STOCHASTIC CONVERGENCE OF A NONCONFORMING FINITE ELEMENT METHOD FOR THE THIN PLATE SPLINE SMOOTHER FOR OBSERVATIONAL DATA

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Abstract. The thin plate spline smoother is a classical model for finding a smooth function from the knowledge of its observation at scattered locations which may have random noises. We consider a nonconforming Morley finite element method to approximate the model. We prove the stochastic convergence of the finite element method which characterizes the tail property of the probability distribution function of the finite element error. We also propose a self-consistent iterative algorithm to determine the smoothing parameter based on our theoretical analysis. Numerical examples are included to confirm the theoretical analysis and to show the competitive performance of the self-consistent algorithm for finding the smoothing parameter.

Key words. Thin plate spline, Morley element, stochastic convergence, optimal parameter choice.

1. Introduction. The thin plate spline smoother is a classical mathematical model for finding a smooth function from the knowledge of its observation at scattered locations which may be subject to random noises. Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ $(d \le 3)$ and $u_0 \in H^2(\Omega)$ be the unknown smooth function. Let $\{x_i\}_{i=1}^n \subset \Omega$ be the scattered locations in the domain where the observations are taken. We want to approximate u_0 from the noisy data $y_i = u_0(x_i) + e_i,\ 1 \le i \le n$, where $\{e_i\}_{i=1}^n$ are independent and identically distributed random variables on some probability space $(\mathfrak{X}, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{E}[e_i] = 0$ and $\mathbb{E}[e_i^2] \le \sigma^2$. Here and in the following $\mathbb{E}[X]$ denotes the expectation of the random variable X. The thin plate spline smoother is defined to be the unique solution of the following variational problem

$$\min_{u \in H^2(\Omega)} \frac{1}{n} \sum_{i=1}^n (u(x_i) - y_i)^2 + \lambda_n |u|^2_{H^2(\Omega)}, \tag{1.1}$$

where $\lambda_n > 0$ is the smoothing parameter.

The spline model for scattered data has been extensively studied in the literature. For $\Omega = \mathbf{R}^d$ and when the minimizer is sought in $D^{-2}L^2(\mathbf{R}^d) = \{u : D^\alpha u \in L^2(\mathbf{R}^d), |\alpha| = 2\}$, [10] proved that (1.1) has a unique solution when the set $\mathbb{T} = \{x_i : i = 1, 2, \cdots, n\}$ is not collinear (i.e. the points in $\mathbb{T}$ are not on the same plane). An explicit formula of the solution is constructed in [10] based on radial basis functions. [15] derived the convergence rate for the expectation of the error $|u_n - u_0|^2_{H^j(\Omega)}$, $j = 0, 1, 2$. Under the assumption that e_i, $i = 1, 2, \cdots, n$, are also sub-Gaussian random variables, [17] proved the stochastic convergence of the error in terms of the empirical norm $\|u_n - u_0\|_n := (n^{-1} \sum_{i=1}^n |u_n(x_i) - u_0(x_i)|^2)^{1/2}$ when $d = 1$. The stochastic convergence which provides additional tail information about

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probability distribution function for the random error is very desirable for the approximation of random variables. We refer to [19] for further information about thin plate spline smoothers.

It is well-known that the numerical method based on radial basis functions to solve the thin plate spline smoother requires the solution of a symmetric indefinite dense linear system of equations of size $O(n)$, which is challenging for applications with very large data sets [13]. Conforming finite element methods for the solution of the thin plate model are studied in [2, 3, 4] and the references therein. In [13] a mixed finite element method for estimating ∇u_n is proposed and the expectation of the finite element error is proved. The advantage of the mixed finite element method in [13] lies in that one can use simple H^1 conforming finite element spaces. The H^1 smoother in [13] that the mixed finite element method aims to approximate is not equivalent to the thin plate spline model (1.1).

In this paper we consider the nonconforming finite element approximation to the problem (1.1). We use the Morley element [12, 14, 16] which is of particular interest for solving fourth order PDEs since it has the least number of degrees of freedom (6 in 2D and 10 in 3D) on each element. As a comparison, the H^2 conforming Argris finite element for fourth order PDEs requires 21 degrees of freedom in 2D and 220 degrees of freedom in 3D on each element. The difficulty of the finite element analysis for the thin plate smoother is the low stochastic regularity of the solution u_n. One can only prove the boundedness of $\mathbb{E}[|u_n|^2_{H^2(\Omega)}]$ (see Theorem 2.2 below). This difficulty is overcome by a smoothing operator based on the C^1-element for any Morley finite element functions. We also prove the probability distribution function of the empirical norm of the finite element error has an exponentially decaying tail. For that purpose we also prove the convergence of the error $\|u_n - u_0\|_n$ in terms of the Orlicz ψ_2 norm (see Theorem 4.8 below) which improves the result in [17].

One of the central issues in the application of the thin plate model is the choice of the smoothing parameter λ_n. In the literature it is usually made by the method of cross validation [19]. The analysis in this paper suggests the optimal choice should be

$$\lambda_n^{1/2+d/8} = \sigma n^{-1/2}(|u_0|_{H^2(\Omega)} + \sigma n^{-1/2})^{-1}. \tag{1.2}$$

Since one does not know u_0 and the standard deviation σ in practical applications, we propose a self-consistent algorithm to determine λ_n from the natural initial guess $\lambda_n = n^{-\frac{4}{4+d}}$. Our numerical experiments show that this self-consistent algorithm convergences very fast and it is very robust with respect to the noises.

The layout of the paper is as follows. In section 2 we recall some preliminary properties of the thin plate model. In section 3 we introduce the nonconforming finite element method and show the convergence of the finite element solution in terms of the expectation of Sobolev norms. In section 4 we study the tail property of the probability distribution function for the finite element error based on the theory of empirical process for sub-Gaussian noises. In section 5 we introduce our self-consistent algorithm for finding the smoothing parameter λ_n and show several numerical examples to support the analysis in this paper. In the appendix we prove a technical lemma on the smoothing operator.

2. The thin plate model. In this section we collect some preliminary results about the thin plate smoother (1.1). In this paper, we will always assume that Ω is a bounded Lipschitz domain satisfying the uniform cone condition (see e.g. [15, P.6]). We will also assume that $\mathbb{T}$ are uniformly distributed in the sense that [15] there exists a constant $B > 0$ such that $\frac{h_{\max}}{h_{\min}} \le B$, where

$$h_{\max} = \sup_{x\in\Omega} \inf_{1\le i\le n} |x - x_i|, \qquad h_{\min} = \inf_{1\le i\ne j\le n} |x_i - x_j|.$$

It is easy to see that there exist constants B_1, B_2 such that $B_1 n^{-1/d} \le h_{\max} \le B h_{\min} \le B_2 n^{-1/d}$.

We write the empirical inner product between the data and any function $v \in C(\bar\Omega)$ as $(y, v)_n = \frac{1}{n}\sum_{i=1}^n y_i v(x_i)$. We also write $(u, v)_n = \frac{1}{n}\sum_{i=1}^n u(x_i)v(x_i)$ for any $u, v \in C(\bar\Omega)$ and the empirical norm $\|u\|_n = (\frac{1}{n}\sum_{i=1}^n u^2(x_i))^{1/2}$ for any $u \in C(\bar\Omega)$. By [15, Theorems 3.3-3.4], there exists a constant $C > 0$ depending only on Ω, B such that for any $u \in H^2(\Omega)$ and sufficiently small $h_{\max}$,

$$\|u\|_{L^2(\Omega)} \le C(\|u\|_n + h_{\max}^2 |u|_{H^2(\Omega)}), \quad \|u\|_n \le C(\|u\|_{L^2(\Omega)} + h_{\max}^2 |u|_{H^2(\Omega)}). \tag{2.1}$$

It follows from (2.1) and the Lax-Milgram lemma that the minimization problem (1.1) has a unique solution $u_n \in H^2(\Omega)$.

Define the bilinear form $a : H^2(\Omega) \times H^2(\Omega) \to \mathbf{R}$ as

$$a(u, v) = \sum_{1\le i,j\le d} \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx \qquad \forall u, v \in H^2(\Omega). \tag{2.2}$$

It is obvious that $|u|^2_{H^2(\Omega)} = a(u, u)$ for any $u \in H^2(\Omega)$. We also denote $(\cdot, \cdot)$ the inner product of $L^2(\Omega)$. The following lemma from [15] plays a key role in studying the convergence of u_n to u_0.

Lemma 2.1. *Let $\mu_1 \le \mu_2 \le \cdots \le \mu_k \le \cdots$ be the eigenvalues of the problem*

$$a(\psi, v) = \mu(\psi, v) \qquad \forall v \in H^2(\Omega). \tag{2.3}$$

Then $\mu_1 = \mu_2 = \cdots = \mu_{d+1} = 0$ and there exist constants $C_1, C_2 > 0$ independent of k such that $C_1(k-1)^{4/d} \le \mu_k \le C_2 k^{4/d}, k = d+2, \cdots$.

Proof. The eigenvalue problem (2.3) obviously has $d+1$ zero eigenvalues with linear polynomials as the eigenfunctions. The rest of the argument is the same as that in the proof of [15, Theorem 5.3]. The key is to use a general theorem in Agmon [1, Theorem 14.6], which concludes that the number of eigenvalues of the biharmonic operator Δ^2 less than or equal to $C_+^{-d/4} k^{4/d}$ is $k(1 + o(1))$. The constant C_+ is independent of k and $o(1)$ goes to zero as $k \to \infty$. The remainder of the proof follows by using some simple argument in [15, Theorem 5.3]. ∎

Theorem 2.2. *Let $u_n \in H^2(\Omega)$ be the unique solution of (1.1). Then there exist constants $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda_n \le \lambda_0$,*

$$\mathbb{E}\big[\|u_n - u_0\|_n^2\big] \le C\lambda_n |u_0|^2_{H^2(\Omega)} + \frac{C\sigma^2}{n\lambda_n^{d/4}}, \tag{2.4}$$

$$\mathbb{E}\big[|u_n|^2_{H^2(\Omega)}\big] \le C|u_0|^2_{H^2(\Omega)} + \frac{C\sigma^2}{n\lambda_n^{1+d/4}}. \tag{2.5}$$

This theorem is proved in [15] when $\Omega = \mathbf{R}^d$ and the minimizer is sought in $D^{-2}L^2(\mathbf{R}^d) = \{u : D^\alpha u \in L^2(\mathbf{R}^d), |\alpha| = 2\}$ in the problem (1.1). Here we give a simpler proof based on Lemma 2.1.

Proof. It is clear that $u_n \in H^2(\Omega)$ satisfies the following variational equation

$$\lambda_n a(u_n, v) + (u_n, v)_n = (y, v)_n \quad \forall v \in H^2(\Omega). \tag{2.6}$$

For any $v \in H^2(\Omega)$, denote the energy norm $|||v|||_{\lambda_n}^2 := \lambda_n a(v,v) + \|v\|_n^2$. By taking $v = u_n - u_0$ in (2.6) one obtains easily

$$|||u_n - u_0|||_{\lambda_n} \leq \lambda_n^{1/2} |u_0|_{H^2(\Omega)} + \sup_{v \in H^2(\Omega)} \frac{(e,v)_n}{|||v|||_{\lambda_n}}, \tag{2.7}$$

where e represents the random error vector. Let $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_k \leq \cdots$ be the eigenvalues of the problem

$$a(\psi, v) = \rho(\psi, v)_n \quad \forall v \in H^2(\Omega). \tag{2.8}$$

It is clear that $\rho_1 = \cdots = \rho_{d+1} = 0$ with the linear polynomials as the eigenfunctions. By using (2.1) and the max-min principle of the Rayleigh quotient for the eigenvalues, one obtains easily

$$\mu_k \leq C \frac{\rho_k}{1 + n^{-4/d}\rho_k} \leq C\rho_k, \quad k = d+2, \cdots. \tag{2.9}$$

Let $\{\psi_k\}_{k=1}^\infty$ be the eigenfunctions of (2.8) corresponding to eigenvalues $\{\rho_k\}_{k=1}^\infty$ satisfying $(\psi_k, \psi_l)_n = \delta_{kl}$, where δ_{kl} is the Kronerker delta function, $k, l = 1, 2, \cdots$. $\{\psi_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$ in the inner product $(\cdot,\cdot)_n$. Now for any $v \in H^2(\Omega)$, we have the expansion $v(x) = \sum_{k=1}^\infty v_k \psi_k(x)$, where $v_k = (v, \psi_k)_n$, $k = 1, 2, \cdots$. Thus $|||v|||_{\lambda_n}^2 = \sum_{k=1}^\infty (\lambda_n \rho_k + 1) v_k^2$. By the Cauchy-Schwarz inequality

$$(e,v)_n^2 \leq \frac{1}{n^2} \sum_{k=1}^\infty (1 + \lambda_n \rho_k) v_k^2 \cdot \sum_{k=1}^\infty (1 + \lambda_n \rho_k)^{-1} \left(\sum_{i=1}^n e_i \psi_k(x_i) \right)^2.$$

