ON THE LAWRENCE-DONIACH MODEL FOR LAYERED SUPERCONDUCTORS

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Abstract. The Lawrence-Doniach model for superconducting layer compounds in which Ginzburg-Landau order parameters in adjacent layers are coupled by Josephson tunneling is considered. The main purpose of this paper is to demonstrate that the time dependent Ginzburg-Landau model is the limit of the Lawrence-Doniach model as the layer spacing goes to zero.

Key words. layered superconductors, Lawrence-Doniach model, Ginzburg-Landau model

AMS(MOS) subject classifications. 35K55, 82D55

1. Introduction. Recently, following the discovery of the high temperature superconductors, there has been a considerable interest in the properties of superconductors. One popular macroscopic model which later was deduced by Gor'kov [9] theoretically from an averaging of the microscopic Bardeen-Cooper-Schrieffer (BCS) theory in [1] was first introduced by Ginzburg and Landau [7] in their phenomenological approach to superconductivity close to the transition temperature. The time dependent Ginzburg-Landau (TDGL) model was deduced by Gor'kov and Eliashberg [10] from the microscopic BCS theory and later this model was studied by many authors. For more physical backgrounds on the microscopic and macroscopic theories of superconductivity we refer to the recent survey articles [3] and [6] and the references therein.

Superconducting layer compounds are materials in which metallic monolayers of transition-metal dichalcogenides are stacked intrinsically (intrinsic layer compounds) or that organic molecules are intercalated between the above metallic layers (intercalated layer compounds). Some examples of such metallic layers are those of TaS_2, $TaSe_2$, NbS_2, $NbSe_2$ and so on. In this paper we shall consider the Lawrence-Doniach (LD) model [13] in which the Ginzburg-Landau order parameter in adjacent layers is coupled by Josephson tunneling. For more information concerning the LD model we refer also to [2], [11], [15] and the references therein.

In this paper we first describe the stationary LD model in section 2 and prove an existence result. Then in section 3 we introduce the time dependent Lawrence-Doniach (TDLD) model and show the existence and uniqueness of the strong solutions of the TDLD model. In section 4 we show the main result of this paper, i.e. we shall show that the TDGL model is the limit of the TDLD model in section 2 as the layer spacing goes to zero. The last section is devoted to some concluding remarks.

In the remainder of this section we introduce some notations. Throughout the paper

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we will assume that Ω is a plane domain with the boundary $\partial\Omega \in C^{2,1}$. Since we are dealing with the plane, it is convenient to introduce two curl operators:

$$\operatorname{curl}\mathbf{A} = \frac{\partial \mathbf{A}_2}{\partial x_1} - \frac{\partial \mathbf{A}_1}{\partial x_2} \quad \text{and} \quad \mathbf{curl}\,\psi = \Big(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1}\Big)^T.$$

We also need the divergence and gradient operators defined by

$$\operatorname{div}\mathbf{A} = \frac{\partial \mathbf{A}_1}{\partial x_1} + \frac{\partial \mathbf{A}_2}{\partial x_2} \quad \text{and} \quad \nabla\psi = \Big(\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2}\Big)^T,$$

respectively. From these definitions, it is easy to check that

$$-\Delta\mathbf{A} = \mathbf{curl}\operatorname{curl}\mathbf{A} - \nabla\operatorname{div}\mathbf{A}.$$

If B denotes some Banach space of real scalar functions, the corresponding space of complex scalar functions will be denoted by its calligraphic form $\mathcal{B}$ and the corresponding space of real vector-valued functions, each of its components belonging to B, will be denoted by its boldfaced form $\mathbf{B}$. However, the norms on $\mathcal{B}$ and $\mathbf{B}$ will be denoted still by $\|\cdot\|_B$ instead of $\|\cdot\|_{\mathcal{B}}$ and $\|\cdot\|_{\mathbf{B}}$. We will make use of the following subspace of $\mathbf{H}^1(\Omega)$:

$$\mathbf{H}_n^1(\Omega) = \big\{\mathbf{Q} \in \mathbf{H}^1(\Omega) \,:\, \mathbf{Q}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\big\}$$

with the norm on this space defined by

$$\|\mathbf{Q}\|_{\mathbf{H}_n^1(\Omega)} = \Big[\|\operatorname{div}\mathbf{Q}\|^2_{L^2(\Omega)} + \|\operatorname{curl}\mathbf{Q}\|^2_{L^2(\Omega)} + \|\mathbf{Q}\|^2_{L^2(\Omega)}\Big]^{1/2}$$

which is equivalent to the usual norm $\|\mathbf{Q}\|_{H^1(\Omega)}$ (cf. e.g. [8]). We will also need the subspace of $\mathbf{H}_n^1(\Omega)$

$$\mathbf{H}_n^1(\operatorname{div};\Omega) = \big\{\mathbf{Q} \in \mathbf{H}_n^1(\Omega) \,:\, \operatorname{div}\mathbf{Q} = 0 \text{ in } \Omega\big\}.$$

2. The stationary model. Let Ω be the domain in $\mathbb{R}^2$, $\partial\Omega \in C^2$ its boundary and $\mathbf{n}$ the unit outer normal vector to $\partial\Omega$. We choose the x_3 direction perpendicular to the layers. The Gibbs free energy functional of the whole system F may be given by the sum of the Gibbs free energies of the individual layers F_j and the Josephson coupling energies of the adjacent metallic layers $\delta\,E_{j,j+1}$. After doing appropriate nondimensionalization, we have

$$\begin{aligned} F &= \sum_{j=1}^{N} F_j + \sum_{j=1}^{N-1} \delta\,E_{j,j+1} \\ &= \sum_{j=1}^{N} s\int_\Omega \Big[-|\psi_j|^2 + \frac{1}{2}|\psi_j|^4 + \Big|-\frac{i}{\kappa}\nabla\psi_j - \mathbf{A}_j\psi_j\Big|^2 + |\operatorname{curl}\mathbf{A}_j - H_j|^2\Big]\,dx \\ &\quad + \sum_{j=1}^{N-1} s\int_\Omega \frac{1}{\kappa^2 s^2}\big|\psi_j - \psi_{j+1}\exp(-i\kappa s\chi_j)\big|^2 + \big|\nabla\chi_j - \delta\,\mathbf{A}_{j+1}\big|^2\Big]\,dx, \end{aligned}$$

where N is the number of the layers, s the layer spacing, κ the Ginzburg-Landau parameter and

$$\delta\,\mathbf{A}_j = \frac{\mathbf{A}_j - \mathbf{A}_{j-1}}{s}, \ \forall j = 1, \cdots, N.$$

The quantity ψ_j is the complex order parameter of the jth metallic layer, $\mathbf{A}_j$ is the magnetic vector potential of the jth layer which is parallel to the layers and χ_j is the magnetic vector potential perpendicular to the layers. $(0, 0, H_j)$ is the applied external magnetic field in the jth layer which is in the x_3 direction. The unknowns of the systems are

$$\{\psi_j\}_{j=1}^{N}, \ \{\mathbf{A}_j\}_{j=1}^{N} \quad \text{and} \quad \{\chi_j\}_{j=1}^{N-1}.$$

We note that in writing the Gibbs free energy functional F we have assumed that the x_3-component of the magnetic vector potential is piecewise constant, which is different from the original LD functional in [11], [13]. Thus the solutions of the LD model in this paper might not yield the pancake vortices that are believed to be described by the original LD model. However, when the layer spacing s is small, the LD model studied in this paper do provide a good approximation to the original LD model. It is also not difficult by employing the method in this paper to prove that the LD model in [11], [13] tends to the TDGL model in §4.1 when the layer spacing s goes to zero.

In this paper we will always assume that

$$\chi_0 = \chi_N = 0, \ \mathbf{A}_0 = \mathbf{A}_1, \ \mathbf{A}_{N+1} = \mathbf{A}_N, \ \psi_0 = \psi_1, \ \psi_{N+1} = \psi_N.$$

These extension conditions are imposed only for mathematical convenience, in the limit $s \to 0$ they will give the correct boundary conditions on the Ginzburg-Landau equations.

Minimizing the total Gibbs free energy with respect to $\overline{\psi}_j$, the conjugate of ψ_j, $\mathbf{A}_j$ for $j = 1, \cdots, N$ and χ_j for $j = 1, \cdots, N-1$, we obtain

$$\begin{aligned}
&\Big(-\frac{i}{\kappa}\nabla - \mathbf{A}_j\Big)^2 \psi_j + \frac{1}{\kappa^2 s^2}\Big[2\psi_j - \psi_{j+1}\exp\big(-i\kappa s\chi_j\big) - \psi_{j-1}\exp\big(i\kappa s\chi_{j-1}\big)\Big] \\
&\qquad\qquad + \big(|\psi_j|^2 - 1\big)\psi_j = 0, \ j = 1, \cdots, N \\
&\mathbf{curl}\,\mathrm{curl}\,\mathbf{A}_j + \delta\,(\nabla\chi_j - \delta\,\mathbf{A}_{j+1}) = -\mathrm{Re}\,\Big[\big(\frac{i}{\kappa}\nabla\psi_j + \mathbf{A}_j\psi_j\big)\overline{\psi}_j\Big] + \mathbf{curl}\,H_j, \ j = 1, \cdots, N \\
&-\Delta\chi_j + \delta\,\mathrm{div}\mathbf{A}_{j+1} = \mathrm{Re}\,\Big[\frac{i}{\kappa s}\psi_j\overline{\psi}_{j+1}\exp\big(i\kappa s\chi_j\big)\Big], \ j = 1, \cdots, N-1
\end{aligned}$$

where $\delta\,(\delta\,\mathbf{A}_{j+1}) = (\mathbf{A}_{j+1} - 2\mathbf{A}_j + \mathbf{A}_{j-1})/s^2$, $j = 1, \cdots, N$. The boundary conditions are

$$\Big(\frac{i}{\kappa}\nabla\psi_j + \mathbf{A}_j\psi_j\Big)\cdot\mathbf{n} = 0, \ \mathrm{curl}\,\mathbf{A}_j = H_j, \ j = 1, \cdots, N \tag{2.1}$$

$$\big(\nabla\chi_j - \delta\,\mathbf{A}_{j+1}\big)\cdot\mathbf{n} = 0, \ j = 1, \cdots, N-1. \tag{2.2}$$

