where $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ are determined from $\bar{\sigma}$ by $\tilde{\sigma}_j = (m+1)\bar{\sigma}/d_2$, $j = 1, 2$.

We use central finite difference scheme on a uniform $n \times n$ grid to discretize the PML problem (2.6)-(2.7) in B_L. The number of points in each dimension n should be proportional to the wave number k since a constant number of points is required for each wavelength. The grid spacing is h so that $L = (n+1)h/2$. Let $d_2 = (p+1/2)h$ and $\Delta\zeta = ch$ which requires $c = (n - 2p)/N$, N is the number of layers (see Figure 3.1 for an example of 5 layers, $c = 3$ and $p = 1$). The PML equations in algorithms 3.1 and 3.2 are discretized using the central finite difference scheme. The function β_i in Ω_i in the definition of the source transfer algorithm is discretized by setting β_i being one in the first c_1 rows near Γ_i and zero in the last c_2 rows near Γ_{i+1} ($c_1, c_2 > 0$ and $c_1 + c_2 = c$). $\hat{\Psi}_{i+1}(\hat{f}_i)$ is then discretized by using a central finite difference scheme. We use our STDDM as the preconditioner of the GMRES method for solving the discrete problem of (2.6)-(2.7). The relative residue tolerance in GMRES solver is set to be 10^{-10}.

We set $\lambda = 2\pi/k_{\max}$, where $k_{\max} = \max(k_1, k_2)$, and choose $\bar{\sigma}$ such that the exponentially decaying factor

$$e^{-\frac{1}{2}k_{\min}\gamma\bar{\sigma}} = e^{-\frac{k_{\min}\lambda\bar{\sigma}}{2\sqrt{2\lambda^2+2\lambda+1}}} \leq 10^{-10}.$$

We discretize one direction with $n = k_{\max}q/2\pi$ points and thus the number of unknowns is $(\max(k_1, k_2)q/2\pi)^2$. We set $c = q$, $p = q$, $c_1 = 4$ and $c_2 = c - 4$. We set the external force $f(x)$ to be a narrow Gaussian point source located at $(r_1, r_2) = (-0.25, -0.25)$:

$$f(x_1, x_2) = e^{-(\frac{4k}{\pi})^2((x_1-r_1)^2+(x_2-r_2)^2)}.$$

We test two cases $k_2 = k_1/3$ and $k_2 = k_1/10$. The numerical results are listed in Table 1 and Table 2. We show the real part of the solutions in these two cases in Figure 3.2. The results show clearly that STDDM method works very well for two-layered media Helmholtz problems.

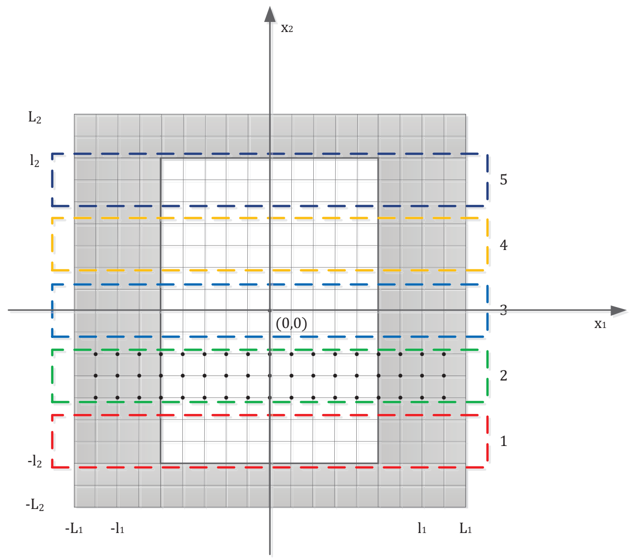

Figure 3.1: An example of discretization grid in 2D.

4. Helmholtz scattering problems

In this section, we extend the STDDM to solve the following Helmholtz scattering problems:

$$\Delta u + k^2 u = f \quad \text{in } R^2 \backslash \overline{D}, \tag{4.1}$$

$$\frac{\partial u}{\partial n_D} = -g \quad \text{on } \Gamma_D, \tag{4.2}$$

$$r^{1/2}\left(\frac{\partial u}{\partial r} - \mathbf{i}ku\right) \rightarrow 0 \quad \text{as } r = |x| \rightarrow \infty. \tag{4.3}$$

Here $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, $g \in H^{-1/2}(\Gamma_D)$ is determined by the incoming wave, and n_D is the unit outer normal to D. We remark that the results can be extended to solve the scattering problems with other boundary conditions, such as Dirichlet or the impedance boundary condition on Γ_D.