This implies

$$\begin{aligned}
\mathbb{E}\left[\sup_{v \in H^2(\Omega)} \frac{(e,v)_n^2}{|||v|||_{\lambda_n}^2} \right] &\leq \frac{1}{n^2} \sum_{k=1}^\infty (1 + \lambda_n \rho_k)^{-1} \mathbb{E}\left(\sum_{i=1}^n e_i \psi_k(x_i) \right)^2 \\
&= \sigma^2 n^{-1} \sum_{k=1}^\infty (1 + \lambda_n \rho_k)^{-1},
\end{aligned}$$

where we have used the fact that $\|\psi_k\|_n = 1$. Now by Lemma 2.1 and (2.9) we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{v \in H^2(\Omega)} \frac{(e,v)_n^2}{|||v|||_{\lambda_n}^2} \right] &\leq C\sigma^2 n^{-1} \sum_{k=1}^\infty (1 + \lambda_n k^{4/d})^{-1} \\
&\leq C\sigma^2 n^{-1} \int_0^\infty (1 + \lambda_n t^{4/d})^{-1} dt \\
&= C \frac{\sigma^2}{n \lambda_n^{d/4}}.
\end{aligned}$$

This completes the proof by using (2.7). □

Theorem 2.1 suggests that an optimal choice of the parameter λ_n is such that $\lambda_n^{1+d/4} = O((\sigma^2 n^{-1}) |u_0|_{H^2(\Omega)}^{-2})$.

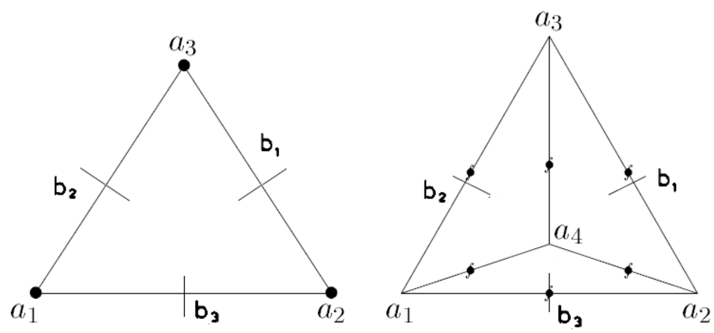


FIG. 3.1. *The degrees of freedom of 2D (left) and 3D (right) Morley element.*

3. Nonconforming finite element method. In this section we consider the nonconforming finite element approximation to the thin plate model (1.1) whose solution $u_n \in H^2(\Omega)$ satisfies the following weak formulation

$$\lambda_n a(u_n, v) + (u_n, v)_n = (y, v)_n \quad \forall v \in H^2(\Omega). \tag{3.1}$$

We assume Ω is a polygonal or polyhedral domain in $\mathbf{R}^d$ $(d = 2, 3)$ in the reminder of this paper. Let $\mathcal{M}_h$ be a family of shape regular and quasi-uniform finite element meshes over the domain Ω. We will use the Morley element [12] for 2D, [16] for 3D to define our nonconforming finite element method. The Morley element is a triple (K, P_K, Σ_K), where $K \in \mathcal{M}_h$ is a simplex in $\mathbf{R}^d$, $P_K = P_2(K)$ is the set of second order polynomials in K, and Σ_K is the set of the degrees of freedom. In 2D, for the element K with vertices $a_i, 1 \le i \le 3$, and mid-points b_i of the edge opposite to the vertex a_i, $1 \le i \le 3$, $\Sigma_K = \{p(a_i), \partial_\nu p(b_i), 1 \le i \le 3, \forall p \in C^1(K)\}$. In 3D, for the element K with edges S_{ij} which connects the vertices a_i, a_j, $1 \le i < j \le 4$, and faces F_j opposite to a_j, $1 \le j \le 4$, $\Sigma_K = \{\frac{1}{|S_{ij}|}\int_{S_{ij}} p, 1 \le i < j \le 4, \frac{1}{|F_j|}\int_{F_j} \partial_\nu p, 1 \le j \le 4, \forall p \in C^1(K)\}$. Here $\partial_\nu p$ is the normal derivative of p of the edges (2D) or faces (3D) of the element. We refer to Figure 3.1 for the illustration of the degrees of freedom of the Morley element.

Let V_h be the Morley finite element space

$$V_h = \{v_h : v_h|_K \in P_2(K)\ \forall K \in \mathcal{M}_h, f(v_h|_{K_1}) = f(v_h|_{K_2})\ \forall f \in \Sigma_{K_1} \cap \Sigma_{K_2}\}.$$

The functions in V_h may not be continuous in Ω. Given a set $G \subset \mathbf{R}^2$, let $\mathcal{M}_h(G) = \{K \in \mathcal{M}_h : G \cap K \ne \emptyset\}$ and $N(G)$ the number of elements in $\mathcal{M}_h(G)$. For any $v_h \in V_h$, we define

$$\hat{v}_h(x_i) = \frac{1}{N(x_i)} \sum_{K \in \mathcal{M}_h(x_i)} (v_h|_K)(x_i), \quad i = 1, 2, \cdots, n. \tag{3.2}$$

Notice that if x_i is located inside some element K, then $\mathcal{M}_h(x_i) = \{K\}$ and $\hat{v}_h(x_i) = v_h(x_i)$, $i = 1, 2, \cdots, n$. With this definition we know that $(\hat{v}_h, \hat{w}_h)_n$ and $(e, \hat{w}_h)_n$ are well-defined for any $v_h, w_h \in V_h$. We recall that $(e, \hat{w}_h)_n = n^{-1}\sum_{i=1}^n e_i \hat{w}_h(x_i)$.

Let

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{M}_h} \sum_{1 \le i,j \le d} \int_K \frac{\partial^2 u_h}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j} dx \quad \forall u_h, v_h \in V_h.$$

The finite element approximation of the problem (3.1) is to find $u_h \in V_h$ such that

$$\lambda_n a_h(u_h, v_h) + (\hat{u}_h, \hat{v}_h)_n = (y, \hat{v}_h)_n \quad \forall v_h \in V_h. \tag{3.3}$$

Since the sampling point set $\mathbb{T}$ is not collinear, by the Lax-Milgram lemma, the problem (3.3) has a unique solution. Here we recall that $(y, \hat{v}_h)_n = n^{-1}\sum_{i=1}^n y_i \hat{v}_h(x_i)$.

The following theorem is the main result of this section.

THEOREM 3.1. *Let $u_n \in H^2(\Omega)$ be the unique solution of (3.1) and $u_h \in V_h$ be the solution of (3.3). Then there exist constants $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda_n \le \lambda_0$,*

$$\mathbb{E}\big[\|u_0 - \hat{u}_h\|_n^2\big] \le C(\lambda_n + h^4)|u_0|_{H^2(\Omega)}^2 + C\left[1 + \frac{h^4}{\lambda_n} + \left(\frac{h^4}{\lambda_n}\right)^{1-d/4}\right]\frac{\sigma^2}{n\lambda_n^{d/4}}. \tag{3.4}$$

In particular, if $h^4 \le C\lambda_n$, we have

$$\mathbb{E}\big[\|u_0 - \hat{u}_h\|_n^2\big] \le C\lambda_n|u_0|_{H^2(\Omega)}^2 + \frac{C\sigma^2}{n\lambda_n^{d/4}}. \tag{3.5}$$

This theorem suggests that one should take the mesh size $h = O(\lambda_n^{1/4})$ to achieve the optimal balance of controlling the finite element error and the modeling error due to λ_n. The proof depends on several lemmas that follow.

Let $I_K : H^2(K) \to P_2(K)$ be the canonical local nodal value interpolant of Morley element [14, 16] and $I_h : L^2(\Omega) \to V_h$ be the global nodal value interpolant such that $(I_h u)|_K = I_K u$ for any $K \in \mathcal{M}_h$ and piecewise $H^2(K)$ functions $u \in L^2(\Omega)$.

LEMMA 3.2. *We have*

$$|u - I_K u|_{H^m(K)} \le Ch_K^{2-m}|u|_{H^2(K)} \quad \forall u \in H^m(K), 0 \le m \le 2, \tag{3.6}$$

$$\|u - \widehat{I_h u}\|_n \le Ch^2|u|_{H^2(\Omega)} \quad \forall u \in H^2(\Omega), \tag{3.7}$$

where h_K is the diameter of the element K and $h = \max_{K\in\mathcal{M}_h} h_K$.

Proof. Since $I_K p = p$ for any $p \in P_2(K)$ [16], the estimate (3.6) follows from the standard interpolation theory for finite element method [9]. Moreover, we have, by local inverse estimates and the standard interpolation estimates

$$\begin{aligned}\|u - I_K u\|_{L^\infty(K)} &\le \inf_{p\in P_2(K)}\Big[\|u - p\|_{L^\infty(K)} + |K|^{-1/2}\|I_K(u-p)\|_{L^2(K)}\Big]\\ &\le Ch_K^{2-d/2}|u|_{H^2(K)}.\end{aligned}$$

Let $\mathbb{T}_K = \{x_i \in \mathbb{T} : x_i \in K, 1 \le i \le n\}$. By the assumption $\mathbb{T}$ is uniformly distributed and the mesh is quasi-uniform, we know that the cardinal $\#\mathbb{T}_K \le Cnh^d$. Thus

$$\|u - \widehat{I_h u}\|_n^2 \le \frac{1}{n}\sum_{K\in\mathcal{M}_h}\#\mathbb{T}_K\|u - I_K u\|_{L^\infty(K)}^2 \le Ch^4|u|_{H^2(\Omega)}^2.$$

This proves (3.7). ☐

LEMMA 3.3. *Let $K, K' \in \mathcal{M}_h$ and $F = K \cap K'$. There exists a constant C independent of h such that for any $v_h \in V_h$, $|\alpha| \le 2$,*

$$\|\partial^\alpha(v_h|_K - v_h|_{K'})\|_{L^\infty(F)} \le Ch^{2-|\alpha|-d/2}(|v_h|_{H^2(K)} + |v_h|_{H^2(K')}).$$

Proof. By [16, Lemma 5] we know that

$$\|v_h|_K - v_h|_{K'}\|_{L^2(F)} \le Ch^{3/2}(|v_h|_{H^2(K)} + |v_h|_{H^2(K')}).$$

By using the inverse estimate we then obtain

$$\begin{aligned}\|\partial^\alpha(v_h|_K - v_h|_{K'})\|_{L^\infty(F)} &\le Ch^{-|\alpha|}\|v_h|_K - v_h|_{K'}\|_{L^\infty(F)}\\ &\le Ch^{-|\alpha|-(d-1)/2}\|v_h|_K - v_h|_{K'}\|_{L^2(F)}\\ &\le Ch^{2-|\alpha|-d/2}(|v_h|_{H^2(K)} + |v_h|_{H^2(K')}).\end{aligned}$$

This proves the lemma. □

Let $\widehat{\mathcal{M}}_h \subset \mathcal{M}_h$ be a collection of elements $K \in \mathcal{M}_h$, we introduce the mesh dependent semi-norm $|\cdot|_{m,\widehat{\mathcal{M}}_h}$, $m \ge 0$,

$$|v|_{m,\widehat{\mathcal{M}}_h} = \left(\sum_{K\in\widehat{\mathcal{M}}_h} |v|^2_{H^m(K)}\right)^{1/2}, \tag{3.8}$$

for any function $v|_K \in H^m(K)$, $K \in \mathcal{M}_h$. We set $|v|_{m,h} = |v|_{m,\mathcal{M}_h}$ for $m = 0, 1, 2$. The following technical lemma will be proved in the appendix of this paper.

Lemma 3.4. *There exists a linear operator* $\Pi_h : V_h \to H^2(\Omega)$ *such that*

$$|v_h - \Pi_h v_h|_{m,h} \le Ch^{2-m}|v_h|_{2,h}, \quad m = 0, 1, 2, \quad \forall v_h \in V_h, \tag{3.9}$$

$$\|\hat{v}_h - \Pi_h v_h\|_n \le Ch^2|v_h|_{2,h} \quad \forall v_h \in V_h. \tag{3.10}$$

where the constant C *is independent of* h.