As a consequence, we see that the current $\mathbf{J}_j$ in the jth layer and the current $J_{j,j+1}$ across the jth and $j+1$th layers are given by

$$\mathbf{J}_j = \mathbf{curl}\,\mathrm{curl}\,\mathbf{A}_j + \delta\,(\nabla\chi_j - \delta\,\mathbf{A}_{j+1}), \ j = 1, \cdots, N \tag{2.3}$$

$$J_{j,j+1} = -\Delta\chi_j + \delta\,\mathrm{div}\mathbf{A}_{j+1}, \ j = 1, \cdots, N-1. \tag{2.4}$$

For any $\phi = \{\phi_j\}_{j=1}^N \in \left[H^2(\Omega)\right]^N$, let the linear transformation G_ϕ from $\left[\mathcal{H}^1(\Omega)\right]^N \times \left[\mathbf{H}^1(\Omega)\right]^N \times \left[H^1(\Omega)\right]^{N-1}$ into itself be defined by

$$G_\phi(\psi, \mathbf{A}, \chi) = (\widetilde{\psi}, \widetilde{\mathbf{A}}, \widetilde{\chi})$$

if

$$\widetilde{\psi}_j = \psi_j \exp(i\kappa\phi_j),\ \ \widetilde{\mathbf{A}}_j = \mathbf{A}_j + \nabla\phi_j,\ \ j = 1, \cdots, N$$

and

$$\widetilde{\chi}_j = \chi_j + \delta\, \phi_{j+1},\ \ j = 1, \cdots, N-1.$$

$(\psi, \mathbf{A}, \chi)$ and $(\widetilde{\psi}, \widetilde{\mathbf{A}}, \widetilde{\chi})$ are said to be equivalent if and only if there exists a $\phi \in \left[H^2(\Omega)\right]^N$ such that $(\widetilde{\psi}, \widetilde{\mathbf{A}}, \widetilde{\chi}) = G_\phi(\psi, \mathbf{A}, \chi)$. It is easy to see that any $(\psi, \mathbf{A}, \chi) \in \left[\mathcal{H}^1(\Omega)\right]^N \times \left[\mathbf{H}^1(\Omega)\right]^N \times \left[H^1(\Omega)\right]^{N-1}$ is gauge equivalent to an element of $\left[\mathcal{H}^1(\Omega)\right]^N \times \left[\mathbf{H}^1_n(\mathrm{div};\Omega)\right]^N \times \left[H^1(\Omega)\right]^{N-1}$ and the functional F is invariant under the gauge transformation G_ϕ. Concerning the existence of minimizers of F we have the following theorem which can be proved by standard minimization arguments.

THEOREM 1. *The functional F has at least one minimizer belonging to $\left[\mathcal{H}^1(\Omega)\right]^N \times \left[\mathbf{H}^1_n(\mathrm{div};\Omega)\right]^N \times \left[H^1(\Omega)\right]^{N-1}$.*

3. The nonstationary model. Let $(0, T)$ be the time interval and $Q_T = \Omega \times (0, T)$. After proper nondimensionalization, the dynamics of layered superconductors can be phenomenologically described by the following equations

$$\begin{aligned}
&\eta\frac{\partial\psi_j}{\partial t} + i\eta\kappa\Phi_j\psi_j + \Big(-\frac{i}{\kappa}\nabla - \mathbf{A}_j\Big)^2\psi_j + \big(|\psi_j|^2 - 1\big)\psi_j \\
&\quad + \frac{1}{\kappa^2 s^2}\Big[2\psi_j - \psi_{j+1}\exp\big(-i\kappa s\chi_j\big) - \psi_{j-1}\exp\big(i\kappa s\chi_{j-1}\big)\Big] = 0,\ \ j = 1, \cdots, N \qquad (3.1)
\end{aligned}$$

$$\mathbf{J}_j = -\Big(\frac{\partial\mathbf{A}_j}{\partial t} + \nabla\Phi_j\Big) - \mathrm{Re}\Big[\big(\frac{i}{\kappa}\nabla\psi_j + \mathbf{A}_j\psi_j\big)\overline{\psi}_j\Big] + \mathbf{curl}\, H_j,\ \ j = 1, \cdots, N \qquad (3.2)$$

$$J_{j,j+1} = -\Big(\frac{\partial\chi_j}{\partial t} + \delta\,\Phi_{j+1}\Big) + \mathrm{Re}\Big[\frac{i}{\kappa s}\psi_j\overline{\psi}_{j+1}\exp\big(i\kappa s\chi_j\big)\Big],\ \ j = 1, \cdots, N-1 \qquad (3.3)$$

where the current $\mathbf{J}_j$ in the jth layer and the current $J_{j,j+1}$ across the jth and $j+1$th layers are given by (2.3)-(2.4). The quantity Φ_j is the electric potential in the jth layer. The boundary conditions are given in (2.1)-(2.2). The unknowns of the equations are

$$\psi = \{\psi_j\}_{j=1}^N,\ \ \mathbf{A} = \{\mathbf{A}_j\}_{j=1}^N,\ \ \Phi = \{\Phi_j\}_{j=1}^N \quad \text{and} \quad \chi = \{\chi_j\}_{j=1}^{N-1}.$$

For any $\phi = \{\phi_j\}_{j=1}^N \in \left[W_2^{2,1}(Q_T)\right]^N$ with $\phi_j(x, 0) = 0, j = 1, \cdots, N$, let the linear transformation G_ϕ from

$$\left[L^2(0,T;\mathcal{H}^1(\Omega))\right]^N \times \left[L^2(0,T;\mathbf{H}^1(\Omega))\right]^N \times \left[L^2(Q_T)\right]^N \times \left[L^2(0,T;H^1(\Omega))\right]^{N-1}$$

into itself be defined by

$$G_\phi(\psi, \mathbf{A}, \Phi, \chi) = (\widetilde{\psi}, \widetilde{\mathbf{A}}, \widetilde{\Phi}, \widetilde{\chi})$$

where

$$\widetilde{\psi}_j = \psi_j \exp(i\kappa\phi_j),\ \ \widetilde{\mathbf{A}}_j = \mathbf{A}_j + \nabla\phi_j,\ \ \widetilde{\Phi}_j = \Phi_j - \frac{\partial\phi_j}{\partial t},\ \ j = 1, \cdots, N$$

and

$$\widetilde{\chi}_j = \chi_j + \delta\,\phi_{j+1},\ \ j = 1, \cdots, N-1.$$

It is easy to check that the equations (3.1)-(3.3) are invariant under the gauge transformation G_ϕ. For any solution $(\widetilde{\psi}, \widetilde{\mathbf{A}}, \widetilde{\Phi}, \widetilde{\chi})$ of the equations (3.1)-(3.3) with the boundary conditions (2.1)-(2.2), let $\phi = \{\phi_j\}_{j=1}^N$ be the solution of the parabolic problem

$$\begin{aligned}
&\frac{\partial\phi_j}{\partial t} - \Delta\phi_j - \delta^2\phi_{j+1} = -\widetilde{\Phi}_j - \mathrm{div}\widetilde{\mathbf{A}}_j - \delta\,\widetilde{\chi}_j,\ \ j = 1, \cdots, N \\
&\nabla\phi_j \cdot \mathbf{n} = \widetilde{\mathbf{A}}_j \cdot \mathbf{n},\ \ \phi_j(x, 0) = 0,\ \ j = 1, \cdots, N-1
\end{aligned}$$

where we let $\phi_{N+1} = \phi_N$, $\phi_0 = \phi_1$, then

$$(\psi, \mathbf{A}, \Phi, \chi) = G_\phi(\widetilde{\psi}, \widetilde{\mathbf{A}}, \widetilde{\Phi}, \widetilde{\chi})$$

is also the solution of the equations (3.1)-(3.3) with the boundary conditions (2.1)-(2.2) and satisfies

$$\Phi_j = -\mathrm{div}\mathbf{A}_j - \delta\,\chi_j \quad \text{in } \Omega,\ \ j = 1, \cdots, N \tag{3.4}$$

$$\mathbf{A}_j \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,\ \ j = 1, \cdots, N. \tag{3.5}$$

Substitute the gauge relations (3.4)-(3.5) and the definition of $\mathbf{J}_j$ and $J_{j,j+1}$ in (2.3)-(2.4) into the equations (3.1)-(3.3) and the boundary conditions (2.1)-(2.2), we obtain the following system of nonlinear parabolic system of equations whose properties will be studied in this paper

$$\begin{aligned}
&\eta\frac{\partial\psi_j}{\partial t} - i\eta\kappa(\mathrm{div}\mathbf{A}_j + \delta\,\chi_j)\psi_j + \Big(-\frac{i}{\kappa}\nabla - \mathbf{A}_j\Big)^2\psi_j + \big(|\psi_j|^2 - 1\big)\psi_j \\
&\qquad + \frac{1}{\kappa^2 s^2}\Big[2\psi_j - \psi_{j+1}\exp\big(-i\kappa s\chi_j\big) - \psi_{j-1}\exp\big(i\kappa s\chi_{j-1}\big)\Big] = 0,\ \ j = 1, \cdots, N
\end{aligned} \tag{3.6}$$

$$\frac{\partial\mathbf{A}_j}{\partial t} - \Delta\mathbf{A}_j - \delta^2\mathbf{A}_{j+1} + \mathrm{Re}\Big[\big(\frac{i}{\kappa}\nabla\psi_j + \mathbf{A}_j\psi_j\big)\overline{\psi}_j\Big] = \mathbf{curl}\,H_j,\ \ j = 1, \cdots, N \tag{3.7}$$

$$\frac{\partial\chi_j}{\partial t} - \Delta\chi_j - \delta^2\chi_{j+1} = \mathrm{Re}\Big[\frac{i}{\kappa s}\psi_j\overline{\psi}_{j+1}\exp\big(i\kappa s\chi_j\big)\Big],\ \ j = 1, \cdots, N-1 \tag{3.8}$$

$$\nabla\psi_j \cdot \mathbf{n} = 0,\ \ \mathbf{A}_j \cdot \mathbf{n} = 0,\ \ \mathrm{curl}\,\mathbf{A}_j = H_j \quad \text{on } \partial\Omega \times (0, T),\ \ j = 1, \cdots, N \tag{3.9}$$

$$\nabla\chi_j \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, T),\ \ j = 1, \cdots, N-1 \tag{3.10}$$

$$\psi_j(x, 0) = \psi_0^j(x),\ \ \mathbf{A}_j(x, 0) = \mathbf{A}_0^j(x) \quad \text{on } \Omega,\ \ j = 1, \cdots, N \tag{3.11}$$

$$\chi_j(x, 0) = \chi_0^j(x) \quad \text{on } \Omega,\ \ j = 1, \cdots, N-1. \tag{3.12}$$