Let B_l contains scatterer D. By using the same PML settings and the notation in section 2, we have the corresponding PML problem for (4.1)-(4.3):

$$\nabla \cdot (A \nabla \hat{u}) + k^2 J \hat{u} = f \quad \text{in } B_L \backslash \overline{D}, \tag{4.4}$$

$$\frac{\partial \hat{u}}{\partial n_D} = -g \quad \text{on } \Gamma_D, \tag{4.5}$$

$$\hat{u} = 0 \quad \text{on } \partial B_L. \tag{4.6}$$

Let $-l_2 = \zeta_1 < \zeta_2 < \cdots < \zeta_M, \zeta_M = -\zeta_{M+1}, \zeta_{M+1} < \zeta_{M+2} < \cdots < \zeta_{N+1} = l_2$, $N \geq 5$, be a division of the interval $(-l_2, l_2)$. We assume $\zeta_i = \zeta_1 + (i-1)\Delta\zeta$ with $\Delta\zeta =$

$k_1/2\pi$	q	NOF	N_{iter}	T_{solve}
50	10	500^2	5	5.69
100	10	1000^2	6	28.71
200	10	2000^2	8	151.12
400	10	4000^2	10	773.37

Table 1: Numerical results for the case $k_2 = k_1/3$.

$k_1/2\pi$	q	NOF	N_{iter}	T_{solve}
50	10	500^2	7	9.95
100	10	1000^2	8	47.84
200	10	2000^2	9	220.10
400	10	4000^2	12	1209.98

Table 2: Numerical results for the case $k_2 = k_1/10$.

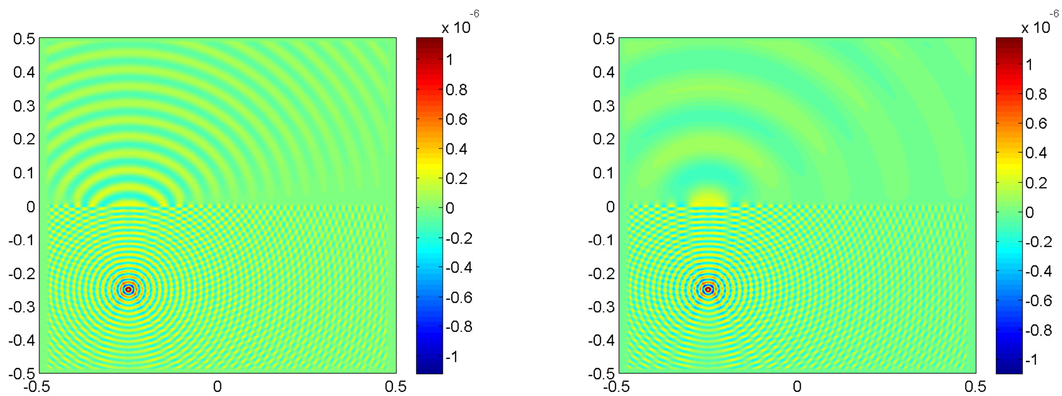

Figure 3.2: The real part of the solution for different choices of k_1 and k_2 when $q = 10$. Left: $k_1/2\pi = 60$, $k_2 = k_1/3$; Right: $k_1/2\pi = 60$, $k_2 = k_1/10$.

$(l_2+\zeta_M)/(M-1)$ for $i \le M$ and $\zeta_i = \zeta_{M+1} + (i-M-1)\Delta\zeta$ with $\Delta\zeta = (l_2+\zeta_M)/(N-M)$ for $i \ge M+1$. Again we denote $\Omega_i = \Omega(\zeta_i, \zeta_{i+1})$ for $i = 1, \cdots, N+1$.

Let the scatterer D be included in Ω_M, then we can directly use the STDDM (Algorithms 3.1 and 3.2) to solve PML equation (4.4)-(4.6). The only difference is that in the first step of Algorithm 3.2, we should solve $\hat{v}_M$ such that

$$J_M^{-1}\nabla\cdot(A_M\nabla\hat{v}_M) + k^2\hat{v}_M = f_M + \hat{f}_{M-1} + \hat{f}_{M+1} \quad \text{in} \quad \hat{\Omega}_M^{\rm PML}\backslash\bar{D}, \tag{4.7}$$

$$\frac{\partial v_M}{\partial n_D} = -g \quad \text{on } \Gamma_D, \tag{4.8}$$

$$\hat{v}_M = 0 \quad \text{on} \quad \partial\hat{\Omega}_M^{\rm PML}, \tag{4.9}$$

where $\hat{\Omega}_M^{\rm PML} = (-l_1 - d_1, l_1 + d_1) \times (\zeta_{M-1} - d_2, \zeta_{M+1} + d_2)$.

