Target Detection and Characterization from Electromagnetic Induction Data*

Habib Ammari† Junqing Chen‡ Zhiming Chen§
Josselin Garnier¶ Darko Volkov‖

September 19, 2012

Abstract

This paper aims to advance the field of nondestructive testing by eddy currents. It provides a mathematical and numerical framework for imaging small volume conductive inclusions of arbitrary shapes from electromagnetic induction data. The effect of measurement noise on the localization and characterization approach developed in this paper is investigated.

Mathematics Subject Classification (MSC2000): 35R30, 35B30
Keywords: eddy current, imaging, induction data, asymptotic formula, detection test, localization, characterization, Hadamard technique, measurement noise

1 Introduction

Nondestructive testing by eddy currents is a technology of choice in the assessment of the structural integrity of a variety of materials such as, for instance, aircrafts or metal beams, see [10]. In this paper we introduce a new eddy current reconstruction method relying on the assumption that the objects to be imaged are small. This present study is related to the theory of small volume perturbations of Maxwell's equations, see [8]. It is, however, specific to eddy currents and to the particular lengthscales relevant to that case.

We first note that in the eddy current regime a diffusion equation is used for modeling electromagnetic fields. The characteristic length is the skin depth of the conductive object

*This work was supported by ERC Advanced Grant Project MULTIMOD–267184.

†Department of Mathematics and Applications, Ecole Normale Supérieure, 45 Rue d'Ulm, 75005 Paris, France (habib.ammari@ens.fr).

‡Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China. This author was supported in part by China NSF under the grant 11001150 and 11171040. (jqchen@math.tsinghua.edu.cn).

§LSEC, Institute of Computational Mathematics Chinese Academy of Sciences, Beijing 100190, China. This author was supported in part by National Basic Research Project under the grant 2011CB309700 and China NSF under the grant 11021101. (zmchen@lsec.cc.ac.cn)

¶Laboratoire de Probabilités et Modèles Aléatoires & Laboratoire Jacques-Louis Lions, Université Paris VII, 75205 Paris Cedex 13, France (garnier@math.jussieu.fr).

‖Department of Mathematical Sciences, Stratton Hall, 100 Institute Road, Worcester, MA 01609-2280, USA (darko@wpi.edu).

to be imaged [10]. We consider the regime where the skin depth is comparable to the characteristic size of the conductive inclusion.

Using the $\boldsymbol{E}$-formulation for the eddy current problem, we first establish energy estimates. We start from integral representation formulas for the electromagnetic fields arising in the presence of a small conductive inclusion to derive an asymptotic expansion for the magnetic part of the field.

Based on that asymptotic formula we are then able to construct a localization method for the conductive inclusion. That method involves a response matrix data. A MUSIC (which stands for MUltiple Signal Classification) imaging functional is proposed for locating the target. It uses the projection of onto the image space of the response matrix. Once the location is found, geometric features of the inclusion can be reconstructed using a least-squares method. These geometric features together with material parameters (electric conductivity and magnetic permeability) are incorporated in polarization tensors.

The so called Hadamard measurement sampling technique is applied in order to reduce the impact of noise in measurements. Finally we provide statistical distributions for the singular values of the response matrix in the presence of measurement noise and we simulate our localization technique on a test example.

The paper is organized as follows. Section 2 is devoted to a variational formulation of the eddy current equations. Section 3 contains the main contributions of this paper. It provides a rigorous derivation of the effect of a small conductive inclusion on the magnetic field measured away from the inclusion. Section 4 applies MUSIC-type localization to eddy current model. Section 5 discusses the effect of noise on the inclusion detection and proposes a detection test based on the significant eigenvalues of the response matrix. Section 6 illustrates numerically on test examples our main findings in this paper. A few concluding remarks are given in the last section.

2 Eddy Current Equations

Suppose that there is an electromagnetic inclusion in $\mathbb{R}^3$ of the form $B_\alpha = \boldsymbol{z} + \alpha B$, where $B \subset \mathbb{R}^3$ is a bounded, smooth domain containing the origin. Let Γ and Γ_α denote the boundary of B and B_α. Let μ_0 denote the magnetic permeability of the free space. Let μ_* and σ_* denote the permeability and the conductivity of the inclusion which are also assumed to be constant. We introduce the piecewise constant magnetic permeability and electric conductivity

$$\mu_\alpha(\boldsymbol{x}) = \begin{cases} \mu_* & \text{in } B_\alpha, \\ \mu_0 & \text{in } B_\alpha^c = \mathbb{R}^3 \backslash \bar{B}_\alpha, \end{cases} \qquad \sigma_\alpha(\boldsymbol{x}) = \begin{cases} \sigma_* & \text{in } B_\alpha, \\ 0 & \text{in } B_\alpha^c. \end{cases}$$

Let $(\boldsymbol{E}_\alpha, \boldsymbol{H}_\alpha)$ denote the eddy current fields in the presence of the electromagnetic inclusion B_α and a source current $\boldsymbol{J}_0$ located outside the inclusion. Moreover, we suppose that $\boldsymbol{J}_0$ has a compact support and is divergence free: $\nabla \cdot \boldsymbol{J}_0 = 0$ in $\mathbb{R}^3$. The fields $\boldsymbol{E}_\alpha$ and $\boldsymbol{H}_\alpha$ are the solutions of the following eddy current equations:

$$\begin{cases} \nabla \times \boldsymbol{E}_\alpha = \mathbf{i}\omega\mu_\alpha \boldsymbol{H}_\alpha & \text{in } \mathbb{R}^3, \\ \nabla \times \boldsymbol{H}_\alpha = \sigma_\alpha \boldsymbol{E}_\alpha + \boldsymbol{J}_0 & \text{in } \mathbb{R}^3, \\ \boldsymbol{E}_\alpha(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}), \quad \boldsymbol{H}_\alpha(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) & \text{as } |\boldsymbol{x}| \to \infty. \end{cases} \tag{2.1}$$

Eliminate $\boldsymbol{H}_\alpha$ in (2.1) we obtain the following $\boldsymbol{E}$-formulation of the eddy current problem (2.1):

$$\begin{cases} \nabla\times\mu_\alpha^{-1}\nabla\times\boldsymbol{E}_\alpha - \mathbf{i}\omega\sigma_\alpha\boldsymbol{E}_\alpha = \mathbf{i}\omega\boldsymbol{J}_0 \quad \text{in } \mathbb{R}^3, \\ \nabla\cdot\boldsymbol{E}_\alpha = 0 \quad \text{in } B_\alpha^c, \\ \boldsymbol{E}_\alpha(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) \quad \text{as } |\boldsymbol{x}|\to\infty. \end{cases} \tag{2.2}$$

We will use the function space

$$\mathbf{X}_\alpha(\mathbb{R}^3) = \left\{\boldsymbol{u} : \frac{\boldsymbol{u}}{\sqrt{1+|\boldsymbol{x}|^2}} \in L^2(\mathbb{R}^3), \nabla\times\boldsymbol{u}\in L^2(\mathbb{R}^3), \nabla\cdot\boldsymbol{u} = 0 \text{ in } B_\alpha^c\right\},$$

and the sesquilinear form on $\mathbf{X}_\alpha(\mathbb{R}^3)\times\mathbf{X}_\alpha(\mathbb{R}^3)$

$$a_\alpha(\boldsymbol{E},\boldsymbol{v}) = (\mu_\alpha^{-1}\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{v})_{\mathbb{R}^3} - \mathbf{i}\omega\sigma_*(\boldsymbol{E},\boldsymbol{v})_{B_\alpha},$$

where $(\cdot,\cdot)_D$ stands for the L^2 inner product on the domain $D\subseteq\mathbb{R}^3$. The weak formulation of the $\boldsymbol{E}$-formulation (2.2) is: Find $\boldsymbol{E}_\alpha\in\mathbf{X}_\alpha(\mathbb{R}^3)$ such that

$$a_\alpha(\boldsymbol{E}_\alpha,\boldsymbol{v}) = \mathbf{i}\omega(\boldsymbol{J}_0,\boldsymbol{v})_{B_\alpha^c}, \quad \forall\boldsymbol{v}\in\mathbf{X}_\alpha(\mathbb{R}^3). \tag{2.3}$$

The uniqueness and existence of solution of the problem (2.3) is known (cf., e.g., Ammari et al. [1] and Hiptmair [13]).

Let $\boldsymbol{E}_0$ be the solution of the problem

$$\begin{cases} \nabla\times\mu_0^{-1}\nabla\times\boldsymbol{E}_0 = \mathbf{i}\omega\boldsymbol{J}_0 \quad \text{in } \mathbb{R}^3, \\ \nabla\cdot\boldsymbol{E}_0 = 0 \text{ in } \mathbb{R}^3, \\ \boldsymbol{E}_0(\boldsymbol{x}) = O(|\boldsymbol{x}|^{-1}) \quad \text{as } |\boldsymbol{x}|\to\infty. \end{cases} \tag{2.4}$$

The field $\boldsymbol{E}_0$ is uniquely existent and satisfies

$$(\mu_0^{-1}\nabla\times\boldsymbol{E}_0,\nabla\times\boldsymbol{v})_{\mathbb{R}^3} = \mathbf{i}\omega(\boldsymbol{J}_0,\boldsymbol{v})_{\mathbb{R}^3}, \quad \forall\boldsymbol{v}\in\mathbf{H}_{-1}(\mathbf{curl};\mathbb{R}^3), \tag{2.5}$$

where $\mathbf{H}_{-1}(\mathbf{curl};\mathbb{R}^3) = \left\{\boldsymbol{u} : \frac{\boldsymbol{u}}{\sqrt{1+|\boldsymbol{x}|^2}} \in L^2(\mathbb{R}^3)^3, \nabla\times\boldsymbol{u}\in L^2(\mathbb{R}^3)^3\right\}$.

3 Derivation of the Asymptotic Formulas

In this section we will derive the asymptotic formula for $\boldsymbol{H}_\alpha$ when the inclusion is small. Let $k = \omega\mu_0\sigma_*$. We are interested in the asymptotic range when $\alpha\to 0$ and

$$\nu := k\alpha^2 \tag{3.1}$$

is of order one.

In eddy current testing the wave equation is converted into the diffusion equation, where the characteristic length is the skin depth δ, given by $\delta = \sqrt{2/k}$. Hence, in the regime $\nu = O(1)$, the skin depth δ is of order of the characteristic size α of the inclusion.

We will always denote C the generic constant which depends possibly on μ_*/μ_0, the upper bound of $\omega\mu_0\sigma_*\alpha^2$, the domain B, but is independent of $\omega,\sigma_*,\mu_0,\mu_*$. Let $\mu_r = \mu_*/\mu_0$.

3.1 Energy Estimates

We start with the following estimate.

Lemma 3.1 *There exists a constant C such that*

$$\|\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0)\|_{L^2(\mathbb{R}^3)}+\sqrt{k}\|\boldsymbol{E}_\alpha-\boldsymbol{E}_0\|_{L^2(B_\alpha)}\le C\alpha^{3/2}(\sqrt{k}\|\boldsymbol{E}_0\|_{L^\infty(B_\alpha)}+\|\nabla\times\boldsymbol{E}_0\|_{L^\infty(B_\alpha)}).$$

Proof. By (2.3) and (2.5), we know that

$$\begin{aligned}&(\mu_\alpha^{-1}\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0),\nabla\times\boldsymbol{v})_{\mathbb{R}^3}-\mathbf{i}\omega(\sigma_\alpha(\boldsymbol{E}_\alpha-\boldsymbol{E}_0),\boldsymbol{v})_{B_\alpha}\\&\quad=(\mu_0^{-1}-\mu_*^{-1})(\nabla\times\boldsymbol{E}_0,\nabla\times\boldsymbol{v})_{B_\alpha}+\mathbf{i}\omega(\sigma_\alpha\boldsymbol{E}_0,\boldsymbol{v})_{B_\alpha},\quad\forall\boldsymbol{v}\in\mathbf{X}(\mathbb{R}^3).\end{aligned}\tag{3.2}$$

Since

$$|(\nabla\times\boldsymbol{E}_0,\nabla\times\boldsymbol{v})_{B_\alpha}|\le C\alpha^{3/2}\|\nabla\times\boldsymbol{E}_0\|_{L^\infty(B_\alpha)}\|\nabla\times\boldsymbol{v}\|_{L^2(B_\alpha)}$$

and

$$|(\sigma_\alpha\boldsymbol{E}_0,\boldsymbol{v})|\le C\alpha^{3/2}\sigma_*\|\boldsymbol{E}_0\|_{L^\infty(B_\alpha)}\|\boldsymbol{v}\|_{L^2(B_\alpha)},$$

by taking $\boldsymbol{v}=\boldsymbol{E}_\alpha-\boldsymbol{E}_0\in\mathbf{X}_\alpha(\mathbb{R}^3)$ in (3.2) and multiplying the obtained equation by μ_0 we have that

$$\begin{aligned}&\mu_r^{-1}\|\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0)\|^2_{L^2(\mathbb{R}^3)}+k\|\boldsymbol{E}_\alpha-\boldsymbol{E}_0\|^2_{L^2(B_\alpha)}\\\le\;&C\alpha^{3/2}\left(\|\nabla\times\boldsymbol{E}_0\|_{L^\infty(B_\alpha)}\|\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0)\|_{L^2(B_\alpha)}+k\|\boldsymbol{E}_0\|_{L^\infty(B_\alpha)}\|\boldsymbol{E}_\alpha-\boldsymbol{E}_0\|_{L^2(B_\alpha)}\right).\end{aligned}$$