For any function v which is piecewise in $H^2(K)$ for any $K \in \mathcal{M}_h$, we use the convenient discrete energy norm

$$|||v|||_{h,\lambda_n} = \left(\lambda_n|v|^2_{2,h} + \|\hat{v}\|^2_n\right)^{1/2}.$$

Here $\hat{v}(x_i)$, $i = 1, 2, \cdots, n$, is defined as in (3.2), that is, $\hat{v}(x_i)$ is the local average of all $v|_K(x_i)$, where $K \in \mathcal{M}_h$ such that $x_i \in K$.

Lemma 3.5. *Let* $u_n \in H^2(\Omega)$ *be the unique solution of* (3.1) *and* $u_h \in V_h$ *be the solution of* (3.3)*. Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}\lambda_n^{1/2}|u_h|_{2,h} + \|\hat{u}_h - u_0\|_n \le C(h^2 + \lambda_n^{1/2})|u_n|_{H^2(\Omega)} + C\frac{h^2}{\lambda_n^{1/2}}\|u_n - u_0\|_n\\ + C\sup_{0\ne v_h\in V_h}\frac{|(e, \hat{v}_h - \Pi_h v_h)_n|}{|||v_h|||_{h,\lambda_n}}.\end{aligned} \tag{3.11}$$

Proof. Since $u_h \in V_h$ satisfies (3.3), we have

$$\begin{aligned}\lambda_n a_h(u_h - v_h, w_h) + (\hat{u}_h - \hat{v}_h, \hat{w}_h)_n = \lambda_n a_h(u_n - v_h, w_h) + (u_n - \hat{v}_h, \hat{w}_h)_n\\ - \lambda_n a_h(u_n, w_n) - (u_n - y, \hat{w}_h)_n \quad \forall v_h, w_h \in V_h.\end{aligned}$$

By taking $w_h = u_h - v_h$, one obtains easily the following Strang lemma [9]

$$\begin{aligned}|||u_n - u_h|||_{h,\lambda_n} \le C\inf_{v_h\in V_h}|||u_n - v_h|||_{h,\lambda_n}\\ + C\sup_{0\ne v_h\in V_h}\frac{|\lambda_n a_h(u_n, v_h) + (u_n - y, \hat{v}_h)_n|}{|||v_h|||_{h,\lambda_n}}.\end{aligned} \tag{3.12}$$

By Lemma 3.2 we have

$$\inf_{v_h\in V_h} |||u_n-\hat v_h|||_{h,\lambda_n} \le C(\lambda_n^{1/2}+h^2)|u_n|_{H^2(\Omega)}. \tag{3.13}$$

Since for any $v_h\in V_h$, $\Pi_h v_h\in H^2(\Omega)$, by (3.1), and the fact that $y_i=u_0(x_i)+e_i$, $i=1,2,\cdots n$, we obtain

$$\begin{aligned}
&\lambda_n a_h(u_n,v_h)+(u_n-y,\hat v_h)_n\\
&=\lambda_n a_h(u_n,v_h-\Pi_h v_h)+(u_n-y,\hat v_h-\Pi_h v_h)_n\\
&\le \lambda_n|u_n|_{H^2(\Omega)}|v_h-\Pi_h v_h|_{2,h}+\|u_n-u_0\|_n\|\hat v_h-\Pi_h v_h\|_n+(e,\hat v_h-\Pi_h v_h)_n\\
&\le C\lambda_n^{1/2}|u_n|_{H^2(\Omega)}|||v_h|||_{n,\lambda_n}+C\frac{h^2}{\lambda_n^{1/2}}\|u_n-u_0\|_n|||v_h|||_{n,\lambda_n}+|(e,\hat v_h-\Pi_h v_h)_n|,
\end{aligned}$$

where we have used Lemma 3.4 in the last inequality. This completes the proof by inserting the above estimate and (3.13) into (3.12). □

Now we are ready to prove the main result of this section.

Proof of Theorem 3.1. From Lemma 3.5 and Theorem 2.2 we know that we are left to estimate the expectation of the last term in (3.11). By using Lemma 3.4 again we obtain

$$\sup_{0\ne v_h\in V_h}\frac{|(e,\hat v_h-\Pi_h v_h)_n|}{|||v_h|||_{h,\lambda_n}} \le C\frac{h^2}{\lambda_n^{1/2}}\sup_{0\ne w_h\in W_h}\frac{|(e,\hat w_h)_n|}{\|w_h\|_{L^2(\Omega)}}, \tag{3.14}$$

where $W_h=\{w_h=v_h-\Pi_h v_h: v_h\in V_h\}$ and we set $\widehat{\Pi_h v_h}(x_i)=\Pi_h v_h(x_i)$ which is legible since $\Pi_n v_h$ is a continuous function. Now we use a general argument in [18, P.334]. Let N_h be the dimension of W_h and $\{\psi_j\}_{j=1}^{N_h}$ be the orthonormal basis of W_h in the $L^2(\Omega)$ inner product. Then for any $w_h=\sum_{j=1}^{N_h}(w_h,\psi_j)\psi_j$, by Cauchy-Schwarz inequality

$$\begin{aligned}
|(e,\hat w_h)_n|^2 &\le \frac{1}{n^2}\sum_{j=1}^{N_h}(w_h,\psi_j)^2\cdot\sum_{j=1}^{N_h}\left(\sum_{i=1}^n e_i\psi_j(x_i)\right)^2\\
&=\frac{1}{n^2}\|w_h\|^2_{L^2(\Omega)}\cdot\sum_{j=1}^{N_h}\left(\sum_{i=1}^n e_i\psi_j(x_i)\right)^2.
\end{aligned}$$

Since e_i, $i=1,2,\cdots,n$, are independent and identically distributed random variables, we have then

$$\mathbb{E}\left[\sup_{0\ne w_h\in W_h}\frac{|(e,\hat w_h)_n|^2}{\|w_h\|^2_{L^2(\Omega)}}\right]\le\frac{\sigma^2}{n^2}\sum_{j=1}^{N_h}\sum_{i=1}^n\psi_j(x_i)^2. \tag{3.15}$$

Now since $\psi_j\in W_h$ is piecewise polynomial, we obtain by inverse estimate that

$$\sum_{j=1}^{N_h}\sum_{i=1}^n\psi_j(x_i)^2\le\sum_{j=1}^{N_h}\sum_{K\in\mathcal{M}_h}\#\mathbb{T}_K\|\psi_j\|^2_{L^\infty(K)}\le\sum_{j=1}^{N_h}Cn\|\psi_j\|^2_{L^2(\Omega)}\le Cnh^{-d},$$

where $\mathbb{T}_K=\mathbb{T}\cap K$ and we have used the fact that $\#\mathbb{T}_K\le Cnh^d$, $\|\psi_j\|_{L^2(\Omega)}=1$, and $N_h\le Ch^{-d}$ because the mesh is quasi-uniform. This implies by (3.14)-(3.15) that

$$\mathbb{E}\left[\sup_{0\ne v_h\in V_h}\frac{|(e,v_h-\Pi_h v_h)_n|^2}{|||v_h|||^2_{h,\lambda_n}}\right]\le C\frac{\sigma^2h^{4-d}}{n\lambda_n}. \tag{3.16}$$

Now from Lemma 3.5 and (3.16) we obtain

$$\mathbb{E}\big[\|u_n - \hat{u}_h\|_n^2\big] \le C(\lambda_n + h^4)\mathbb{E}\big[|u_n|_{H^2(\Omega)}^2\big] + C\frac{h^4}{\lambda_n}\mathbb{E}\big[\|u_n - u_0\|_n^2\big] + C\frac{\sigma^2 h^{4-d}}{n\lambda_n}.$$

This completes the proof by using Theorem 2.2. □

4. Stochastic convergence. In this section we study the stochastic convergence of the error $\|u_0 - \hat{u}_h\|_n$ which characterizes the tail property of $\mathbb{P}(\|u_0 - \hat{u}_h\|_n \ge z)$ for $z > 0$. We assume the noises e_i, $i = 1, 2, \cdots, n$, are independent and identically distributed sub-Gaussian random variables with parameter $\sigma > 0$. A random variable X is sub-Gaussian with parameter σ if it satisfies

$$\mathbb{E}\left[e^{\lambda(X - \mathbb{E}[X])}\right] \le e^{\frac{1}{2}\sigma^2\lambda^2} \quad \forall \lambda \in \mathbf{R}. \tag{4.1}$$

The probability distribution function of a sub-Gaussian random variable has a exponentially decaying tail, that is, if X is a sub-Gaussian random variable, then

$$\mathbb{P}(|X - \mathbb{E}[X]| \ge z) \le 2e^{-\frac{1}{2}z^2/\sigma^2} \quad \forall z > 0. \tag{4.2}$$

In fact, by Markov inequality, for any $\lambda > 0$,

$$\mathbb{P}(X - \mathbb{E}[X] \ge z) = \mathbb{P}(\lambda(X - \mathbb{E}[X]) \ge \lambda z) \le e^{-\lambda z}\mathbb{E}[e^{\lambda(X - \mathbb{E}[X])}] \le e^{-\lambda z + \frac{1}{2}\sigma^2\lambda^2}.$$

By taking $\lambda = z/\sigma^2$ yields $\mathbb{P}(X - \mathbb{E}[X] \ge z) \le e^{-\frac{1}{2}z^2/\sigma^2}$. Similarly, one can prove $\mathbb{P}(X - \mathbb{E}[X] \le -z) \le e^{-\frac{1}{2}z^2/\sigma^2}$. This shows (4.2).

The following theorem is the main result of this section.

THEOREM 4.1. *Let $u_h \in V_h$ be the solution of* (3.3). *Denote by $\rho_0 = |u_0|_{H^2(\Omega)} + \sigma n^{-1/2}$. If we take $h = O(\lambda_n^{1/4})$ and $\lambda_n^{1/2+d/8} = O(\sigma n^{-1/2}\rho_0^{-1})$, then there exists a constant $C > 0$ such that for any $z > 0$,*

$$\mathbb{P}(\|\hat{u}_h - u_0\|_n \ge \lambda_n^{1/2}\rho_0 z) \le 2e^{-Cz^2}, \quad \mathbb{P}(|u_h|_{2,h} \ge \rho_0 z) \le 2e^{-Cz^2}. \tag{4.3}$$

This theorem indicates that the probability distribution function of the random error $\|\hat{u}_h - u_0\|_n$ decays exponentially as $n \to \infty$. In terms of the terminology of the stochastic convergence order, we have $\|\hat{u}_h - u_0\|_n = O_p(\lambda_n^{1/2})\rho_0$. The proof of this theorem will be given in the subsection 4.2 after we study the stochastic convergence of $\|u_n - u_0\|_n$ in the next subsection.

4.1. Stochastic convergence of the thin plate splines. We will use several tools from the theory of empirical processes [18, 17] for our analysis. We start by recalling the definition of Orlicz norm. Let ψ be a monotone increasing convex function satisfying $\psi(0) = 0$. Then the Orilicz norm $\|X\|_\psi$ of a random variable X is defined as

$$\|X\|_\psi = \inf\left\{C > 0 : \mathbb{E}\left[\psi\left(\frac{|X|}{C}\right)\right] \le 1\right\}. \tag{4.4}$$

By using Jensen's inequality, it is easy to check $\|X\|_\psi$ is a norm. If $\psi(t) = t^p, p > 1$, the Orilicz norm is exactly the L^p norm of the random variable. In the following we will use the $\|X\|_{\psi_2}$ norm with $\psi_2(t) = e^{t^2} - 1$ for any $t > 0$. By definition, for any $C > \|X\|_{\psi_2}$, we have $\mathbb{E}[e^{|X|^2/C^2}] \leq 2$. Thus, for any $z > 0$, $e^{z^2/C^2}\mathbb{P}(|X| \geq z) \leq \mathbb{E}[e^{|X|^2/C^2}] \leq 2$, which yields by letting $C \to \|X\|_{\psi_2}$ that

$$\mathbb{P}(|X| \geq z) \leq 2\, e^{-z^2/\|X\|_{\psi_2}^2} \qquad \forall z > 0. \tag{4.5}$$

Our strategy to prove Theorem 4.1 is to estimate the Orlicz ψ_2-norm of $\|\hat{u}_h - u_0\|_n$, $|u_h|_{2,h}$ and then use (4.5). The following lemma from [18, Lemma 2.2.1] is the inverse of (4.5).