We will also impose the extension rules

$$\chi_0=\chi_N=0,\ \psi_0=\psi_1,\ \psi_N=\psi_{N+1},\ \mathbf{A}_0=\mathbf{A}_1,\ \mathbf{A}_N=\mathbf{A}_{N+1}. \tag{3.13}$$

Let $W(0,T) = L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$, $\mathcal{W}(0,T) = L^2(0,T;\mathcal{H}^1(\Omega))\cap H^1(0,T;(\mathcal{H}^1(\Omega))')$ and $\mathbf{W}_n(0,T) = L^2(0,T;\mathbf{H}^1_n(\Omega)) \cap H^1(0,T;(\mathbf{H}^1_n(\Omega))')$, where B' denotes the dual space of B. The weak formulation of the Problem (3.6) -(3.12) can be defined as following.

Problem (P^N). Find $\psi_j \in \mathcal{W}(0,T), \mathbf{A}_j \in \mathbf{W}_n(0,T)$, for $j=1,\cdots,N$ and $\chi_j \in W(0,T)$, for $j=1,\cdots,N-1$ such that

$$\psi_j(x,0)=\psi_0^j(x),\ \mathbf{A}_j(x,0)=\mathbf{A}_0^j(x), \chi_j(x,0)=\chi_0^j(x). \tag{3.14}$$

and

$$\begin{aligned}
&\sum_{j=1}^N \int_0^T \int_\Omega \Big[\eta \frac{\partial \psi_j}{\partial t}\omega_j - i\eta\kappa(\mathrm{div}\mathbf{A}_j + \delta\,\chi_j)\psi_j\omega_j + (|\psi_j|^2-1)\psi_j\omega_j\Big]\,dx\,dt \\
&+\sum_{j=1}^N \int_0^T \int_\Omega \Big(-\frac{i}{\kappa}\nabla\psi_j - \mathbf{A}_j\psi_j\Big)\Big(\frac{i}{\kappa}\nabla\omega_j - \mathbf{A}_j\omega_j\Big)\,dx\,dt \\
&+\frac{1}{\kappa^2 s^2}\sum_{j=1}^N \int_0^T \int_\Omega \big[\psi_j - \psi_{j+1}\exp(-i\kappa s\chi_j)\big]\big[\omega_j - \omega_{j+1}\exp(i\kappa s\chi_j)\big]\,dx\,dt \\
=&0, \qquad \forall \omega_1,\cdots,\omega_N \in L^2(0,T;\mathcal{H}^1(\Omega)),\ \omega_{N+1}=\omega_N, \omega_0=\omega_1
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\sum_{j=1}^N \int_0^T \int_\Omega \Big[\frac{\partial \mathbf{A}_j}{\partial t}\mathbf{B}_j + \mathrm{div}\mathbf{A}_j\mathrm{div}\mathbf{B}_j + \mathrm{curl}\,\mathbf{A}_j\mathrm{curl}\,\mathbf{B}_j + \delta\,\mathbf{A}_{j+1}\delta\,\mathbf{B}_{j+1}\Big]\,dx\,dt \\
&+\sum_{j=1}^N \int_0^T \int_\Omega \mathrm{Re}\Big[\Big(\frac{i}{\kappa}\nabla\psi_j + \mathbf{A}_j\psi_j\Big)\overline{\psi}_j\Big]\mathbf{B}_j\,dx\,dt - \sum_{j=1}^N \int_0^T \int_\Omega H_j\mathrm{curl}\,\mathbf{B}_j\,dx\,dt \\
=&0 \qquad \forall \mathbf{B}_1,\cdots,\mathbf{B}_N \in L^2(0,T;\mathbf{H}^1_n(\Omega)),\ \mathbf{B}_{N+1}=\mathbf{B}_N, \mathbf{B}_0=\mathbf{B}_1
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\sum_{j=1}^{N-1} \int_0^T \int_\Omega \Big[\frac{\partial \chi_j}{\partial t}\zeta_j + \nabla\chi_j\nabla\zeta_j\Big]\,dx\,dt + \sum_{j=1}^N \int_0^T \int_\Omega \delta\,\chi_j\delta\,\zeta_j\,dx\,dt \\
&-\sum_{j=1}^{N-1} \int_0^T \int_\Omega \frac{1}{\kappa s}\mathrm{Re}\Big[i\psi_j\overline{\psi}_{j+1}\exp(i\kappa s\chi_j)\Big]\zeta_j\,dx\,dt \\
=&0, \qquad \forall \zeta_1,\cdots,\zeta_{N-1} \in L^2(0,T;H^1(\Omega)),\ \zeta_0=\zeta_N=0.
\end{aligned} \tag{3.17}$$

Theorem 2. *Let $\psi_0^j \in \mathcal{H}^1(\Omega)$, $\mathbf{A}_0^j \in \mathbf{H}^1_n(\Omega)$ and $\chi_0^j \in H^1(\Omega)$ such that $|\psi_0^j| \le 1$ a.e. on Ω, $j=1,\cdots,N$. Furthermore, let $H_j \in H^1(0,T;H^1(\Omega)), j=1,\cdots,N$. Then the Problem ($\mathrm{P}^N$) has a unique solution $(\psi,\mathbf{A},\chi) \in \big[\mathcal{W}_2^{2,1}(Q_T)\big]^N \times \big[\mathbf{W}_2^{2,1}(Q_T)\big]^N \times \big[W_2^{2,1}(Q_T)\big]^{N-1}$ satisfying $|\psi_j| \le 1$ a.e. in Q_T.*

The uniqueness of the solutions can be proved by standard arguments. The existence can be proved by using Leray-Schauder's fixed point theorem as in [5] with minor modifications. Since the method to deduce the necessary *a priori* estimates is similar to that given in next section for proving the convergence, we omit here the details of the proof of Theorem 2.

4. The relation to Ginzburg-Landau model. In this section we are going to show that the three-dimensional TDGL model is indeed the limit of the TDLD model for layered superconductors in section 3 as the layer spacing $s \to 0$. At first we recall briefly the TDGL model.

4.1. The TDGL *model.* Let $\Omega^* = \Omega \times (0, L) \subset \mathbb{R}^3$. Denote by $x^* = (x, x_3) = (x_1, x_2, x_3)$ and ∇^* the gradient operator in $\mathbb{R}^3$. Let $\mathcal{W}^*(0,T) = L^2(0,T;\mathcal{H}^1(\Omega^*)) \cap H^1(0,T;(\mathcal{H}^1(\Omega^*))')$ and $\mathbf{W}^*_n(0,T) = L^2(0,T;\mathbf{H}^1_n(\Omega^*)) \cap H^1(0,T;(\mathbf{H}^1_n(\Omega^*))')$. Let Ψ be the complex order parameter and Θ the three dimensional magnetic potential. The weak formulation of the phenomenological TDGL model can be formulated as follows, where we have chosen the gauge such that the electric potential $\Phi = -\nabla^* \cdot \Theta$. For the details about the TDGL model we refer to [5] and the references therein.

PROBLEM (P). Find $(\Psi, \Theta) \in \mathcal{W}^*(0,T) \times \mathbf{W}^*_n(0,T)$ such that

$$\Psi(x^*, 0) = \Psi_0(x^*), \quad \Theta(x^*, 0) = \Theta_0(x^*) \tag{4.1}$$

and

$$\begin{aligned}
&\eta \int_0^T \int_{\Omega^*} \frac{\partial \Psi}{\partial t} \omega \, dx^* \, dt - i\eta\kappa \int_0^T \int_{\Omega^*} (\nabla^* \cdot \Theta)\Psi\omega \, dx^* \, dt \\
&+ \int_0^T \int_{\Omega^*} \Big(-\frac{i}{\kappa}\nabla^*\Psi - \Theta\Psi\Big)\Big(\frac{i}{\kappa}\nabla^*\omega - \Theta\omega\Big) \, dx^* \, dt \\
&+ \int_0^T \int_{\Omega^*} \big(|\Psi|^2 - 1\big)\Psi\omega \, dx^* \, dt \\
&=0, \qquad \forall \omega \in L^2(0,T;\mathcal{H}^1(\Omega^*))
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\int_0^T \int_{\Omega^*} \frac{\partial \Theta}{\partial t} \mathbf{B} \, dx^* \, dt + \int_0^T \int_{\Omega^*} \Big[(\nabla^* \cdot \Theta)(\nabla^* \cdot \mathbf{B}) + (\nabla^* \times \Theta)(\nabla^* \times \mathbf{B})\Big] \, dx^* \, dt \\
&+ \int_0^T \int_{\Omega^*} \mathrm{Re}\Big[\Big(\frac{i}{\kappa}\nabla^*\Psi + \Theta\Psi\Big)\overline{\Psi}\Big]\mathbf{B} \, dx^* \, dt \\
&= \int_0^T \int_{\Omega^*} \mathbf{H}(\nabla^* \times \mathbf{B}) \, dx^* \, dt, \qquad \forall \mathbf{B} \in L^2(0,T;\mathbf{H}^1_n(\Omega^*)).
\end{aligned} \tag{4.3}$$

Here we assume that the applied external magnetic field $\mathbf{H} = (0, 0, H)$ is in the x_3 direction. When the boundary $\partial\Omega^*$ is smooth (say C^2), the existence and uniqueness of strong solutions (Ψ, Θ) has been proved in [5]. In the present case, however, since $\partial\Omega^* \notin C^2$, we can not apply the results in [5] directly. Nevertheless, we still have the following theorem.

