RECENT ADVANCES OF A POSTERIORI ERROR ANALYSIS AND ADAPTIVE FINITE ELEMENT METHODS

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Abstract. We report the recent progress in deriving sharp a posteriori error estimates and developing efficient adaptive finite element methods for linear and nonlinear partial differential equations. We first consider a model problem: the linear elliptic equation with strongly discontinuous coefficients. We derive a new a posteriori error estimator which depends locally on the oscillations of the coefficients around singular points. Some numerical experiments are reported to explain the importance and efficiency of the underlying adaptive finite element methods. We next use a direct energy estimate method to develop an efficient and reliable a posteriori error estimator for linear parabolic equations which does not depend on any regularity assumption on the underlying elliptic operator. A convergent adaptive algorithm with variable time-step sizes and space meshes is proposed and studied which, at each time step, delays the mesh coarsening until the final iteration of the adaptive procedure, allowing only mesh and time-step size refinements before. The key ingredient in the convergence analysis is a new coarsening strategy. Finally we report the results obtained for solving the continuous casting problem, a convection-dominated nonlinearly degenerate diffusion problem.

1 Introduction

A posteriori error estimates are computable quantities in terms of the discrete solution and data that measure the actual discrete errors without the knowledge of limit solutions. They are essential in designing algorithms for mesh and time-step size modification which equi-distribute the computational effort and optimize the computation. Ever since the pioneering work of Babuška and Rheinboldt [5], the adaptive finite element methods based on a posteriori error estimates have become a central theme in scientific and engineering computations. We refer to the book of Verfürth [23] for an extensive review on residual type a posteriori estimators and the adaptive procedures, and refer to Chen and Nochetto [9], Chen and Dai [6], Chen, Nochetto and Schmidt [10], and Nochetto, Schmidt and Verdi [18] for solving nonlinear partial differential equations arising from physical and industrial processes.

We first consider a model problem: the linear elliptic equation with strongly discontinuous coefficients

$$-\mathrm{div}(a(x)\nabla u) = f \qquad \text{in} \qquad \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on} \qquad \partial\Omega. \tag{1.2}$$

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Here Ω is a polyhedral domain in $\mathbf{R}^d (d = 2, 3)$, the source function f is assumed to be in $L^2(\Omega)$, and the coefficient $a(x)$ is positive and piecewise constant. We derive a new a posteriori error estimator which depends locally on the oscillations of the coefficients around singular points. We also report some numerical experiments to demonstrate the efficiency of the underlying adaptive finite element methods.

The main tool in deriving a posteriori error estimates for parabolic problems in [6], [10], [13], [14], [18] is the analysis of linear dual problems of the corresponding *error* equations. The derived a posteriori error estimates, however, depend on the H^2 regularity assumption on the underlying elliptic operator. The next objective of the paper is to remove such a rather restrictive assumption and derive an a posteriori error estimate which bounds the full energy error of the approximate solution for a linear parabolic problem with discontinuous coefficients. Moreover, a local lower bound is proved for the associated local a posteriori error estimator. The method to prove the upper bound is a direct energy estimate argument which has been used in Chen and Nochetto [9] for elliptic obstacle problem and in Chen, Nochetto and Schmidt [11] for the phase relaxation model, a system of one parabolic equation coupled with one variational inequality. Based on this error estimator, we develop a convergent adaptive algorithm with variable time-step sizes and space meshes which, at each time step, delays the mesh coarsening until the final iteration of the adaptive procedure, allowing only mesh and time-step size refinements before.

The continuous casting problem is a convection-dominated nonlinearly degenerate diffusion problem. It is discretized implicitly in time via the method of characteristics, and in space via continuous piecewise linear finite elements. In the final part of the paper we show the numerical results obtained by using the adaptive methods based on a posteriori error estimates. The purpose here to show the ability and the potential of the methodology discussed in this note for solving complicated nonlinear problems.

2 A posteriori error analysis for elliptic problems with discontinuous coefficients

We begin with introducing some notation. For any open set $G \subset \mathbf{R}^d$, we denote by $H^1(G)$ the standard Sobolev space of functions in $L^2(G)$ whose first derivatives are also in $L^2(G)$. The coefficient $a(x)$ is assumed to be piecewise constant and positive. Thus the semi-norm $\|\!|\cdot|\!\|_G$ defined by $\|\!|v|\!\|_G^2 = (a\nabla v, \nabla v)_G$ is equivalent to the $H_0^1(\Omega)$ norm when $G = \Omega$. Here $(\cdot,\cdot)_G$ stands for the $L^2(G)$ inner product and we will write $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$. For any $f \in L^2(\Omega)$, the weak formulation of the problem (1.1)-(1.2) reads as follows

$$u \in H_0^1(\Omega): \quad (a\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{2.1}$$

Let $\mathcal{M}_H$ be a regular triangulation over Ω such that $a(x)$ is constant on each element K, and $\mathcal{B}_H$ the collection of all inter-element sides of $\mathcal{M}_H$. Let V_H be the space of conforming linear finite element space over $\mathcal{M}_H$, and $V_0^H = V^H \cap H_0^1(\Omega)$. Let $u_H \in V_0^H$ be the solution of the discrete problem

$$u_H \in V_0^H: \quad (a\nabla u_H, \nabla v_H) = (f, v_H) \quad \forall v_H \in V_0^H. \tag{2.2}$$