LEMMA 4.2. *If there exist positive constants C, K such that $\mathbb{P}(|X| > z) \leq Ke^{-Cz^2}$, $\forall z > 0$, then $\|X\|_{\psi_2} \leq \sqrt{(1+K)/C}$.*

Let T be a semi-metric space with the semi-metric d and $\{X_t : t \in T\}$ be a random process indexed by T. The random process $\{X_t : t \in T\}$ is called sub-Gaussian if

$$\mathbb{P}(|X_s - X_t| > z) \leq 2e^{-\frac{1}{2}z^2/\mathsf{d}(s,t)^2} \qquad \forall s, t \in T, \quad z > 0. \tag{4.6}$$

For a semi-metric space (T, d) and $\varepsilon > 0$, the covering number $N(\varepsilon, T, \mathsf{d})$ is the minimum number of ε-balls that cover T and $\log N(\varepsilon, T, \mathsf{d})$ is called the covering entropy which is an important quantity to characterize the complexity of the set T. The following maximal inequality [18, Section 2.2.1] plays an important role in our analysis.

LEMMA 4.3. *If $\{X_t : t \in T\}$ is a separable sub-Gaussian random process, then*

$$\|\sup_{s,t \in T} |X_s - X_t|\|_{\psi_2} \leq K \int_0^{\operatorname{diam} T} \sqrt{\log N\Big(\frac{\varepsilon}{2}, T, \mathsf{d}\Big)}\, d\varepsilon.$$

Here $K > 0$ is some constant.

The following result on the estimation of the covering entropy of Sobolev spaces is due to Birman-Solomyak [8].

LEMMA 4.4. *Let Q be the unit square in $\mathbf{R}^d$ and $SW^{\alpha,p}(Q)$ be the unit sphere of the Sobolev space $W^{\alpha,p}(Q)$, where $\alpha > 0, p \geq 1$. Then for $\varepsilon > 0$ sufficient small, the entropy*

$$\log N(\varepsilon, SW^{\alpha,p}(Q), \|\cdot\|_{L^q(Q)}) \leq C\varepsilon^{-d/\alpha},$$

where if $\alpha p > d$, $1 \leq q \leq \infty$, otherwise if $\alpha p \leq d$, $1 \leq q \leq q^*$ with $q^* = p(1-\alpha p/d)^{-1}$.

For any $\delta > 0, \rho > 0$, define

$$S_{\delta,\rho}(\Omega) := \{u \in H^2(\Omega) : \|u\|_n \leq \delta, |u|_{H^2(\Omega)} \leq \rho\}. \tag{4.7}$$

The following lemma estimates the entropy of the set $S_{\delta,\rho}(\Omega)$.

LEMMA 4.5. *There exists a constant C independent of $\delta, \rho, \varepsilon$ such that*

$$\log N(\varepsilon, S_{\delta,\rho}(\Omega), \|\cdot\|_{L^\infty(\Omega)}) \leq C\left(\frac{\rho + \delta}{\varepsilon}\right)^{d/2}.$$

Proof. By (2.1) we have for any $u \in S_{\delta,\rho}(\Omega)$, $\|u\|_{H^2(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + |u|_{H^2(\Omega)}) \leq C(\|u\|_n + |u|_{H^2(\Omega)}) \leq C(\delta + \rho)$, where we have used the fact that $h_{\max} \leq Cn^{-1/d} \leq C$. The lemma now follows from Lemma 4.4. □

The following lemma is proved by the argument in [18, Lemma 2.2.7].

LEMMA 4.6. *$\{E_n(u) := (e, u)_n : u \in H^2(\Omega)\}$ is a sub-Gaussian random process with respect to the semi-distance $\mathsf{d}(u, v) = \|u - v\|_n^*$, where $\|u\|_n^* := \sigma n^{-1/2}\|u\|_n$.*

Proof. By definition $E_n(u)-E_n(v)=\sum_{i=1}^n c_ie_i$, where $c_i=\frac{1}{n}(u-v)(x_i)$. Since e_i is a sub-Gaussian random variable with parameter σ and $\mathbb{E}[e_i]=0$, by (4.1), $\mathbb{E}[e^{\lambda e_i}]\le e^{\frac12\sigma^2\lambda^2}, \forall\lambda>0$. Thus, since e_i, $i=1,2,\cdots,n$, are independent random variables,

$$\mathbb{E}\left[e^{\lambda\sum_{i=1}^n c_ie_i}\right]\le e^{\frac12\sigma^2\lambda^2\sum_{i=1}^n c_i^2}=e^{\frac12\sigma^2 n^{-1}\lambda^2\|u-v\|_n^2}=e^{\frac12\mathsf{d}(u,v)^2\lambda^2}.$$

This shows $E_n(u)-E_n(v)$ is a sub-Gaussian random variable with parameter $\mathsf{d}(u,v)$. By (4.2) we have

$$\mathbb{P}(|E_n(u)-E_n(v)|\ge z)\le 2e^{-\frac12 z^2/\mathsf{d}(u,v)^2},\quad\forall z>0.$$

This shows the lemma by the definition of sub-Gaussian random process (4.6). ∎

The following lemma which improves Lemma 4.2 will be used in our subsequent analysis.

LEMMA 4.7. *If X is a random variable which satisfies*

$$\mathbb{P}(|X|>\alpha(1+z))\le C_1e^{-z^2/K_1^2},\quad\forall\alpha>0,z\ge1,$$

where C_1,K_1 are some positive constants, then $\|X\|_{\psi_2}\le C(C_1,K_1)\alpha$ for some constant $C(C_1,K_1)$ depending only on C_1,K_1.

Proof. If $y\ge2\alpha$, then $z=(y/\alpha)-1\ge1$. Thus

$$\mathbb{P}(|X|>y)=\mathbb{P}(|X|>\alpha(1+z))\le C_1\exp\left[-\frac{1}{K_1^2}\left(\frac{y}{\alpha}-1\right)^2\right].$$

Since $(\frac{y}{\alpha}-1)^2\ge\frac12(\frac{y}{\alpha})^2-1$ by Cauchy-Schwarz inequality, we obtain

$$\mathbb{P}(|X|>y)\le C_1e^{\frac{1}{K_1^2}}e^{-\frac{y^2}{2K_1^2\alpha^2}}=C_1e^{\frac{1}{K_1^2}}e^{-\frac{y^2}{K_2^2}},$$

where $K_2:=\sqrt{2}\alpha K_1$. On the other hand, if $y<2\alpha$, then $y^2/K_2^2<4\alpha^2/(2\alpha^2K_1^2)=2/K_1^2$. Thus

$$\mathbb{P}(|X|>y)\le e^{\frac{y^2}{K_2^2}}e^{-\frac{y^2}{K_2^2}}\le e^{\frac{2}{K_1^2}}e^{-\frac{y^2}{K_2^2}}.$$

Therefore, $\mathbb{P}(|X|>y)\le C_2e^{-y^2/K_2^2}$, $\forall y>0$, where $C_2=\max(C_1e^{1/K_1^2},e^{2/K_1^2})$. This implies by Lemma 4.2,

$$\|X\|_{\psi_2}\le\sqrt{1+C_2}K_2=C(C_1,K_1)\alpha,\quad\text{where } C(C_1,K_1)=\sqrt{2}K_1\sqrt{1+C_2}.$$

This completes the proof. ∎

THEOREM 4.8. *Let $u_n\in H^2(\Omega)$ be the solution of (3.1). Denote by $\rho_0=|u_0|_{H^2(\Omega)}+\sigma n^{-1/2}$. If we take $\lambda_n^{1/2+d/8}=O(\sigma n^{-1/2}\rho_0^{-1})$, then there exists a constant $C>0$ such that*

$$\mathbb{P}(\|u_n-u_0\|_n\ge\lambda_n^{1/2}\rho_0z)\le 2e^{-Cz^2},\quad\mathbb{P}(|u_n|_{H^2(\Omega)}\ge\rho_0z)\le 2e^{-Cz^2}.$$

Proof. By (4.5) we only need to prove

$$\|\|u_n-u_0\|_n\|_{\psi_2}\le C\lambda_n^{1/2}\rho_0,\quad\||u_n|\|_{\psi_2}\le C\rho_0. \tag{4.8}$$

We will only prove the first estimate in (4.8) by the peeling argument. The other estimate can be proved in a similar way. It follows from (2.6) that

$$\|u_n - u_0\|_n^2 + \lambda_n |u_n|_{H^2(\Omega)}^2 \le 2(e, u_n - u_0)_n + \lambda_n |u_0|_{H^2(\Omega)}^2. \tag{4.9}$$

Let $\delta > 0,\ \rho > 0$ be two constants to be determined later, and

$$A_0 = [0, \delta), A_i = [2^{i-1}\delta, 2^i\delta), \quad B_0 = [0, \rho), B_j = [2^{j-1}\rho, 2^j\rho), \qquad i, j \ge 1. \tag{4.10}$$

For $i, j \ge 0$, define

$$F_{ij} = \{v \in H^2(\Omega) :\ \|v\|_n \in A_i\ ,\ |v|_{H^2(\Omega)} \in B_j\}.$$

Then we have

$$\mathbb{P}(\|u_n - u_0\|_n > \delta) \le \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \mathbb{P}(u_n - u_0 \in F_{ij}). \tag{4.11}$$

Now we estimate $\mathbb{P}(u_n - u_0 \in F_{ij})$. By Lemma 4.6, $\{(e, v)_n : v \in H^2(\Omega)\}$ is a sub-Gaussian random process with respect to the semi-distance $\mathsf{d}(u, v) = \sigma n^{-1/2}\|u - v\|_n$. It is easy to see that

$$\operatorname{diam} F_{ij} \le \sigma n^{-1/2} \sup_{u-u_0, v-u_0 \in F_{ij}} (\|u - u_0\|_n + \|v - u_0\|_n) \le 2\sigma n^{-1/2} \cdot 2^i \delta.$$

Then by the maximal inequality in Lemma 4.3 we have

$$\begin{aligned}
\|\sup_{u-u_0\in F_{ij}} |(e, u - u_0)_n|\|_{\psi_2} &\le K \int_0^{\sigma n^{-1/2}\cdot 2^{i+1}\delta} \sqrt{\log N\left(\frac{\varepsilon}{2}, F_{ij}, \mathsf{d}\right)}\, d\varepsilon \\
&= K \int_0^{\sigma n^{-1/2}\cdot 2^{i+1}\delta} \sqrt{\log N\left(\frac{\varepsilon}{2\sigma n^{-1/2}}, F_{ij}, \|\cdot\|_n\right)}\, d\varepsilon.
\end{aligned}$$

By Lemma 4.5 we have the estimate for the entropy

$$\begin{aligned}
\log N\left(\frac{\varepsilon}{2\sigma n^{-1/2}}, F_{ij}, \|\cdot\|_n\right) &\le \log N\left(\frac{\varepsilon}{2\sigma n^{-1/2}}, F_{ij}, \|\cdot\|_{L^\infty(\Omega)}\right) \\
&\le C\left(\frac{2\sigma n^{-1/2}\cdot(2^i\delta + 2^j\rho)}{\varepsilon}\right)^{d/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\sup_{u-u_0\in F_{ij}} |(e, u - u_0)_n|\|_{\psi_2} &\le K \int_0^{\sigma n^{-1/2}\cdot 2^{i+1}\delta} \left(\frac{2\sigma n^{-1/2}\cdot(2^i\delta + 2^j\rho)}{\varepsilon}\right)^{d/4} d\varepsilon \\
&= C\sigma n^{-1/2}(2^i\delta + 2^j\rho)^{d/4}(2^i\delta)^{1-d/4} \\
&\le C\sigma n^{-1/2}[2^i\delta + (2^i\delta)^{1-d/4}(2^j\rho)^{d/4}].
\end{aligned} \tag{4.12}$$

By (4.9) and (4.5) we have for $i, j \ge 1$:

$$\begin{aligned}
\mathbb{P}(u_n - u_0 \in F_{ij}) &\le \mathbb{P}(2^{2(i-1)}\delta^2 + \lambda_n 2^{2(j-1)}\rho^2 \le 2 \sup_{u-u_0\in F_{ij}} |(e, u - u_0)_n| + \lambda_n \rho_0^2) \\
&= \mathbb{P}(2 \sup_{u-u_0\in F_{ij}} |(e, u - u_0)_n| \ge 2^{2(i-1)}\delta^2 + \lambda_n 2^{2(j-1)}\rho^2 - \lambda_n\rho_0^2) \\
&\le 2\exp\left[-\frac{1}{C\sigma^2 n^{-1}}\left(\frac{2^{2(i-1)}\delta^2 + \lambda_n 2^{2(j-1)}\rho^2 - \lambda_n\rho_0^2}{2^i\delta + (2^i\delta)^{1-d/4}(2^j\rho)^{d/4}}\right)^2\right].
\end{aligned}$$