This completes the proof. $\square$

Let $H^1(B_\alpha):=\{\varphi\in L^2(B_\alpha),\nabla\varphi\in L^2(B_\alpha)^3\}$. Let $\phi_0\in H^1(B_\alpha)$ be the solution of the problem

$$-\Delta\phi_0=-\nabla\cdot\boldsymbol{F}\text{ in }B_\alpha,\quad-\partial_{\boldsymbol{n}}\phi_0=(\boldsymbol{E}_0(\boldsymbol{x})-\boldsymbol{F}(\boldsymbol{x}))\cdot\boldsymbol{n}\text{ on }\Gamma_\alpha,\quad\int_{B_\alpha}\phi_0\,d\boldsymbol{x}=0,\tag{3.3}$$

where

$$\boldsymbol{F}(\boldsymbol{x})=\frac{1}{2}(\nabla\times\boldsymbol{E}_0)(\boldsymbol{z})\times(\boldsymbol{x}-\boldsymbol{z})+\frac{1}{3}[D(\nabla\times\boldsymbol{E}_0)](\boldsymbol{z})(\boldsymbol{x}-\boldsymbol{z})\times(\boldsymbol{x}-\boldsymbol{z}).\tag{3.4}$$

Here $D(\nabla\times\boldsymbol{E}_0)$ is the gradient matrix of $\nabla\times\boldsymbol{E}_0$. Let tr denote the trace. Since $\mathrm{tr}[D(\nabla\times\boldsymbol{E}_0)]=\nabla\cdot(\nabla\times\boldsymbol{E}_0)=0$, we know that

$$\nabla\times\boldsymbol{F}(\boldsymbol{x})=(\nabla\times\boldsymbol{E}_0)(\boldsymbol{z})+[D(\nabla\times\boldsymbol{E}_0)](\boldsymbol{z})(\boldsymbol{x}-\boldsymbol{z}).\tag{3.5}$$

Note that since $\boldsymbol{E}_0$ is smooth in $\bar{B}_\alpha$ we have

$$\|\nabla\times\boldsymbol{E}-\nabla\times\boldsymbol{F}\|_{L^\infty(B_\alpha)}\le C\alpha^2\|\nabla\times\boldsymbol{E}_0\|_{W^{2,\infty}(B_\alpha)}.\tag{3.6}$$

Denote by $\boldsymbol{H}_0=(\mathbf{i}\omega\mu_0)^{-1}\nabla\times\boldsymbol{E}_0$ and introduce $\boldsymbol{w}\in\mathbf{X}_\alpha(\mathbb{R}^3)$ as the solution of the problem

$$a_\alpha(\boldsymbol{w},\boldsymbol{v})=\mathbf{i}\omega\mu_0(\mu_0^{-1}-\mu_*^{-1})(\boldsymbol{H}_0(\boldsymbol{z}),\nabla\times\boldsymbol{v})_{B_\alpha}+\mathbf{i}\omega(\sigma_\alpha\boldsymbol{F},\boldsymbol{v})_{B_\alpha},\quad\forall\boldsymbol{v}\in\mathbf{X}_\alpha.\tag{3.7}$$

The following lemma provides a higher-order correction of the error estimate in Lemma 3.1.

Lemma 3.2 *There exists a constant C such that*

$$\|\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\boldsymbol{w})\|_{L^2(\mathbb{R}^3)}\le C\alpha^{5/2}(|1-\mu_r^{-1}|+\alpha\nu)\|\nabla\times\boldsymbol{E}_0\|_{W^{2,\infty}(B_\alpha)}, \quad (3.8)$$

$$\|\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\nabla\phi_0-\boldsymbol{w}\|_{L^2(B_\alpha)}\le C\alpha^{7/2}(|1-\mu_r^{-1}|+\alpha\nu)\|\nabla\times\boldsymbol{E}_0\|_{W^{2,\infty}(B_\alpha)}, \quad (3.9)$$

where $\nu=k\alpha^2$ is defined in (3.1).

Proof. First we notice that by taking $\boldsymbol{v}=\nabla\psi$ in (2.3), where $\psi\in H^1(\mathbb{R}^3)$ with compact support containing B_α such that $\psi=0$ on the support of $\boldsymbol{J}_0$,

$$i\omega(\sigma_\alpha\boldsymbol{E}_\alpha,\nabla\psi)_{B_\alpha}=0,\quad \forall\psi\in H^1(B_\alpha).$$

This yields $\nabla\cdot\boldsymbol{E}_\alpha=0$ in B_α and $\boldsymbol{E}_\alpha\cdot\boldsymbol{n}=0$ on Γ_α. Similarly, we know from (3.7) that $\boldsymbol{w}\cdot\boldsymbol{n}=-\boldsymbol{F}(\boldsymbol{x})\cdot\boldsymbol{n}$ on Γ_α and $\nabla\cdot\boldsymbol{w}=-\nabla\cdot\boldsymbol{F}$ in B_α. From (3.3) we also know that $\nabla\cdot(\boldsymbol{E}_0+\nabla\phi_0)=\nabla\cdot\boldsymbol{F}$ in B_α and $(\boldsymbol{E}_0+\nabla\phi_0)\cdot\boldsymbol{n}=\boldsymbol{F}(\boldsymbol{x})\cdot\boldsymbol{n}$ on Γ_α. Thus

$$\nabla\cdot(\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\nabla\phi_0-\boldsymbol{w})=0\ \text{ in } B_\alpha,\qquad (\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\nabla\phi_0-\boldsymbol{w})\cdot\boldsymbol{n}=0\ \text{ on }\Gamma_\alpha,$$

which implies by scaling argument and the embedding theorem that

$$\begin{aligned}\|\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\nabla\phi_0-\boldsymbol{w}\|_{L^2(B_\alpha)} &\le C\alpha\|\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\nabla\phi_0-\boldsymbol{w})\|_{L^2(B_\alpha)}\\ &= C\alpha\|\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\boldsymbol{w})\|_{L^2(B_\alpha)},\end{aligned}$$

for some constant C independent of α and σ_*. Therefore, (3.9) follows from (3.8).

To show (3.8), we define $\tilde\phi_0$ as the solution of the exterior problem

$$-\Delta\tilde\phi_0=0 \text{ in } B_\alpha^c,\quad \tilde\phi_0=\phi_0 \text{ on }\Gamma_\alpha,\quad \tilde\phi_0\to 0 \text{ as } |\boldsymbol{x}|\to\infty.$$

The existence of $\tilde\phi_0$ in $W^{1,-1}(B_\alpha^c)=\left\{\varphi:\dfrac{\varphi}{\sqrt{1+|\boldsymbol{x}|^2}}\in L^2(B_\alpha^c),\nabla\varphi\in L^2(B_\alpha^c)^3\right\}$ is known (cf., e.g., Nédélec [16]).

Define $\mathbf{\Phi}_0=\nabla\phi_0$ in B_α, $\mathbf{\Phi}_0=\nabla\tilde\phi_0$ in B_α^c, then $\mathbf{\Phi}_0\in\mathbf{X}_\alpha(\mathbb{R}^3)$. It follows from (3.2) and (3.7) that

$$\begin{aligned}&(\mu_\alpha^{-1}\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\mathbf{\Phi}_0-\boldsymbol{w}),\nabla\times\boldsymbol{v})_{\mathbb{R}^3}-\mathbf{i}\omega(\sigma_\alpha(\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\mathbf{\Phi}_0-\boldsymbol{w}),\boldsymbol{v})_{B_\alpha}\\ &\quad=\mathbf{i}\omega\mu_0(\mu_0^{-1}-\mu_*^{-1})(\boldsymbol{H}_0-\boldsymbol{H}_0(\boldsymbol{z}),\nabla\times\boldsymbol{v})_{B_\alpha}+\mathbf{i}\omega(\sigma_\alpha(\boldsymbol{E}_0+\mathbf{\Phi}_0-\boldsymbol{F}),\boldsymbol{v})_{B_\alpha}.\end{aligned}$$

Multiply the above equation by μ_0 we have then

$$\begin{aligned}&(\mu_0\mu_\alpha^{-1}\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\mathbf{\Phi}_0-\boldsymbol{w}),\nabla\times\boldsymbol{v})_{\mathbb{R}^3}-\mathbf{i}k(\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\mathbf{\Phi}_0-\boldsymbol{w},\boldsymbol{v})_{B_\alpha}\\ &\quad=\mathbf{i}\omega\mu_0(1-\mu_r^{-1})(\boldsymbol{H}_0(\boldsymbol{x})-\boldsymbol{H}_0(\boldsymbol{z}),\nabla\times\boldsymbol{v})_{B_\alpha}+\mathbf{i}k(\boldsymbol{E}_0+\mathbf{\Phi}_0-\boldsymbol{F},\boldsymbol{v})_{B_\alpha}.\end{aligned} \quad (3.10)$$

It is easy to check that

$$\begin{aligned}|\mathbf{i}\omega\mu_0(\boldsymbol{H}_0-\boldsymbol{H}_0(\boldsymbol{z}),\nabla\times\boldsymbol{v})_{B_\alpha}| &\le C\alpha^{5/2}\|\mathbf{i}\omega\mu_0\boldsymbol{H}_0\|_{W^{1,\infty}(B_\alpha)}\|\nabla\times\boldsymbol{v}\|_{L^2(B_\alpha)}\\ &= C\alpha^{5/2}\|\nabla\times\boldsymbol{E}_0\|_{W^{1,\infty}(B_\alpha)}\|\nabla\times\boldsymbol{v}\|_{L^2(B_\alpha)}.\end{aligned}$$

Now taking $\boldsymbol{v}=\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\mathbf{\Phi}_0-\boldsymbol{w}\in\mathbf{X}_\alpha(\mathbb{R}^3)$ in (3.10), since $\nabla\times\mathbf{\Phi}_0=0$ in $\mathbb{R}^3$ and $\mathbf{\Phi}_0=\nabla\phi_0$ in B_α, we obtain that

$$\begin{aligned}&\|\nabla\times(\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\boldsymbol{w})\|_{L^2(\mathbb{R}^3)}^2+k\|\boldsymbol{E}_\alpha-\boldsymbol{E}_0-\nabla\phi_0-\boldsymbol{w}\|_{L^2(B_\alpha)}^2\\ \le\ & C\alpha^{5/2}|1-\mu_r^{-1}|\|\nabla\times\boldsymbol{E}_0\|_{W^{1,\infty}(B_\alpha)}\|\nabla\times\boldsymbol{v}\|_{L^2(B_\alpha)}+k\|\boldsymbol{E}_0-\boldsymbol{F}+\nabla\phi_0\|_{L^2(B_\alpha)}\|\boldsymbol{v}\|_{L^2(B_\alpha)}\\ \le\ & C\alpha^{5/2}(|1-\mu_r^{-1}|+\alpha\nu)\|\nabla\times\boldsymbol{E}_0\|_{W^{2,\infty}(B_\alpha)}\|\nabla\times\boldsymbol{v}\|_{L^2(B_\alpha)},\end{aligned}$$

where $\nu = k\alpha^2$. Here, we have used

$$\|\boldsymbol{E}_0 - \boldsymbol{F} + \nabla\phi_0\|_{L^2(B_\alpha)} \le C\alpha\|\nabla\times(\boldsymbol{E}_0 - \boldsymbol{F})\|_{L^2(B_\alpha)} \le C\alpha^{9/2}\|\nabla\times\boldsymbol{E}_0\|_{W^{2,\infty}(B_\alpha)} \quad (3.11)$$

and $\|\boldsymbol{v}\|_{L^2(B_\alpha)} \le C\alpha\|\nabla\times\boldsymbol{v}\|_{L^2(B_\alpha)}$, since $\boldsymbol{E}_0 - \boldsymbol{F} + \nabla\phi_0$ and $\boldsymbol{v}$ are divergence free in B_α and have vanishing normal traces on Γ_α. This shows (3.8) and completes the proof. □

We notice that by Green's formula,

$$\begin{aligned}(\mu_0^{-1} - \mu_*^{-1})(\boldsymbol{H}_0(\boldsymbol{z}), \nabla\times\boldsymbol{v})_{B_\alpha} &= (\mu_0^{-1} - \mu_*^{-1})\int_{\Gamma_\alpha}(\boldsymbol{H}_0(\boldsymbol{z})\times\boldsymbol{n})\cdot\boldsymbol{v} \\ &= \int_{\Gamma_\alpha}[\mu_\alpha^{-1}\boldsymbol{H}_0(\boldsymbol{z})\times\boldsymbol{n}]_{\Gamma_\alpha}\cdot\boldsymbol{v},\end{aligned}$$

where $[\cdot]_{\Gamma_\alpha}$ stands for the jump of the function across Γ. Let $\hat{\boldsymbol{w}}(\boldsymbol{\xi}) = \boldsymbol{w}(\boldsymbol{z}+\alpha\boldsymbol{\xi})$, we know from (3.7) that, $\forall\boldsymbol{v}\in\mathbf{X}_1(\mathbb{R}^3)$,

$$(\mu^{-1}\nabla\times\hat{\boldsymbol{w}}, \nabla\times\boldsymbol{v})_{\mathbb{R}^3} - \mathbf{i}\omega\alpha^2(\sigma\hat{\boldsymbol{w}}, \boldsymbol{v})_B = \mathbf{i}\alpha\omega\mu_0\int_\Gamma[\mu^{-1}\boldsymbol{H}_0(\boldsymbol{z})\times\boldsymbol{n}]_\Gamma\cdot\boldsymbol{v} + \mathbf{i}\omega\alpha^2(\sigma_\alpha\boldsymbol{F}(\boldsymbol{z}+\alpha\boldsymbol{\xi}), \boldsymbol{v})_B$$

where $\mu(\boldsymbol{x}) = \mu_*$ in B, $\mu(\boldsymbol{x}) = \mu_0$ in B^c and $\sigma(\boldsymbol{x}) = \sigma_*$ in B, $\sigma(\boldsymbol{x}) = 0$ in B^c.