For any inter-element side $e \in \mathcal{B}_H$, let Ω_e be the collection of two elements sharing e and define the local error indicator η_e as

$$\eta_e^2 := \sum_{K \in \Omega_e} \| H_K f \|_{L^2(K)}^2 + \| H_e^{1/2} J_e \|_{L^2(e)}^2, \tag{2.3}$$

where $H_K := \text{diam}(K)$, $H_e := \text{diam}(e)$, and $J_e := [\![a\nabla u_H]\!]_e \cdot \nu$ stands for the jump of flux across side e which is independent of the orientation of the unit normal ν to e. The following a posteriori error estimate is well-known, see Babuška and Miller [1], Verfürth [23],

$$\| u - u_H \|_{H^1(\Omega)}^2 \leq C \sum_{e \in \mathcal{B}^n} \eta_e^2. \tag{2.4}$$

The constant C in (2.4) depends on the coefficient $a(x)$, more precisely, it is proportional to $\max_{x\in\Omega} a(x)/\min_{x\in\Omega} a(x)$. This poor dependence on the coefficient $a(x)$ is rather inaccurate when $a(x)$ has large variations. In a recent paper [2], by working on the energy norm $|\!|\!| v |\!|\!|_\Omega = \| a^{1/2}\nabla v \|_{L^2(\Omega)}\Omega$, Bernardi and Verfürth derived the following a posteriori error estimate

$$|\!|\!| u - u_H |\!|\!|_\Omega^2 \leq C \sum_{e \in \mathcal{B}^n} \eta_e^2, \tag{2.5}$$

where the local error indicator η_e is given by

$$\eta_e^2 := \sum_{K \in \Omega_e} \| H_K a_K^{-1/2} f \|_{L^2(K)}^2 + \| H_e^{1/2} a_e^{-1/2} J_e \|_{L^2(e)}^2. \tag{2.6}$$

Here a_K is the constant value of $a(x)$ on $K \in \mathcal{M}_H$ and $a_e = \max_{K\in\Omega_e}(a_K)$. Let the domain Ω be split into disjoint open subdomains $\Omega_l, 1 \leq l \leq L$, such that $a(x)$ is constant on each Ω_l. The constant C in (2.5) can be made independent of the ratio $\max_{x\in\Omega} a(x)/\min_{x\in\Omega} a(x)$ under the assumption that for any two different subdomains $\bar{\Omega}_l$ and $\bar{\Omega}_k$ sharing at least one point, there is a connected path passing from $\bar{\Omega}_l$ to $\bar{\Omega}_k$ through adjacent subdomains such that the function $a(x)$ is monotone along this path (Hypothesis 2.7 in [2]). This result was extended to the quasi-monotone coefficients in Petzoldt [19]. The condition of quasi-monotone coefficients, however, rules out some interesting physical cases like the checkerboard discontinuous coefficients.

For any node x, ω_x is referred to the union of all elements sharing the common vertex x. Roughly speaking, the distribution of coefficients around certain interior node x is quasi-monotone if the trace of $a(x)$ on a small sphere $B \subset \omega_x$ around x has only one local maximum. A local maximum is attained on $B \cap K_i$, $K_i \in \mathcal{M}_H$, if $a_{K_i} > a_{K_j}$ for all elements K_j in ω_x sharing one common side with K_i. If x is a boundary node, then it is required for quasi-monotone condition that each local maximum is adjacent to the boundary $\partial\Omega$. In this paper, the nodes with respect to which the quasi-monotone condition for the distribution of coefficients is violated will be called singular nodes. For any $K \in \mathcal{M}_H$, denote by $\omega_K := \cup\{K' \in \mathcal{M}_H : \bar{K} \cap \bar{K}' \neq \emptyset\}$. If K has one singular node, we set $\Lambda_K := \max_{K'\subset\omega_K}(a_K/a_{K'})$; otherwise, we set $\Lambda_K = 1$. In Chen and Dai [7] we proved the following a posteriori error estimate for the elliptic problem (1.1)-(1.2):

$$|\!|\!| u - u_H |\!|\!|_\Omega^2 \leq C_1 \sum_{e \in \mathcal{B}^n} \eta_e^2, \tag{2.7}$$

where the constant C_1 depends only on the minimum angle of the triangulation $\mathcal{M}_H$, and the local error indicator η_e is given by

$$\eta_e^2 := \sum_{K\in\Omega_e} \Lambda_K \| H_K a_K^{-1/2} f \|_{L^2(K)}^2 + \Lambda_e \| H_e^{1/2} a_e^{-1/2} J_e \|_{L^2(e)}^2, \tag{2.8}$$

where $\Lambda_e = \max_{K\in\Omega_e} \Lambda_K$. Note that Λ_K is equal to one in the elementsaway from the singular nodes and only active around the singular nodes. This is the major improvement of our new local error estimator.

Now we report some numerical results to demostrate the efficiency of the adaptive finite element methods based on above error estimator (2.7). In [7] we discussed in more detail the influence of a posteriori error estimator, refinement/coarsening strategy, and the determination of various parameters on the successful implementation of adaptive procedures.