We will use STDDM to solve the PML equation (4.7)-(4.9). First we introduce some

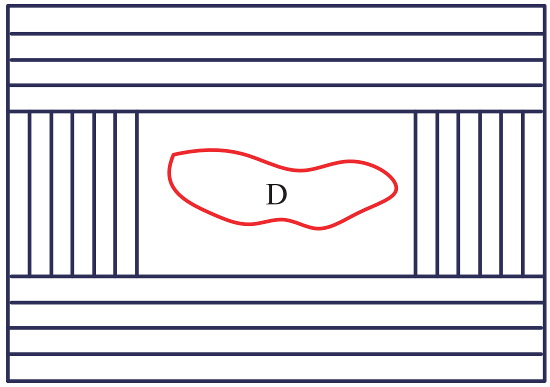

Figure 4.3: An example of STDDM for Helmholtz scattering problems.

notation. Let $-l_1=\bar\zeta_1<\bar\zeta_2<\cdots<\bar\zeta_{\bar M},\bar\zeta_{\bar M}=-\bar\zeta_{\bar M+1},\bar\zeta_{\bar M+1}<\bar\zeta_{\bar M+2}<\cdots<\bar\zeta_{\bar N+1}=l_1$, $\bar N\geq 5$, be a division of the interval $(-l_1,l_1)$. For simplicity, we assume $\bar\zeta_i=\bar\zeta_1+(i-1)\bar\Delta\zeta$ with $\bar\Delta\zeta=(l_1+\bar\zeta_{\bar M})/(\bar M-1)$ for $i\leq\bar M$ and $\bar\zeta_i=\bar\zeta_{\bar M+1}+(i-\bar M-1)\bar\Delta\zeta$ with $\bar\Delta\zeta=(l_1+\bar\zeta_{\bar M})/(\bar N-\bar M)$ for $i\geq\bar M+1$. Let $\breve\Omega_i=(\bar\zeta_i,\bar\zeta_{i+1})\times(\zeta_{M-1}-d_2,\zeta_{M+1}+d_2)$ and $\breve\Gamma_i=\{x\in\mathbb{R}^2:x_1=\bar\zeta_i\}$. We assume $\bar D\subset\breve\Omega_{\bar M}$.

Define $F=f_M+\hat f_{M-1}+\hat f_{M+1}$. Similar to the definition of f_i and β_i, we can define F_i and $\bar\beta_i$ as follows. Let $F_i=F|_{\breve\Omega_i}$ in $\breve\Omega_i$ and $F_i=0$ in $\hat\Omega_M^{\rm PML}\backslash\overline{\breve\Omega_i}$ and $\bar\beta_i=\bar\beta_i(x_1)$ a smooth function defined in $\breve\Omega_i$ such that

$$\bar\beta_i=1,\bar\beta_i'=0\ \text{ on }\breve\Gamma_i,\quad \bar\beta_i=\bar\beta_i'=0\ \text{ on }\breve\Gamma_{i+1},\quad |\bar\beta_i'|\leq C(\bar\Delta\zeta)^{-1}\ \text{ in }(\bar\zeta_i,\bar\zeta_{i+1}). \tag{4.10}$$

We define the PML complex coordinate stretching outside the domain $(\bar\zeta_i,\bar\zeta_{i+2})\times(\zeta_{M-1}-d_2,\zeta_{M+1}+d_2)$ as $\breve x_i(x)=(\breve x_i(x_1),\breve x_i(x_2))^T$, $i\neq\bar M$, where $\breve x_i(x_2)=\tilde x_M(x_2)$ and

$$\breve x_i(x_1)=\begin{cases} x_1+\mathbf{i}\int_{\bar\zeta_{i+2}}^{x_1}\sigma_1(t+\bar\zeta_{\bar N}-\bar\zeta_{i+2})dt & \text{if } x_1>\bar\zeta_{i+2},\\ x_1 & \text{if } \bar\zeta_i\leq x_1\leq\bar\zeta_{i+2},\\ x_1+\mathbf{i}\int_{\bar\zeta_i}^{x_1}\sigma_1(t-\bar\zeta_i+\bar\zeta_1)dt & \text{if } x_1<\bar\zeta_i.\end{cases} \tag{4.11}$$

We denote $\breve A_i(x)=\operatorname{diag}\left(\dfrac{\breve x_i(x_2)'}{\breve x_i(x_1)'},\dfrac{\breve x_i(x_1)'}{\breve x_i(x_2)'}\right)$ and $\breve J_i(x)=\breve x_i(x_1)'\breve x_i(x_2)'$.

For $i=\bar M$, we define the PML complex coordinate stretching outside the domain $(\bar\zeta_{\bar M-1},\bar\zeta_{\bar M+2})\times(\zeta_{M-1}-d_2,\zeta_{M+1}+d_2)$ as $\breve x_{\bar M}(x)=(\breve x_{\bar M}(x_1),\breve x_{\bar M}(x_2))^T$, where $\breve x_{\bar M}(x_2)=\tilde x_M(x_2)$, and

$$\breve x_{\bar M}(x_1)=\begin{cases} x_1+\mathbf{i}\int_{\bar\zeta_{\bar M+2}}^{x_1}\sigma_1(t+\bar\zeta_{\bar N}-\bar\zeta_{\bar M+2})dt & \text{if } x_1>\bar\zeta_{\bar M+2},\\ x_1 & \text{if } \bar\zeta_{\bar M-1}\leq x_1\leq\bar\zeta_{\bar M+2},\\ x_1+\mathbf{i}\int_{\bar\zeta_{\bar M-1}}^{x_1}\sigma_1(t-\bar\zeta_{\bar M-1}+\bar\zeta_1)dt & \text{if } x_1<\bar\zeta_{\bar M-1}.\end{cases} \tag{4.12}$$

We denote $\breve A_{\bar M}(x)=\operatorname{diag}\left(\dfrac{\breve x_{\bar M}(x_2)'}{\breve x_{\bar M}(x_1)'},\dfrac{\breve x_{\bar M}(x_1)'}{\breve x_{\bar M}(x_2)'}\right)$ and $\breve J_{\bar M}(x)=\breve x_{\bar M}(x_1)'\breve x_{\bar M}(x_2)'$.