This motivates us to define the interface problem

$$\left\{\begin{array}{l}\nabla\times\mu^{-1}\nabla\times\boldsymbol{w}_0 - \mathbf{i}\omega\sigma\alpha^2\boldsymbol{w}_0 = \mathbf{i}\omega\sigma\alpha^2\left[\alpha^{-1}\boldsymbol{F}(\boldsymbol{z}+\alpha\boldsymbol{\xi})\right] \text{ in } B\cup B^c, \\ [\boldsymbol{w}_0\times\boldsymbol{n}]_\Gamma = 0, \quad [\mu^{-1}\nabla\times\boldsymbol{w}_0\times\boldsymbol{n}]_\Gamma = -\mathbf{i}\omega(1-\mu_r^{-1})\boldsymbol{H}_0(\boldsymbol{z})\times\boldsymbol{n} \text{ on } \Gamma, \\ \boldsymbol{w}_0(\boldsymbol{\xi}) = O(|\boldsymbol{\xi}|^{-1}) \text{ as } |\boldsymbol{\xi}|\to\infty.\end{array}\right. \quad (3.12)$$

It is easy to check that $\boldsymbol{w}(\boldsymbol{x}) = \alpha\boldsymbol{w}_0\Big(\dfrac{\boldsymbol{x}-\boldsymbol{z}}{\alpha}\Big)$.

The following theorem which is the main result of this section now follows from directly Lemma 3.2.

Theorem 3.1 *There exists a constant C such that*

$$\Big\|\nabla\times\Big(\boldsymbol{E}_\alpha - \boldsymbol{E}_0 - \alpha\boldsymbol{w}_0(\frac{\boldsymbol{x}-\boldsymbol{z}}{\alpha})\Big)\Big\|_{L^2(B_\alpha)} \le C\alpha^{5/2}(|1-\mu_r^{-1}| + \alpha\nu)\|\nabla\times\boldsymbol{E}_0\|_{W^{1,\infty}(B_\alpha)},$$

$$\Big\|\boldsymbol{E}_\alpha - \boldsymbol{E}_0 - \nabla\phi_0 - \alpha\boldsymbol{w}_0\Big(\frac{\boldsymbol{x}-\boldsymbol{z}}{\alpha}\Big)\Big\|_{L^2(B_\alpha)} \le C\alpha^{7/2}(|1-\mu_r^{-1}| + \alpha\nu)\|\nabla\times\boldsymbol{E}_0\|_{W^{1,\infty}(B_\alpha)}.$$

To conclude this section we remark that

$$\begin{aligned}\alpha^{-1}\boldsymbol{F}(\boldsymbol{z}+\alpha\boldsymbol{\xi}) &= \mathbf{i}\omega\mu_0\left(\frac{1}{2}\boldsymbol{H}_0(\boldsymbol{z})\times\boldsymbol{\xi} + \frac{\alpha}{3}D\boldsymbol{H}_0(\boldsymbol{z})\boldsymbol{\xi}\times\boldsymbol{\xi}\right) \\ &= \mathbf{i}\omega\mu_0\left(\frac{1}{2}\sum_{i=1}^3\boldsymbol{H}_0(\boldsymbol{z})_i\boldsymbol{e}_i\times\boldsymbol{\xi} + \frac{\alpha}{3}\sum_{i,j=1}^3 D\boldsymbol{H}_0(\boldsymbol{z})_{ij}\boldsymbol{e}_i\boldsymbol{e}_j^T\boldsymbol{\xi}\times\boldsymbol{\xi}\right),\end{aligned} \quad (3.13)$$

where $D\boldsymbol{H}_0(\boldsymbol{z})_{ij}$ is the (i,j)-th element of the matrix $D\boldsymbol{H}_0(\boldsymbol{z})$. Thus

$$\boldsymbol{w}_0(\boldsymbol{\xi}) = \mathbf{i}\omega\mu_0\left(\frac{1}{2}\sum_{i=1}^3\boldsymbol{H}_0(\boldsymbol{z})_i\boldsymbol{\theta}_i(\boldsymbol{\xi}) + \frac{\alpha}{3}\sum_{i,j=1}^3 D\boldsymbol{H}_0(\boldsymbol{z})_{ij}\boldsymbol{\Psi}_{ij}(\boldsymbol{\xi})\right), \quad (3.14)$$

where $\boldsymbol{\theta}_i(\boldsymbol{\xi})$ is the solution of the following interface problem

$$\begin{cases} \nabla\times\mu^{-1}\nabla\times\boldsymbol{\theta}_i - \mathbf{i}\omega\sigma\alpha^2\boldsymbol{\theta}_i = \mathbf{i}\omega\sigma\alpha^2\boldsymbol{e}_i\times\boldsymbol{\xi} \text{ in } B\cup B^c, \\ [\boldsymbol{\theta}_i\times\boldsymbol{n}]_\Gamma = 0, \quad [\mu^{-1}\nabla\times\boldsymbol{\theta}_i\times\boldsymbol{n}]_\Gamma = -[\mu^{-1}]_\Gamma\boldsymbol{e}_i\times\boldsymbol{n} \text{ on } \Gamma, \\ \boldsymbol{\theta}_i(\boldsymbol{\xi}) = O(|\boldsymbol{\xi}|^{-1}) \text{ as } |\boldsymbol{\xi}|\to\infty, \end{cases} \tag{3.15}$$

and $\boldsymbol{\Psi}_{ij}$ is the solution of

$$\begin{cases} \nabla\times\mu^{-1}\nabla\times\boldsymbol{\Psi}_{ij} - \mathbf{i}\omega\sigma\alpha^2\boldsymbol{\Psi}_{ij} = \mathbf{i}\omega\sigma\alpha^2\xi_j\boldsymbol{e}_i\times\boldsymbol{\xi} \text{ in } B\cup B^c, \\ [\boldsymbol{\Psi}_{ij}\times\boldsymbol{n}]_\Gamma = 0, \quad [\mu^{-1}\nabla\times\boldsymbol{\Psi}_{ij}\times\boldsymbol{n}]_\Gamma = 0, \\ \boldsymbol{\Psi}_{ij}(\boldsymbol{\xi}) = O(|\boldsymbol{\xi}|^{-1}) \text{ as } |\boldsymbol{\xi}|\to\infty. \end{cases} \tag{3.16}$$

Here $\boldsymbol{e}_i$ is unit vector in the x_i direction.

We impose $\nabla\cdot\boldsymbol{\theta}_i = 0$ outside B to make the solution $\boldsymbol{\theta}_i$ unique outside B. In this case by [1, Proposition 3.1] that $\boldsymbol{\theta}_i = O(|\boldsymbol{\xi}|^{-2})$ and $\nabla\times\boldsymbol{\theta}_i = O(|\boldsymbol{\xi}|^{-3})$ as $|\boldsymbol{\xi}|\to\infty$. Similarly, by imposing $\nabla\cdot\boldsymbol{\Psi}_{ij} = 0$ outside B we know that $\nabla\times\boldsymbol{\Psi}_{ij} = O(|\boldsymbol{\xi}|^{-3})$ which implies by integrating (3.16) over B that

$$\begin{aligned} \mathbf{i}\omega\sigma_*\alpha^2\int_B(\boldsymbol{\Psi}_{ij}+\xi_j\boldsymbol{e}_i\times\boldsymbol{\xi})d\boldsymbol{\xi} &= \int_{\partial B}\boldsymbol{n}\times\mu^{-1}\nabla\times\boldsymbol{\Psi}_{ij}d\boldsymbol{\xi} \\ &= \int_{\partial B_R}\boldsymbol{n}\times\mu^{-1}\nabla\times\boldsymbol{\Psi}_{ij}d\boldsymbol{\xi} \\ &\to 0 \text{ as } R\to+\infty, \end{aligned}$$

where B_R is a ball of radius R so that $B\subset B_R$. Thus we obtain

$$\int_B(\boldsymbol{\Psi}_{ij}+\xi_j\boldsymbol{e}_i\times\boldsymbol{\xi})d\boldsymbol{\xi} = 0. \tag{3.17}$$

3.2 Integral Representation Formulas

The integral representation is similar to the Stratton-Chu formula for time harmonic Maxwell equations (cf., e.g., Nédélec [16]).

Lemma 3.3 *Let D be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary Γ_D whose unit outer normal is $\boldsymbol{n}$. For any $\boldsymbol{E}\in\mathbf{H}_{-1}(\mathbf{curl};\mathbb{R}^3\backslash\bar{D})$ satisfying $\nabla\times\nabla\times\boldsymbol{E} = 0, \nabla\cdot\boldsymbol{E} = 0$ in $\mathbb{R}^3\backslash\bar{D}$, we have, for any $\boldsymbol{x}\in\mathbb{R}^3\backslash\bar{D}$,*

$$\boldsymbol{E}(\boldsymbol{x}) = -\nabla\times\int_{\Gamma_D}(\boldsymbol{E}\times\boldsymbol{n})G(\boldsymbol{x},\boldsymbol{y}) - \int_{\Gamma_D}(\nabla\times\boldsymbol{E}\times\boldsymbol{n})G(\boldsymbol{x},\boldsymbol{y}) - \nabla\int_{\Gamma_D}(\boldsymbol{E}\cdot\boldsymbol{n})G(\boldsymbol{x},\boldsymbol{y}),$$

where $G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$ is the fundamental solution of the Laplace equation.

