ON THE AUGMENTED LAGRANGIAN APPROACH TO SIGNORINI ELASTIC CONTACT PROBLEM

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Abstract. The Signorini problem describes the contact of a linearly elastic body with a rigid frictionless foundation. It is transformed into a saddle point problem of some augmented Lagrangian functional and then discretized by finite element methods. Optimal error estimates are obtained for general smooth domains which are not necessarily convex. The key ingredient in the analysis is a discrete inf-sup condition which guarantees the existence of the saddle point.

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1 Introduction

The Signorini problem describing the contact of a linearly elastic body with a rigid frictionless foundation plays an important role in formulating the contact problems in solid mechanics into variational inequalities. The efficient numerical solution of Signorini problem is crucial in solving more realistic frictional contact problems for small deformations in a elastic media via incremental methods [15], [14], [18], [20]. Let Ω be the domain in $\mathbf{R}^3$ occupied by a body unilaterally supported by a frictionless rigid foundation, and subject to body forces $\mathbf{f}$ and surface tractions $\mathbf{t}$ applied to a portion Γ_F of the body's surface Γ. The body is fixed along a portion Γ_D of its boundary. We denote by Γ_C the part of the boundary which contains the actual contact surface. Let $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_F \cup \bar{\Gamma}_C$ and for simplicity, we assume that $\text{meas}(\Gamma_D) > 0$. The case when $\Gamma_D = \emptyset$ can be treated similarly by combining with the results in [21] for semicoercive variational inequalities. Denote by the constraint set

$$\mathbf{K} = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v} = 0 \text{ a.e. on } \Gamma_D \text{ and } \mathbf{v}\cdot\boldsymbol{\nu} - g \le 0 \text{ a.e. on } \Gamma_C\},$$

where $\boldsymbol{\nu}$ is the unit outer normal to Γ. The Signorini problem can be formulated as the following variational inequality: find $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \ge L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{K}, \tag{1.1}$$

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where the bilinear form $a(\cdot,\cdot)$ and the linear form $L(\cdot)$ are defined by

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega E_{ijkl}u_{k,l}v_{i,j}dx \quad \forall \mathbf{u},\mathbf{v}\in H^1(\Omega)^3, \tag{1.2}$$

$$L(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}dx + \int_{\Gamma_F}\mathbf{t}\cdot\mathbf{v}ds \quad \forall \mathbf{v}\in H^1(\Omega)^3 \tag{1.3}$$

with $u_{i,j} = \partial u_i/\partial u_j, 1\le i,j\le 3$, and $E_{ijkl}(x)$ the components of Hooke's tensor at x having the symmetry properties

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij}, \quad 1\le i,j,k,l\le 3. \tag{1.4}$$

Throughout the paper the usual summation convention for repeated indices is used. Here $\mathbf{u}$ stands for the displacement field of the body, and g is the initial gap so that the constraint $\mathbf{u}\cdot\boldsymbol{\nu} - g \le 0$ on Γ_C represents the nonpenetration condition. We refer to the book [18] for more information on the physical background of the model and its various extensions.

The existence of unique solution to (1.1) follows from the general theory of convex analysis [9], Korn's inequality and the assumption meas$(\Gamma_D) > 0$. The finite element approximations to (1.1) have been considered in [15], [14] and [18] which lead to the following quadratic programming problems: find the minimum of the function

$$J(y) = \frac{1}{2}(y, Ay) - (f, y)$$

subject to the constraint

$$\mathcal{K} = \{y\in \mathbf{R}^n : By \le d\},$$

where A is an $n\times n$ stiffness matrix, B is generally an $m\times n$ rectangular matrix, and f, d are given vectors in $\mathbf{R}^n$. This programming problem can be solved by various optimization methods of projection type or the penalty method [15], [14],[18]. However, the former methods suffer the drawback that the projection to $\mathcal{K}$ is not easy to find since the matrix B is not diagonal, and the latter method yields systems of linear equations of poor condition number. The finite element error estimates of order $O(h^{3/4})$ are obtained in [15], [14] and [18] for polyhedral or convex smooth domains Ω. We also refer to the earlier works [10], [3] and [13] on the finite element methods and error analysis for general variational inequalities.