Then we have the following STDDM for solving PML equation (4.7)-(4.9).

Algorithm 4.1. (SOURCE TRANSFER FOR HELMHOLTZ SCATTERING PROBLEM)
1° *Let* $\hat{F}_1 = F_1$ *in* $\breve{\Omega}_1$ *and* $\hat{F}_{\bar{N}} = F_{\bar{N}}$ *in* $\breve{\Omega}_N$*;*
2° *For* $i = 1, \cdots, \bar{M} - 2$*, compute* $\hat{F}_{i+1} = F_{i+1} + \bar{\Psi}_{i+1}(\hat{F}_i)$ *in* $\breve{\Omega}_{i+1}$*, where*

$$\bar{\Psi}_{i+1}(\hat{F}_i) = \begin{cases} \breve{J}_i^{-1}\nabla\cdot(\breve{A}_i\nabla(\bar{\beta}_{i+1}\hat{u}_i)) + k^2(\bar{\beta}_{i+1}\hat{u}_i) & \text{in } \breve{\Omega}_{i+1}, \\ 0 & \text{elsewhere}, \end{cases}$$

and $\hat{u}_i$ *solves the problem*

$$\begin{aligned} \breve{J}_i^{-1}\nabla\cdot(\breve{A}_i\nabla\hat{u}_i) + k^2\hat{u}_i &= -\hat{F}_i \quad \text{in } \breve{\Omega}_i^{\mathrm{PML}}, &(4.13)\\ \hat{u}_i &= 0 \quad \text{on } \partial\breve{\Omega}_i^{\mathrm{PML}}, &(4.14)\end{aligned}$$

where $\breve{\Omega}_i^{\mathrm{PML}} = (\bar{\zeta}_i - d_1, \bar{\zeta}_{i+2}) \times (\zeta_{M-1} - d_2, \zeta_{M+1} + d_2)$.
3° *For* $i = N, \cdots, \bar{M} + 2$*, compute* $\hat{F}_{i-1} = F_{i-1} + \bar{\Psi}_{i-1}(\hat{F}_i)$ *in* $\breve{\Omega}_{i-1}$*, where*

$$\bar{\Psi}_{i-1}(\hat{F}_i) = \begin{cases} \breve{J}_{i-1}^{-1}\nabla\cdot(\breve{A}_{i-1}\nabla(\beta_{i-1}\hat{u}_i)) + k^2(\bar{\beta}_{i-1}\hat{u}_i) & \text{in } \breve{\Omega}_{i-1}, \\ 0 & \text{elsewhere}, \end{cases}$$

and $\hat{u}_i$ *solves the problem*

$$\begin{aligned} \breve{J}_{i-1}^{-1}\nabla\cdot(\breve{A}_{i-1}\nabla\hat{u}_i) + k^2\hat{u}_i &= -\hat{F}_i \quad \text{in } \breve{\Omega}_i^{\mathrm{PML}}, &(4.15)\\ \hat{u}_i &= 0 \quad \text{on } \partial\breve{\Omega}_i^{\mathrm{PML}}, &(4.16)\end{aligned}$$

where $\breve{\Omega}_i^{\mathrm{PML}} = (\bar{\zeta}_{i-1}, \bar{\zeta}_{i+1} + d_1) \times (\zeta_{M-1} - d_2, \zeta_{M+1} + d_2)$.

Algorithm 4.2. (WAVE EXPANSION FOR HELMHOLTZ SCATTERING PROBLEM)
1° *Solve* $\bar{v}_{\bar{M}}$ *such that*

$$\begin{aligned} \breve{J}_{\bar{M}}^{-1}\nabla\cdot(\breve{A}_{\bar{M}}\nabla\bar{v}_{\bar{M}}) + k^2\bar{v}_{\bar{M}} &= F_{\bar{M}} + \hat{F}_{\bar{M}-1} + \hat{F}_{\bar{M}+1} \quad \text{in} \quad \breve{\Omega}_{\bar{M}}^{\mathrm{PML}}, &(4.17)\\ \bar{v}_{\bar{M}} &= 0 \qquad \text{on} \quad \partial\breve{\Omega}_{\bar{M}}^{\mathrm{PML}}, &(4.18)\end{aligned}$$