THEOREM 3. *Let $\Psi_0 \in \mathcal{H}^1(\Omega^*)$, $\Theta_0 \in \mathbf{H}^1_n(\Omega^*)$ such that $|\Psi_0| \leq 1$ a.e. on Ω. Let further $H \in H^1(0,T;L^2(\Omega^*))$. Then the Problem (P) has a unique weak solution (Ψ,Θ) such that*

$$\Psi \in H^1(0,T;\mathcal{L}^2(\Omega^*)) \cap L^\infty(0,T;\mathcal{H}^1(\Omega^*)) \tag{4.4}$$

$$\Theta \in H^1(0,T;\mathbf{L}^2(\Omega^*)) \cap L^\infty(0,T;\mathbf{H}^1_n(\Omega^*)) \tag{4.5}$$

and $|\Psi| \leq 1$ a.e. in $Q_T^* = \Omega^* \times (0,T)$.

Proof. That $|\Psi| \leq 1$ a.e. in Q_T^* can be proved by taking $\omega = (|\Psi|^2-1)_+\overline{\Psi}\chi_{(0,t)}$, where $\chi_{(0,t)}$ is the characteristic function of the interval $(0,t)$, as the test function in (4.2) and using the method in [5]. The existence of a weak solution (Ψ,Θ) satisfying (4.4)-(4.5) is a direct consequence of the convergence result to be proved in section 4.3. The uniqueness of the solution (Ψ,Θ) having the regularity properties (4.4)-(4.5) can be proved by the standard method. Here we omit the details of the proof. □

4.2. The estimates. In this section we are going to show some estimates for the solution $\left(\{\psi_j\}_{j=1}^N, \{\mathbf{A}_j\}_{j=1}^N, \{\chi_j\}_{j=1}^{N-1}\right)$ of the Problem (P^N) which will be used in section 4.3 for the proof of main result in this paper. In the following we shall always denote by C the generic constants which are independent of the number of layers N and the layer spacing s.

LEMMA 1. *If $|\psi_j(x,0)| \leq 1$, a.e. in Ω, $j = 1,\cdots,N$, then we have*

$$|\psi_j(x,t)| \leq 1 \ \text{a.e. in } \Omega \times (0,T), \ \forall j = 1,\cdots,N.$$

Proof. Let $U_j = (|\psi_j|^2 - 1)$. Denote by $U_j^+ = \max(U_j,0)$ and $U_j^- = \min(U_j,0)$, then $U_j^+ \geq 0, U_j^- \leq 0$ and $U_j = U_j^+ + U_j^-$. Choose $\omega_j = (|\psi_j|^2-1)_+\overline{\psi}_j\chi_{(0,t)}$ as the test function in (3.15) and take the real part of the equation, then since

$$\begin{aligned}
&\mathrm{Re}\left[\eta\int_0^t\int_\Omega \frac{\partial\psi_j}{\partial t}\omega_j\,dx\,d\tau\right] = \frac{\eta}{4}\int_\Omega (|\psi_j(x,t)|^2-1)_+^2\,dx;\\
&\mathrm{Re}\left[-i\eta\kappa\int_0^t\int_\Omega (\mathrm{div}\mathbf{A}_j + \delta\,\chi_j)\psi_j\omega_j\,dx\,d\tau\right] = 0;\\
&\mathrm{Re}\left[\int_0^t\int_\Omega \Big(-\frac{i}{\kappa}\nabla\psi_j - \mathbf{A}_j\psi_j\Big)\Big(\frac{i}{\kappa}\nabla\omega_j - \mathbf{A}_j\omega_j\Big)\,dx\,d\tau\right]\\
=&\mathrm{Re}\left[\int_0^t\int_{\Omega_+^j(\tau)} \Big(-\frac{i}{\kappa}\nabla\psi_j - \mathbf{A}_j\psi_j\Big)\Big(\frac{i}{\kappa}\nabla|\psi_j|^2\Big)\overline{\psi}_j\,dx\,d\tau\right]\\
&+\left[\int_0^t\int_{\Omega_+^j(\tau)} \Big|-\frac{i}{\kappa}\nabla\psi_j - \mathbf{A}_j\psi_j\Big|^2(|\psi_j|^2-1)_+\,dx\,d\tau\right]\\
=&\frac{1}{2\kappa^2}\int_0^t\int_{\Omega_+^j(\tau)}\Big|\nabla|\psi_j|^2\Big|^2\,dx\,d\tau + \int_0^t\int_\Omega \Big|-\frac{i}{\kappa}\nabla\psi_j - \mathbf{A}_j\psi_j\Big|^2(|\psi_j|^2-1)_+\,dx\,d\tau\\
\geq&\frac{1}{2\kappa^2}\int_0^t\int_{\Omega_+^j(\tau)}\Big|\nabla|\psi_j|^2\Big|^2\,dx\,d\tau;
\end{aligned}$$

where $\Omega_+^j(\tau)=\{x\in\Omega:\ |\psi_j(x,\tau)|\geq 1\}$, and

$$\mathrm{Re}\Big[\int_0^t\int_\Omega (|\psi_j|^2-1)\psi_j\omega_j\,dx\,d\tau\Big]=\int_0^t\int_\Omega (|\psi_j|^2-1)_+^2|\psi_j|^2\,dx\,d\tau\geq 0;$$

$$\begin{aligned}
&\mathrm{Re}\Big\{\int_0^t\int_\Omega \big[\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)\big]\big[\omega_j-\omega_{j+1}\exp(i\kappa s\chi_j)\big]\,dx\,d\tau\Big\}\\
=&\mathrm{Re}\int_0^t\int_\Omega \Big[|\psi_j|^2U_j^+-\psi_j\overline{\psi}_{j+1}U_{j+1}^+\exp(i\kappa s\chi_j)\\
&\qquad -\overline{\psi}_j\psi_{j+1}U_j^+\exp(-i\kappa s\chi_j)+|\psi_{j+1}|^2U_{j+1}^+\Big]\,dx\,d\tau\\
\geq&\int_0^t\int_\Omega\Big[|\psi_j|^2U_j^+-|\psi_j||\psi_{j+1}|U_{j+1}^+-|\psi_j||\psi_{j+1}|U_j^++|\psi_{j+1}|^2U_{j+1}^+\Big]\,dx\,d\tau\\
\geq&\int_0^t\int_\Omega\Big[|\psi_j|^2U_j^+-\frac{1}{2}|\psi_j|^2U_{j+1}^+-\frac{1}{2}|\psi_{j+1}|^2U_{j+1}^+\\
&\qquad -\frac{1}{2}|\psi_j|^2U_j^+-\frac{1}{2}|\psi_{j+1}|^2U_j^++|\psi_{j+1}|^2U_{j+1}^+\Big]\,dx\,d\tau\\
=&\frac{1}{2}\int_0^t\int_\Omega\Big[|\psi_j|^2U_j^+-|\psi_j|^2U_{j+1}^+-|\psi_{j+1}|^2U_j^++|\psi_{j+1}|^2U_{j+1}^+\Big]\,dx\,d\tau\\
=&\frac{1}{2}\int_0^t\int_\Omega\big(|\psi_j|^2-|\psi_{j+1}|^2\big)\big(U_j^+-U_{j+1}^+\big)\,dx\,d\tau\\
=&\frac{1}{2}\int_0^t\int_\Omega\big(U_j-U_{j+1}\big)\big(U_j^+-U_{j+1}^+\big)\,dx\,d\tau\\
=&\frac{1}{2}\int_0^t\int_\Omega\Big[\big(U_j^+-U_{j+1}^+\big)^2+\big(U_j^--U_{j+1}^-\big)\big(U_j^+-U_{j+1}^+\big)\Big]\,dx\,d\tau\\
=&\frac{1}{2}\int_0^t\int_\Omega\Big[\big(U_j^+-U_{j+1}^+\big)^2-\big(U_j^-U_{j+1}^++U_{j+1}^-U_j^+\big)\Big]\,dx\,d\tau\\
\geq&\frac{1}{2}\int_0^t\int_\Omega\big(U_j^+-U_{j+1}^+\big)^2\,dx\,d\tau,
\end{aligned}$$

since $U_j^-,U_{j+1}^-\leq 0$, $U_j^+,U_{j+1}^+\geq 0$, we obtain

$$\frac{\eta}{4}\sum_{j=1}^N\int_\Omega |U_j^+(x,t)|^2\,dx+\frac{1}{2\kappa^2}\sum_{j=1}^N\int_0^t\int_\Omega|\nabla U_j|^2\,dx\,d\tau\leq 0.$$

This implies $U_j^+=0$ a.e. in Q_T, thus $|\psi_j|\leq 1$ a.e. in Q_T, $\forall j=1,\cdots,N$. $\square$

LEMMA 2. *We have*

$$\max_{0\le t\le T}\sum_{j=1}^{N}\int_\Omega |\psi_j(x,t)|^2\,dx+\sum_{j=1}^{N}\int_0^T\int_\Omega\Big|-\frac{i}{\kappa}\nabla\psi_j-\mathbf{A}_j\psi_j\Big|^2\,dx\,dt$$
$$+\sum_{j=1}^{N}\int_0^T\int_\Omega\Big[|\psi_j|^4+\frac{1}{\kappa^2s^2}|\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)|^2\Big]\,dx\,dt$$
$$\le C\sum_{j=1}^{N}\int_\Omega|\psi_j(x,0)|^2\,dx.$$

Proof. By choosing $\omega_j=\overline{\psi}_j\chi_{(0,t)}$ in (3.15) and taking the real part of the equation, we can prove the lemma by the standard argument. □