Now we take

$$\delta^2 = \lambda_n \rho_0^2 (1+z)^2, \ \rho = \rho_0, \quad \text{where } z \geq 1. \tag{4.13}$$

Since by assumption $\lambda_n^{1/2+d/8} = O(\sigma n^{-1/2} \rho_0^{-1})$ and $\sigma n^{-1/2} \rho_0^{-1} \leq 1$, we have $\lambda_n \leq C$ for some constant. By some simple calculation we have for $i, j \geq 1$,

$$\mathbb{P}(u_n - u_0 \in F_{ij}) \leq 2 \exp \left[-C \left(\frac{2^{2(i-1)} z(1+z) + 2^{2(j-1)}}{2^i(1+z) + (2^i(1+z))^{1-d/4} (2^j)^{d/4}} \right)^2 \right].$$

By using the elementary inequality $ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$ for any $a, b > 0, p, q > 1, p^{-1} + q^{-1} = 1$, we have $(2^i(1+z))^{1-d/4}(2^j)^{d/4} \leq (1+z)2^i + 2^j$. Thus

$$\mathbb{P}(u_n - u_0 \in F_{ij}) \leq 2 \exp \left[-C(2^{2i} z^2 + 2^{2j}) \right].$$

Similarly, one can prove for $i \geq 1, j = 0$,

$$\mathbb{P}(u_n - u_0 \in F_{i0}) \leq 2 \exp \left[-C(2^{2i} z^2) \right].$$

Therefore, since $\sum_{j=1}^{\infty} e^{-C(2^{2j})} \leq e^{-C} < 1$ and $\sum_{i=1}^{\infty} e^{-C(2^{2i} z^2)} \leq e^{-Cz^2}$, we obtain finally

$$\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \mathbb{P}(u_n - u_0 \in F_{ij}) \leq 2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} e^{-C(2^{2i} z^2 + 2^{2j})} + 2 \sum_{i=1}^{\infty} e^{-C(2^{2i} z^2)} \leq 4 e^{-Cz^2}.$$

Now inserting the estimate to (4.11) we have

$$\mathbb{P}(\|u_n - u_0\|_n > \lambda_n^{1/2} \rho_0 (1+z)) \leq 4 e^{-Cz^2} \quad \forall z \geq 1. \tag{4.14}$$

This implies by using Lemma 4.7 that $\| \|u_n - u_0\|_n \|_{\psi_2} \leq C \lambda_n^{1/2} \rho_0$, which is the first estimate in (4.8). This completes the proof. □

In terms of the terminology of the stochastic convergence order, we have $\|u_n - u_0\|_n = O_p(\lambda_n^{1/2}) \rho_0$ which by the assumption of Theorem 4.8 yields

$$\|u_n - u_0\|_n = O_p(n^{-\frac{2}{4+d}}) \sigma^{\frac{4}{4+d}} \rho_0^{-\frac{4}{4+d}}.$$

This estimate is proved in [17, Section 10.1.1] when $d = 1$. Our result in Theorem 4.8 is stronger in the sense that it also provides the tail property of the probability distribution function of the random error $\|u_n - u_0\|_n$.

We remark that Theorem 4.8 can be proved by using the theory of concentration inequalities [5, 7] under stronger assumptions on the random noise $e_i, i = 1, \cdots, n$. In fact, one can prove Theorem 4.8 by using McDiarmid bounded difference equality [7, Theorem 6.2] when e_i, $i = 1, 2, \cdots, n$, are bounded random variables or by using the Gaussian concentration inequality [7, Theorem 5.6] when e_i, $i = 1, 2, \cdots, n$, is the Gaussian random variable. One may also derive upper bounds for higher centered moments of the error $\|u_n - u_0\|_n$ by using the moments inequalities [6], [7, Chapter 15] if the random variables $e_i, i = 1, 2, \cdots, n$, are not sub-Gaussian.

4.2. Proof of Theorem 4.1. We start by recalling the following lemma in [17, Corollary 2.6] about the estimation of the covering entropy of finite dimensional subsets.

LEMMA 4.9. *Let G be a finite dimensional subspace of $L^2(\Omega)$ of dimension $N > 0$ and $G_R = \{f \in G : \|f\|_{L^2(\Omega)} \le R\}$. Then*

$$N(\varepsilon, G_R, \|\cdot\|_{L^2(\Omega)}) \le (1 + 4R/\varepsilon)^N, \quad \forall \varepsilon > 0.$$

LEMMA 4.10. *Let $G_h := \{v_h \in V_h : |||v_h|||_{h,\lambda_n} = (\lambda_n |v_h|_{2,h}^2 + \|\hat{v}_h\|_n^2)^{1/2} \le 1\}$. Assume that $h = O(\lambda_n^{1/4})$ and $n\lambda_n^{d/4} \ge 1$. Then*

$$\| \sup_{v_h \in G_h} |(e, \hat{v}_h - \Pi_h v_h)_n| \|_{\psi_2} \le C\sigma n^{-1/2} \lambda_n^{-d/8}.$$

Proof. Similar to the proof of Lemma 4.6 we know that $\{\hat{E}_n(v_h) := (e, \hat{v}_h - \Pi_h v_h)_n \ \forall v_h \in G_h\}$ is a sub-Gaussian random process with respect to the semi-distance $\hat{\mathsf{d}}(v_h, w_h) = \sigma n^{-1/2} \|(\hat{v}_h - \Pi_h v_h) - (\hat{w}_h - \Pi_h w_h)\|_n$. By Lemma 3.4, for any $v_h \in G_h$, $\|\hat{v}_h - \Pi_h v_h\|_n \le Ch^2 |v_h|_{2,h} \le Ch^2 \lambda_n^{-1/2} \le C$, where we have used the assumption $h = O(\lambda_n^{1/4})$ in the last inequality. This implies that the diameter of G_h is bounded by $C\sigma n^{-1/2}$. Now by the maximal inequality in Lemma 4.3

$$\| \sup_{v_h \in G_h} |(e, \hat{v}_h - \Pi_h v_h)_n| \|_{\psi_2} \le K \int_0^{C\sigma n^{-1/2}} \sqrt{\log N\left(\frac{\varepsilon}{2}, G_h, \hat{\mathsf{d}}\right)}\, d\varepsilon. \tag{4.15}$$

By Lemma 3.4 and the inverse estimate,

$$\hat{\mathsf{d}}(v_h, w_h) \le C\sigma n^{-1/2} h^2 |v_h - w_h|_{2,h} \le C\sigma n^{-1/2} \|v_h - w_h\|_{L^2(\Omega)} \quad \forall v_h, w_h \in V_h.$$

Thus

$$\log N\left(\frac{\varepsilon}{2}, G_h, \hat{\mathsf{d}}\right) = \log N\left(\frac{\varepsilon}{C\sigma n^{-1/2}}, G_h, \|\cdot\|_{L^2(\Omega)}\right). \tag{4.16}$$

Our next goal is to show that G_h is a bounded set in the $L^2(\Omega)$ norm so that we can use Lemma 4.9 to estimate the covering entropy of G_h. For any $v_h \in G_h$, we have $|v_h|_{2,h} \le \lambda_n^{-1/2}, \|v_h\|_n \le 1$. By Lemma 3.4, $\Pi_h v_h \in H^2(\Omega)$ and thus by (2.1)

$$\begin{aligned}
\|\Pi_h v_h\|_{L^2(\Omega)} &\le C(h_{\max}^2 |\Pi_h v_h|_{H^2(\Omega)} + \|\Pi_h v_h\|_n) \\
&\le C(n^{-2/d} \lambda_n^{-1/2} + \|\Pi_h v_h - \hat{v}_h\|_n + \|\hat{v}_h\|_n) \\
&\le C(n^{-2/d} \lambda_n^{-1/2} + Ch^2 \lambda_n^{-1/2} + 1) \\
&\le C \quad\quad \forall v_h \in G_h,
\end{aligned}$$

where we have used $h \le C\lambda_n^{1/4}$ and $n\lambda_n^{d/4} \ge 1$ in the last inequality. This yields

$$\|v_h\|_{L^2(\Omega)} \le \|v_h - \Pi_h v_h\|_{L^2(\Omega)} + \|\Pi_h v_h\|_{L^2(\Omega)} \le Ch^2 \lambda_n^{-1/2} + C \le C \ \ \forall v_h \in G_h. \tag{4.17}$$

Since the dimension of V_h is bounded by Ch^{-d}, we obtain then by using Lemma 4.9 and (4.16) that

$$\log N\left(\frac{\varepsilon}{2}, G_h, \hat{\mathsf{d}}\right) \le Ch^{-d}(1 + \sigma n^{-1/2}/\varepsilon).$$

Inserting this estimate to (4.15)

$$\| \sup_{v_h\in G_h} |(e, \hat{v}_h - \Pi_h v_h)_n| \|_{\psi_2} \le C \int_0^{C\sigma n^{-1/2}} \sqrt{Ch^{-d}(1+\sigma n^{-1/2}/\varepsilon)}\, d\varepsilon$$
$$\le Ch^{-d/2}\sigma n^{-1/2}.$$

This completes the proof since $h = O(\lambda_n^{1/4})$. □

Now we are in the position to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. By Lemma 3.5 we have

$$\begin{aligned}
&\|\|\hat{u}_h - u_0\|_n\|_{\psi_2} + \lambda_n^{1/2}\||u_h|_{2,h}\|_{\psi_2}\\
&\le C(h^2+\lambda_n^{1/2})\||u_n|_{H^2(\Omega)}\|_{\psi_2} + C\frac{h^2}{\lambda_n^{1/2}}\|\|u_n - u_0\|_n\|_{\psi_2}\\
&+ C\| \sup_{v_h\in G_h} \||(e, \hat{v}_h - \Pi_h v_h)_n|\|_{\psi_2}.
\end{aligned}$$

The theorem follows from (4.8), Lemma 4.10, the assumption $\sigma n^{-1/2} \le C\lambda_n^{1/2+d/8}\rho_0$, and (4.5). This completes the proof. □

5. Numerical examples. In this section, we present several examples to confirm the theoretical results in this paper. From Theorem 4.1 we know that the mesh size should be comparable with $\lambda_n^{1/4}$. The smoothing parameter λ_n is usually determined by the cross-validation in the literature [19]. Here we propose a self-consistent algorithm to determine the parameter λ_n based on the equation

$$\lambda_n^{1/2+d/8} = \sigma n^{-1/2}(|u_0|_{H^2(\Omega)} + \sigma n^{-1/2})^{-1}, \tag{5.1}$$

which is indicated from Theorem 4.1. In the algorithm we estimate $|u_0|_{H^2(\Omega)}$ by $|u_h|_{2,h}$ and σ by $\|u_h - y\|_n$ since $\|u_0 - y\|_n = \|e\|_n$ provides a good estimation of the variance by the law of large numbers.

ALGORITHM 5.1. (SELF-CONSISTENT ALGORITHM FOR FINDING λ_n)

1° *Given an initial guess of $\lambda_{n,0}$;*

2° *For $k \ge 0$ and $\lambda_{n,k}$ known, compute u_h with the parameter $\lambda_{n,k}$ over a quasi-uniform mesh with the element width $h = \lambda_{n,k}^{1/4}$;*

3° *Compute $\lambda_{n,k+1}^{1/2+d/8} = \|\hat{u}_h - y\|_n n^{-1/2}(|u_h|_{2,h} + \|\hat{u}_h - y\|_n n^{-1/2})^{-1}$.*

A natural choice of the initial guess is $\lambda_{n,0} = n^{-\frac{4}{d+4}}$ which is used in our all computations. We will always take $\Omega = (0,1)\times(0,1)$ and $\{x_i\}_{i=1}^n$ being uniformly distributed over Ω. The finite element mesh of Ω is constructed by first dividing the domain into $h^{-1}\times h^{-1}$ uniform rectangles and then connecting the lower left and upper right vertices of each rectangle. We set $e_i,\ i = 1,2,\cdots,n$, being independent normal random variables with variance σ.