where $\breve{\Omega}_{\bar{M}}^{\mathrm{PML}} = (\bar{\zeta}_{\bar{M}-1} - d_1, \bar{\zeta}_{\bar{M}+2} + d_1) \times (\zeta_{M-1} - d_2, \zeta_{M+1} + d_2)$.
2° *For* $i = \bar{M} - 1, \cdots, 2$*, find* $\bar{v}_i$ *such that*

$$\begin{aligned} \breve{J}_{i-1}^{-1}\nabla\cdot(\breve{A}_{i-1}\nabla\bar{v}_i) + k^2\bar{v}_i &= F_i + \hat{F}_{i-1} \quad \text{in } \breve{D}_i^{\mathrm{PML}}, &(4.19)\\ \bar{v}_i &= \bar{v}_{i+1} \qquad \text{on } \partial\breve{D}_i^{\mathrm{PML}} \cap \breve{\Gamma}_{i+1}, &(4.20)\\ \bar{v}_i &= 0 \qquad \text{on } \partial\breve{D}_i^{\mathrm{PML}} \backslash \breve{\Gamma}_{i+1}, &(4.21)\end{aligned}$$

where $\breve{D}_i^{\mathrm{PML}} = (\bar{\zeta}_{i-1} - d_1, \bar{\zeta}_{i+1}) \times (\zeta_{M-1} - d_2, \zeta_{M+1} + d_2)$.
3° *For* $i = \bar{M} + 1, \cdots, N - 1$*, find* $\bar{v}_i$ *such that*

$$\begin{aligned} \breve{J}_i^{-1}\nabla\cdot(\breve{A}_i\nabla\bar{v}_i) + k^2\bar{v}_i &= F_i + \hat{F}_{i+1} \quad \text{in } \breve{D}_i^{\mathrm{PML}}, &(4.22)\\ \bar{v}_i &= \bar{v}_{i-1} \qquad \text{on } \partial\breve{D}_i^{\mathrm{PML}} \cap \breve{\Gamma}_i, &(4.23)\\ \bar{v}_i &= 0 \qquad \text{on } \partial\breve{D}_i^{\mathrm{PML}} \backslash \breve{\Gamma}_i, &(4.24)\end{aligned}$$

where $\breve{D}_i^{\mathrm{PML}} = (\bar{\zeta}_i, \bar{\zeta}_{i+2} + d_1) \times (\zeta_{M-1} - d_2, \zeta_{M+1} + d_2)$.

$k/2\pi$	q	NOF	N_{iter}	T_{solve}
40	10	400^2-40^2	6	4.63
80	10	800^2-40^2	7	21.33
160	10	1600^2-40^2	8	98.95
320	10	3200^2-40^2	10	511.73

Table 3: Numerical results for different wave numbers k with external force $f(x)$ located at $(-0.25,-0.25)$.

$k/2\pi$	q	NOF	N_{iter}	T_{solve}
40	10	400^2-40^2	6	4.60
80	10	800^2-40^2	7	21.20
160	10	1600^2-40^2	8	99.70
320	10	3200^2-40^2	9	509.68

Table 4: Numerical results for different wave numbers k with external force $f(x)$ located at $(-0.25,0.0)$.

Let $\check{v}_M=\bar{v}_{\bar{M}}$ in $\breve{\Omega}_{\bar{M}}$, $\check{v}_M=\bar{v}_i$ in $\breve{\Omega}_i$ for all $i=3,\cdots,\bar{M}-1$, $\check{v}_M=\bar{v}_2$ in $(-l_1-d_1,\bar{\zeta}_3)\times(\zeta_{M-1}-d_2,\zeta_{M+1}+d_2)$, $\check{v}_M=\bar{v}_i$ in $\breve{\Omega}_i$ for all $i=\bar{M}+1,\cdots,\bar{N}-2$ and $\check{v}_M=\bar{v}_{\bar{N}-1}$ in $(\bar{\zeta}_{\bar{N}-1},l_1+d_1)\times(\zeta_{M-1}-d_2,\zeta_{M+1}+d_2)$. We obtain $\check{v}_M$ as the approximation of the solution $\hat{v}_M$ of the problem (4.7)-(4.9)can get $\check{v}_M$. By combining Algorithms 3.1 to 4.2, we can solve the PML equation (4.4)-(4.6)by STDDM.

Example 4.1. Let $L:=l_1+d_1=l_2+d_2=0.5$ and choose the thickness of the PML layer $d_1=d_2=\lambda$, where $\lambda=2\pi/k$ is the wavelength. We suppose the scatterer is a box located in the center of domain $[-0.5,0.5]^2$. The length and width of the box are both 4 wavelengths. In the following tests, we suppose that there is no incoming wave, which means $g(x)=0$. We test two cases: the external force $f(x)$ is a narrow Gaussian point source located at $(x,y)=(-0.25,-0.25)$ and $(x,y)=(-0.25,0.0)$.