The augmented Lagrangian methods have been widely used in nonlinear constrained optimization problems and nonlinear boundary value problems to relax complicated constraints or difficult couplings among nonlinear and nonsmooth terms or to enhance convexities of the objective functions (cf. [2], [12], [11]). They have also been successfully applied to solve the parameter identification problems by combining the output least squares and the error equation formulation in [16], [5]. In particular, the augmented Lagrangian method, together with finite element discretization, is proposed in [12, Chapter 7] to solve a 2D Signorini problem but without convergence analysis.

The purpose of the paper is twofolds: we first present a careful analysis of the augmented Lagrangian method solving the full 3D problem (1.1) and its finite element discretizations, and secondly we derive finite element error estimates of suboptimal convergence order $O(h^{3/4})$. We also prove the optimal error estimates $O(h)$ under the further assumption that the free boundary where the constraint changes from contact to noncontact having finite length. It turns out that in order to show the discrete augmented Lagrangian functional exists a saddle point, the appropriate inf-sup condition must be satisfied. The conforming piecewise linear element for the displacement and the piecewise constant element for the Lagrangian multiplier, which is used in [12] for 2D problem, do *not* satisfy this inf-sup condition in 3D case (see Remark 3.2 for details). Instead we show that if the piecewise constant element for the multiplier is replaced by piecewise linear element, the inf-sup condition can be established after using suitable numerical integration formula. In the finite element error analysis we remove the convexity assumption on the smooth domain Ω by making full use of the Lagrangian multiplier formulation and extending the results in [19] for finite element error analysis on general smooth domains. We finally remark that Lagrangian multiplier technique also plays an important role in the a posteriori error analysis for elliptic variational inequalities [4].

The paper is organized as follows: In §2 we transform the Signorini problem into the saddle point problem of some augmented Lagrangian functional. The existence and uniqueness of saddle points are proved. In §3 we discretize the saddle point problem by finite element methods and prove the discrete saddle point problem exists a unique solution. Here appropriate discrete inf-sup condition is important. We also characterize the saddle points and give a convergent decomposition Uzawa's algorithm for the discrete saddle point problem. In §4 we carry out the finite element error analysis. Finally in §5 we give some concluding remarks.

2 The Augmented Lagrangian Formulation

Let Ω be a domain in $\mathbf{R}^3$ with boundary $\Gamma \in C^{2,1}$. Assume that the body is fixed along a portion Γ_D of Γ and the surface traction $\mathbf{t}$ is applied to a portion Γ_F of the boundary. Let Γ_C be the part of the boundary which is a candidate contact surface. Let $\Sigma = \mathrm{int}(\Gamma - \Gamma_D)$ and assume that $\bar{\Gamma}_C \subset \Sigma$, $\Gamma_F \subset \Sigma$ and $\Gamma_F \cap \Gamma_C = \emptyset$. Denote by $\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v} = 0 \text{ on } \Gamma_D\}$ and $\mathbf{K}$ the subset

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} : \ \mathbf{v} \cdot \boldsymbol{\nu} - g \leq 0 \text{ a.e. on } \Gamma_C\}.$$

For any $\mathbf{v} \in \mathbf{V}$ we can define its normal and tangential components

$$\gamma^0_{\Sigma n}(\mathbf{v}) = \mathbf{v} \cdot \boldsymbol{\nu}, \quad \gamma^0_{\Sigma T}(\mathbf{v}) = \mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nu})\boldsymbol{\nu}.$$

It is known [18, Theorem 5.6] that $\gamma^0_{\Sigma n} : \mathbf{V} \to H^{1/2}_{00}(\Sigma)$ and $\gamma^0_{\Sigma T} : \mathbf{V} \to \mathbf{H}^{1/2}_{T00}(\Sigma)$ are surjective linear continuous mappings, where $H^{1/2}_{00}(\Sigma)$ is the subspace of $H^{1/2}(\Sigma)$ introduced in [17, Chapter 1] and $\mathbf{H}^{1/2}_{T00}(\Sigma)$ the subspace of $H^{1/2}_{00}(\Sigma)^3$ given by

$$\mathbf{H}^{1/2}_{T00}(\Sigma) = \{\mathbf{v} \in H^{1/2}_{00}(\Sigma)^3 : \ \mathbf{v} \cdot \boldsymbol{\nu} = 0 \text{ a.e. on } \Sigma\}.$$

Given $\mathbf{f} \in L^2(\Omega)^3$ and $\mathbf{t} \in L^2(\Gamma_F)^3$, we consider in this paper the numerical solutions of the following variational problem.