EXAMPLE 5.1. *In this example we show that the choice of the smoothing parameter λ_n by (5.1) is optimal. Let $u_0 = \sin(2\pi x^2 + 3\pi y)e^{x^3+y}$ whose surface plot is depicted in Figure 5.1. Table 5.1 shows the relative empirical error $\|\hat{u}_h - u_0\|_n/\|u_0\|_n$ for different choices of $\lambda_n = 10^{-k}, k = 2,\cdots,9$, when $\sigma n^{-1/2} = 1/10, 1/50$, 1/500, respectively. We observe that the relative empirical errors are always the smallest when λ_n is closest to the optimal choice given by* (5.1). *In this example we choose $h = 0.002$ sufficiently small so that the finite element errors are negligible.*

EXAMPLE 5.2. *In this example we show the convergence of the finite element method. We fix $\lambda_n = 10^{-8}$ and choose σ, n according to* (5.1). *Figure 5.2 shows the*

σ \ λ_n	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
1	0.9170	0.8453	0.6325	0.4474	0.5540	0.6394	0.6530	0.6544
2	0.9582	0.8764	0.5528	0.3340	0.4559	0.7588	1.0274	1.0838
4	0.9785	0.8997	0.6007	0.3290	0.4715	0.8055	1.4036	2.0460
10	0.9826	0.8951	0.5875	0.3466	0.4647	0.7741	1.3284	2.3813

σ \ λ_n	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
1	0.9745	0.8935	0.5877	0.2333	0.1125	0.1636	0.3009	0.4837
2	0.9807	0.8964	0.5878	0.2343	0.1062	0.1563	0.2859	0.5243
4	0.9838	0.8997	0.5908	0.2359	0.1151	0.1684	0.2885	0.5061
10	0.9858	0.8999	0.5934	0.2459	0.1207	0.1680	0.2855	0.5008

σ \ λ_n	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
0.1	0.9745	0.8928	0.5840	0.2211	0.0573	0.0207	0.0306	0.0490
0.2	0.9810	0.8974	0.5887	0.2267	0.0556	0.0184	0.0281	0.0516
0.4	0.9840	0.8996	0.5916	0.2309	0.0575	0.0202	0.0284	0.0501
1	0.9858	0.9009	0.5935	0.2328	0.0577	0.0191	0.0280	0.0503

TABLE 5.1

The relative empirical error $\|\hat{u}_h - u_0\|_n / \|u_0\|_n$ for different choices of the parameter λ_n. In the first table, n is chosen such that $\sigma n^{-1/2} = 1/10$. The error is smallest when $\lambda_n = 10^{-5}$ which agrees with the optimal choice $\lambda_n \approx 2.1 \times 10^{-5}$ by (5.1). In the second table, n is chosen such that $\sigma n^{-1/2} = 1/50$. The error is smallest when $\lambda_n = 10^{-6}$ which agrees with the optimal choice $\lambda_n \approx 2.4 \times 10^{-6}$ by (5.1). In the third table, n is chosen such that $\sigma n^{-1/2} = 1/500$. The error is smallest when $\lambda_n = 10^{-7}$ which agrees with the optimal choice $\lambda_n \approx 1.1 \times 10^{-7}$ by (5.1).

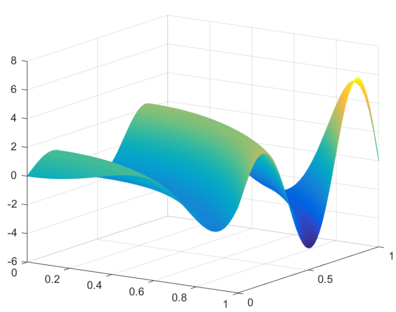
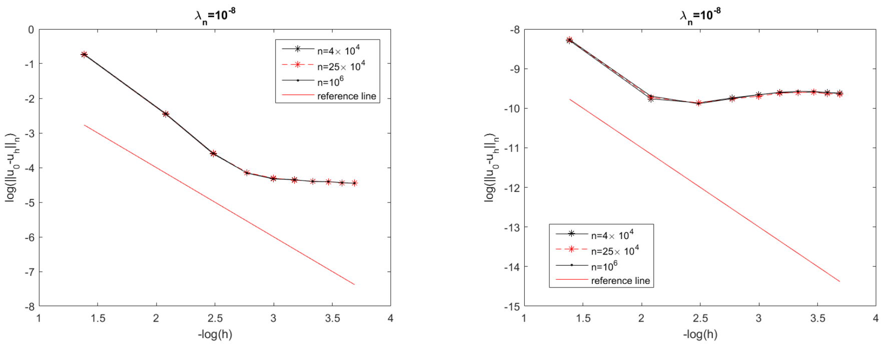


FIG. 5.1. The surface plot of the exact solution $u_0 = \sin(2\pi x^2 + 3\pi y)e^{x^3+y}$.

finite element convergence rate when $u_0 = \sin(2\pi x^2 + 3\pi y)e^{x^3+y}$ and $u_0 = (xy)^{1.5+\alpha}$ with $\alpha = 0.001$. The H^2 norm of $(xy)^{1.5+\alpha}$ blows up when $\alpha \to 0$. We observe that before the finite element error reaches the modeling error controlled by λ_n, the convergence rate is about h^2, which conforms with our theoretical analysis. We also remark that the first test function is very smooth and the second test function is almost only in H^2. This explains the difference between the two graphs in Figure 5.2.

EXAMPLE 5.3. *We first show the empirical error $\|u_0 - \hat{u}_h\|_n$ depends linearly on*



(a) (b)

FIG. 5.2. (a) The convergence of the finite element when $u_0 = \sin(2\pi x^2 + 3\pi y)e^{x^3+y}$. (b) The convergence of the finite element method when $u_0 = (xy)^{1.5+\alpha}$ with $\alpha = 0.001$. The reference line is the line with the slope -2 corresponding to the convergence rate h^2.

$\lambda_n^{1/2}$ to confirm (4.3). Let $u_0 = \sin(2\pi x^2 + 3\pi y)e^{x^3+y}$. We choose $\sigma = 10, 5, 1, 0.5$ and n varying from 2500 to 25×10^4. We use (5.1) to determine the parameter λ_n and take the mesh size $h = \lambda_n^{1/4}$. Figure 5.3 shows clearly the linear dependence of the empirical error on $\lambda_n^{1/2}$.

Next we show that the probability density function of the empirical error $\|\hat{u}_h - u_0\|_n$ has an exponentially decaying tail as indicated by (4.3). We set the variance $\sigma = 0.2$, $n = 10^4$, and the mesh size $h = \lambda_n^{1/4}$. We take 10,000 samples and compute the empirical error $\|\hat{u}_h - u_0\|_n$ for each sample. Figure 5.4(a) shows the histogram plot of the empirical errors for $u_0 = \sin(2\pi x^2 + 3\pi y)e^{x^3+y}$.

Figure 5.4(b) plots the quantile-quantile (Q-Q) plot to compare the sample distribution of the empirical error with the standard normal distribution. The Q-Q plot is a standard graphic tool in statistics to check the data distribution [11]. If the sample distribution is indeed normal, the Q-Q plot will give a scattered plot in which the points show a linear relationship between the sample and the theoretical quantiles. We observe from Figure 5.4(b) that almost all the points are concentrated around the dotted line, which implies that the overall distribution of the error is very close to a normal distribution. Moreover, the points around the two ends are also not far from the line, which indicates that the tail distribution of the error is also close to a Gaussian tail. The probability density function is computed by the Matlab function 'qqplot'.

In above examples, we have verified the optimality of the choice λ_n in (5.1), the convergence of the finite element method, and the stochastic convergence in Theorem 4.1. In practical computations, one usually does not know the exact solution and the variance of the noise. Our next example shows the efficiency of Algorithm 5.1 to determine λ_n without knowing u_0 and σ.

EXAMPLE 5.4. *We test the efficiency of the Algorithm 5.1 to estimate the smoothing parameter λ_n. Algorithm 5.1 is terminated when $|\lambda_{n,k} - \lambda_{n,k+1}| \le 10^{-10}$. We show the predicted parameter λ_n^{alg} and the corresponding relative empirical error er^{alg} for different choices of σ and n in Table 5.2 for $u_0 = \sin(2\pi x^2 + 3\pi y)e^{x^3+y}$ and Table 5.3 for $u_0 = \sin(2\pi x)\sin(2\pi y)$. For the ease of comparison, we also show the optimal parameter λ_n^{opt} by (5.1) and the corresponding relative empirical error er^{opt} in the tables.*

Algorithm 5.1 usually stops within 20 iterations. We observe that when $\sigma n^{-1/2}$

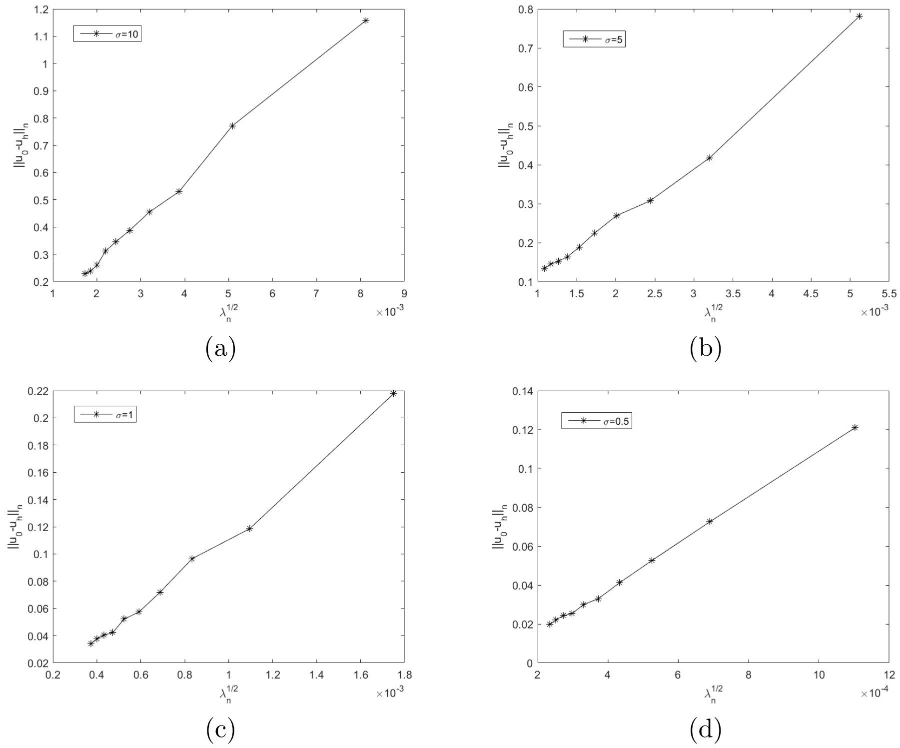


FIG. 5.3. The linear dependence of the empirical error $\|u_0 - \hat{u}_h\|_n$ on $\lambda_n^{1/2}$ for different choices of σ. (a)-(d) refer to $\sigma = 10, 5, 1, 0.5$, respectively.