Proof. For the sake of completeness we sketch the proof. Since $\boldsymbol{E}\in\mathbf{H}_{-1}(\mathbf{curl};\mathbb{R}^3\backslash\bar{D})$, for any $\boldsymbol{F}$ such that $\boldsymbol{F} = O(|\boldsymbol{x}|^{-1})$ as $|\boldsymbol{x}|\to\infty$, we can obtain by integrating by parts, the conditions $\nabla\times\nabla\times\boldsymbol{E} = 0, \nabla\cdot\boldsymbol{E} = 0$ in $\mathbb{R}^3\backslash\bar{D}$, and using standard argument that

$$\begin{aligned} (\boldsymbol{E}, -\Delta\boldsymbol{F})_{\mathbb{R}^3\backslash\bar{D}} &= (\boldsymbol{E}, \nabla\times\nabla\times\boldsymbol{F} - \nabla\nabla\cdot\boldsymbol{F})_{\mathbb{R}^3\backslash\bar{D}} \\ &= -\int_{\Gamma_D}(\boldsymbol{E}\times\boldsymbol{n})\cdot\nabla\times\boldsymbol{F} - \int_{\Gamma_D}\nabla\times\boldsymbol{E}\times\boldsymbol{n}\cdot\boldsymbol{F} + \int_{\Gamma_D}(\boldsymbol{E}\cdot\boldsymbol{n})\nabla\cdot\boldsymbol{F}. \end{aligned}$$

Now for $\boldsymbol{x} \in \mathbb{R}^3 \backslash \bar{D}, \boldsymbol{y} \in \Gamma_D$, we choose $\boldsymbol{F} = G(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{e}_j$ and thus $-\Delta F = \delta(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{e}_j$. Then we have

$$\begin{aligned}
& \boldsymbol{E}_j(\boldsymbol{x}) \\
= & -\int_{\Gamma_D} (\boldsymbol{E} \times \boldsymbol{n}) \cdot \nabla_{\boldsymbol{y}} \times (G(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{e}_j) - \int_{\Gamma_D} (\nabla\times \boldsymbol{E} \times \boldsymbol{n})_j G(\boldsymbol{x}, \boldsymbol{y}) + \int_{\Gamma_D} (\boldsymbol{E} \cdot \boldsymbol{n}) \frac{\partial G(\boldsymbol{x}.\boldsymbol{y})}{\partial y_j} \\
= & -\left(\nabla\times \int_{\Gamma_D} (\boldsymbol{E} \times \boldsymbol{n}) G(\boldsymbol{x}, \boldsymbol{y})\right)_j - \int_{\Gamma_D} (\nabla\times \boldsymbol{E} \times \boldsymbol{n})_j G(\boldsymbol{x}, \boldsymbol{y}) - \frac{\partial}{\partial x_j} \int_{\Gamma_D} (\boldsymbol{E} \cdot \boldsymbol{n}) G(\boldsymbol{x}, \boldsymbol{y}),
\end{aligned}$$

where we have used the fact that

$$(\boldsymbol{E}(\boldsymbol{y}) \times \boldsymbol{n}) \cdot \nabla_{\boldsymbol{x}} \times (G(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{e}_j) = -(\nabla_{\boldsymbol{x}} \times (G(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{E}(\boldsymbol{y}) \times \boldsymbol{n}))_j.$$

This completes the proof. □

The following lemma will be useful in deriving the asymptotic formula in next subsection. Recall that $\boldsymbol{H}_\alpha = \frac{1}{\mathbf{i}\omega\mu_\alpha} \nabla\times \boldsymbol{E}_\alpha, \boldsymbol{H}_0 = \frac{1}{\mathbf{i}\omega\mu_0} \nabla\times \boldsymbol{E}_0$.

Lemma 3.4 *Let $\tilde{\boldsymbol{H}}_\alpha = \boldsymbol{H}_\alpha - \boldsymbol{H}_0$. Then we have, for $\boldsymbol{x} \in B_\alpha^c$,*

$$\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{x}) = \int_{B_\alpha} \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) \times \nabla\times \tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y})\, d\boldsymbol{y} + (1 - \frac{\mu_*}{\mu_0}) \int_{B_\alpha} (\boldsymbol{H}_\alpha \cdot \nabla_{\boldsymbol{y}}) \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}.$$

Proof. It is easy to check that $\nabla\times \tilde{\boldsymbol{H}}_\alpha = 0$ and $\nabla\cdot \tilde{\boldsymbol{H}}_\alpha = 0$ in B_α^c. By the representation formula in Lemma 3.3 we have

$$\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{x}) = -\nabla\times \int_{\Gamma_\alpha} (\tilde{\boldsymbol{H}}_\alpha^+ \times \boldsymbol{n}) G(\boldsymbol{x}, \boldsymbol{y}) - \nabla \int_{\Gamma_\alpha} (\tilde{\boldsymbol{H}}_\alpha^+ \cdot \boldsymbol{n}) G(\boldsymbol{x}, \boldsymbol{y}),$$

where $\tilde{\boldsymbol{H}}_\alpha^+ = \tilde{\boldsymbol{H}}_\alpha|_{B_\alpha^c}$. Denote $\tilde{\boldsymbol{H}}_\alpha^- = \tilde{\boldsymbol{H}}_\alpha|_{B_\alpha}$. Similar notation applies to $\boldsymbol{E}_\alpha^\pm$. By the interface condition $[\tilde{\boldsymbol{E}}_\alpha \times \boldsymbol{n}]_{\Gamma_\alpha} = 0$, we have

$$\begin{aligned}
\tilde{\boldsymbol{H}}_\alpha^+ \cdot \boldsymbol{n} = \frac{1}{\mathbf{i}\omega\mu_0} \nabla \times \boldsymbol{E}_\alpha^+ \cdot \boldsymbol{n} - \boldsymbol{H}_0 \cdot \boldsymbol{n} \quad &= \quad \frac{1}{\mathbf{i}\omega\mu_0} \mathrm{div}_{\Gamma_\alpha} (\boldsymbol{E}_\alpha^+ \times \boldsymbol{n}) - \boldsymbol{H}_0 \cdot \boldsymbol{n} \\
&= \quad \frac{\mu_*}{\mu_0} \boldsymbol{H}_\alpha^- \cdot \boldsymbol{n} - \boldsymbol{H}_0 \cdot \boldsymbol{n}.
\end{aligned}$$

Then since $[\tilde{\boldsymbol{H}}_\alpha \times \boldsymbol{n}]_{\Gamma_\alpha} = 0$, we have

$$\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{x}) = -\nabla\times \int_{\Gamma_\alpha} (\tilde{\boldsymbol{H}}_\alpha^- \times \boldsymbol{n}) G(\boldsymbol{x}, \boldsymbol{y}) - \nabla \int_{\Gamma_\alpha} (\frac{\mu_*}{\mu_0} \boldsymbol{H}_\alpha^- \cdot \boldsymbol{n} - \boldsymbol{H}_0 \cdot \boldsymbol{n}) G(\boldsymbol{x}, \boldsymbol{y}). \tag{3.18}$$

For the first term,

$$\begin{aligned}
& -\nabla\times \int_{\Gamma_\alpha} (\tilde{\boldsymbol{H}}_\alpha^- \times \boldsymbol{n}) G(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y} \\
= \quad & \nabla\times \int_{B_\alpha} \nabla_{\boldsymbol{y}} \times (\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y}))\, d\boldsymbol{y} \\
= \quad & \nabla\times \int_{B_\alpha} (G(\boldsymbol{x}, \boldsymbol{y}) \nabla\times \tilde{\boldsymbol{H}}_\alpha + \nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) \times \tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y}))\, d\boldsymbol{y} \\
= \quad & \int_{B_\alpha} \Big(\nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) \times \nabla\times \tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y}) + (\tilde{\boldsymbol{H}}_\alpha \cdot \nabla_{\boldsymbol{x}}) \nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y})\Big)\, d\boldsymbol{y}, \qquad (3.19)
\end{aligned}$$

where we have used the identity

$$\nabla\times(\boldsymbol{u}\times\boldsymbol{v})=\boldsymbol{u}(\nabla\cdot\boldsymbol{v})-(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}-\boldsymbol{v}(\nabla\cdot\boldsymbol{u}),$$

and the fact that $\nabla_{\boldsymbol{x}}\cdot\nabla_{\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})=-\Delta_{\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})=0$. For the second term, we first notice that

$$-\nabla\int_{\Gamma_\alpha}(\frac{\mu_*}{\mu_0}\boldsymbol{H}_\alpha^-\cdot\boldsymbol{n}-\boldsymbol{H}_0\cdot\boldsymbol{n})G(\boldsymbol{x},\boldsymbol{y})=-\frac{\mu_*}{\mu_0}\nabla\int_{\Gamma_\alpha}\tilde{\boldsymbol{H}}_\alpha^-\cdot\boldsymbol{n}G(\boldsymbol{x},\boldsymbol{y})+(1-\frac{\mu_*}{\mu_0})\int_{\Gamma_\alpha}\boldsymbol{H}_0\cdot\boldsymbol{n}G(\boldsymbol{x},\boldsymbol{y}).$$

By integration by parts we have

$$\begin{aligned}\nabla\int_{\Gamma_\alpha}\tilde{\boldsymbol{H}}_\alpha^-\cdot\boldsymbol{n}G(\boldsymbol{x},\boldsymbol{y}) &= \nabla\int_{B_\alpha}\nabla_{\boldsymbol{y}}\cdot(G(\boldsymbol{x},\boldsymbol{y})\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y}))\,d\boldsymbol{y}\\ &= \nabla\int_{B_\alpha}\nabla_{\boldsymbol{y}}G(\boldsymbol{x},\boldsymbol{y})\cdot\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y})+G(\boldsymbol{x},\boldsymbol{y})\nabla\cdot\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y})\,d\boldsymbol{y}\\ &= \int_{B_\alpha}(\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}})\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}.\end{aligned}$$

Similarly

$$\nabla\int_{\Gamma_\alpha}(\boldsymbol{H}_0\cdot\boldsymbol{n})G(\boldsymbol{x},\boldsymbol{y})=\int_{B_\alpha}(\boldsymbol{H}_0(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}})\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}.$$

Thus

$$\begin{aligned}&-\nabla\int_{\Gamma_\alpha}(\frac{\mu_*}{\mu_0}\boldsymbol{H}_\alpha^-\cdot\boldsymbol{n}-\boldsymbol{H}_0\cdot\boldsymbol{n})G(\boldsymbol{x},\boldsymbol{y})\\ =\;&-\frac{\mu_*}{\mu_0}\int_{B_\alpha}(\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}})\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\\ &+(1-\frac{\mu_*}{\mu_0})\int_{B_\alpha}(\boldsymbol{H}_0(\boldsymbol{y})\cdot\nabla_{\boldsymbol{y}})\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}.\end{aligned}\tag{3.20}$$

This completes the proof by substituting (3.19)-(3.20) into (3.18). □

3.3 Asymptotic Formulas

In this subsection we prove the following theorem which is the main result of this section.

Theorem 3.2 *Let $\nu=k\alpha^2$ be of order one. For $\boldsymbol{x}$ away from the location $\boldsymbol{z}$ of the inclusion, we have*

$$\begin{aligned}\boldsymbol{H}_\alpha(\boldsymbol{x})-\boldsymbol{H}_0(\boldsymbol{x}) &= \mathbf{i}\nu\alpha^2\left[\frac{1}{2}\sum_{i=1}^3\boldsymbol{H}_0(\boldsymbol{z})_i\int_B\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\times(\boldsymbol{\theta}_i+\boldsymbol{e}_i\times\boldsymbol{\xi})d\boldsymbol{\xi}\right]\\ &+ \mathbf{i}\nu\alpha^3\left[\frac{1}{2}\sum_{i=1}^3\boldsymbol{H}_0(\boldsymbol{z})_i\int_B D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\xi}\times(\boldsymbol{\theta}_i+\boldsymbol{e}_i\times\boldsymbol{\xi})d\boldsymbol{\xi}\right]\\ &+ \alpha^3\Big(1-\frac{\mu_0}{\mu_*}\Big)\left[\sum_{i=1}^3\boldsymbol{H}_0(\boldsymbol{z})_i\int_B D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\Big(\boldsymbol{e}_i+\frac{1}{2}\nabla\times\boldsymbol{\theta}_i\Big)d\boldsymbol{\xi}\right]+R,\end{aligned}$$

where

$$|R| \le C\nu\alpha^3|1-\mu_r^{-1}|\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)} + C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)},$$

uniformly in $\boldsymbol{x}$ *in any compact set away from* $\boldsymbol{z}$.

Proof. The proof starts from the integral representation formula in Lemma 3.4. We first consider the first term in the integral representation in Lemma 3.4. By Theorem 3.1 we know that

$$\|\boldsymbol{E}_\alpha - \boldsymbol{E}_0 - \nabla\phi_0 - \alpha\boldsymbol{w}_0(\frac{\boldsymbol{x}-\boldsymbol{z}}{\alpha})\|_{L^2(B_\alpha)} \le C\alpha^{7/2}(|1-\mu_r^{-1}|+\alpha\nu)\|\nabla\times\boldsymbol{E}_0\|_{W^{2,\infty}(B_\alpha)}. \quad (3.21)$$

With the help that $\nabla\times\boldsymbol{H}_0 = 0$ and $\nabla\times\boldsymbol{H}_\alpha = \sigma\boldsymbol{E}_\alpha$ in B_α, we have

$$\begin{aligned}
& \int_{B_\alpha} \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y}) \times \nabla\times\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y})\,d\boldsymbol{y} \\
= \;& \sigma_* \int_{B_\alpha} \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y}) \times \boldsymbol{E}_\alpha(\boldsymbol{y})\,d\boldsymbol{y} \\
= \;& \sigma_* \int_{B_\alpha} \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y}) \times \Big(\boldsymbol{E}_\alpha(\boldsymbol{y}) - \boldsymbol{E}_0(\boldsymbol{y}) - \nabla\phi_0(\boldsymbol{y}) - \alpha\boldsymbol{w}_0(\frac{\boldsymbol{y}-\boldsymbol{z}}{\alpha})\Big)\,d\boldsymbol{y} \\
+ \;& \sigma_* \int_{B_\alpha} \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y}) \times (\boldsymbol{E}_0(\boldsymbol{y}) + \nabla\phi_0(\boldsymbol{y}) - \boldsymbol{F}(\boldsymbol{y}))\,d\boldsymbol{y} \\
+ \;& \sigma_* \int_{B_\alpha} (\nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y}) - \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{z}) - D^2_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})) \times \Big(\boldsymbol{F}(\boldsymbol{y}) + \alpha\boldsymbol{w}_0(\frac{\boldsymbol{y}-\boldsymbol{z}}{\alpha})\Big)\,d\boldsymbol{y} \\
+ \;& \sigma_* \int_{B_\alpha} (\nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{z}) + D^2_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})) \times \Big(\boldsymbol{F}(\boldsymbol{y}) + \alpha\boldsymbol{w}_0(\frac{\boldsymbol{y}-\boldsymbol{z}}{\alpha})\Big)\,d\boldsymbol{y} \\
=: \;& \mathrm{I}_1 + \cdots + \mathrm{I}_4.
\end{aligned}$$