Problem (P). Find $\mathbf{u} \in \mathbf{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{K}. \tag{2.1}$$

Here the bilinear form $a : \mathbf{V} \times \mathbf{V} \to \mathbf{R}$ and the linear form $L : \mathbf{V} \to \mathbf{R}$ are defined according to (1.2)-(1.3). The elasticities E_{ijkl} are assumed to be functions in $L^\infty(\Omega)$ satisfying the symmetric relations in (1.4) and the property that there exists a constant $m > 0$ such that a.e. on Ω

$$E_{ijkl}(x)\varepsilon_{ij}\varepsilon_{kl} \geq m\varepsilon_{ij}\varepsilon_{ij} \tag{2.2}$$

for any $\varepsilon = [\varepsilon_{ij}] \in \mathbf{R}^{3\times 3}$ with $\varepsilon_{ij} = \varepsilon_{ji}$. It is well-known that the the unique solution $\mathbf{u} \in \mathbf{K}$ of Problem (P) can be characterized as (cf.e.g. [18, Theorems 6.2-6.3])

$$\begin{aligned}
-\sigma_{ij}(\mathbf{u})_{,j} &= f_i && \text{in} \quad L^2(\Omega), && (2.3)\\
\sigma(\mathbf{u})\boldsymbol{\nu} &= \mathbf{t} && \text{on} \quad \Gamma_F \text{ in } L^2(\Sigma)^3, && (2.4)\\
\gamma^0_{\Sigma n}(\sigma(\mathbf{u})\boldsymbol{\nu}) &\leq 0 && \text{on} \quad \Gamma_C \text{ in } (H^{1/2}_{00}(\Sigma))', && (2.5)\\
\langle \gamma^0_{\Sigma n}(\sigma(\mathbf{u})\boldsymbol{\nu}), \mathbf{u}\cdot\boldsymbol{\nu} - g\rangle_{\Gamma_C} &= 0 && \text{on} \quad \Gamma_C, && (2.6)\\
\gamma^0_{\Sigma T}(\sigma(\mathbf{u})\boldsymbol{\nu}) &= 0 && \text{on} \quad \Gamma_C \text{ in } (\mathbf{H}^{1/2}_{00}(\Sigma))', && (2.7)
\end{aligned}$$

where $\sigma(\mathbf{u}) = [\sigma_{ij}(\mathbf{u})] \in \mathbf{R}^{3\times 3}$, $\sigma_{ij}(\mathbf{u}) = E_{ijkl}u_{k,l}$ and $\langle \cdot, \cdot \rangle_{\Gamma_C}$ denote by the duality pairing between $(H^{1/2}(\Gamma_C))'$ and $H^{1/2}(\Gamma_C)$ or the inner product in $L^2(\Gamma_C)$.

Now we are in the position to introduce the augmented Lagrangian approach to solve Problem (P). Let $H = L^2(\Gamma_C)$, we define the energy functional $G : \mathbf{V} \to \mathbf{R}$ and the indicator functional $F : H \to \bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ as follows

$$G(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \tag{2.8}$$

$$F(q) = \begin{cases} 0 & \text{if } q \leq 0 \text{ a.e. on } \Gamma_C; \\ +\infty & \text{otherwise.} \end{cases} \tag{2.9}$$

Obviously the Problem (P) is equivalent to the minimization problem

$$\begin{aligned}
&\text{minimize } [F(q) + G(\mathbf{v})] \\
&\text{subject to } (\mathbf{v}, q) \in \mathbf{V} \times H \text{ and } q = \mathbf{v}\cdot\boldsymbol{\nu} - g.
\end{aligned}$$