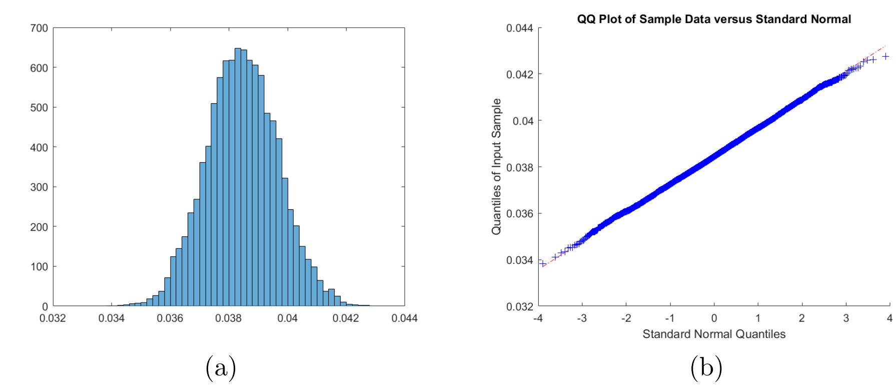


FIG. 5.4. (a) The histogram plot of the empirical errors $\|\hat{u}_h - u_0\|_n$ of $10,000$ samples when $u_0 = \sin(2\pi x^2 + 3\pi y)e^{x^3+y}$. (b) The quantile-quantile plot of the $10,000$ sample empirical error distribution versus the standard normal distribution.

is small, the outcome of Algorithm 5.1 agrees quite well with the optimal choice of the parameter. However, when $\sigma n^{-1/2}$ is large (e.g. larger than 0.1 *in Table 5.2), the predicted parameters λ_n^{alg} are much larger than the optimal choice from* (5.1)*. The corresponding relative empirical errors with both optimal choice* (5.1) *and the predicted parameter from Algorithm 5.1 are quite large. Intuitively, in these cases,*

σ / n		0.1	1	5	10
$n = 900$	λ_n^{opt}	2.2434e-07	4.8326e-06	4.1295e-05	1.0399e-04
	λ_n^{alg}	1.4122e-07	9.9249e-06	0.7528	0.8393
	er^{opt}	0.0303	0.1754	0.5059	0.9895
	er^{alg}	0.0300	0.2169	0.9883	1.0898
$n = 2500$	λ_n^{opt}	1.1353e-07	2.4457e-06	2.0904e-05	5.2652e-05
	λ_n^{alg}	9.3216e-08	3.7356e-06	0.6697	0.8948
	er^{opt}	0.0203	0.1301	0.3760	0.5136
	er^{alg}	0.0202	0.1465	0.9920	0.9900
$n = 10^4$	λ_n^{opt}	4.5054e-08	9.7062e-07	8.2973e-06	2.0904e-05
	λ_n^{alg}	4.0800e-08	1.1661e-06	2.4146e-05	0.7256
	er^{opt}	0.0130	0.0772	0.2603	0.4340
	er^{alg}	0.0130	0.0805	0.3616	0.9968
$n = 25 \times 10^4$	λ_n^{opt}	5.2695e-09	1.1353e-07	9.7062e-07	2.4457e-06
	λ_n^{alg}	4.8469e-09	1.1312e-07	1.1480e-06	3.5483e-06
	er^{opt}	0.0039	0.0210	0.0686	0.1184
	er^{alg}	0.0040	0.0210	0.0714	0.1352

TABLE 5.2

Algorithm 5.1: The predicted parameter λ_n^{alg}, the corresponding relative empirical error er^{alg}, the optimal parameter λ_n^{opt}, and the corresponding relative empirical error er^{opt} for different choices of σ and n. $u_0 = \sin(2\pi x^2 + 3\pi y)e^{x^3+y}$.

the exact solution is strongly polluted by the noises and more information (e.g. more measurements) is needed for recovering the exact solution. It is interesting to observe that even when the predicted parameter from Algorithm 5.1 is quite far from the optimal choice, the corresponding empirical errors with the predicted parameter are of the same order as the errors with the optimal choice of λ_n , see e.g., the case when $\sigma = 5, n = 900$. This suggests that Algorithm 5.1 is a good alternative for finding the regularization parameter in practical applications.

6. Appendix: Proof of Lemma 3.4. We first prove the lemma for the case $d = 2$. The case of $d = 3$ will be briefly discussed later. We will construct $\Pi_h v_h$ by using the Agyris element. We recall [9, P.71] that for any $K \in \mathcal{M}_h$, Agyris element is a triple (K, P_K, Λ_K), where $P_K = P_5(K)$ and the set of degrees of freedom, with the notation in Figure 6.1, $\Lambda_K = \{p(a_i), Dp(a_i)(a_j - a_i), D^2p(a_i)(a_j - a_i, a_k - a_i), \partial_\nu p(b_i), 1 \le i,j,k \le 3, j \ne i, k \ne i, \forall p \in C^2(K)\}$. Let X_h be the Agyris finite element space

$$X_h = \{v_h : v_h|_K \in P_5(K), \forall K \in \mathcal{M}_h, f(v_h|_{K_1}) = f(v_h|_{K_2}), \forall f \in \Lambda_{K_1} \cap \Lambda_{K_2}\}.$$

It is known that $X_h \subset H^2(\Omega)$.

We define the operator Π_h as follows. For any $v_h \in V_h$, $w_h := \Pi_h v_h \in X_h$ such that for any $K \in \mathcal{M}_h$, $w_h|_K \in P_5(K)$ and

$$\partial^\alpha(w_h|_K)(a_i) = \frac{1}{N(a_i)} \sum_{K' \in \mathcal{M}_h(a_i)} \partial^\alpha(v_h|_{K'})(a_i), \quad 1 \le i \le 3, \quad |\alpha| \le 2, \qquad (6.1)$$

$$\partial_\nu(w_h|_K)(b_i) = \partial_\nu(v_h|_K)(b_i), \quad 1 \le i \le 3. \qquad (6.2)$$

n \ σ		0.1	1	5	10
$n = 900$	λ_n^{opt}	3.7573e-06	8.0865e-05	6.8826e-04	0.0017
	λ_n^{alg}	3.8646e-06	5.6663e-04	0.5203	0.8795
	er^{opt}	0.0530	0.3220	0.8352	0.9461
	er^{alg}	0.0533	0.5418	1.1182	1.1760
$n = 2500$	λ_n^{opt}	1.9015e-06	4.0941e-05	3.4909e-04	8.7667e-04
	λ_n^{alg}	1.8521e-06	1.2797e-04	0.5817	0.8009
	er^{opt}	0.0405	0.2784	0.6589	0.9869
	er^{alg}	0.0403	0.3936	1.0158	1.2421
$n = 10^4$	λ_n^{opt}	7.5464e-07	1.6253e-05	1.3877e-04	3.4909e-04
	λ_n^{alg}	7.2955e-07	2.1903e-05	0.0027	0.7171
	er^{opt}	0.0224	0.1211	0.4207	0.7499
	er^{alg}	0.0223	0.1351	0.7220	1.0268
$n = 25 \times 10^4$	λ_n^{opt}	8.8265e-08	1.9015e-06	1.6253e-05	4.0941e-05
	λ_n^{alg}	8.2733e-08	1.9686e-06	2.3759e-05	1.0877e-04
	er^{opt}	0.0067	0.0386	0.1379	0.2478
	er^{alg}	0.0067	0.0388	0.1596	0.3523

TABLE 5.3

Algorithm 5.1: The predicted parameter λ_n^{alg}, the corresponding relative empirical error er^{alg}, the optimal parameter λ_n^{opt}, and the corresponding relative empirical error er^{opt} for different choices of σ and n. $u_0 = \sin(2\pi x) \times \sin(2\pi y)$.

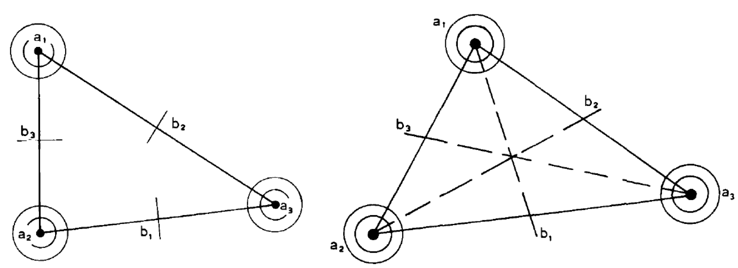


FIG. 6.1. *The degrees of freedom of Agyris element (left) and Hermite triangle of type (5) (right).*

Here $\mathcal{M}_h(a_i)$ and $N(a_i)$ are defined above (3.2). To show the estimate (3.9) we follow an idea in [9, Theorem 6.1.1] and use the H^1 conforming but affine equivalent element Hermite triangle of type (5) [9, P.102], which is a triple (K, P_K, Θ_K), where $P_K = P_5(K)$ and the set of degrees of freedom $\Theta_K = \{p(a_i), Dp(a_i)(a_j - a_i), D^2p(a_i)(a_j - a_i, a_k - a_i), Dp(b_i)(a_i - b_i), 1 \le i, j, k \le 3, j \ne i, k \ne i, \forall p \in C^2(K)\}$. For any $K \in \mathcal{M}_h$, denote by $p_i, p_{ij}, p_{ijk}, q_i$ the basis functions associated with the degrees of freedom $p(a_i), Dp(a_i)(a_j - a_i), D^2p(a_i)(a_j - a_i, a_k - a_i), Dp(b_i)(a_i - b_i)$, $1 \le i, j, k \le 3, j \ne i, k \ne i$.

For any $v_h \in V_h$, we also define a linear operator $q_h := \Lambda_h v_h$ as follows: for any $K \in \mathcal{M}_h$, $q_h|_K \in P_5(K)$ and

$$\partial^\alpha(q_h|_K)(a_i) = \frac{1}{N(a_i)} \sum_{K' \in \mathcal{M}_h(a_i)} \partial^\alpha(v_h|_{K'})(a_i), \quad 1 \le i \le 3, \quad |\alpha| \le 2, \qquad (6.3)$$

$$D(q_h|_K)(b_i)(a_i - b_i) = D(v_h|_K)(b_i)(a_i - b_i), \quad 1 \le i \le 3. \qquad (6.4)$$

Then from the definition of Morley element and Hermite triangle of type (5), we know that $\phi_h|_K := (v_h - q_h)|_K \in P_5(K)$ satisfies

$$\phi_h(x) = \sum_{i,j=1,2,3,j\neq i} D(\phi_h|_K)(a_i)(a_j - a_i)p_{ij}(x) + \sum_{i,j,k=1,2,3,j\neq i,k\neq i} D^2(\phi_h|_K)(a_i)(a_j - a_i, a_k - a_i)p_{ijk}(x).$$

Since a regular family of Hermite triangle of type (5) is affine-equivalent, by standard scaling argument [9, Theorem 3.1.2], we obtain easily $|q_i|_{H^m(K)} + |p_i|_{H^m(K)} + |p_{ij}|_{H^m(K)} + |p_{ijk}|_{H^m(K)} \leq Ch_K^{1-m}$, $m = 0, 1, 2$. Thus, for $m = 0, 1, 2$,

$$|\phi_h|_{H^m(K)} \leq Ch_K^{1-m} \left(\sum_{i=1}^{3} \sum_{1\leq|\alpha|\leq 2} h^{|\alpha|} |\partial^\alpha(v_h|_K)(a_i) - \partial^\alpha(q_h|_K)(a_i)|^2 \right)^{1/2}. \tag{6.5}$$

By Lemma 3.3 and the fact that $\partial^\alpha(q_h|_K)(a_i)$ is the local average of $\partial^\alpha v_h$ over elements around a_i in (6.3)

$$|\partial^\alpha(v_h|_K)(a_i) - \partial^\alpha(q_h|_K)(a_i)| \leq Ch^{1-|\alpha|}|v_h|_{2,\mathcal{M}_h(a_i)} \quad \forall 1 \leq |\alpha| \leq 2,$$

we recall from the notation in (3.8) that $|v_h|_{2,\mathcal{M}_h(a_i)} = \left(\sum_{K\in\mathcal{M}_h(a_i)} |v_h|^2_{H^2(K)}\right)^{1/2}$. Inserting above estimate into (6.5), we get

$$|v_h - q_h|_{H^m(K)} \leq Ch^{2-m}|v_h|_{2,\mathcal{M}_h(K)}, \quad m = 0, 1, 2. \tag{6.6}$$

By (6.1)-(6.4) we know that $q_h - w_h \in P_5(K)$ and satisfies

$$q_h(x) - w_h(x) = \sum_{i=1}^{3} D(q_h|_K - w_h|_K)(b_i)(a_i - b_i)q_i(x).$$

On the other hand, for $1 \leq i \leq 3$,

$$D(q_h|_K - w_h|_K)(b_i)(a_i - b_i) = \partial_\nu(q_h|_K - v_h|_K)(b_i)[(a_i - b_i)\cdot\nu],$$

since $\partial_\nu(w_h|_K)(b_i) = \partial_\nu(v_h|_K)(b_i)$ by (6.2) and the tangential derivative of $(q_h|_K - w_h|_K)$ vanishes as a consequence of (6.1) and (6.3). Since $|q_i|_{H^m(K)} \leq Ch_K^{1-m}$ for $m = 0, 1, 2$, we obtain then

$$\begin{aligned} |q_h - w_h|_{H^m(K)} &\leq Ch^{2-m} \left(\sum_{i=1}^{3} |\partial_\nu(q_h|_K - v_h|_K)(b_i)|^2 \right)^{1/2} \\ &\leq Ch^{2-m}|v_h|_{2,\mathcal{M}_h(K)}, \quad m = 0, 1, 2, \end{aligned} \tag{6.7}$$

where in the second inequality we have used the fact that by the inverse estimate and (6.6),

$$|\partial_\nu(q_h|_K - v_h|_K)(b_i)| \leq |q_h - v_h|_{W^{1,\infty}(K)} \leq Ch_K^{-1}|q_h - v_h|_{H^1(K)} \leq C|v_h|_{2,\mathcal{M}_h(K)}.$$

Combining (6.6) and (6.7) shows (3.9).