We take the medium property σ_j, $j=1,2$, the same as that of Example 3.1. We use the same numerical settings as in Example 3.1 in Algorithms 3.1 and 3.2. In Algorithms 4.1 and 4.2, we also use finite difference scheme and the similar settings as that in Example 3.1. We use STDDM as the preconditioner of the GMRES method for solving the discrete problem of (4.4)-(4.6). The relative residue tolerance in GMRES solver is set to be 10^{-10}. We choose $\bar{\sigma}$ such that the exponentially decaying factor

$$e^{-\frac{1}{2}k\gamma\bar{\sigma}}=e^{-\frac{k\lambda\bar{\sigma}}{2\sqrt{\lambda^2+1}}}\leq 10^{-10}.$$

We use central finite difference scheme on a uniform $n\times n$ grid to discretize the PML problem (4.4)-(4.6) in B_L. We discretize one wavelength with q points, such that one direction with $n=kq/2\pi$ points. So the number of unknowns is $(kq/2\pi)^2-(4q)^2$. We list the results of these two cases in Tables 3 and 4. The real part of the solutions are given in Figure 4.4. The results also confirm that STDDM works very well for the Helmholtz scattering problems.

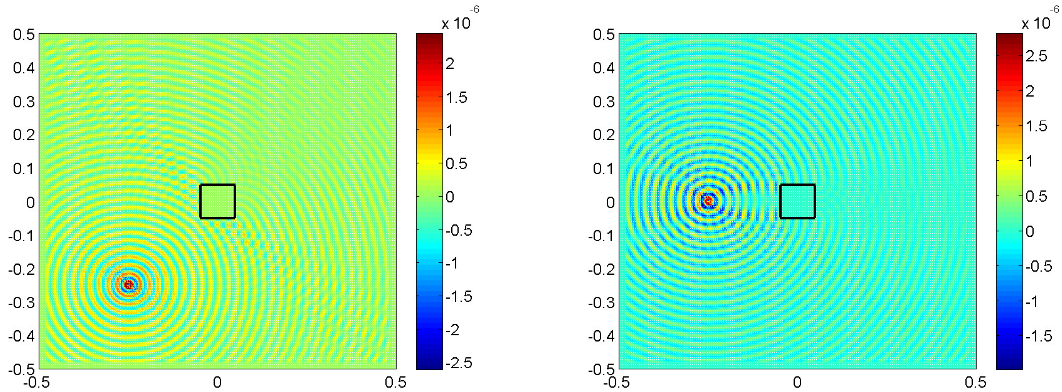

Figure 4.4: The real part of the solutions of scattering problem. $k/2\pi = 40$, $q = 10$. Left: The external force located at $(0.25, 0.25)$; Right: The external force located at $(0.25, 0.5)$.

5. Helmholtz problems in 3D

In this section we consider the STDDM method for the following 3D Helmholtz problems:

$$\Delta u + k^2 u = f \quad \text{in } \mathbb{R}^3, \tag{5.1}$$

$$r\left(\frac{\partial u}{\partial r} - \mathbf{i}ku\right) \rightarrow 0 \quad \text{as } r = |x| \rightarrow \infty, \tag{5.2}$$

where $k > 0$ is the wave number and $f \in H^{-1}_{\text{comp}}(\mathbb{R}^3)$, that is, $f \in H^{-1}(\mathbb{R}^3)$ and has compact support.

Let the source f be supported inside $B_l = \{x \in \mathbb{R}^3 : |x_1| < l_1, |x_2| < l_2, |x_3| < l_3\}$ and $\alpha_3(x_3) = 1 + \mathbf{i}\sigma_3(x_3)$, $\sigma_3 = \sigma_1$. Similar to the 2D problems, we have that the PML equation in unbounded domain for (5.1)-(5.2)

$$J^{-1}\nabla \cdot (A\nabla \tilde{u}) + k^2 \tilde{u} = f \quad \text{in } \mathbb{R}^3, \tag{5.3}$$

where $A(x) = \text{diag}\left(\frac{\alpha_2(x_2)\alpha_3(x_3)}{\alpha_1(x_1)}, \frac{\alpha_1(x_1)\alpha_3(x_3)}{\alpha_2(x_2)}, \frac{\alpha_1(x_1)\alpha_2(x_2)}{\alpha_3(x_3)}\right)$ is a diagonal matrix and $J(x) = \alpha_1(x_1)\alpha_2(x_2)\alpha_3(x_3)$.