By (3.21), we have

$$\begin{aligned}
|\,\mathrm{I}_1| &\le C\alpha^5(|1-\mu_r^{-1}|+\alpha\nu)\sigma_*\|\nabla\times\boldsymbol{E}_0\|_{W^{2,\infty}(B_\alpha)} \\
&\le Ck\alpha^5|1-\mu_r^{-1}|\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)} + C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}.
\end{aligned}$$

By (3.13) we have $|\,\mathrm{I}_2| \le C\alpha^6\sigma_*\|\nabla\times\boldsymbol{E}_0\|_{W^{2,\infty}(B_\alpha)} \le C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}$. Similarly, by using (3.4) and (3.14) we can show $|\mathrm{I}_3| \le C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}$. For the remaining term we first observe that

$$\mathrm{I}_4 = \mathbf{i}\alpha^4\sigma_* \int_B (\nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{z}) + \alpha D^2_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\xi}) \times (\alpha^{-1}\boldsymbol{F}(z+\alpha\boldsymbol{\xi}) + \boldsymbol{w}_0(\boldsymbol{\xi}))\,d\boldsymbol{\xi}.$$

On the other hand,

$$\begin{aligned}
\alpha^{-1}\boldsymbol{F}(z+\alpha\boldsymbol{\xi}) + \boldsymbol{w}_0(\boldsymbol{\xi}) \;&=\; \mathbf{i}\omega\mu_0\Big[\frac{1}{2}\sum_{i=1}^{3}\boldsymbol{H}_0(\boldsymbol{z})_i(\boldsymbol{e}_i\times\boldsymbol{\xi}+\boldsymbol{\theta}_i) \\
&+\; \frac{\alpha}{3}\sum_{i,j=1}^{3} D\boldsymbol{H}_0(z)_{ij}(\xi_j\boldsymbol{e}_i\times\boldsymbol{\xi}+\boldsymbol{\Psi}_{ij})\Big],
\end{aligned}$$

which implies after using (3.17)

$$\begin{aligned}
\mathrm{I}_4 &= \mathbf{i}k\alpha^4\left[\frac{1}{2}\sum_{i=1}^{3}\boldsymbol{H}_0(\boldsymbol{z})_i\int_B \nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\times(\boldsymbol{e}_i\times\boldsymbol{\xi}+\boldsymbol{\theta}_i)d\,\boldsymbol{\xi}\right]\\
&+ \mathbf{i}k\alpha^5\left[\frac{1}{2}\sum_{i=1}^{3}\boldsymbol{H}_0(\boldsymbol{z})_i\int_B D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\xi}\times(\boldsymbol{e}_i\times\boldsymbol{\xi}+\boldsymbol{\theta}_i)d\,\boldsymbol{\xi}\right]+R_1,
\end{aligned}$$

where $|R_1|\le C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}$. This shows that

$$\begin{aligned}
&\int_{B_\alpha}\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\times\nabla\times\tilde{\boldsymbol{H}}_\alpha(\boldsymbol{y})d\,\boldsymbol{y}\\
=\ & \mathbf{i}k\alpha^4\left[\frac{1}{2}\sum_{i=1}^{3}\boldsymbol{H}_0(\boldsymbol{z})_i\int_B \nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\times(\boldsymbol{e}_i\times\boldsymbol{\xi}+\boldsymbol{\theta}_i)d\,\boldsymbol{\xi}\right]\\
+\ & \mathbf{i}k\alpha^5\left[\frac{1}{2}\sum_{i=1}^{3}\boldsymbol{H}_0(\boldsymbol{z})_i\int_B D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\xi}\times(\boldsymbol{e}_i\times\boldsymbol{\xi}+\boldsymbol{\theta}_i)d\,\boldsymbol{\xi}\right]+R_2,
\end{aligned}\tag{3.22}$$

where $|R_2|\le Ck\alpha^5|1-\mu_r^{-1}|\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}+C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}$.

Now we turn to the second term in Lemma 3.4. From Theorem 3.1 we know that

$$\left\|\boldsymbol{H}_\alpha-\frac{\mu_0}{\mu_*}\boldsymbol{H}_0-\frac{\alpha}{\mathbf{i}\omega\mu_*}\nabla_{\boldsymbol{x}}\times\boldsymbol{w}_0(\frac{\boldsymbol{x}-\boldsymbol{z}}{\alpha})\right\|_{L^2(B_\alpha)}\le C\alpha^{5/2}\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}.\tag{3.23}$$

Let $\boldsymbol{H}_0^*(\boldsymbol{\xi})=\dfrac{1}{\mathbf{i}\omega\mu_0}\nabla_{\boldsymbol{\xi}}\times\boldsymbol{w}_0(\boldsymbol{\xi})$. Then

$$\begin{aligned}
&\int_{B_\alpha}(\boldsymbol{H}_\alpha\cdot\nabla_{\boldsymbol{y}})\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\\
=\ & -\int_{B_\alpha}D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\boldsymbol{H}_\alpha(\boldsymbol{y})\,d\boldsymbol{y}\\
=\ & -\int_{B_\alpha}D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\left(\boldsymbol{H}_\alpha(\boldsymbol{y})-\frac{\mu_0}{\mu_*}\boldsymbol{H}_0(\boldsymbol{y})-\frac{\mu_0}{\mu_*}\boldsymbol{H}_0^*(\frac{\boldsymbol{y}-\boldsymbol{z}}{\alpha})\right)d\boldsymbol{y}\\
& -\frac{\mu_0}{\mu_*}\int_{B_\alpha}(D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})-D^2_{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{z}))(\boldsymbol{H}_0(\boldsymbol{y})+\boldsymbol{H}_0^*(\frac{\boldsymbol{y}-\boldsymbol{z}}{\alpha}))\,d\boldsymbol{y}\\
& -\frac{\mu_0}{\mu_*}\int_{B_\alpha}D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})(\boldsymbol{H}_0(\boldsymbol{y})-\boldsymbol{H}_0(\boldsymbol{z}))\,d\boldsymbol{y}\\
& -\frac{\mu_0}{\mu_*}\int_{B_\alpha}D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})(\boldsymbol{H}_0(\boldsymbol{z})+\boldsymbol{H}_0^*(\frac{\boldsymbol{y}-\boldsymbol{z}}{\alpha}))\,d\boldsymbol{y}\\
=:\ & \mathrm{II}_1+\cdots+\mathrm{II}_4.
\end{aligned}$$

It is easy to see from (3.23) that $|\mathrm{II}_1|\le C\alpha^4\|\boldsymbol{H}_0\|_{W^{1,\infty}(B_\alpha)}$. By (3.14) we know that

$$\|H_0^*(\frac{\boldsymbol{y}-\boldsymbol{z}}{\alpha})\|_{L^2(B_\alpha)}\le C\alpha^{3/2}\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)},$$

which implies $|\,\mathrm{II}_2| \le C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}$. Similarly, we have $|\,\mathrm{II}_3| \le C\alpha^4\|\boldsymbol{H}_0\|_{W^{1,\infty}(B_\alpha)}$. Finally, by (3.14), we have

$$\mathrm{II}_4 = -\frac{\mu_0}{\mu_*}\alpha^3\sum_{i=1}^3 \boldsymbol{H}_0(z)_i\int_B D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\Big(\boldsymbol{e}_i+\frac{1}{2}\nabla\times\boldsymbol{\theta}_i\Big)d\,\boldsymbol{\xi}+R_3,$$

where $|R_3| \le C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}$. Therefore,

$$\begin{aligned}&\int_{B_\alpha}(\boldsymbol{H}_\alpha\cdot\nabla_{\boldsymbol{y}})\nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\\ =\;& -\frac{\mu_0}{\mu_*}\alpha^3\sum_{i=1}^3 \boldsymbol{H}_0(z)_i\int_B D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\Big(\boldsymbol{e}_i+\frac{1}{2}\nabla\times\boldsymbol{\theta}_i\Big)d\,\boldsymbol{\xi}+R_4 \qquad (3.24)\end{aligned}$$

with $|R_4| \le C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}$. This completes the proof by substituting (3.24) and (3.22) into the integral representation formula in Lemma 3.4. □

Assume that $\mu_0=\mu_*$. Similar argument leading to (3.17) yields from (3.15) that $\int_B(\boldsymbol{\theta}_i+\boldsymbol{e}_i\times\boldsymbol{\xi})d\,\boldsymbol{\xi}=0$. Thus the asymptotic formula derived in Theorem 3.2 reduces in this case to

$$\begin{aligned}\boldsymbol{H}_\alpha(\boldsymbol{x})-\boldsymbol{H}_0(\boldsymbol{x}) &\simeq \mathbf{i}k\alpha^5\left[\frac{1}{2}\sum_{i=1}^3\boldsymbol{H}_0(\boldsymbol{z})_i\int_B D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\xi}\times(\boldsymbol{e}_i\times\boldsymbol{\xi}+\boldsymbol{\theta}_i)d\,\boldsymbol{\xi}\right]\\ &= \mathbf{i}k\alpha^5 D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\mathbb{M}^T\boldsymbol{H}_0(z). \qquad (3.25)\end{aligned}$$

The remainder now satisfies $|R| \le C\alpha^4\|\boldsymbol{H}_0\|_{W^{2,\infty}(B_\alpha)}$. Here $\mathbb{M}$ is the 3×3 matrix whose column vectors are $\frac{1}{2}\int_B\boldsymbol{\xi}\times(\boldsymbol{\theta}_1+\boldsymbol{e}_1\times\boldsymbol{\xi})d\boldsymbol{\xi}$, $\frac{1}{2}\int_B\boldsymbol{\xi}\times(\boldsymbol{\theta}_2+\boldsymbol{e}_2\times\boldsymbol{\xi})d\boldsymbol{\xi}$, and $\frac{1}{2}\int_B\boldsymbol{\xi}\times(\boldsymbol{\theta}_3+\boldsymbol{e}_3\times\boldsymbol{\xi})d\boldsymbol{\xi}$.

Now we assume that $\boldsymbol{J}_0$ is a dipole source whose position is denoted by $\boldsymbol{s}$

$$\boldsymbol{J}_0(\boldsymbol{x}) = \nabla\times\big(\boldsymbol{p}\,G(\boldsymbol{x},\boldsymbol{s})\big), \qquad (3.26)$$

where the unit vector $\boldsymbol{p}$ is the direction of the magnetic dipole. In the absence of any inclusion, the magnetic field $\boldsymbol{H}_0$ due to $\boldsymbol{J}_0(\boldsymbol{x})$ is given by

$$\boldsymbol{H}_0(\boldsymbol{x}) = \nabla\times\nabla\times(\boldsymbol{p}G(\boldsymbol{x},\boldsymbol{s})) = D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{s})\boldsymbol{p}. \qquad (3.27)$$

The asymptotic formula (3.25) can be rewritten as

$$\boldsymbol{q}\cdot(\boldsymbol{H}_\alpha-\boldsymbol{H}_0)(\boldsymbol{x}) \simeq \mathbf{i}k\alpha^5\left(D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\boldsymbol{q}\right)^T\mathbb{M}^T\left(D^2_{\boldsymbol{x}}G(\boldsymbol{z},\boldsymbol{s})\boldsymbol{p}\right). \qquad (3.28)$$

On the other hand, $\mathbb{M}^T$ is a complex tensor. Writing

$$\mathbb{M}^T = \Re e\mathbb{M}^T + i\Im m\mathbb{M}^T,$$

we obtain

$$\Re e(\boldsymbol{q}\cdot(\boldsymbol{H}_\alpha-\boldsymbol{H}_0)(\boldsymbol{x})) \simeq -k\alpha^5\left(D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\boldsymbol{q}\right)^T(\Im m\,\mathbb{M}^T)\left(D^2_{\boldsymbol{x}}G(\boldsymbol{z},\boldsymbol{s})\boldsymbol{p}\right),$$

and

$$\Im m(\boldsymbol{q}\cdot(\boldsymbol{H}_\alpha-\boldsymbol{H}_0)(\boldsymbol{x})) \simeq k\alpha^5\left(D^2_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{z})\boldsymbol{q}\right)^T(\Re e\,\mathbb{M}^T)\left(D^2_{\boldsymbol{x}}G(\boldsymbol{z},\boldsymbol{s})\boldsymbol{p}\right).$$

4 Localization and Characterization

Let the $N \times M$ response matrix $\mathbf{A} = (A_{nm})_{n=1,\dots,N, m=1,\dots,M}$ be defined by

$$A_{nm} := \big(\boldsymbol{H}_\alpha^{(m)}(\boldsymbol{r}_n) - \boldsymbol{H}_0^{(m)}(\boldsymbol{r}_n)\big) \cdot \boldsymbol{q}.$$

We assume that $N \geq P$, i.e., there are more receivers than sources. As in [4], in order to locate the conductive inclusion $\boldsymbol{z} + \alpha B$ we can use the MUSIC imaging functional. We focus on formula (3.28) and define the MUSIC imaging functional for a search point $\boldsymbol{z}^S$ by

$$\mathcal{I}_{\mathrm{MU}}(\boldsymbol{z}^S) := \frac{1}{\sum_{l=1}^{3} \| \left(\mathbf{I}_N - \mathbf{P}^{\mathrm{imag}}\right) \left(D_{\boldsymbol{x}}^2 G(\boldsymbol{r}_1, \boldsymbol{z}^s)\boldsymbol{q} \cdot \boldsymbol{e}_l, \dots, D_{\boldsymbol{x}}^2 G(\boldsymbol{r}_N, \boldsymbol{z})\boldsymbol{q} \cdot \boldsymbol{e}_l)^T \|},$$

where $\mathbf{P}^{\mathrm{imag}}$ is the orthogonal projection on the image of $\mathbf{A}$ and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is an orthonormal basis of $\mathbb{R}^3$.