To show (3.10), we use the notation in the proof of Lemma 3.2, the inverse estimate and (3.9) to get

$$\|\hat{v}_h - w_h\|_n^2 \le \frac{C}{n} \sum_{K\in\mathcal{M}_h} \#\mathbb{T}_K \|v_h - w_h\|_{L^\infty(K)}^2 \le C\|v_h - w_h\|_{L^2(\Omega)}^2 \le Ch^4|v_h|_{2,h}^2.$$

Now we prove the 3D case which is very similar to the proof for 2D case above. We will construct $\Pi_h v_h$ by using the three dimensional C^1 element of Zhang constructed in [20] which simplifies an earlier construction of Zenisek [21]. For any tetrahedron $K \in \mathcal{M}_h$, the $C^1 - P_9$ element in [20] is a triple (K, P_K, Λ_K), where $P_K = P_9(K)$ and the set of degrees of freedom Λ_K consists of the following 220 functionals: for any $p \in C^2(K)$,

1° The nodal values of $p(a_i), Dp(a_i)(a_j - a_i), D^2p(a_i)(a_j - a_i, a_k - a_i), D^3(a_i)(a_j - a_i, a_k - a_i, a_l - a_i), D^4p(a_i)(a_j - a_i, a_k - a_i, a_l - a_i, a_n - a_i), 1 \le i \le 4, 1 \le j \le k \le l \le n \le 4, i \notin \{j,k,l,n\}$, where $\{a_i\}_{i=1}^4$ are the vertices of K; (120 functionals)

2° The 2 first order normal derivatives $\partial_{\nu_k} p(a_{ij})$ and 3 second order normal derivatives $\partial^2_{\nu_k\nu_l} p(b_{ij}), \partial^2_{\nu_k\nu_l} p(c_{ij})$ on the edge with vertices a_i, a_j, $1 \le i \ne j \le 4$, where $\nu_k, k = 1,2$, are unit vectors perpendicular to the edge, and $a_{ij} = (a_i + a_j)/2$, $b_{ij} = (2a_i + a_j)/3$, $c_{ij} = (a_i + 2a_j)/3$; (48 functionals)

3° The nodal value $p(a_{ijk})$ and 6 normal derivatives $\partial_\nu p(a^n_{ijk})$ on the face with vertices a_i, a_j, a_k, $1 \le i,j,k \le 4, i \ne j, j \ne k, k \ne i$, $n = 1,2,\cdots,6$, where a_{ijk} is the barycenter of the face and $a^1_{ijk} = (2a_i + a_j + a_k)/4, a^2_{ijk} = (a_i + 2a_j + a_k)/4, a^3_{ijk} = (a_i + a_j + 2a_k)/4, a^4_{ijk} = (4a_i + a_j + a_k)/6, a^5_{ijk} = (a_i + 4a_j + a_k)/6, a^6_{ijk} = (a_i + a_j + 4a_k)/6$; (24 functionals)

4° The nodal values $p(d_i), 1 \le i \le 4$, at internal points $d_1 = (2a_1 + a_2 + a_3 + a_4)/5, d_2 = (a_1 + 2a_2 + a_3 + a_4)/5, d_3 = (a_1 + a_2 + 2a_3 + a_4)/5, d_4 = (a_1 + a_2 + a_3 + 2a_4)/5$. (4 functionals)

Let X_h be the finite element space

$$X_h = \{v_h : v_h|_K \in P_9(K), \forall K \in \mathcal{M}_h, f(v_h|_{K_1}) = f(v_h|_{K_2}), \forall f \in \Lambda_{K_1} \cap \Lambda_{K_2}\}.$$

It is known that $X_h \subset H^2(\Omega)$. We define the operator Π_h as follows. For any $v_h \in V_h$, $w_h := \Pi_h v_h \in X_h$ such that for any $K \in \mathcal{M}_h$, $w_h|_K \in P_9(K)$, for the degrees of freedom at vertices a_i, $1 \le i \le 4$,

$$\partial^\alpha(w_h|_K)(a_i) = \frac{1}{N(a_i)} \sum_{K'\in\mathcal{M}_h(a_i)} \partial^\alpha(v_h|_{K'})(a_i), \quad |\alpha| \le 4, \tag{6.8}$$

for the degrees of freedom on the edge with vertices a_i, a_j, $1 \le i \ne j \le 4$,

$$\partial_{\nu_k}(w_h|_K)(a_{ij}) = \frac{1}{N(a_{ij})} \sum_{K'\in\mathcal{M}_h(a_{ij})} \partial_{\nu_k}(v_h|_{K'})(a_{ij}), \quad k = 1,2, \tag{6.9}$$

$$\partial_{\nu_k\nu_l}(w_h|_K)(b_{ij}) = \frac{1}{N(b_{ij})} \sum_{K'\in\mathcal{M}_h(b_{ij})} \partial_{\nu_k\nu_l}(v_h|_{K'})(b_{ij}), \quad k,l = 1,2, \tag{6.10}$$

$$\partial_{\nu_k\nu_l}(w_h|_K)(c_{ij}) = \frac{1}{N(c_{ij})} \sum_{K'\in\mathcal{M}_h(c_{ij})} \partial_{\nu_k\nu_l}(v_h|_{K'})(c_{ij}), \quad k,l = 1,2, \tag{6.11}$$

for the degrees of freedom on the faces with vertices $a_i, a_j, a_k,\ 1 \le i, j, k \le 4, i \ne j, j \ne k, k \ne i$,

$$(w_h|_K)(a_{ijk}) = \frac{1}{N(a_{ijk})} \sum_{K' \in \mathcal{M}_h(a_{ijk})} (v_h|_{K'})(a_{ijk}), \tag{6.12}$$

$$\partial_\nu (w_h|_K)(a^n_{ijk}) = \frac{1}{N(a^n_{ijk})} \sum_{K' \in \mathcal{M}_h(a^n_{ijk})} \partial_\nu (v_h|_{K'})(a^n_{ijk}), \quad n = 1, 2 \cdots, 6, \tag{6.13}$$

and finally for the degrees of freedom at the interior points $d_i, 1 \le i \le 4$,

$$(w_h|_K)(d_i) = (v_h|_K)(d_i). \tag{6.14}$$

To show the desired estimate (3.9) in 3D we use the C^0-P_9 element in [20] which is a triple (K, P_K, Θ_K), where $P_K = P_9(K)$ and the set of degrees of freedom Θ_K is defined by replacing some of the degrees of freedom of the $C^1 - P_9$ element Λ_K as follows:

1° For the edge with vertices $a_i, a_j,\ 1 \le i \ne j \le 4$, replace the 2 edge first order normal derivatives by $Dp(a_{ij})(a_k - a_{ij}), Dp(a_{ij})(a_l - a_{ij})$ and denote the corresponding nodal basis functions $p^k_{ij}(x), p^l_{ij}(x)$, where a_k, a_l are the other 2 vertices of K other than a_i, a_j;

2° For the edge with vertices $a_i, a_j,\ 1 \le i \ne j \le 4$, replace the 3 edge second order normal derivatives by $D^2p(b_{ij})(a_k - b_{ij}, a_l - b_{ij}), D^2p(c_{ij})(a_k - b_{ij}, a_l - b_{ij})$ and denote the corresponding nodal basis functions $p^{kl}_{ij}(x), q^{kl}_{ij}(x)$, where a_k, a_l are the other 2 vertices of K other than a_i, a_j;

3° For the face with vertices $a_i, a_j, a_k,\ 1 \le i, j, k \le 4, i \ne j, j \ne k, k \ne i$, replace the face normal derivatives by $Dp(a^n_{ijk})(a_l - a^n_{ijk})$ and denote the corresponding nodal basis functions $p^n_{ijk}(x)$, where a_l is the vertex of K other than a_i, a_j, a_k, $n = 1, 2 \cdots, 6$.

A regular family of this $C^0 - P_9$ element is affine-equivalent. For any $v_h \in V_h$, we also define an operator $q_h := \Lambda_h v_h$ in a similar way as the definition of Π_h by replacing the average normal derivatives in (6.9)-(6.11) and (6.13) by the corresponding directional derivatives in the definition of degrees of freedom for the $C^0 - P_9$ element. By the same argument as that in the proof of 2D case in section 3 we have

$$|v_h - q_h|_{H^m(K)} \le C h^{2-m} |v_h|_{2, \mathcal{M}_h(K)}, \quad m = 0, 1, 2. \tag{6.15}$$

Next we expend $q_h - w_h \in P_9(K)$ in terms of the nodal basis functions of the $C^0 - P_9$ element. From the definition of the $C^1 - P_9$ and $C^0 - P_9$ elements, we have $q_h - w_h = \phi_e + \phi_f$ in K, where the edge part of the function $q_h - w_h$ is

$$\begin{aligned}
\phi_e(x) = & \sum_{\substack{1 \le i \ne j \le 4 \\ \{k,l\} \in \{1,2,3,4\} \setminus \{i,j\}, k \ne l}} \Big[D(q_h|_K - w_h|_K)(a_{ij})(a_k - a_{ij}) p^k_{ij}(x) \\
& \qquad + D(q_h|_K - w_h|_K)(a_{ij})(a_l - a_{ij}) p^l_{ij}(x) \Big] \\
& + \sum_{\substack{1 \le i \ne j \le 4 \\ \{k,l\} \in \{1,2,3,4\} \setminus \{i,j\}, k \le l}} \Big[D^2(q_h|_K - w_h|_K)(b_{ij})(a_k - b_{ij}, a_l - b_{ij}) p^{kl}_{ij}(x) \\
& \qquad + D^2(q_h|_K - w_h|_K)(c_{ij})(a_k - c_{ij}, a_l - c_{ij}) q^{kl}_{ij}(x) \Big],
\end{aligned}$$

and the face part of the function $q_h - w_h$ is

$$\phi_f(x) = \sum_{\substack{1\le i,j,k\le 4, i\ne j, j\ne k, k\ne i\\ \{l\}\in\{1,2,3,4\}\setminus\{i,j,k\}}} \sum_{n=1}^{6} D(q_h|_K - w_h|_K)(a_{ijk}^n)(a_l - a_{ijk}^n)p_{ijk}^n(x).$$

Since the tangential derivatives of $q_h - w_h$ along the edges vanish, we obtain by the same argument as that in the proof of 2D case in section 3 that

$$|\phi_e|_{H^m(K)} \le Ch^{2-m}|v_h|_{2,\mathcal{M}_h(K)}, \quad m = 0, 1, 2. \tag{6.16}$$

On any face F of K, $q_h - w_h - \phi_e \in P_9(F)$ and its nodal values at 3 vertices up to 4th order derivatives vanish, its first order normal derivative at the midpoint and two second order normal derivatives at two internal trisection points on 3 edges vanish, and the nodal value at the barycenter also vanishes. This implies $q_h - w_h - \phi_e = 0$ on any face of the element K. Let τ_{ijk}^n be the tangential unit vector on the face of vertices a_i, a_j, a_k such that

$$a_l - a_{ijk}^n = [(a_l - a_{ijk}^n)\cdot \tau_{ijk}^n]\tau_{ijk}^n + [(a_l - a_{ijk}^n)\cdot \nu]\nu.$$

Now by (6.11), (6.15)-(6.16), and the inverse estimate we have

$$\begin{aligned} &|D(q_h|_K - w_h|_K)(a_{ijk}^n)(a_l - a_{ijk}^n)| \\ \le\ & |[(a_l - a_{ijk}^n)\cdot\tau_{ijk}^n]D\phi_e(a_{ijk}^n)\tau_{ijk}^n| + |[(a_l - a_{ijk}^n)\cdot\nu]D(q_h|_K - w_h|_K)(a_{ijk}^n)\nu| \\ \le\ & Ch^{1/2}|v_h|_{2,\mathcal{M}_h(K)}. \end{aligned} \tag{6.17}$$

Since a regular family of $C^0 - P_9$ element is affine-equivalent, we have $|p_{ijk}^n|_{H^m(K)} \le Ch^{3/2-m}$, $m = 0, 1, 2$. Therefore, by (6.17) we obtain

$$|\phi_f|_{H^m(K)} \le Ch^{2-m}|v_h|_{2,\mathcal{M}_h(K)}, \quad m = 0, 1, 2. \tag{6.18}$$

Combining (6.15), (6.16), (6.18) yields the desired estimate (3.9) in 3D since $v_h - w_h = (v_h - q_h) + \phi_e + \phi_f$ in K. The estimate (3.10) can be proved in the same way as the proof for the 2D case in section 3. This completes the proof. ☐

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