We choose the numerical example used in [17, §5.3] to run our experiments. An exact solution of the elliptic problem with piecewise constant coefficients and vanishing right-hand side f is constructed by using the formulas derived by Kellogg [15]. Let $\Omega = (-1,1)^2$, $a(x) = R$ in the first and third quadrants, and $a(x) = 1$ in the second and fourth quadrants. An exact weak solution u of (1.1) for $f = 0$ is given in polar coordinates by $u(r,\vartheta) = r^\gamma \mu(\vartheta)$, where

$$\mu(\vartheta) = \begin{cases} \cos((\pi/2-\sigma)\gamma)\cdot\cos((\vartheta-\pi/2+\rho)\gamma) & \text{if} \quad 0\le\vartheta\le\pi/2 \\ \cos(\rho\gamma)\cdot\cos((\vartheta-\pi+\sigma)\gamma) & \text{if} \quad \pi/2\le\vartheta\le\pi \\ \cos(\sigma\gamma)\cdot\cos((\vartheta-\pi-\rho)\gamma) & \text{if} \quad \pi\le\vartheta\le 3\pi/2 \\ \cos((\pi/2-\rho)\gamma)\cdot\cos((\vartheta-3\pi/2-\sigma)\gamma) & \text{if} \quad 3\pi/2\le\vartheta\le 2\pi, \end{cases}$$

and the numbers γ, ρ, σ, R satisfy the nonlinear relations

$$\begin{cases} R = -\tan((\pi/2-\sigma)\gamma)\cdot\cot(\rho\gamma) \\ 1/R = -\tan(\rho\gamma)\cdot\cot(\sigma\gamma) \\ R = -\tan(\sigma\gamma)\cdot\cot((\pi/2-\rho)\gamma) \\ 0<\gamma<2 \\ \max(0,\pi\gamma-\pi) < 2\gamma\rho < \min(\pi\gamma,\pi) \\ \max(0,\pi-\pi\gamma) < -2\gamma\sigma < \min(\pi,2\pi-\pi\gamma). \end{cases} \tag{2.9}$$

This solution u is in Sobolev space $H^{1+\delta}(\Omega)$ with $\delta < \gamma$ and thus is very singular for small parameter γ. For example, when $\gamma = 0.1$, (2.9) is solved in [17] to obtain

$$R \approx 161.4476387975881, \qquad \rho = \pi/4, \qquad \sigma \approx -14.92256510455152. \tag{2.10}$$

The surface plot of the resultant solution u is depicted in Fig.1. Let u_n be the finite element approximation of u on a uniform mesh constructed by first dividing the domain into $n \times n$ subdomains and then connecting the lower left and upper right corners of each subdomain. The surface plot of the relative error $(u - u_{1024})/\| u \|_{L^\infty(\Omega)}$ is depicted in Fig.2. The relative error is about 23.68 percent around the singularity point. The energy error $|\!|\!| u - u_{1024} |\!|\!|_\Omega$ is 0.6954, a very slow decrease from $|\!|\!| u - u_{128} |\!|\!|_\Omega = 0.8547$. Since the solution u is in Sobolev space $H^{1+\delta}(\Omega)(\delta < 0.1)$, we know from standard finite element a priori error analysis that to bring the energy error below 0.1, we must introduce 10^{11} nodes in each space direction. Therefore, the standard finite element method with uniform mesh is not appropriate for solving such elliptic problems with strong singularities.

However, our adaptive method requires only 2793 nodes to achieve an energy error of 0.07303. The surface plot of the relative error $(u - u_h)/\| u \|_{L^\infty(\Omega)}$ is depicted in Fig.3. The maximum of the relative error is 0.011719, less than 2 percent around the singularity point.

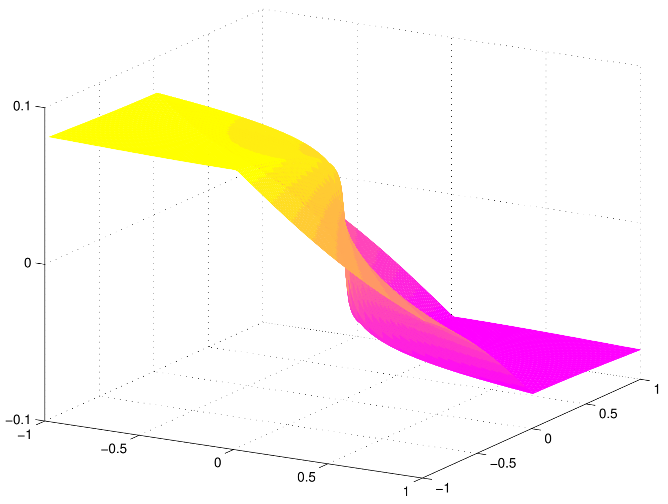

Figure 1: The surface plot of the exact solution u for $\gamma = 0.1$.