Let $B_L = (-l_1 - d_1, l_1 + d_1) \times (-l_2 - d_2, l_2 + d_2) \times (-l_3 - d_3, l_3 + d_3)$, $d_1, d_2, d_3 > 0$, be the box that includes B_l. Then the PML problem whose solution approximates $\tilde{u}$ in (5.3) in the bounded domain B_L is defined by

$$\nabla \cdot (A\nabla \hat{u}) + k^2 J\hat{u} = f \quad \text{in } B_L, \tag{5.4}$$

$$\hat{u} = 0 \quad \text{on } \partial B_L. \tag{5.5}$$

Throughout this section we assume that the medium property satisfies

(H2) $l_1 = l_2 = l_3$, $d_1 = d_2 = d_3$, and

$$\int_{l_1}^{l_1+d_2} \sigma_1(t)dt \quad = \quad \int_{l_2}^{l_2+d_2} \sigma_2(t)dt = \int_{l_3}^{l_3+d_3} \sigma_3(t)dt = \bar{\sigma}.$$

For any $a, b \in R$, we denote $\Omega(a,b) = \{x \in \mathbb{R}^3 : a < x_3 < b\}$. We also use the notation $\Omega(-\infty, b)$ for the half-space $\{x \in \mathbb{R}^3 : x_3 < b\}$. Let $-l_3 = \zeta_1 < \zeta_2 < \cdots < \zeta_{N+1} = l_3$, $N > 1$, be a division of the interval $(-l_3, l_3)$. We also assume $\zeta_i = \zeta_1 + (i-1)\Delta\zeta$,where $\Delta\zeta = 2l_3/N$. Let $\Omega_i = \Omega(\zeta_i, \zeta_{i+1})$ and $\Gamma_i = \{x \in \mathbb{R}^3 : x_3 = \zeta_i\}$. We define $f_i = f|_{\Omega_i}$ in Ω_i and $f_i = 0$ in $\mathbb{R}^3 \backslash \bar{\Omega}_i$. Denote by $\beta_i = \beta_i(x_3)$ a smooth function defined in Ω_i such that

$$\beta_i = 1, \beta_i' = 0 \;\text{ on } \Gamma_i, \;\; \beta_i = \beta_i' = 0 \;\text{ on } \Gamma_{i+1}, \;\; |\beta_i'| \le C(\Delta\zeta)^{-1} \;\text{ in } (\zeta_i, \zeta_{i+1}). \tag{5.6}$$

We also define the PML complex coordinate stretching outside the domain $(-l_1, l_1) \times (-l_2, l_2) \times (\zeta_i, \zeta_{i+2})$ as $\tilde{x}_{i+1}(x) = (\tilde{x}_{i+1}(x_1), \tilde{x}_{i+1}(x_2), \tilde{x}_{i+1}(x_3))^T$, where $\tilde{x}_{i+1}(x_1) = \tilde{x}(x_1)$, $\tilde{x}_{i+1}(x_2) = \tilde{x}(x_2)$ and

$$\tilde{x}_{i+1}(x_3) = \begin{cases} x_3 + \mathbf{i}\int_{\zeta_{i+2}}^{x_3} \sigma_3(t + \zeta_N - \zeta_{i+2})dt & \text{if } x_3 > \zeta_{i+2}, \\ x_3 & \text{if } \zeta_i \le x_3 \le \zeta_{i+2}, \\ x_3 + \mathbf{i}\int_{\zeta_i}^{x_3} \sigma_3(t - \zeta_i + \zeta_1)dt & \text{if } x_3 < \zeta_i. \end{cases} \tag{5.7}$$

We denote $A_i(x) = \operatorname{diag}\left(\frac{\tilde{x}_i(x_2)'\tilde{x}_i(x_3)'}{\tilde{x}_i(x_1)'}, \frac{\tilde{x}_i(x_1)'\tilde{x}_i(x_3)'}{\tilde{x}_i(x_2)'}, \frac{\tilde{x}_i(x_1)'\tilde{x}_i(x_2)'}{\tilde{x}_i(x_3)'}\right)$ and $J_i(x) = \tilde{x}_i(x_1)'\tilde{x}_i(x_2)'\tilde{x}_i(x_3)'$.

To obtain the STDDM for Helmholtz problems in 3D, we just need to set $\Omega_i^{\text{PML}} = (-l_1 - d_1, l_1 + d_1) \times (-l_2 - d_2, l_2 + d_2) \times (\zeta_i - d_3, \zeta_{i+2} + d_3)$ in Algorithm 2.1 and $D_i^{\text{PML}} = (-l_1 - d_1, l_1 + d_1) \times (-l_2 - d_2, l_2 + d_2) \times (\zeta_{i-1} - d_3, \zeta_{i+1})$ in Algorithm 2.2 to obtain new algorithms for the source transfer and wave expansion algorithms.

We remark that there exists another strategy for the 3D problem. In the new Algorithms 2.1 and 2.2, we need to solve a subproblem defined in domain $(-l_1 - d_1, l_1 + d_1) \times (-l_2 - d_2, l_2 + d_2) \times (a, b)$, where a, b is different in each subproblem. This subproblem can be solved again by the STDDM method along x_1 or x_2 direction, just like the method we use in solving the scattering problem in section 4. Here we omit the details.