LEMMA 3. *We have*

$$\max_{0\le t\le T}\sum_{j=1}^{N}\int_\Omega|\mathbf{A}_j(x,t)|^2\,dx+\max_{0\le t\le T}\sum_{j=1}^{N-1}\int_\Omega|\chi_j(x,t)|^2\,dx$$
$$+\sum_{j=1}^{N}\int_0^T\int_\Omega\Big[|\mathrm{curl}\,\mathbf{A}_j|^2+|\mathrm{div}\mathbf{A}_j|^2+|\delta\,\mathbf{A}_{j+1}|^2+|\delta\,\chi_j|^2\Big]\,dx\,dt$$
$$+\sum_{j=1}^{N-1}\int_0^T\int_\Omega|\nabla\chi_j|^2\,dx\,dt$$
$$\le C\sum_{j=1}^{N}\int_\Omega\Big[|\psi_j(x,0)|^2+|\mathbf{A}_j(x,0)|^2+|\chi_j(x,0)|^2\Big]\,dx+C\sum_{j=1}^{N}\int_0^T\int_\Omega|H_j|^2\,dx\,dt.$$

Proof. Take $\mathbf{B}_j=\mathbf{A}_j\chi_{(0,t)}$ in (3.16) and $\zeta_j=\chi_j\chi_{(0,t)}$ in (3.17), then

$$\sum_{j=1}^{N}\frac{1}{2}\int_\Omega|\mathbf{A}_j(x,t)|^2\,dx+\sum_{j=1}^{N}\int_0^t\int_\Omega\Big[|\mathrm{curl}\,\mathbf{A}_j|^2+|\mathrm{div}\mathbf{A}_j|^2+|\delta\,\mathbf{A}_{j+1}|^2\Big]\,dx\,d\tau$$
$$+\frac{1}{2}\sum_{j=1}^{N}\int_\Omega|\chi_j(x,t)|^2\,dx+\sum_{j=1}^{N-1}\int_0^t\int_\Omega|\nabla\chi_j|^2\,dx\,d\tau+\sum_{j=1}^{N}\int_0^t\int_\Omega|\delta\,\chi_j|^2\,dx\,d\tau$$
$$=\frac{1}{2}\sum_{j=1}^{N}\int_\Omega|\mathbf{A}_j(x,0)|^2\,dx-\sum_{j=1}^{N}\int_0^t\int_\Omega\mathrm{Re}\,\Big[(\frac{i}{\kappa}\nabla\psi_j+\mathbf{A}_j\psi_j)\overline{\psi}_j\Big]\mathbf{A}_j\,dx\,d\tau$$
$$+\sum_{j=1}^{N}\int_0^t\int_\Omega H_j\mathrm{curl}\,\mathbf{A}_j\,dx\,d\tau+\frac{1}{2}\sum_{j=1}^{N-1}\int_\Omega|\chi_j(x,0)|^2\,dx$$
$$+\sum_{j=1}^{N-1}\frac{1}{\kappa s}\int_0^t\int_\Omega\mathrm{Re}\,\Big[i\psi_j\overline{\psi}_{j+1}\exp(i\kappa s\chi_j)\Big]\chi_j\,dx\,d\tau$$
$$=(\mathrm{I})_1+\cdots+(\mathrm{I})_5.$$

But

$$
\begin{aligned}
(\mathrm{I})_2 \leq & \frac{1}{2}\sum_{j=1}^{N}\int_0^t\int_\Omega \Big|\frac{i}{\kappa}\nabla\psi_j+\mathbf{A}_j\psi_j\Big|^2\,dx\,d\tau+\frac{1}{2}\sum_{j=1}^{N}\int_0^t\int_\Omega |\mathbf{A}_j|^2\,dx\,d\tau;\\
(\mathrm{I})_3 \leq & \sum_{j=1}^{N}\int_0^t\int_\Omega |H_j|^2\,dx\,d\tau+\frac{1}{4}\sum_{j=1}^{N}\int_0^t\int_\Omega |\mathrm{curl}\,\mathbf{A}_j|^2\,dx\,d\tau;\\
(\mathrm{I})_5 = & \sum_{j=1}^{N-1}\int_0^t\int_\Omega \mathrm{Re}\Big[\frac{i}{\kappa s}\big(\psi_j\overline{\psi}_{j+1}\exp(i\kappa s\chi_j)-|\psi_{j+1}|^2\big)\Big]\chi_j\,dx\,d\tau\\
\leq & \frac{1}{2}\sum_{j=1}^{N-1}\int_0^t\int_\Omega |\chi_j|^2\,dx\,d\tau+\frac{1}{2\kappa^2 s^2}\sum_{j=1}^{N-1}\int_0^t\int_\Omega |\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)|^2\,dx\,d\tau.
\end{aligned}
$$

We can prove the lemma by using Lemma 2 and Gronwall's inequality. □

LEMMA 4. *We have*

$$
\begin{aligned}
& \max_{0\leq t\leq T}\sum_{j=1}^{N}\int_\Omega \Big[|\mathrm{curl}\,\mathbf{A}_j(x,t)|^2+|\mathrm{div}\mathbf{A}_j(x,t)|^2+|\delta\,\mathbf{A}_{j+1}(x,t)|^2+|\delta\,\chi_j(x,t)|^2\Big]\,dx\\
& +\sum_{j=1}^{N}\int_0^T\int_\Omega \Big|\frac{\partial\mathbf{A}_j}{\partial t}\Big|^2\,dx\,dt\\
\leq & C\sum_{j=1}^{N}\int_\Omega \Big[|\psi_j(x,0)|^2+|\nabla\mathbf{A}_j(x,0)|^2+|\delta\,\mathbf{A}_{j+1}(x,0)|^2+|H_j(x,0)|^2\Big]\,dx\\
& +C\sum_{j=1}^{N}\int_0^T\int_\Omega |\frac{\partial H_j}{\partial t}|^2\,dx\,dt.
\end{aligned}
$$

Proof. Take $\mathbf{B}_j=\frac{\partial\mathbf{A}_j}{\partial t}\chi_{(0,t)}$ in (3.16), then

$$
\begin{aligned}
& \sum_{j=1}^{N}\int_0^t\int_\Omega \Big|\frac{\partial\mathbf{A}_j}{\partial t}\Big|^2\,dx\,d\tau+\frac{1}{2}\sum_{j=1}^{N}\int_\Omega \Big[|\mathrm{curl}\,\mathbf{A}_j(x,t)|^2+|\mathrm{div}\mathbf{A}_j(x,t)|^2+|\delta\,\mathbf{A}_{j+1}(x,t)|^2\Big]\,dx\\
= & \frac{1}{2}\sum_{j=1}^{N}\int_\Omega \Big[|\mathrm{curl}\,\mathbf{A}_j(x,0)|^2+|\mathrm{div}\mathbf{A}_j(x,0)|^2+|\delta\,\mathbf{A}_{j+1}(x,0)|^2\Big]\,dx\\
& -\sum_{j=1}^{N}\int_0^t\int_\Omega \mathrm{Re}\Big[\big(\frac{i}{\kappa}\nabla\psi_j+\mathbf{A}_j\psi_j\big)\overline{\psi}_j\Big]\frac{\partial\mathbf{A}_j}{\partial t}\,dx\,d\tau+\sum_{j=1}^{N}\int_0^t\int_\Omega H_j\mathrm{curl}\,\frac{\partial\mathbf{A}_j}{\partial t}\,dx\,d\tau\\
= & (\mathrm{II})_1+\cdots+(\mathrm{II})_3.
\end{aligned}
$$

But

$$
\begin{aligned}
(\mathrm{II})_2 \leq & \frac{1}{2}\sum_{j=1}^{N}\int_0^t\int_\Omega\Big|\frac{\partial \mathbf{A}_j}{\partial t}\Big|^2\,d\,x\,d\,\tau+\frac{1}{2}\sum_{j=1}^{N}\int_0^t\int_\Omega\Big|-\frac{i}{\kappa}\nabla\psi_j-\mathbf{A}_j\psi_j\Big|^2\,d\,x\,d\,\tau\\
(\mathrm{II})_3 = & -\sum_{j=1}^{N}\int_0^t\int_\Omega\frac{\partial H_j}{\partial t}\mathrm{curl}\,\mathbf{A}_j\,d\,x\,d\,\tau\\
& +\sum_{j=1}^{N}\int_\Omega\Big[H_j(x,t)\mathrm{curl}\,\mathbf{A}_j(x,t)-H_j(x,0)\mathrm{curl}\,\mathbf{A}_j(x,0)\Big]\,d\,x\\
\leq & \frac{1}{2}\sum_{j=1}^{N}\int_\Omega\Big|\mathrm{curl}\,\mathbf{A}_j(x,t)\Big|^2\,d\,x+\frac{1}{2}\sum_{j=1}^{N}\int_0^t\int_\Omega\Big|\mathrm{curl}\,\mathbf{A}_j\Big|^2\,d\,x\,d\,\tau\\
& +C\sum_{j=1}^{N}\int_\Omega\Big[|H_j(x,0)|^2+|\mathrm{curl}\,\mathbf{A}_j(x,0)|^2\Big]\,d\,x+C\sum_{j=1}^{N}\int_0^T\int_\Omega\Big|\frac{\partial H_j}{\partial t}\Big|^2\,d\,x\,d\,t.
\end{aligned}
$$

We can prove the lemma by using Lemma 2 and Lemma 3. $\square$

Similarly, by taking $\zeta_j=\frac{\partial\chi_j}{\partial t}\chi_{(0,t)}$ in (3.17), we have the following result.