Once the inclusion is located we can compute from the response matrix $\mathbf{A}$ the tensor $\mathbb{M}$ associated with the inclusion by a least-square method. Given the location of the inclusion, we minimize the discrepancy between the computed and the measured response matrices.

5 Noisy Measurements

In this section we consider that there are M sources and N receivers. The measures are noisy, which means that the magnetic field measured by a receiver is corrupted by an additive noise that can be described in terms of a complex Gaussian random variable with mean zero and variance $2\sigma_{\mathrm{n}}^2$ (in other words, the real and imaginary parts of the measurement noise are independent and follow a Gaussian distribution with mean zero and variance σ_{n}^2). The recorded noises are independent from each other.

5.1 Hadamard Technique

Standard acquisition. In the standard acquisition scheme, the response matrix is measured during a sequence of M experiments. In the mth experience, $m = 1, \dots, M$, the mth source (located at $\boldsymbol{s}_m$) generates the magnetic dipole $\boldsymbol{J}_0^{(m)}(\boldsymbol{r}) = \nabla \times (\boldsymbol{p}G(\boldsymbol{r}, \boldsymbol{s}_m))$ and the N receivers (located at $\boldsymbol{r}_n$, $n = 1, \dots, N$) record the magnetic field in the $\boldsymbol{q}$ direction which means that they measure

$$A_{\mathrm{meas},nm} = A_{0,nm} + W_{nm}, \quad n = 1, \dots, N, \quad m = 1, \dots, M,$$

which gives the matrix

$$\mathbf{A}_{\mathrm{meas}} = \mathbf{A}_0 + \mathbf{W}, \tag{5.1}$$

where $\mathbf{A}_0$ is the unperturbed response matrix

$$A_{0nm} := \Big(\boldsymbol{H}_\alpha^{(m)}(\boldsymbol{r}_n) - \boldsymbol{H}_0^{(m)}(\boldsymbol{r}_n)\Big) \cdot \boldsymbol{q},$$

and W_{nm} are independent Gaussian random variables with mean zero and variance σ_{n}^2. Here, $\boldsymbol{H}_\alpha^{(m)}(\boldsymbol{r}_n)$ and $\boldsymbol{H}_0^{(m)}(\boldsymbol{r}_n)$ are the magnetic fields generated by a magnetic dipole at $\boldsymbol{s}_m$ and measured at the receiver $\boldsymbol{r}_n$, respectively in the presence and absence of the inclusion.

The Hadamard technique is a noise reduction technique in the presence of additive noise that uses the structure of Hadamard matrices.

Definition 5.1 *A Hadamard matrix* $\mathbf{H}$ *of order* M *is a* $M \times M$ *matrix whose elements are* -1 *or* $+1$ *and such that* $\mathbf{H}^T\mathbf{H} = M\mathbf{I}$. *Here* $\mathbf{I}$ *is the* 3×3 *identity matrix.*

Hadamard matrices do not exist for all M. A necessary condition for the existence is that $M = 1, 2$ or a multiple of 4. A sufficient condition is that M is a power of two. Explicit examples are known for all M multiple of 4 up to $M = 664$ [19].

Hadamard acquisition. In the Hadamard acquisition scheme, the response matrix is measured during a sequence of M experiments. In the mth experience, $m = 1, \ldots, M$, all sources generate magnetic dipoles, the m' source generating $H_{mm'}\boldsymbol{J}_0^{(m')}(\boldsymbol{r})$. This means that we use all sources to their maximal emission capacity with a specific coding of their signs. The N receivers record the magnetic field in the $\boldsymbol{q}$ direction, which means that they measure

$$B_{\text{meas},nm} = \sum_{m'=1}^{M} H_{mm'}A_{0,nm'} + W_{nm} = (\mathbf{A}_0\mathbf{H}^T)_{nm} + W_{nm}, \quad n = 1, \ldots, N, \quad m = 1, \ldots, M,$$

which gives the matrix

$$\mathbf{B}_{\text{meas}} = \mathbf{A}_0\mathbf{H}^T + \mathbf{W},$$

where $\mathbf{A}_0$ is the unperturbed response matrix and W_{nm} are independent Gaussian random variables with mean zero and variance σ_n^2. The measured response matrix $\mathbf{A}_{\text{meas}}$ is obtained by right multiplying the matrix $\mathbf{B}_{\text{meas}}$ by the matrix $\frac{1}{M}\mathbf{H}$:

$$\mathbf{A}_{\text{meas}} := \frac{1}{M}\mathbf{B}_{\text{meas}}\mathbf{H} = \frac{1}{M}\mathbf{A}_0\mathbf{H}^T\mathbf{H} + \frac{1}{M}\mathbf{W}\mathbf{H},$$

which gives

$$\mathbf{A}_{\text{meas}} = \mathbf{A}_0 + \widetilde{\mathbf{W}}, \qquad \widetilde{\mathbf{W}} = \frac{1}{M}\mathbf{W}\mathbf{H}. \tag{5.2}$$

The interest of the Hadamard technique is that the new noise matrix $\widetilde{\mathbf{W}}$ has independent entries with Gaussian statistics, mean zero, and variance σ_n^2/M:

$$\begin{aligned}
\mathbb{E}[\widetilde{W}_{nm}\widetilde{W}_{n'm'}] &= \frac{1}{M^2}\sum_{q,q'=1}^{M} H_{qm}H_{q'm'}\mathbb{E}[W_{nq}W_{n'q'}] = \frac{\sigma_\text{n}^2}{M^2}\sum_{q,q'=1}^{M} H_{qm}H_{q'm'}\delta_{nn'}\delta_{qq'} \\
&= \frac{\sigma_\text{n}^2}{M^2}\sum_{q=1}^{M} H_{qm}(H^T)_{m'q}\delta_{nn'}\frac{\sigma_\text{n}^2}{M^2}(\mathbf{H}^T\mathbf{H})_{m'm}\delta_{nn'} \\
&= \frac{\sigma_\text{n}^2}{M}\delta_{mm'}\delta_{nn'},
\end{aligned}$$

where $\mathbb{E}$ stands for the expectation and δ_{mn} is the Kronecker delta symbol. This gain of a factor M in the signal-to-noise ratio is called the Hadamard advantage.

5.2 Singular Values of a Noisy Matrix

We consider here the situation in which the measured response matrix consists of independent noise coefficients with mean zero and variance σ_n^2/M and the number of receivers is larger than the number of sources $N \geq M$. This is the case when the response matrix is acquired with the Hadamard technique and there is no inclusion in the medium.

We denote by $\sigma_1^{(M)} \geq \sigma_2^{(M)} \geq \sigma_3^{(M)} \geq \cdots \geq \sigma_M^{(M)}$ the singular values of the response matrix $\mathbf{A}$ sorted by decreasing order and by $\Lambda^{(M)}$ the corresponding integrated density of states defined by

$$\Lambda^{(M)}([a,b]) := \frac{1}{M}\mathrm{Card}\Big\{l = 1,\dots,M\,,\,\sigma_l^{(M)} \in [a,b]\Big\}, \quad \text{for any } a < b.$$

$\Lambda^{(M)}$ is a counting measure which consists of a sum of Dirac masses:

$$\Lambda^{(M)} = \frac{1}{M}\sum_{j=1}^{M}\delta_{\sigma_j^{(M)}}.$$

For large N and M with $N/M = \gamma \geq 1$ fixed we have the following results.

Proposition 5.1 a) *The random measure $\Lambda^{(M)}$ almost surely converges to the deterministic absolutely continuous measure Λ with compact support:*

$$\Lambda([\sigma_u,\sigma_v]) = \int_{\sigma_u}^{\sigma_v}\frac{1}{\sigma_{\mathrm{n}}}\rho_\gamma\Big(\frac{\sigma}{\sigma_{\mathrm{n}}}\Big)d\sigma, \qquad 0 \leq \sigma_u \leq \sigma_v \tag{5.3}$$

where ρ_γ is the deformed quarter-circle law given by

$$\rho_\gamma(\sigma)\begin{cases}\dfrac{1}{\pi\sigma}\sqrt{\big((\gamma^{1/2}+1)^2-\sigma^2\big)\big(\sigma^2-(\gamma^{1/2}-1)^2\big)} & \text{if } \gamma^{1/2}-1 < \sigma \leq \gamma^{1/2}+1,\\ 0 & \text{otherwise.}\end{cases} \tag{5.4}$$

b) *The normalized l^2-norm of the singular values satisfies*

$$M\Big[\frac{1}{M}\sum_{j=1}^{M}(\sigma_j^{(M)})^2 - \gamma\sigma_{\mathrm{n}}^2\Big] \overset{M\to\infty}{\longrightarrow} \sqrt{2\gamma}\sigma_{\mathrm{n}}^2 Z \text{ in distribution,} \tag{5.5}$$

where Z follows a Gaussian distribution with mean zero and variance one.

c) *The maximal singular value satisfies*

$$\sigma_1^{(M)} \simeq \sigma_{\mathrm{n}}\Big[\gamma^{1/2}+1+\frac{1}{2M^{2/3}}\big(1+\gamma^{-1/2}\big)^{1/3}Z_1 + o\big(\frac{1}{M^{2/3}}\big)\Big] \text{ in distribution,} \tag{5.6}$$

where Z_1 follows a type-1 Tracy Widom distribution.

The type-1 Tracy-Widom distribution has the pdf p_{TW1}:

$$\mathbb{P}(Z_1 \leq z) = \int_{-\infty}^{z} p_{\mathrm{TW1}}(x)dx = \exp\Big(-\frac{1}{2}\int_z^\infty \varphi(x) + (x-z)\varphi^2(x)dx\Big),$$

where φ is the solution of the Painlevé equation

$$\varphi''(x) = x\varphi(x) + 2\varphi(x)^3, \qquad \varphi(x) \overset{x\to+\infty}{\simeq} \mathrm{Ai}(x), \tag{5.7}$$

Ai being the Airy function. The expectation of Z_1 is $\mathbb{E}[Z_1] \simeq -1.21$ and its variance is $\mathrm{Var}(Z_1) \simeq 1.61$.