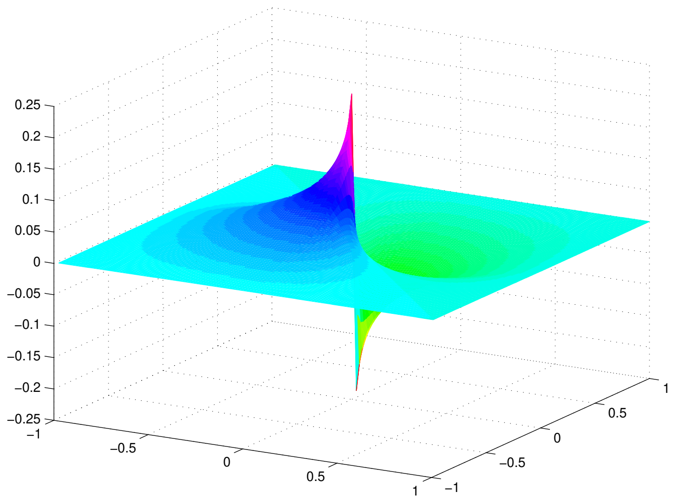

Figure 2: The surface plot of the relative error $(u - u_{1024})/\| u \|_{L^\infty(\Omega)}$. The maximum of the relative error is 0.2368.

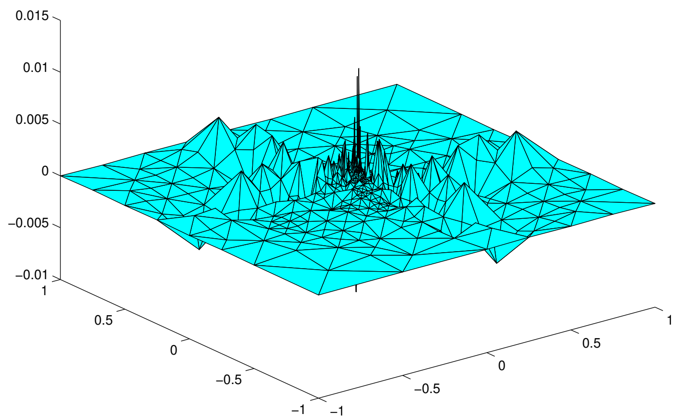

Figure 3: The surface plot of the relative error $(u-u_h)/\| u \|_{L^\infty(\Omega)}$ of the adaptive solution u_h on an adaptive mesh of 2793 nodes. The maximum of the relative error is 0.011719.

3 Reliable and efficient a posteriori error estimate for a linear parabolic problem

Let Ω be a polyhedron domain in $\mathbf{R}^d (d=1,2,3)$, $\Gamma=\partial\Omega$ and $T>0$, In this section we consider the following linear parabolic equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \operatorname{div}(a(x)\nabla u) = f \quad \text{ in } \Omega\times(0,T), \\ &u=0 \quad \text{ on } \Gamma\times(0,T), \quad u(x,0)=u_0(x) \quad \text{ in } \Omega, \end{aligned} \tag{3.1}$$

where $f\in L^2(0,T;L^2(\Omega))$, $u_0\in L^2(\Omega)$ and $a(x)$ is a piecewise constant function. Let τ_n be the n-th time-step size and $t^n=\sum_{i=1}^n \tau_i$. Denote by $\mathcal{M}^n$ a uniformly regular partition of Ω into simplices such that $a(x)$ is a constant on each element $K\in\mathcal{M}^n$. We use refinement/coarsening procedures based on the bisection algorithm, which lead to compatible consecutive meshes whose minimum angles are bounded uniformly away from zero. Let V^n indicate the usual space of conforming linear finite elements over $\mathcal{M}^n$ and $V_0^n = V^n\cap H_0^1(\Omega)$. In this section, we consider the following simple fully discrete finite element scheme for solving (3.1): For $n=1,2,\cdots$, find $U_h^n\in V_0^n$ such that

$$\left\langle \frac{U_h^n - U_h^{n-1}}{\tau_n}, v\right\rangle + \langle a\nabla U_h^n, \nabla v\rangle = \langle \bar{f}^n, v\rangle \quad \forall v\in V_0^n, \tag{3.2}$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product on $L^2(\Omega)$, and

$$\bar{f}^n = \frac{1}{\tau_n}\int_{t^{n-1}}^{t^n} f(x,t)\, dt.$$

Let $U^n\in H_0^1(\Omega)$ be the solution of the following semi-discrete problem

$$\left\langle \frac{U^n - U^{n-1}}{\tau_n}, \varphi\right\rangle + \langle a\nabla U^n, \nabla\varphi\rangle = \langle \bar{f}^n, \varphi\rangle \quad \forall \varphi\in H_0^1(\Omega). \tag{3.3}$$

We observe that by modifying the time-step size τ_n, we are essentially controlling the error between $u^n = u(x,t^n)$ and U^n, not between u^n and U_h^n. Moreover, let $U_*^n \in H_0^1(\Omega)$ be the solution of the following auxiliary problem

$$\Big\langle \frac{U_*^n - U_h^{n-1}}{\tau_n}, \varphi \Big\rangle + \langle a\nabla U_*^n, \nabla\varphi\rangle = \langle \bar{f}^n, \varphi\rangle \quad \forall \varphi \in H_0^1(\Omega), \tag{3.4}$$

then we also observe that for fixed time-step size τ_n, by adapting the mesh $\mathcal{M}^n$ we are essentially controlling the error between U_h^n and U_*^n, not between U_h^n and U^n (or the real solution u). These two observations play an important role in the subsequent analysis to prove the local lower bound for the space a posteriori error estimator and in the development of a convergent refinement/coarsening strategy.