LEMMA 5. *We have*

$$
\begin{aligned}
&\max_{0\leq t\leq T}\sum_{j=1}^{N}\int_\Omega\Big[|\nabla\chi_j(x,t)|^2+|\delta\,\chi_j(x,t)|^2\Big]\,d\,x+\sum_{j=1}^{N}\int_0^T\int_\Omega\Big|\frac{\partial\chi_j}{\partial t}\Big|^2\,d\,x\,d\,t\\
\leq & C\sum_{j=1}^{N}\int_\Omega\Big[|\nabla\chi_j(x,0)|^2+|\delta\,\chi_j(x,0)|^2+|\psi_j(x,0)|^2\Big]\,d\,x.
\end{aligned}
$$

LEMMA 6. *We have*

$$
\begin{aligned}
&\max_{0\leq t\leq T}\sum_{j=1}^{N}\int_\Omega\Big[|\nabla\psi_j(x,t)|^2+|\delta\,\psi_j(x,t)|^2\Big]\,d\,x+\sum_{j=1}^{N}\int_0^T\int_\Omega\Big|\frac{\partial\psi_j}{\partial t}\Big|^2\,d\,x\,d\,t\\
\leq & C\sum_{j=1}^{N}\Big[\|\psi_j(\cdot,0)\|^2_{H^1(\Omega)}+\|\mathbf{A}_j(\cdot,0)\|^2_{H^1(\Omega)}+\|\chi_j(\cdot,0)\|^2_{H^1(\Omega)}+\|\psi_j(\cdot,0)\|^4_{L^4(\Omega)}\Big]\\
& +C\sum_{j=1}^{N}\Big[\|\delta\,\mathbf{A}_{j+1}(\cdot,0)\|^2_{L^2(\Omega)}+\|\delta\,\psi_{j+1}(\cdot,0)\|^2_{L^2(\Omega)}+\|\delta\,\chi_j(\cdot,0)\|^2_{L^2(\Omega)}\Big]\\
& +C\sum_{j=1}^{N}\|H_j\|^2_{H^1(0,T;L^2(\Omega))}.
\end{aligned}
$$

Proof. Choose $\omega_j=\frac{\partial\overline{\psi}_j}{\partial t}\chi_{(0,t)}$ in (3.15) and take the real part of the equation, then

$$
\eta\sum_{j=1}^{N}\int_0^t\int_\Omega\Big|\frac{\partial\psi_j}{\partial t}\Big|^2\,d\,x\,d\,\tau+\frac{1}{2}\sum_{j=1}^{N}\int_\Omega\Big|-\frac{i}{\kappa}\nabla\psi_j-\mathbf{A}_j\psi_j\Big|^2(x,t)\,d\,x
$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{j=1}^{N}\int_\Omega |\psi_j(x,t)|^4\,dx+\frac{1}{2\kappa^2 s^2}\sum_{j=1}^{N}\int_\Omega |\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)|^2(x,t)\,dx\\
=&\mathrm{Re}\Big[i\eta\kappa\sum_{j=1}^{N}\int_0^t\int_\Omega(\mathrm{div}\mathbf{A}_j+\delta\,\chi_j)\psi_j\frac{\partial\overline{\psi}_j}{\partial t}\,dx\,d\tau\Big]\\
&+\frac{1}{2}\sum_{j=1}^{N}\int_\Omega\Big|-\frac{i}{\kappa}\nabla\psi_j-\mathbf{A}_j\psi_j\Big|^2(x,0)\,dx\\
&+\mathrm{Re}\Big[\sum_{j=1}^{N}\int_0^t\int_\Omega\big(-\frac{i}{\kappa}\nabla\psi_j-\mathbf{A}_j\psi_j\big)\overline{\psi}_j\frac{\partial\mathbf{A}_j}{\partial t}\,dx\,d\tau\Big]\\
&+\frac{1}{4}\sum_{j=1}^{N}\int_\Omega|\psi_j(x,0)|^4\,dx+\frac{1}{2}\sum_{j=1}^{N}\int_\Omega\Big[|\psi_j(x,t)|^2-|\psi_j(x,0)|^2\Big]\,dx\\
&+\frac{1}{2\kappa^2 s^2}\sum_{j=1}^{N}\int_\Omega|\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)|^2(x,0)\,dx\\
&-\frac{1}{\kappa^2 s^2}\sum_{j=1}^{N}\int_0^t\int_\Omega\mathrm{Re}\Big\{\big(\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)\big)\overline{\psi}_{j+1}\frac{\partial}{\partial t}\Big[\exp(i\kappa s\chi_j)\Big]\Big\}\,dx\,d\tau\\
=&(\mathrm{III})_1+\cdots+(\mathrm{III})_7.
\end{aligned}
$$

But

$$
\begin{aligned}
(\mathrm{III})_1\le&\frac{1}{2}\sum_{j=1}^{N}\int_0^t\int_\Omega\Big|\frac{\partial\psi_j}{\partial t}\Big|^2\,dx\,d\tau+C\sum_{j=1}^{N}\int_0^t\int_\Omega\Big[|\mathrm{div}\mathbf{A}_j|^2+|\delta\,\chi_j|^2\Big]\,dx\,d\tau;\\
(\mathrm{III})_2\le&C\sum_{j=1}^{N}\int_\Omega\Big[|\nabla\psi_j(x,0)|^2+|\mathbf{A}_j(x,0)|^2\Big]\,dx;\\
(\mathrm{III})_3\le&\sum_{j=1}^{N}\int_0^t\int_\Omega\Big[\Big|-\frac{i}{\kappa}\nabla\psi_j-\mathbf{A}_j\psi_j\Big|^2+\Big|\frac{\partial\mathbf{A}_j}{\partial t}\Big|^2\Big]\,dx\,d\tau;\\
(\mathrm{III})_6\le&\sum_{j=1}^{N}\frac{1}{\kappa^2 s^2}\int_\Omega\Big[|\psi_j-\psi_{j+1}|^2(x,0)+|\psi_{j+1}(1-\exp(-i\kappa s\chi_j))|^2(x,0)\Big]\,dx\\
\le&C\sum_{j=1}^{N}\int_\Omega\Big[|\delta\,\psi_{j+1}(x,0)|^2+|\chi_j(x,0)|^2\Big]\,dx;\\
(\mathrm{III})_7\le&\frac{1}{\kappa^2 s^2}\sum_{j=1}^{N}\int_0^t\int_\Omega\Big|\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)\Big|\Big|i\kappa s\frac{\partial\chi_j}{\partial t}\Big|\,dx\,d\tau\\
\le&\frac{1}{\kappa^2 s^2}\sum_{j=1}^{N}\int_0^t\int_\Omega|\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)|^2\,dx\,d\tau+C\sum_{j=1}^{N}\int_0^t\int_\Omega\Big|\frac{\partial\chi_j}{\partial t}\Big|^2\,dx\,d\tau.
\end{aligned}
$$

Thus, by using Lemma 2-Lemma 5, we get

$$
\begin{aligned}
&\sum_{j=1}^{N}\int_0^t\int_\Omega\Big|\frac{\partial\psi_j}{\partial t}\Big|\,d\,x\,d\,\tau+\max_{0\le t\le T}\sum_{j=1}^{N}\int_\Omega\Big|-\frac{i}{\kappa}\nabla\psi_j-\mathbf{A}_j\psi_j\Big|^2(x,t)\,d\,x\\
&+\max_{0\le t\le T}\frac{1}{\kappa^2 s^2}\sum_{j=1}^{N}\int_\Omega|\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)|^2(x,t)\,d\,x\\
\le& C\sum_{j=1}^{N}\Big[\|\psi_j(\cdot,0)\|^2_{H^1(\Omega)}+\|\mathbf{A}_j(\cdot,0)\|^2_{H^1(\Omega)}+\|\chi_j(\cdot,0)\|^2_{H^1(\Omega)}+\|\psi_j(\cdot,0)\|^4_{L^4(\Omega)}\Big]\\
&+C\sum_{j=1}^{N}\Big[\|\delta\,\mathbf{A}_{j+1}(\cdot,0)\|^2_{L^2(\Omega)}+\|\delta\,\psi_{j+1}(\cdot,0)\|^2_{L^2(\Omega)}+\|\delta\,\chi_j(\cdot,0)\|^2_{L^2(\Omega)}\Big]\\
&+C\sum_{j=1}^{N}\|H_j\|^2_{H^1(0,T;L^2(\Omega))}.
\end{aligned}
$$

Now the lemma follows if we note that

$$
\begin{aligned}
\sum_{j=1}^{N}\int_\Omega|\nabla\psi_j|^2\,d\,x\le&2\kappa^2\sum_{j=1}^{N}\int_\Omega\Big[\Big|-\frac{i}{\kappa}\nabla\psi_j-\mathbf{A}_j\psi_j\Big|^2+|\mathbf{A}_j\psi_j|^2\Big]\,d\,x\\
\le&2\kappa^2\sum_{j=1}^{N}\int_\Omega\Big[\Big|-\frac{i}{\kappa}\nabla\psi_j-\mathbf{A}_j\psi_j\Big|^2+|\mathbf{A}_j|^2\Big]\,d\,x
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{N}\int_\Omega|\delta\,\psi_{j+1}|^2\,d\,x\\
\le&2\kappa^2\sum_{j=1}^{N}\int_\Omega\frac{1}{\kappa^2 s^2}\Big[|\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)|^2+|\psi_{j+1}(1-\exp(-i\kappa s\chi_j))|^2\Big]\,d\,x\\
\le&2\kappa^2\sum_{j=1}^{N}\int_\Omega\Big[\frac{1}{\kappa^2 s^2}|\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j)|^2+2|\chi_j|^2\Big]\,d\,x.
\end{aligned}
$$

This completes the proof. □

4.3. The convergence. Let $\Omega^*=\Omega\times(0,L)$, $Q_T=\Omega\times(0,T)$ and $Q_T^*=\Omega^*\times(0,T)$. Denote by $x=(x_1,x_2)$ and $x^*=(x,x_3)$. For any integer $N\ge1$, let $s=L/N$. We are going to show that the solution of Problem (P^N) converges to the solution of Problem (P) as $N\to\infty$. Let $z_j=js$ for $j=0,\cdots,N$ and $I_1=[z_0,z_1], I_j=(z_{j-1},z_j]$ for $j=2,\cdots,N$. For any function ξ defined in Q_T^*, denote by $\xi_j(x,t)=\xi(x,z_j,t)$ for $j=0,\cdots,N$. If