Example 5.1. Let $L := l_1 + d_1 = l_2 + d_2 = L_3 + d_3 = 0.5$ and choose the thickness of the PML layer $d_1 = d_2 = d_3 = \lambda$, where $\lambda = 2\pi/k$ is the wavelength. We also take that the medium property

$$\sigma_j(t) = \tilde{\sigma}_j \left(\frac{|t| - l_j}{d_2}\right)^m, \; m \ge 1 \text{ integer}, \; \tilde{\sigma}_j > 0 \text{ is a constant}, \; j = 1, 2, 3,$$

where $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, and $\tilde{\sigma}_3$ are determined from $\bar{\sigma}$ by $\tilde{\sigma}_j = (m+1)\bar{\sigma}/d_2$, $j = 1, 2, 3$.

We set $m = 2$ and choose $\bar{\sigma}$ such that the exponentially decaying factor $e^{-\frac{1}{2}k\gamma\bar{\sigma}} \le 10^{-10}$, where $\gamma = \frac{\lambda}{\sqrt{\lambda^2+1}}$.

$k/2\pi$	q	NOF	N_{iter}	T_{solve}
12	8	96^3	8	1447.43
18	8	144^3	10	5266.80
24	8	192^3	11	15706.33

Table 5: Numerical results for different wave numbers k in 3D.

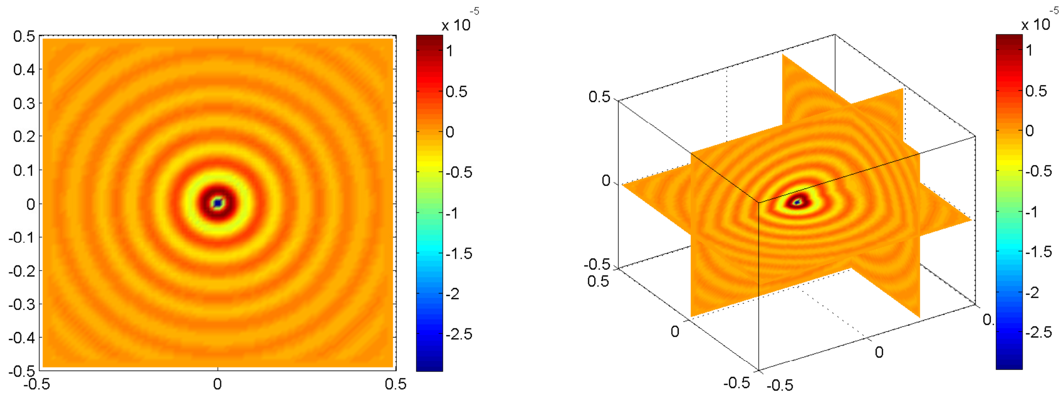

Figure 5.5: The real part of the solution shen $k/2\pi = 12$ and $q = 8$. Left: Near plane $z = 0.0$; Right: Near plane $x = 0.0$, $y = 0.25$ and $z = 0.0$.

We use central finite difference scheme on a uniform $n \times n \times n$ grid to discretize the PML problem (5.4)-(5.5) in B_L. The grid spacing is h so that $L = (n+1)h/2$. Let $d_3 = (p+1/2)h$ and $\Delta\zeta = ch$ which requires $c = (n-2p)/N$, N is the number of layers. The PML equations in the new source transfer and wave expansion algorithm are discretized using the central finite difference scheme. The function β_i in Ω_i in the definition of the source transfer algorithm is discretized by setting β_i being one in the first c_1 planes near Γ_i and zero in the last c_2 planes near Γ_{i+1} ($c_1, c_2 > 0$ and $c_1 + c_2 = c$). $\hat{\Psi}_{i+1}(\hat{f}_i)$ is also discretized by using central finite difference scheme. We use our STDDM as the preconditioner of the GMRES method for solving the discrete problem of (5.4)-(5.5). The relative residue tolerance in GMRES solver is set to be 10^{-12}. We discretize one wavelength with q points, such that one direction with $n = kq/2\pi$ points. Then the number of unknowns is $(qk/2\pi)^3$. We set the external force $f(x)$ is a narrow Gaussian point source located at the origin:

$$f(x_1, x_2, x_3) = e^{-(\frac{4k}{\pi})^2((x_1-r_1)^2+(x_2-r_2)^2+(x_3-r_3)^2)}.$$

We also set $c = q$, $p = q$, $c_1 = 4$ and $c_2 = c - 4$.

From Table 5 we observe that STDDM works quite well in this 3D case. We show the real part of solution when $k/2\pi = 12$ and $q = 8$ in Figure 5.5.

6. Conclusions

In this paper we have extended the STDDM to solve the Helmholtz problems in two-layered media, the Helmholtz scattering problems with bounded scatterer, and Helmholtz problems in 3D unbounded domains. Numerical results indicated clearly the efficiency of STDDM as a preconditioner for solving the discretization problem of the Helmholtz problems considered. We remark that STDDM provides a general framework to decompose the domain and construct the preconditioner in the continuous setting for wave propagation problems in unbounded domains. The method is independent of the special meshes and discretization methods used in the subdomains. We will extend the method to solve electromagnetic and elastic wave propagation problems in unbounded domains in forthcoming works.

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