We define the interior residual

$$R^n := \bar{f}^n - \frac{U_h^n - U_h^{n-1}}{\tau_n},$$

along with the jump residual across $e \in \mathcal{B}^n$

$$J_e^n := [\![a\nabla U_h^n]\!]_e \cdot \nu_e = (a\nabla U_h^n|_{K_1} - a\nabla U_h^n|_{K_2}) \cdot \nu_e, \qquad e = \partial K_1 \cap \partial K_2,$$

using the convention that the unit normal vector ν_e to e points from K_2 to K_1. We observe that integration by parts implies

$$\langle a\nabla U_h^n, \nabla\varphi\rangle = -\sum_{e\in\mathcal{B}^n} \int_e J_e^n \varphi ds \qquad \forall\varphi \in H_0^1(\Omega).$$

Theorem 3.1 (Chen and Jia [8]) *For any integer $1 \le m \le N$, there exists a constant $C > 0$ depending only the minimum angle of meshes $\mathcal{M}^n, n = 1, 2, \cdots, m$, and the coefficient $a(x)$ such that the following a posteriori error estimate holds*

$$\begin{aligned} &\frac{1}{2}\| u^m - U_h^m \|_{L^2(\Omega)}^2 + \sum_{n=1}^m \int_{t^{n-1}}^{t^n} |\!|\!| u - U_h^n |\!|\!|_\Omega^2 dt \\ \le\ & \| u_0 - U_h^0 \|_{L^2(\Omega)}^2 + \sum_{n=1}^m \tau_n \eta_{\text{time}}^n + C\sum_{n=1}^m \tau_n \eta_{\text{space}}^n \\ +\ & 2\Big(\sum_{n=1}^m \int_{t^{n-1}}^{t^n} \| f - \bar{f}^n \|_{L^2(\Omega)} dt\Big)^2, \end{aligned}$$

where the time error estimator η_{time}^n and space error estimator η_{space}^n are given by

$$\eta_{\text{time}}^n = \frac{1}{3}|\!|\!| U_h^n - U_h^{n-1} |\!|\!|_\Omega^2, \qquad \eta_{\text{space}}^n = \sum_{e\in\mathcal{B}^n} \eta_e^n$$

with the local error estimator η_e^n defined as

$$\eta_e^n = \frac{1}{2}\sum_{K\in\Omega_e} h_K^2 \| R^n \|_{L^2(K)}^2 + h_e \| J_e^n \|_{L^2(e)}^2.$$

Here Ω_e is the collection of two elements sharing the common side $e \in \mathcal{B}^n$.

Let $P^n : L^2(\Omega) \to V_0^n$ be the L^2 projection operator. We remark that the coarsening errors involving $U_h^{n-1} - P^n U_h^{n-1}$ which appeared in previous studies [10], [13], [18] are not present in our a posteriori error estimates. They are hidden in the space error term $\| h_n R^n \|_{L^2(\Omega)}^2$ and time error term $\|U_h^n - U_h^{n-1}\|_\Omega^2$. Here h_n is the piecewise constant function which equals to h_K on each $K \in \mathcal{M}^n$.

Now we consider the following local lower bound of the space error estimator which ensures over-refinement will not occur for the refinement strategy based on our space error estimator. For any $K \in \mathcal{M}^n$ and $\varphi \in L^2(\Omega)$, we define $P_K\varphi = \frac{1}{|K|}\int_K \varphi dx$, the average of φ over K.

Theorem 3.2 (Chen and Jia [8]) *Suppose that there exists a constant $\hat{C} > 0$ such that $h_K^2 \leq \hat{C}\tau_n$ for any $K \in \mathcal{M}^n$. Then there exist constants $C_2, C_3 > 0$ depending only on $\hat{C}$, the minimum angle of $\mathcal{M}^n$ and the coefficient $a(x)$ such that for any $e \in \mathcal{B}^n$, the following estimate holds*

$$\begin{aligned} \eta_e^n &\leq C_2 \sum_{K\in\Omega_e} \left(\frac{1}{\tau_n} \| U_*^n - U_h^n \|_{L^2(K)}^2 + \|U_*^n - U_h^n\|_K^2 \right) \\ &+ C_3 \sum_{K\in\Omega_e} h_K^2 \| R^n - P_K R^n \|_{L^2(K)}^2. \end{aligned}$$

Based on the local error indicators, the usual adaptive algorithm solving the parabolic problem (3.1) at the n-th time step reads as follows

$$\text{Solve} \;\to \text{Estimate} \;\to \text{Refine/Coarsen.}$$

Here the refinement/coarsening procedure includes both the mesh and time-step size modifications. There are several possibilities proposed in the literature on the strategies of the adaptive control of meshes and time-step sizes. We refer to Schmidt and Siebert [22] for a nice review on various adaptive algorithms and their implementation details. In this paper, at each time step n, we propose the following algorithm to modify the time-step size τ_n and mesh $\mathcal{M}^n$ starting from the initial time-step size $\tau_{n,0} = \tau_{n-1}$ and initial mesh $\mathcal{M}^{n,0} = \mathcal{M}^{n-1}$:

1. Refine the time-step size $\tau_{n.0}$ and the mesh $\mathcal{M}^{n,0}$ so that for the solution $U_h^n \in V_0^n$ of (3.2) on the final mesh $\mathcal{M}^n$ with final time-step size τ_n, the associated space and time error estimators are less than the prescribed tolerances respectively;

2. Coarsen the mesh $\mathcal{M}^n$ so that for the solution U_H^n on the coarsened mesh $\mathcal{M}_H^n$, the associated coarsening error estimator less than some prescribed tolerance;

3. Enlarge the initial time-step size $\tau_{n+1,0}$ for next time step if the current time error estimator is much less than the tolerance.