$\{\xi_j(x,t)\}_{j=0}^N$ is a family of functions defined in Q_T, we set

$$\widehat{\xi}^N(x,x_3,t)=\sum_{j=1}^N\Big[\frac{z_j-x_3}{z_j-z_{j-1}}\xi_{j-1}(x,t)+\frac{x_3-z_{j-1}}{z_j-z_{j-1}}\xi_j(x,t)\Big]\chi_j(x_3), \tag{4.6}$$

where $\chi_j(x_3)$ is the characteristic function of the interval I_j, $j=1,\cdots,N$. Let $\{\psi_j(x,t)\}_{j=1}^N$, $\{\mathbf{A}_j(x,t)\}_{j=1}^N$ and $\{\chi_j(x,t)\}_{j=1}^{N-1}$ be the solution of Problem (P^N) with the initial functions $\{\psi_0^j(x)\}_{j=1}^N$, $\{\mathbf{A}_0^j(x)\}_{j=1}^N$, $\{\chi_0^j(x)\}_{j=1}^{N-1}$ and the external magnetic function $\{H_j(x,t)\}_{j=1}^N$. We define

$$\psi^N(x,x_3,t)=\sum_{j=1}^N\psi_j(x,t)\chi_j(x_3), \tag{4.7}$$

$$\mathbf{A}^N(x,x_3,t)=\sum_{j=1}^N\mathbf{A}_j(x,t)\chi_j(x_3), \tag{4.8}$$

$$\chi^N(x,x_3,t)=\sum_{j=1}^N\chi_{j-1}(x,t)\chi_j(x_3). \tag{4.9}$$

We define $\widehat{\psi}_0^N(x,x_3)$, $\widehat{\mathbf{A}}_0^N(x,x_3)$, $\widehat{\chi}_0^N(x,x_3)$ as in (4.6) and

$$H^N(x,x_3,t)=\sum_{j=1}^N H_j(x,t)\chi_j(x_3). \tag{4.10}$$

Note that

$$\begin{aligned}
&\frac{1}{\kappa s}(\psi_j-\psi_{j+1}\exp(-i\kappa s\chi_j))\\
=&\frac{1}{\kappa s}(\psi_j-\psi_{j+1})+\frac{1}{\kappa s}(1-\exp(-i\kappa s\chi_j))\psi_{j+1}\\
=&-\frac{1}{\kappa}\delta\,\psi_{j+1}+\frac{1}{\kappa s}\psi_{j+1}\int_{z_j}^{z_{j+1}}\frac{\partial}{\partial x_3}\Big[\exp(-i\kappa\chi_j(z_{j+1}-x_3))\Big]\,dx_3\\
=&-\frac{1}{\kappa}\delta\,\psi_{j+1}+i\chi_j\psi_{j+1}\Big[\frac{1}{s}\int_{z_j}^{z_{j+1}}\exp(-i\kappa\chi_j(z_{j+1}-x_3))\,dx_3\Big]\\
=&-i\Big[\Big(-\frac{i}{\kappa}\delta\,\psi_{j+1}-\chi_j\psi_{j+1}\Big)+g_{j+1}\psi_{j+1}\Big],
\end{aligned}$$

where

$$\begin{aligned}
|g_{j+1}|=&\Big|i\chi_j\Big[\frac{1}{s}\int_{z_j}^{z_{j+1}}\big[\exp(-i\kappa\chi_j(z_{j+1}-x_3))-1\big]\,dx_3\Big]\Big|\\
\le&|\chi_j|\cdot\frac{1}{s}\int_{z_j}^{z_{j+1}}|\exp(-i\kappa\chi_j(z_{j+1}-x_3))-1|\,dx_3\\
\le&|\chi_j|\cdot\frac{1}{s}\int_{z_j}^{z_{j+1}}2|-i\kappa\chi_j(z_{j+1}-x_3)|\,dx_3\\
\le&2\kappa s|\chi_j|^2.
\end{aligned} \tag{4.11}$$

With the same reasons, we obtain

$$\frac{1}{\kappa s}(\omega_j - \omega_{j+1}\exp(i\kappa s\chi_j)) = i\Big[\Big(\frac{i}{\kappa}\delta\,\omega_{j+1} - \chi_j\omega_{j+1}\Big) + f_{j+1}\omega_{j+1}\Big]$$

with

$$|f_{j+1}| \leq 2\kappa s|\chi_j|^2. \tag{4.12}$$

Then, from (3.14)-(3.17), we know that

$$\widehat{\psi}^N(x^*,0) = \widehat{\psi}_0^N(x^*),\ \widehat{\mathbf{A}}^N(x^*,0) = \widehat{\mathbf{A}}_0^N(x^*),\ \widehat{\chi}^N(x^*,0) = \widehat{\chi}_0^N(x^*) \tag{4.13}$$

and

$$\begin{aligned}
&\int_0^T\int_{\Omega^*}\Big[\eta\frac{\partial\psi^N}{\partial t}\omega^N - i\eta\kappa\Big(\mathrm{div}\mathbf{A}^N + \frac{\partial\widehat{\chi}^N}{\partial x_3}\Big)\psi^N\omega^N + (|\psi^N|^2-1)\psi^N\omega^N\Big]\,dx^*\,dt\\
&+\int_0^T\int_{\Omega^*}\Big(-\frac{i}{\kappa}\nabla\psi^N - \mathbf{A}^N\psi^N\Big)\Big(\frac{i}{\kappa}\nabla\omega^N - \mathbf{A}^N\omega^N\Big)\,dx^*\,dt\\
&+\int_0^T\int_{\Omega^*}\Big[\Big(-\frac{i}{\kappa}\frac{\partial\widehat{\psi}^N}{\partial x_3} - \chi^N\psi^N\Big) + g^N\psi^N\Big]\Big[\Big(\frac{i}{\kappa}\frac{\partial\widehat{\omega}^N}{\partial x_3} - \chi^N\omega^N\Big) + f^N\omega^N\Big]\,dx^*\,dt\\
=&0,\quad \forall\omega_1,\cdots,\omega_N \in L^2(0,T;\mathcal{H}^1(\Omega)),\ \omega_0=\omega_1,\ \omega_{N+1}=\omega_N
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\int_0^T\int_{\Omega^*}\frac{\partial\mathbf{A}^N}{\partial t}\mathbf{B}^N\,dx^*\,dt\\
&+\int_0^T\int_{\Omega^*}\Big[\mathrm{curl}\,\mathbf{A}^N\mathrm{curl}\,\mathbf{B}^N + \mathrm{div}\mathbf{A}^N\mathrm{div}\mathbf{B}^N + \frac{\partial\widehat{\mathbf{A}}^N}{\partial x_3}\frac{\partial\widehat{\mathbf{B}}^N}{\partial x_3}\Big]\,dx^*\,dt\\
&+\int_0^T\int_{\Omega^*}\mathrm{Re}\Big[\Big(\frac{i}{\kappa}\nabla\psi^N + \mathbf{A}^N\psi^N\Big)\overline{\psi}^N\Big]\mathbf{B}^N\,dx^*\,dt - \int_0^T\int_{\Omega^*}H^N\mathrm{curl}\,\mathbf{B}^N\,dx^*\,dt\\
=&0,\quad \forall\mathbf{B}_1,\cdots,\mathbf{B}_N \in L^2(0,T;\mathbf{H}_n^1(\Omega)),\ \mathbf{B}_0=\mathbf{B}_1,\ \mathbf{B}_{N+1}=\mathbf{B}_N
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&\int_0^T\int_{\Omega^*}\Big[\frac{\partial\chi^N}{\partial t}\zeta^N + \nabla\chi^N\nabla\zeta^N + \frac{\partial\widehat{\chi}^N}{\partial x_3}\frac{\partial\widehat{\zeta}^N}{\partial x_3}\Big]\,dx^*\,dt\\
&+\int_0^T\int_{\Omega^*}\mathrm{Re}\Big[\Big(\frac{i}{\kappa}\frac{\partial\widehat{\psi}^N}{\partial x_3} + \chi^N\psi^N - g^N\psi^N\Big)\overline{\widetilde{\psi}}^N\Big]\zeta^N\,dx^*\,dt\\
=&0,\quad \forall\zeta_1,\cdots,\zeta_N \in L^2(0,T;H^1(\Omega)),\ \zeta_0=\zeta_1,\zeta_{N+1}=\zeta_N.
\end{aligned} \tag{4.16}$$

where $\widehat{\omega}^N$, $\widehat{\mathbf{B}}^N$, $\widehat{\chi}^N$ are defined as in (4.6), f^N, g^N are defined as in (4.10), $\omega^N, \mathbf{B}^N, \chi^N$ are defined as in (4.7), (4.8), (4.9), respectively, and

$$\widetilde{\psi}^N(x,x_3) = \sum_{j=1}^N \psi_{j-1}(x,t)\chi_j(x_3), \tag{4.17}$$

where $\chi_j(x_3)$ is the characteristic function of the interval I_j. To proceed further, let

$$\widehat{\Theta}^N = (\widehat{\mathbf{A}}^N, \widehat{\chi}^N), \ \Theta^N = (\mathbf{A}^N, \chi^N) \ \text{ and } \ \widehat{\Theta}_0^N = (\widehat{\mathbf{A}}_0^N, \widehat{\chi}_0^N),$$

then we have

$$\widehat{\Theta}^N \in L^2(0,T;\mathbf{H}_n^1(\Omega^*)) \ \text{ and } \ \widehat{\Theta}_0^N \in \mathbf{H}_n^1(\Omega^*).$$

The following theorem is the main result of this paper.