Proof. Point a) is Marcenko-Pastur result [15]. Point b) follows from the expression of the normalized l^2-norm of the singular values in terms of the entries of the matrix:

$$\frac{1}{M}\sum_{j=1}^{M}(\sigma_j^{(M)})^2 = \frac{1}{M}\mathrm{tr}\big(\mathbf{A}^T\mathbf{A}\big) = \frac{1}{M}\sum_{n=1}^{N}\sum_{m=1}^{M}A_{nm}^2,$$

and from the application of the central limit theorem in the regime $M \gg 1$. The third point follows from [14]. $\square$

5.3 Singular Values of the Unperturbed Response Matrix

We consider here the situation in which there is one conductive inclusion in the medium and there is no measurement noise. The response matrix is then the $N \times M$ matrix $\mathbf{A}_0$ defined by

$$(\mathbf{A}_0)_{nm} = ik\alpha^5\big(D_{\boldsymbol{x}}^2 G(\boldsymbol{r}_n,\boldsymbol{z})\boldsymbol{q}\big)^T\mathbb{M}^T\big(D_{\boldsymbol{x}}^2 G(\boldsymbol{z},\boldsymbol{s}_m)\boldsymbol{q}\big). \tag{5.8}$$

Let us introduce the singular value decomposition of the symmetric 3×3 matrix $\Re e\mathbb{M}^T$:

$$\Re e\mathbb{M}^T = (\mathbf{V}^{\mathbb{M}})^T\mathbf{\Sigma}^{\mathbb{M}}\mathbf{V}^{\mathbb{M}},$$

where $\mathbf{\Sigma}^{\mathbb{M}}$ is a diagonal matrix with singular values $\sigma_1^{\mathbb{M}} \geq \sigma_2^{\mathbb{M}} \geq \sigma_3^{\mathbb{M}} > 0$. The matrix $\Im m\mathbf{A}_0$ possesses three nonzero singular values given by

$$\begin{aligned}\sigma_j^{\mathbf{A}_0} &= k\alpha^5\sigma_j^{\mathbb{M}}\Big[\sum_{m=1}^{M}\Big|\big(\mathbf{V}^{\mathbb{M}}\big(D_{\boldsymbol{x}}^2G(\boldsymbol{z},\boldsymbol{s}_m)\boldsymbol{q}\big)\big)_j\Big|^2\Big]^{1/2}\\ &\quad\times\Big[\sum_{n=1}^{N}\Big|\big(\mathbf{V}^{\mathbb{M}}\big(D_{\boldsymbol{x}}^2G(\boldsymbol{r}_n,\boldsymbol{z})\boldsymbol{q}\big)\big)_j\Big|^2\Big]^{1/2}, \quad j=1,2,3.\end{aligned}$$

A similar conclusion holds for $\Re e\mathbf{A}_0$.

5.4 Singular Values of the Perturbed Response Matrix

The response matrix using the Hadamard technique in the presence of an inclusion and in the presence of measurement noise is

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{W}, \tag{5.9}$$

where $\mathbf{A}_0$ is given by (5.8) and $\mathbf{W}$ has independent random entries with Gaussian statistics, mean zero and variance σ_{n}^2/M.

We consider the critical regime in which the singular values of the unperturbed matrix are of the same order as the singular values of the noise, that is to say, $\sigma_1^{\mathbf{A}_0}$, the first singular value of $\Re e\mathbf{A}_0$, is of the same order of magnitude as σ_{n}. We will say a few words about the cases $\sigma_1^{\mathbf{A}_0}$ much larger or much smaller than σ_{n} after the analysis of the critical regime.

Proposition 5.2 a) *The normalized l^2-norm of the singular values satisfies*

$$M\Big[\frac{1}{M}\sum_{j=1}^{M}(\sigma_j^{(M)})^2 - \gamma\sigma_{\mathrm{n}}^2\Big] \overset{M\to\infty}{\longrightarrow} (\sigma_0^{\mathbf{A}_0})^2 + \sqrt{2\gamma}\sigma_{\mathrm{n}}^2 Z \textit{ in distribution,} \tag{5.10}$$

where Z follows a Gaussian distribution with mean zero and variance one and

$$\sigma_0^{\mathbf{A}_0} = \Big[\sum_{j=1}^{3} (\sigma_j^{\mathbf{A}_0})^2 \Big]^{1/2}. \tag{5.11}$$

b1) *If $\sigma_1^{\mathbf{A}_0} < \gamma^{1/4} \sigma_{\rm n}$, then the maximal singular value satisfies*

$$\sigma_1^{(M)} \simeq \sigma_{\rm n} \Big[\gamma^{1/2} + 1 + \frac{1}{2M^{2/3}} \big(1+\gamma^{-1/2}\big)^{1/3} Z_1 + o\big(\frac{1}{M^{2/3}}\big) \Big] \text{ in distribution,} \tag{5.12}$$

where Z_1 follows a type-1 Tracy Widom distribution.

b2) *If $\sigma_1^{\mathbf{A}_0} > \gamma^{1/4} \sigma_{\rm n}$, then*

$$\sigma_1^{(M)} = \sigma_1^{\mathbf{A}_0} \Big(\alpha + O\big(\frac{1}{M^{1/2}}\big) \Big) \text{ in probability,} \tag{5.13}$$

where

$$\alpha = \left(1 + (1+\gamma) \frac{\sigma_{\rm n}^2}{(\sigma_1^{\mathbf{A}_0})^2} + \gamma \frac{\sigma_{\rm n}^4}{(\sigma_1^{\mathbf{A}_0})^4} \right)^{1/2}.$$

If, additionally, $\sigma_1^{\mathbf{A}_0} > \sigma_2^{\mathbf{A}_0}$, then the maximal singular value in the regime $M \gg 1$ has Gaussian distribution with the mean and variance given by

$$\mathbb{E}\big[\sigma_1^{(M)}\big] = \sigma_1^{\mathbf{A}_0} \Big(\alpha + o\big(\frac{1}{M^{1/2}}\big) \Big), \tag{5.14}$$

$$\mathrm{Var}\big(\sigma_1^{(M)}\big) = \frac{\sigma_{\rm n}^2}{M} \Big(\beta + o(1) \Big), \tag{5.15}$$

where

$$\beta = \frac{1 - \gamma \frac{\sigma_{\rm n}^2}{(\sigma_1^{\mathbf{A}_0})^2}}{\left(1 + (1+\gamma) \frac{\sigma_{\rm n}^2}{(\sigma_1^{\mathbf{A}_0})^2} + \gamma \frac{\sigma_{\rm n}^4}{(\sigma_1^{\mathbf{A}_0})^4} \right)^{1/2}}.$$

Proof. Point a follows again from the explicit expression of the l^2-norm of the singular values in terms of the entries of the matrix. Point b in the case $N = M$ is addressed in [11] and the extension to $N \geq M$ is only technical. Note that, in the item b2, if $\sigma_1^{\mathbf{A}_0} \sigma_2^{\mathbf{A}_0} \geq \sigma_3^{\mathbf{A}_0}$, then the fluctuations are not Gaussian anymore, but they can be characterized as shown in [11]. □

5.5 Detection Test

We focus again on formulas (3.28). We consider the response matrix in the presence of measurement noise:

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{W},$$

where $\mathbf{A}_0$ is zero in the absence of a conductive inclusion and equal to (5.8) when there is an inclusion. The matrix $\mathbf{W}$ models additive measurement noise and its entries are independent and identically distributed with Gaussian statistics, mean zero and variance $\sigma_{\rm n}^2/M$.

The objective is to propose a detection test for the conductive inclusion. Since we know that the presence of an inclusion is characterized by the existence of three significant singular values for $\Im m\mathbf{A}_0$ (there are three significant values for $\Re e\mathbf{A}_0$ but of lower order), we propose to use a test of the form $R > r$ for the alarm corresponding to the presence of a conductive inclusion. Here R is the quantity obtained from the measured response matrix defined by

$$R = \frac{\sigma_1^{(M)}}{\left[\frac{1}{M-3(1+\gamma^{-1/2})^2}\sum_{j=4}^M(\sigma_j^{(M)})^2\right]^{1/2}}, \tag{5.16}$$

and the threshold value r has to be chosen by the user. This choice follows from Neyman-Pearson theory as we explain below. It requires the knowledge of the statistical distribution of R which we give in the following proposition.

Proposition 5.3 *In the asymptotic regime $M \gg 1$ the following statements hold.*

a) *In absence of a conductive inclusion we have*

$$R \simeq 1 + \gamma^{-1/2} + \frac{1}{2M^{2/3}}\gamma^{-1/2}\big(1+\gamma^{-1/2}\big)^{1/3}Z_1 + o(\frac{1}{M^{2/3}}), \tag{5.17}$$

where Z_1 follows a type-1 Tracy Widom distribution.

b) *In presence of a conductive inclusion:*

b1) *If $\sigma_1^{\mathbf{A}_0} > \gamma^{1/4}\sigma_{\mathrm{n}}$, then we have*

$$R \simeq \frac{\sigma_1^{\mathbf{A}_0}}{\gamma^{1/2}\sigma_{\mathrm{n}}}\alpha + \frac{1}{\gamma^{1/2}M^{1/2}}\beta^{1/2}Z_0, \tag{5.18}$$

where Z_0 follows a Gaussian distribution with mean zero and variance one.

b2) *If $\sigma_1^{\mathbf{A}_0} < \gamma^{1/4}\sigma_{\mathrm{n}}$, then we have (5.17).*

Proof. In absence of a conductive inclusion, we have on the one hand that the truncated normalized l^2-norm of the singular values satisfies

$$M\Big[\frac{1}{M-3(1+\gamma^{-1/2})^2}\sum_{j=4}^M(\sigma_j^{(M)})^2 - \gamma\sigma_{\mathrm{n}}^2\Big] \stackrel{M\to\infty}{\longrightarrow} \sqrt{2\gamma}\sigma_{\mathrm{n}}^2 Z \text{ in distribution,}$$

where Z follows a Gaussian distribution with mean zero and variance one, which implies that

$$\Big[\frac{1}{M-3(1+\gamma^{-1/2})^2}\sum_{j=4}^M(\sigma_j^{(M)})^2\Big]^{1/2} = \gamma^{1/2}\sigma_{\mathrm{n}} + o(\frac{1}{M^{2/3}}) \text{ in probability,} \tag{5.19}$$

and on the other hand the maximal singular value satisfies (5.6) in distribution. Using Slutsky's theorem, we find the first item of the proposition.

In presence of a conductive inclusion, we have on the one hand that the truncated normalized l^2-norm of the singular values satisfies (5.19). On the other hand the maximal singular value is described by Proposition 5.2 which gives the desired result by Slutsky's theorem. $\square$

The data (i.e. the measured response matrix) gives the value of the ratio R. We propose to use a test of the form $R > r$ for the alarm corresponding to the presence of a conductive inclusion. The quality of this test can be quantified by two coefficients:
- The false alarm rate (FAR) is the probability to sound the alarm while there is no inclusion:

$$\mathrm{FAR} = \mathbb{P}(R > r_\delta | \text{ no inclusion }).$$

- The probability of detection (POD) is the probability to sound the alarm when there is an inclusion:

$$\mathrm{POD} = \mathbb{P}(R > r_\delta | \text{ inclusion }).$$

It is not possible to find a test that minimizes the FAR and maximizes the POD. However, by the Neyman-Pearson lemma, the decision rule of sounding the alarm if and only if $R > r_\delta$ maximizes the POD for a given FAR λ with the threshold

$$r_\delta = 1 + \gamma^{-1/2} + \frac{1}{2M^{2/3}} \gamma^{-1/2} \big(1 + \gamma^{-1/2}\big)^{1/3} \Phi_{\mathrm{TW1}}^{-1}(1 - \delta), \tag{5.20}$$

where Φ_{TW1} is the cumulative distribution function of the Tracy-Widom distribution of type 1. The computation of the threshold r_δ is easy since it depends only on the number of sensors N and M and on the FAR δ. Note that we should use a Tracy-Widom distribution table. We have, for instance, $\Phi_{\mathrm{TW1}}^{-1}(0.9) \simeq 0.45$, $\Phi_{\mathrm{TW1}}^{-1}(0.95) \simeq 0.98$ and $\Phi_{\mathrm{TW1}}^{-1}(0.99) \simeq 2.02$.

The POD of this optimal test (optimal amongst all tests with the FAR δ) depends on the value $\sigma_1^{\mathbf{A}_0}$ and on the noise level σ_{n}. Here we find that the POD is

$$\mathrm{POD} = \Phi\left(\sqrt{M} \frac{\frac{\sigma_1^{\mathbf{A}_0}}{\sigma_{\mathrm{n}}} \alpha - \gamma^{1/2} r_\delta}{\beta^{1/2}} \right),$$

where Φ is the cumulative distribution function of the normal distribution with mean zero and variance one. The theoretical test performance improves very rapidly with M once $\sigma_1^{\mathbf{A}_0} > \gamma^{1/4}\sigma_{\mathrm{n}}$. This result is indeed valid as long as $\sigma_1^{\mathbf{A}_0} > \gamma^{1/4}\sigma_{\mathrm{n}}$. When $\sigma_1^{\mathbf{A}_0} < \gamma^{1/4}\sigma_{\mathrm{n}}$, so that the inclusion is buried in noise (more exactly, the singular values corresponding to the inclusion are buried into the deformed quarter-circle distribution of the other singular values), then we have $\mathrm{POD} = 1 - \Phi_{\mathrm{TW1}}\big(\Phi_{\mathrm{TW1}}^{-1}(1-\delta)\big) = \delta$. Therefore the probability of detection is given by

$$\mathrm{POD} = \max \left\{ \Phi\left(\sqrt{M} \frac{\frac{\sigma_1^{\mathbf{A}_0}}{\sigma_{\mathrm{n}}} \alpha - \gamma^{1/2} r_\delta}{\beta^{1/2}} \right), \delta \right\} \tag{5.21}$$

Remark: The previous results were obtained by an asymptotic analysis assuming that M is large and $\sigma_1^{\mathbf{A}_0}$ and σ_{n} are of the same order. In the case in which $\sigma_1^{\mathbf{A}_0}$ is much larger than σ_{n}, then the proposed test has a POD of 100%. In the case in which $\sigma_1^{\mathbf{A}_0}$ is much smaller than σ_{n}, then it is not possible to detect the inclusion from the singular values of the response matrix and the proposed test has a POD equal to the FAR (as shown above, this is the case as soon as $\sigma_1^{\mathbf{A}_0} < \gamma^{1/4}\sigma_{\mathrm{n}}$).