In [8] we extend the convergence analysis of adaptive finite element methods developed for linear elliptic problems in Dörfler [12] and Morin, Nochetto and Siebert [17] to prove the iteration in Step 1 of above algorithm terminates in finite number of steps. The main difficulty now is the treatment of the so-called data oscillation of the residual $\bar{f}^n - (U_h^n - U_h^{n-1})/\tau_n$ which changes at each refinement procedure.

The choice of the coarsening error estimator and coarsening strategy in Step 2 is a subtle issue. The error incurred due to the over-coarsening can only be reduced by re-refining the coarsened mesh. Thus over-coarsening leads to unnecessary solution of discrete problems, that is usually expensive and undesirable. Our coarsening error indicator is based on a direct control of the error between the coarsened discrete solution and the limit solution U_*^n of (3.4). More precisely, let $\mathcal{M}_H^n$ be a coarsening of $\mathcal{M}^n$, and U_H^n, U_h^n be the corresponding solutions of (3.2) with fixed time-step size τ_n. Since $V_0^{n,H} \subset V_0^n$, we deduce by Galerkin orthogonality

$$\begin{aligned}
& \frac{1}{\tau_n} \| U_*^n - U_H^n \|_{L^2(\Omega)}^2 + \|\!| U_*^n - U_H^n |\!\|_\Omega^2 \\
\leq \; & \frac{1}{\tau_n} \| U_*^n - U_h^n \|_{L^2(\Omega)}^2 + \|\!| U_*^n - U_h^n |\!\|_\Omega^2 \\
+ \; & \frac{1}{\tau_n} \| U_h^n - I_H^n U_h^n \|_{L^2(\Omega)}^2 + \|\!| U_h^n - I_H^n U_h^n |\!\|_\Omega^2,
\end{aligned}$$

where $\|\!|\varphi|\!\|_\Omega = \langle a\nabla\varphi, \nabla\varphi\rangle^{1/2}$ is the energy norm of $\varphi \in H^1(\Omega)$, and $I_H^n : C(\bar{\Omega}) \to V^{n,H}$ is the usual linear finite element interpolant over $\mathcal{M}_H^n$. Our coarsening error indicator is then given by

$$\eta_{\text{coarse}}^n := \frac{1}{\tau_n} \| U_h^n - I_H^n U_h^n \|_{L^2(\Omega)}^2 + \|\!| U_h^n - I_H^n U_h^n |\!\|_\Omega^2,$$

which does not depend on the coarsened solution U_H^n. It is shown in [8] that this direct error control leads to only one coarsening step. The implementation issue of the coarsening startegy is discussed in detail in [8].

4 The continuous casting problem

Let the ingot occupy a cylindrical domain Ω with large aspect ratio. Let $0 < L < +\infty$ be the length of the ingot and $\Gamma \subset \mathbf{R}^d$ for $d = 1$ or 2 be its (polygonal) cross section. We show $\Omega = \Gamma \times (0, L)$ in Fig.4, and hereafter write $x = (y, z) \in \Omega$ with $y \in \Gamma$ and $0 < z < L$.

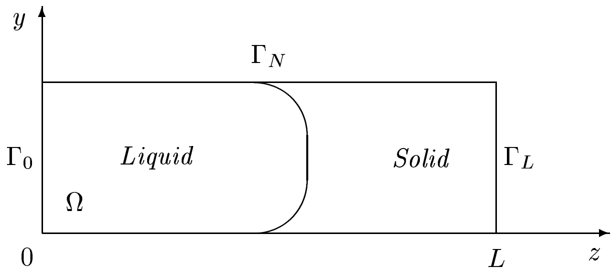

Figure 4: The domain Ω

We study the following *convection-dominated nonlinearly degenerate diffusion* problem

$$\partial_t u + v(t)\partial_z u - \Delta\theta = 0 \quad \text{in} \quad Q_T, \tag{4.1}$$

$$\theta = \beta(u) \quad \text{in} \quad Q_T, \tag{4.2}$$

$$\theta = g_D \quad \text{on} \quad \Gamma_0 \times (0, T), \tag{4.3}$$

$$\partial_\nu \theta + p(\theta - \theta_{\text{ext}}) = 0 \quad \text{on} \quad \Gamma_N \times (0,T), \tag{4.4}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \tag{4.5}$$

where $Q_T = \Omega \times (0,T)$,

$$\Gamma_0 = \Gamma \times \{0\}, \quad \Gamma_L = \Gamma \times \{L\}, \quad \Gamma_N = \partial\Gamma \times (0,L),$$