THEOREM 4. *Let $\Psi_0 \in \mathcal{H}^1(\Omega^*)$, $\Theta_0 \in \mathbf{H}_n^1(\Omega^*)$ such that $|\Psi_0| \le 1$ a.e. on Ω^*. Let further $H \in H^1(0,T;L^2(\Omega^*))$. Then the Problem (P) exists a weak solution (Ψ,Θ) such that*

$$\Psi \in H^1(0,T;\mathcal{L}^2(\Omega^*)) \cap L^\infty(0,T;\mathcal{H}^1(\Omega^*)), \ \ \Theta \in H^1(0,T;\mathbf{L}^2(\Omega^*)) \cap L^\infty(0,T;\mathbf{H}_n^1(\Omega^*)).$$

Moreover, if, as $N \to \infty$,

$$\widehat{\psi}_0^N \rightharpoonup \Psi_0, \ \text{in } \mathcal{H}^1(\Omega^*); \tag{4.18}$$

$$\widehat{\Theta}_0^N \rightharpoonup \Theta_0, \ \text{in } \mathbf{H}_n^1(\Omega^*); \tag{4.19}$$

$$H^N \rightharpoonup H, \ \text{in } H^1(0,T;L^2(\Omega^*)); \tag{4.20}$$

then we have

$$\widehat{\psi}^N \rightharpoonup \Psi, \ \text{in } H^1(0,T;\mathcal{L}^2(\Omega^*)), \quad \widehat{\psi}^N \rightharpoonup^* \Psi, \ \text{in } L^\infty(0,T;\mathcal{H}^1(\Omega^*)); \tag{4.21}$$

$$\widehat{\Theta}^N \rightharpoonup \Theta, \ \text{in } H^1(0,T;\mathbf{L}^2(\Omega^*)), \quad \widehat{\Theta}^N \rightharpoonup^* \Theta, \ \text{in } L^\infty(0,T;\mathcal{H}_n^1(\Omega^*)). \tag{4.22}$$

Here $(\widehat{\psi}^N, \widehat{\Theta}^N)$ is the solution of Problem (P^N) corresponding to the initial functions $(\widehat{\psi}_0^N, \widehat{\Theta}_0^N)$ and external magnetic field H^N.

Proof. It is clear that we only need to show the second part of the theorem. From the assumptions (4.18)-(4.20) and Lemma 1 - Lemma 6 we know that there exists a constant C independent of N and s such that

$$\begin{aligned}
&|\psi^N(x,t)| \le 1, \ |\widehat{\psi}^N(x,t)| \le 1 \quad \text{a.e. in } Q_T^*;\\
&\|\widehat{\psi}^N\|_{L^\infty(0,T;H^1(\Omega^*))} \le C, \ \|\widehat{\psi}^N\|_{H^1(0,T;L^2(\Omega^*))} \le C;\\
&\|\nabla\psi^N\|_{L^\infty(0,T;L^2(\Omega^*))} \le C, \ \|\psi^N\|_{H^1(0,T;L^2(\Omega^*))} \le C;\\
&\|\widehat{\Theta}^N\|_{L^\infty(0,T;H_n^1(\Omega^*))} \le C, \ \|\widehat{\Theta}^N\|_{H^1(0,T;L^2(\Omega^*))} \le C;\\
&\|\nabla\Theta^N\|_{L^\infty(0,T;L^2(\Omega^*))} \le C, \ \|\Theta^N\|_{H^1(0,T;L^2(\Omega^*))} \le C.
\end{aligned}$$

Thus, after possibly extracting subsequences, there exists two functions

$$\Psi \in H^1(0,T;\mathcal{L}^2(\Omega^*)) \cap L^\infty(0,T;\mathcal{H}^1(\Omega^*)), \ \ \Theta \in H^1(0,T;\mathbf{L}^2(\Omega^*)) \cap L^\infty(0,T;\mathbf{H}_n^1(\Omega^*))$$

such that (4.21)-(4.22) fulfill. In the sequel we shall prove that (Ψ, Θ) also satisfies (4.2)-(4.3) which will complete the proof of the theorem. To this purpose, we first note that since

$$\|\widehat{\psi}^N - \psi^N\|_{L^2(Q_T^*)} \to 0 \quad \text{and} \quad \|\widehat{\Theta}^N - \Theta^N\|_{L^2(Q_T^*)} \to 0,$$

without lost of generality, we can also assume that

$$\psi^N \rightharpoonup \Psi \text{ in } H^1(0,T;\mathcal{L}^2(\Omega^*)), \quad \nabla\psi^N \rightharpoonup^* \nabla\Psi \text{ in } L^\infty(0,T;\mathcal{L}^2(\Omega^*)); \tag{4.23}$$

$$\Theta^N \rightharpoonup \Theta \text{ in } H^1(0,T;\mathbf{L}^2(\Omega^*)), \quad \nabla\Theta^N \rightharpoonup^* \nabla\Theta \text{ in } L^\infty(0,T;\mathbf{L}^2(\Omega^*)); \tag{4.24}$$

$$\psi^N \to \Psi \text{ in } \mathcal{L}^4(Q_T^*), \quad \Theta^N \to \Theta \text{ in } \mathbf{L}^2(Q_T^*); \tag{4.25}$$

where (4.25) follows from the fact that $L^\infty(0,T;H^1(\Omega^*)) \cap H^1(0,T;L^2(\Omega^*))$ is compactly embedded into $L^4(Q_T^*)$ (cf. e.g. [14]) and $L^2(Q_T^*)$, and the relations

$$\|\widehat{\psi}^N - \psi^N\|_{L^4(Q_T^*)} \le \sqrt[4]{2}\|\widehat{\psi}^N - \psi^N\|^{1/2}_{L^2(Q_T^*)} \to 0,$$

where we have used Lemma 1. By using the Nirenberg's inequality (cf. e.g. [12])

$$\|w\|_{L^4(\Omega)} \le C\|w\|^{1/2}_{H^1(\Omega)}\|w\|^{1/2}_{L^2(\Omega)} \quad \forall w \in H^1(\Omega),$$

we have

$$\begin{aligned}
\|\chi^N\|^4_{L^4(Q_T^*)} =& \int_0^T \sum_{j=1}^N s \int_\Omega |\chi_{j-1}|^4 \, dx \, dt \\
\le& C \int_0^T \sum_{j=1}^N s\|\chi_{j-1}\|^2_{H^1(\Omega)}\|\chi_{j-1}\|^2_{L^2(\Omega)} \, dt \\
\le& C \int_0^T \Big[\sum_{j=1}^N s\|\chi_{j-1}\|^2_{H^1(\Omega)} \cdot \sum_{j=1}^N s\|\delta\, \chi_j\|^2_{L^2(\Omega)}\Big] \, dt \\
\le& C,
\end{aligned}$$

where we have used Lemma 3 and Lemma 5. For any $\omega \in \mathcal{H}^2(Q_T^*) \cap \mathcal{C}(\overline{Q_T^*})$, let $\omega_j(x,t) = \omega(x, z_j, t)$ and define ω^N, $\widehat{\omega}^N$ as in (4.7) and (4.6), respectively, then we have

$$\|\widehat{\omega}^N - \omega\|_{L^2(0,T;H^1(Q_T^*))} \to 0, \tag{4.26}$$

$$\|\omega^N - \omega\|_{C(\overline{Q_T^*})} \to 0, \ \|\nabla\omega^N - \nabla\omega\|_{L^2(Q_T^*)} \to 0. \tag{4.27}$$

Since

$$\|g^N\psi^N\|_{L^2(Q_T^*)} \le \|g^N\|_{L^2(Q_T^*)} \le 2\kappa s\|\chi^N\|^2_{L^4(Q_T^*)} \to 0$$

and

$$\|f^N\omega^N\|_{L^2(Q_T^*)} \le \|f^N\|_{L^2(Q_T^*)}\|\omega^N\|_{C(Q_T^*)} \le 2\kappa s\|\chi^N\|^2_{L^4(Q_T^*)}\|\omega^N\|_{C(Q_T^*)} \to 0,$$

by letting $N \to \infty$ in (4.14), we conclude

$$
\begin{aligned}
&\int_0^T \int_{\Omega^*} \Big[\eta \frac{\partial \Psi}{\partial t}\omega - i\eta\kappa\big(\mathrm{div}\mathbf{A} + \frac{\partial \chi}{\partial x_3}\big)\Psi\omega + (|\Psi|^2 - 1)\Psi\omega\Big]\, d\,x^*\, d\,t \\
&+ \int_0^T \int_{\Omega^*} \Big(-\frac{i}{\kappa}\nabla\Psi - \mathbf{A}\Psi\Big)\Big(\frac{i}{\kappa}\nabla\omega - \mathbf{A}\omega\Big)\, d\,x^*\, d\,t \\
&+ \int_0^T \int_{\Omega^*} \Big(-\frac{i}{\kappa}\frac{\partial \Psi}{\partial x_3} - \chi\Psi\Big)\Big(\frac{i}{\kappa}\frac{\partial \omega}{\partial x_3} - \chi\omega\Big)\, d\,x^*\, d\,t \\
=&0, \qquad \forall \omega \in \mathcal{H}^2(Q_T^*) \cap \mathcal{C}(\overline{Q_T^*}),
\end{aligned}
$$

where $(\mathbf{A}, \chi) = \Theta$. Thus we see that (4.2) holds for any $\omega \in \mathcal{H}^2(Q_T^*) \cap \mathcal{C}(\overline{Q_T^*})$. By using standard arguments we have that (Ψ, Θ) satisfies (4.2). Similarly, we can show that (Ψ, Θ) satisfies (4.3) by letting $N \to \infty$ in (4.15)-(4.16). This completes the proof. $\square$

5. Conclusion. In this paper it has been proved that the time dependent Ginzburg-Landau model is indeed the limit of the time dependent Lawrence-Doniach model as the layer spacing goes to zero. This indicates that we can approximate the three-dimensional TDGL model by the two-dimensional TDLD model. This gives us a convergent approximation scheme for solving the three-dimensional TDGL equations. It is also interesting to note that the TDLD model is well-suited for parallel computing. We also note that the numerical solutions of two-dimensional TDGL equations has been carried out in [4]. To the authors' best knowledge there have been no serious numerical results on the three-dimensional TDGL equations. These ideas will be pursued in a forthcoming paper.

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