6 Numerical Experiments

In this section, we will give some numerical examples to demonstrate the detecting algorithm. The unperturbed measurement is acquired synthetically by asymptotic formula (3.25) and noisy measurements are given by (5.9). Assume that B_α is an ellipsoid described by equation

$$(x - x_0)^2 + (y - y_0)^2 + \frac{(z - z_0)^2}{4} = \alpha^2$$

where α is characteristic length of the inclusion measured in meters. Then the domain B is characterized by letting $\alpha = 1$ and (x_0, y_0, z_0) be origin. We assume that the inclusion B_α is also located at origin, $\alpha = 0.01$, $\mu^* = \mu_0 = 1.2566 \times 10^{-6}$ H/m and $\sigma = 5.96 \times 10^7$S/m. Letting $\omega = 133.5$ to make $k\alpha^2 = 1$, then (5.8) is of order α^3. We compute the polarization tensor M by an edge element code for (3.15). In this situation, the numerical tensor M is

$$M = \begin{pmatrix} 2.6185 + 0.3501i & 0.0000 + 0.0000i & 0.0001 + 0.0000i \\ 0.0000 + 0.0000i & 2.6180 + 0.3500i & -0.0001 - 0.0000i \\ 0.0001 + 0.0000i & -0.0001 - 0.0000i & 1.6403 + 0.1930i \end{pmatrix}$$

We remark here that the polarization tensor M is computed numerically, one can imagine M is a diagonal matrix for an ellipsoidal inclusion.

The configuration of the detecting system includes coincident transmitter and receiver arrays uniformly distributed on the square $[-2, 2] \times [-2, 2] \times 1$, both consisting of $256(M = N = 16^2)$ vertical dipoles ($\boldsymbol{p} = \boldsymbol{q} = \boldsymbol{e}_3$) emitting or receiving with unit amplitude. The searching domain is a box $[-0.5, 0.5]^3$ below the arrays. It is worth to be mentioned here that the number of transducers should be a multiple of 4 in order to mimic the Hadamard technique in realistic situation.

Mentioned that $\mathbf{H_0}$ is real, after we acquired the multistatic response $\mathbf{A_0}$, we first take its imaginary part to get rid of $\mathbf{H_0}$ as we won't compute the incoming field without the inclusion. Then in what follows, we denote $\mathbf{A_0}$ the imaginary part of unperturbed MSR matrix computed by (3.25).

In the above setting, we calculate the SVD of the unperturbed MSR matrix $\mathbf{A_0}$. Figure 1 displays logarithmic scale plot of the singular values of $\mathbf{A_0}$. It has a good accordance with the previous theoretical analysis: each inclusion according to three singular values. Then we can construct the projection $\mathbf{P}^{\mathrm{imag}}$ with the first three singular vectors corresponding to the first three significant singular values. In the right part of Figure 1, we also plot the magnitude of $\mathcal{I}_{\mathrm{MU}}$ on cross section $z = 0$, which shows that the MUSIC algorithm can detect the inclusion very well.

We test the influence of the noisy measurements by adding a Gaussian noisy matrix with mean zero and variance σ_n^2/M to unperturbed MSR matrix $\mathbf{A_0}$. In our tests, the Gaussian noise is generated by MATLAB function *randn*. The imaging results shown in Figure 2 indicate that with the decreasing of noise level the imaging results become more and more sharp. Then we show the validity of (5.21). Noticing that $M = N$ makes $\gamma = 1$ in our setting. By the analysis in Section 5, for given FAR δ, POD depends on the ratio $\sigma_1^{\mathbf{A_0}}/\sigma_n$. Here we only consider the critical regime in which $\sigma_1^{\mathbf{A_0}}$ is of the same order of σ_n (specially $\sigma_1^{\mathbf{A_0}} > \sigma_n$). Fixing FAR δ, for each ratio $\sigma_1^{\mathbf{A_0}}/\sigma_n$, we generate 1000 Gaussian noisy matrices with mean zero and variance σ_n^2/M and add them to $\mathbf{A_0}$ to get according noisy MSR matrices $\mathbf{A}$. We compute R with the help of SVD for each $\mathbf{A}$ and count the times for $R > r_\delta$ to get the numerical POD. Figure 3 shows the comparisons between numerical POD

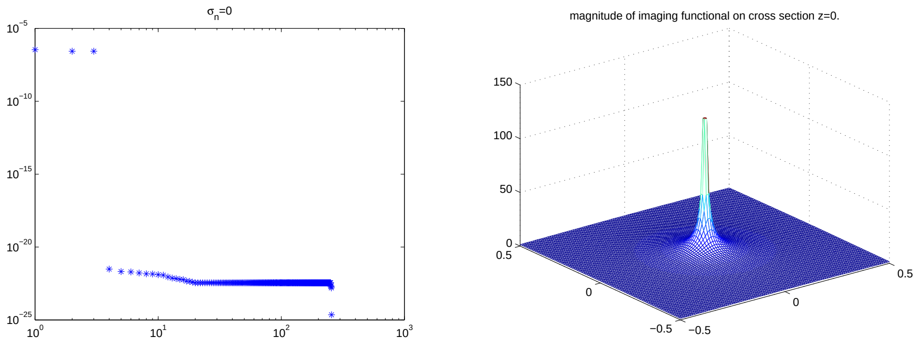

Figure 1: Distribution of singular values of A_0 with $M = N = 256$ and the magnitude of $\mathcal{I}_{\mathrm{MU}}$ on plane $z = 0.0$.

and (5.21) for each δ. We can conclude that the numerical results have a good accordance with (5.21) and the accordance is better when $\sigma_1^{\mathbf{A_0}}/\sigma_n$ becomes bigger.

7 Concluding Remarks

In this paper we have provided an asymptotic expansion for the perturbations of the magnetic field due to the presence of a small conductive inclusion. The characteristic size of the inclusion is of order the depth skin. Our asymptotic expansion is valid for arbitrary shaped inclusions. Based on it, a detection test and a localization method have been provided and tested. It would be very interesting to use our results in this paper for real-time target identification in eddy current imaging using dictionary matching. We also plan to use them for target tracking from induction data. Another interesting problem is to quantitatively estimate the resolution of the direct localization from induction data in the presence of noise. This would be the subject of a forthcoming publication.

References

[1] H. Ammari, A. Buffa, and J.C. Nédélec, A justification of eddy currents model for the Maxwell equations, SIAM J. Appl. Math., 60 (2000), 1805–1823.

[2] H. Ammari, J. Garnier, H. Kang, W.-K. Park, and K. Sølna, Imaging schemes for perfectly conducting cracks, SIAM J. Appl. Math., 71 (2011), 68–91.

[3] H. Ammari, J. Garnier, and K. Sølna, A statistical approach to target detection and localization in the presence of noise, Waves Random Complex Media, 22 (2012), 40–65.

[4] H. Ammari, E. Iakovleva, D. Lesselier, and G. Perrusson, A MUSIC-type electromagnetic imaging of a collection of small three-dimensional inclusions, SIAM J. Sci. Comput., 29 (2007), 674–709.

[5] H. Ammari and H. Kang, *Reconstruction of small inhomogeneities from boundary measurements*, vol. 1846, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2004.

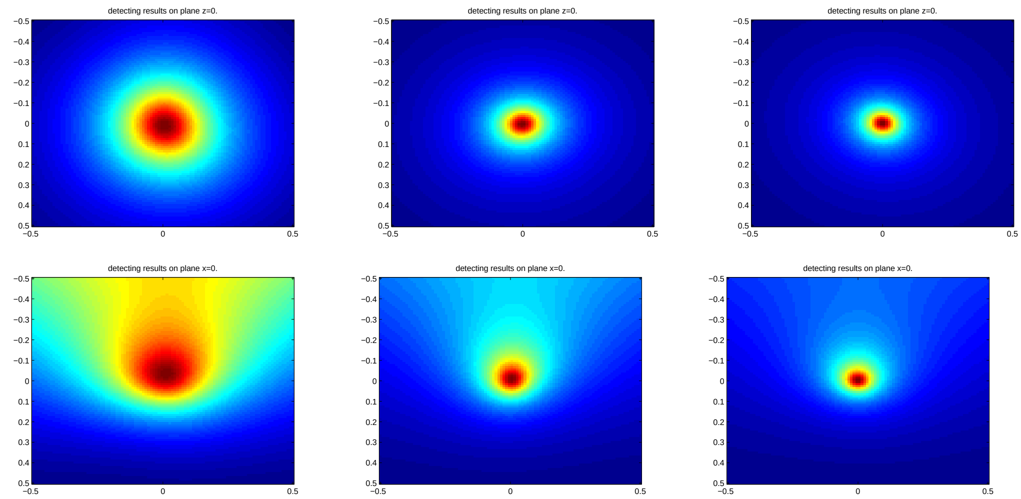

Figure 2: Detecting results on cross sectional plane $z = 0$(top) and $x = 0$(bottom) for different noise level σ_n. $\sigma_1^{\mathbf{A_0}}/\sigma_n = 10, 20, 30$ from left to right.

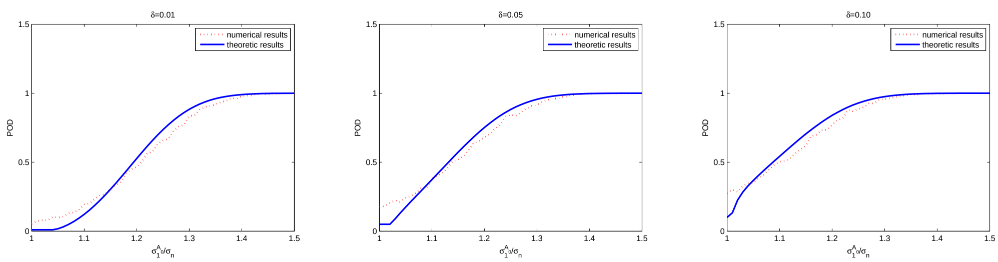

Figure 3: POD with respect to $\sigma_1^{\mathbf{A_0}}/\sigma_n$ for different δ, $\delta = 0.01, 0.05, 0.10$ from left to right.

[6] H. Ammari and H. Kang, *Polarization and Moment Tensors: with Applications to Inverse Problems and Effective Medium Theory*, Applied Mathematical Sciences, Vol. 162, Springer-Verlag, New York, 2007.

[7] H. Ammari and A. Khelifi, Electromagnetic scattering by small dielectric inhomogeneities, J. Math. Pures Appl., 82 (2003), 749–842.

[8] H. Ammari, M. Vogelius, and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations, J. Math. Pures Appl., 80 (2001), 769–814.

[9] H. Ammari and D. Volkov, The leading-order term in the asymptotic expansion of the scattering amplitude of a collection of Finite Number of dielectric inhomogeneities of small diameter, Int. J. Mult. Comput. Eng., 3 (2005), 149–160.

[10] B. A. Auld and J. C. Moulder, Review of advances in quantitative eddy current nondestructive evaluation, J. Nondest. Eval., 18 (1999), 3–36.

[11] M. Capitaine, C. Donati-Martin, and D. Féral, Central limit theorems for eigenvalues of deformations of Wigner matrices, arXiv:0903.4740.

[12] D.J. Cedio-Fengya, S. Moskow, and M.S. Vogelius, Identification of conductivity imperfections of small diameter by boundary measurements: Continuous dependence and computational reconstruction, Inverse Problems, 14 (1998), 553–595.

[13] R. Hiptmair, Symmetric coupling for eddy current problems, SIAM J. Numer. Anal. 40 (2002), 41-65.

[14] I. M. Johnstone, On the distrbution of the largest eigenvalue in principal components analysis, Ann. Statist., 29 (2001), 295–327.

[15] V. A. Marcenko and L. A. Pastur, Distributions of eigenvalues of some sets of random matrices, Math. USSR-Sb., 1 (1967), 507–536.

[16] J.C. Nédélec, *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*, Springer, 2001.

[17] S. J. Norton and I. J. Won, Identification of buried unexploded ordnance from broadband electromagnetic induction data, IEEE Trans. Geoscience Remote Sensing, 39 (2001), 2253–2261.

[18] J. Rosell, R. Casanas, and H. Scharfetter, Sensitivity maps and system requirements for magnetic induction tomography using a planar gradiometer, Physiol. Meas., 22 (2001), 212–130.

[19] J. Seberry, B. J. Wysocki, and T. A. Wysocki, On some applications of Hadamard matrices, Metrika, 62 (2005), 221–239.