and $\theta + \theta_c$ is the absolute temperature, θ_c is the melting temperature, u is the enthalpy, $v(t) > 0$ is the extraction velocity of the ingot, ν is the unit outer normal to $\partial\Omega$, and θ_{ext} is the external temperature. The mapping $\beta : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and monotone increasing; since β is not strictly increasing, (4.1) is *degenerate* parabolic. The missing *outflow* boundary condition on Γ_L is unclear because the ingot moves at the casting speed and is cut shorter from time to time. It is thus evident that any standard boundary condition could only be an approximation. For simplicity, we impose a Dirichlet outflow condition

$$\theta = g_D < 0 \quad \text{on} \quad \Gamma_L \times (0,T). \tag{4.6}$$

A Robin type boundary condition on Γ_L is also possible (see the discussion in [10]). It is convenient to denote by Γ_D the Dirichlet part of $\partial\Omega$, that is $\Gamma_0 \cup \Gamma_L$.

The importance of simulating and controlling the continuous casting process in the production of steel, copper, and other metals is recognized in industry. The extraction velocity $v(t)$ as well as the cooling conditions on the mold and water spray region are known to be decisive in determining material properties of the ingot. Avoiding excessive thermal stresses and material defects is an essential, and rather empirical, aspect of the continuous casting process.

The system (4.1)-(4.5) is a special case of general Stefan problems with prescribed convection Rulla [21]. An outflow Dirichlet condition together with an inflow Neumann condition is assumed in [21] to guarantee uniqueness of weak solutions; our more realistic boundary data (4.3) and (4.6) violate this restriction. Under the additional assumption that the free boundary does not touch the inflow boundary Γ_0, uniqueness of weak solutions to (4.1)-(4.5) and (4.6) is shown in Rodrigues and Yi [20].

We now introduce the fully discrete problem, which combines continuous piecewise linear finite elements in space with characteristic finite differences in time. In fact, we use the method of characteristics to discretize the convection. We denote by τ_n the n-th time step and set

$$t^n := \sum_{i=1}^{n} \tau_i, \quad \varphi^n(\cdot) := \varphi(\cdot, t^n)$$

for any function φ continuous in $(t^{n-1}, t^n]$. Let N be the total number of time steps, that is $t^N \geq T$.

Let $\mathbf{V}_0 = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ and $\mathbf{V}^*$ the dual space of $\mathbf{V}_0$. We denote by $\mathcal{M}^n$ a uniformly regular partition of Ω into simplexes. The mesh $\mathcal{M}^n$ is obtained by refinement/coarsening of $\mathcal{M}^{n-1}$, and thus $\mathcal{M}^n$ and $\mathcal{M}^{n-1}$ are *compatible*. Let $\mathbf{V}^n$ indicate the usual space of C^0 piecewise linear finite elements over $\mathcal{M}^n$ and $\mathbf{V}_0^n = \mathbf{V}^n \cap \mathbf{V}_0$. Let $\{x_k^n\}_{k=1}^{K^n}$ denote the interior nodes of $\mathcal{M}^n$. Let $I^n : C(\bar{\Omega}) \to \mathbf{V}^n$ be the usual Lagrange interpolation operator, namely $(I^n\varphi)(x_k^n) = \varphi(x_k^n)$ for all $1 \leq k \leq K^n$. Finally, let the discrete inner products $\langle \cdot, \cdot \rangle^n$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_E^n$ be the sum over $S \in \mathcal{M}^n$ of the element scalar products

$$\langle \varphi, \chi \rangle_S^n = \int_S I^n \langle \varphi\chi \rangle dx, \quad \langle\!\langle \varphi, \chi \rangle\!\rangle_S^n = \int_{S \cap E} I^n(\varphi\chi) d\sigma,$$

for any piecewise uniformly continuous functions φ, χ. The discrete initial enthalpy $U^0 \in \mathbf{V}^0$ is defined at nodes x_k^0 of $\mathcal{M}^0 = \mathcal{M}^1$ to be

$$U^0(x_k^0) := u_0(x_k^0) \quad \forall\, x_k^0 \in \Omega \backslash F_0, \quad U^0(x_k^0) := 0 \quad \forall\, x_k^0 \in F_0.$$

Hence, U^0 is easy to evaluate in practice.

Discrete Problem. *Given $U^{n-1}, \Theta^{n-1} \in \mathbf{V}^{n-1}$, then $\mathcal{M}^{n-1}$ and τ^{n-1} are modified as described below to get $\mathcal{M}^n$ and τ_n and thereafter $U^n, \Theta^n \in \mathbf{V}^n$ computed according to the following prescription*, for any $\varphi \in \mathbf{V}_0^n$,

$$\Theta^n = I^n \beta(U^n), \quad \Theta^n - I^n g_D^n \in \mathbf{V}_0^n, \quad \bar{U}^{n-1} := U^{n-1}(\bar{x}^{n-1}),$$

$$\frac{1}{\tau_n} \langle U^n - I^n \bar{U}^{n-1}, \varphi \rangle^n + \langle \nabla \Theta^n, \nabla \varphi \rangle + \langle\!\langle p^n(\Theta^n - \theta_{\text{ext}}^n), \varphi \rangle\!\rangle_{\Gamma_N}^n = 0,$$

where $\bar{x}^{n-1} = x - v^{n-1} \tau_n e_z$.

In view of the constitutive relation $\Theta^n = I^n \beta(U^n)$ being enforced only at the nodes, and the use of mass lumping, the discrete problem yields a monotone operator in $\mathbf{R}^{K^n}$ which is easy to implement and solve by nonlinear SOR [18], for example.

The following a posteriori error estimate which depends mildly on the casting velocity V is proved in Chen, Nochetto and Schmidt [10].

Theorem 4.1 *There exists a constant $C > 0$ independent of V and t^m such that the following a posteriori error estimate holds for all $t^m \in [0, T]$,*

$$\int_0^{t^m} \| \beta(u) - \beta(U) \|_{L^1(\Omega)} dt \le C (V t^m)^{1/2} \Big(\mathcal{E}_0 + \sum_{i=5}^{10} \mathcal{E}_i + \Big(\Lambda_m \sum_{i=1}^{4} \mathcal{E}_i \Big)^{1/2} \Big),$$

where

$$\Lambda_m = \Big(\sum_{n=1}^{m} \tau_n \big(1 + \lambda |\Omega| + \| \Theta^n \|_{L^2(\Omega)}^2 \big) \Big)^{1/2}$$

and the error indicators $\mathcal{E}_i$ depending only the discrete solutions U^n, Θ^n and the known data.

Now we briefly report a numerical example from [10] to illustrate the ability of the proposed adaptive method to capture the free boundaries. Further numerical experiments and the implementation details can be found in [10]

After proper scaling with the non-dimensional domain is of size 0.021×2.5 and the slopes of β are $k_s = 0.006$, $k_l = 0.029$. A temperature range $\tilde{\theta} \in (1000, 1800)\,°K$ leads to scaled values $|\beta(u)| = O(10^{-2})$, while the scaled latent heat is $\lambda = 1$. The scaled convection speed is about $v = 225$, so convection is *dominant*. The simulations run for $t \in (0, 0.1)$, which is equivalent to a final time $\tilde{T} = 10000 s \approx 2\frac{3}{4} h$. Initial conditions are chosen piecewise linear in z direction, with a prescribed initial position of the interface at $z = L/10$. We prescribe a variable casting speed

$$\tilde{v}(\tilde{t}) = 0.0175 + 0.005 * \sin(0.00175\, \tilde{t}) \ [m/s],$$

which has a strong influence on the length of the liquid pool inside the slab. The largest velocity is chosen equal to the constant velocity in the problem with stationary casting speed; all other parameters are left unchanged. This guarantees that the liquid pool will not reach the outflow boundary. The variable velocity is shown in Figure 5, together with the number of elements in the adapted meshes $M(t^n) = \#(\mathcal{M}^n)$ and time step sizes. Due to a longer liquid pool (and interface), there are more mesh elements when the velocity is larger and a smaller time step size is needed. Figures 6 and 7 display adaptive meshes and temperature graphs for $t = 0.05$ and $t = 0.07$, corresponding to large and small velocity values. We see from these figures that the meshes are very fine near the free boundaries and they move along with the moving of the free boundaries

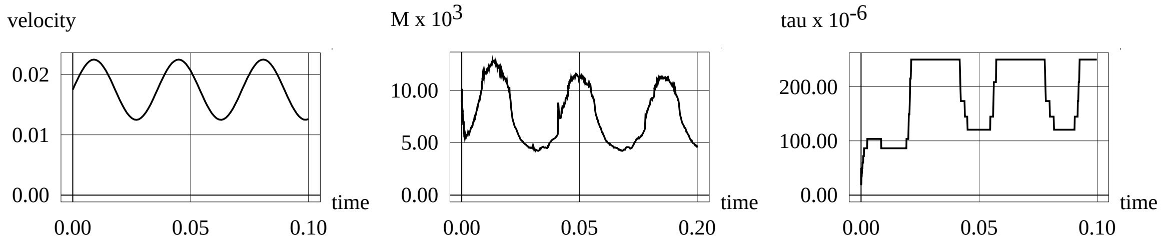

Figure 5: Variable casting speed. Velocity $\tilde{v}(t)$, element count, and time step sizes.

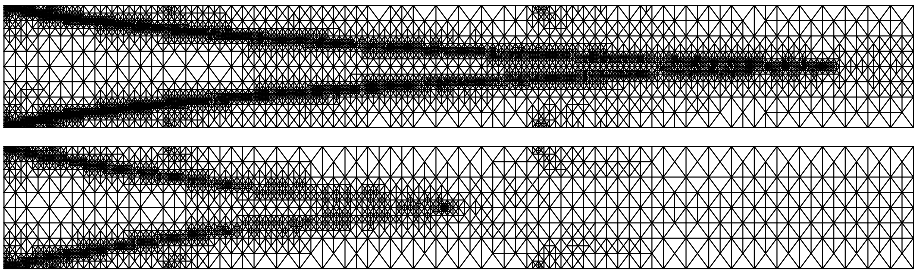

Figure 6: Variable casting speed. Adaptive meshes for $t = 0.05$ (top) and $t = 0.07$ (bottom).

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Figure 7: Variable casting speed. Temperature graphs for $t = 0.05$ (top) and $t = 0.07$ (bottom).

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