For any $r \geq 0$, we define the augmented Lagrangian functional $\mathcal{L}_r : \mathbf{V} \times H \times H \to \bar{\mathbf{R}}$

$$\mathcal{L}_r(\mathbf{v}, q; \mu) = F(q) + G(\mathbf{v}) + \langle \mu, \mathbf{v}\cdot\boldsymbol{\nu} - g - q\rangle_{\Gamma_C} + \frac{r}{2}\| \mathbf{v}\cdot\boldsymbol{\nu} - g - q \|^2_{L^2(\Gamma_C)} \tag{2.10}$$

and introduce the associate saddle problem

Problem (Q). Find $(\mathbf{u}, p; \lambda) \in \mathbf{V} \times H \times H$ such that

$$\mathcal{L}_r(\mathbf{u}, p; \mu) \leq \mathcal{L}_r(\mathbf{u}, p; \lambda) \leq \mathcal{L}_r(\mathbf{v}, q; \lambda) \quad \forall (\mathbf{v}, q; \mu) \in \mathbf{V} \times H \times H. \tag{2.11}$$

The following theorem establishes the relation between Signorini Problem (P) and the saddle point Problem (Q).

Theorem 2.1 *Assume that the unique solution $\mathbf{u} \in \mathbf{K}$ of Problem (P) satisfies $\sigma_n(\mathbf{u}) = \sigma_{ij}(\mathbf{u})\nu_i\nu_j \in L^2(\Gamma_C)$. Then the Lagrangian functional (2.10) exists a unique saddle point $(\mathbf{u}, p; \lambda) \in \mathbf{V} \times H \times H$. Moreover, $(\mathbf{u}, p; \lambda)$ satisfies*

$$p = \mathbf{u} \cdot \boldsymbol{\nu} - g \leq 0, \quad \lambda = -\sigma_n(\mathbf{u}) = -\sigma_{ij}\nu_i\nu_j \geq 0 \quad \text{on } \Gamma_C. \tag{2.12}$$

Proof. 1. We first claim that if $(\mathbf{u}, p; \lambda) \in \mathbf{V} \times H \times H$ is a saddle point of $\mathcal{L}_r$ in (2.10), then $\mathbf{u} \in \mathbf{K}$ is a solution of Problem (P) and (2.12) holds.
To do that, we note from the left inequality of (2.11) that

$$\langle \mu - \lambda, \mathbf{u} \cdot \boldsymbol{\nu} - g - p \rangle_{\Gamma_C} \leq 0 \quad \forall \mu \in L^2(\Gamma_C),$$

which yields $p = \mathbf{u} \cdot \boldsymbol{\nu} - g$ on Γ_C. Now the right inequality in (2.11) implies $p \leq 0$ a.e. on Γ_C and

$$G(\mathbf{u}) \leq G(\mathbf{v}) + F(q) + \langle \lambda, \mathbf{v} \cdot \boldsymbol{\nu} - g - q \rangle_{\Gamma_C} + \frac{r}{2} \| \mathbf{v} \cdot \boldsymbol{\nu} - g - q \|^2_{L^2(\Gamma_C)} \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times H. \tag{2.13}$$

Thus if $\mathbf{v} \in \mathbf{K}$, by taking $q = \mathbf{v} \cdot \boldsymbol{\nu} - g \leq 0$ a.e. on Γ_C in (2.13) we obtain

$$G(\mathbf{u}) \leq G(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{K}.$$

Hence $\mathbf{u} \in \mathbf{K}$ is a solution of Problem (P). It remains to show the last equality in (2.12). By taking $\mathbf{v} = \mathbf{u} + t(\mathbf{w} - \mathbf{u})$, where $t \in (0, 1)$, $q = p$ in (2.13) we get

$$\frac{G(\mathbf{u} + t(\mathbf{w} - \mathbf{u})) - G(\mathbf{u})}{t} + \langle \lambda, (\mathbf{w} - \mathbf{u}) \cdot \boldsymbol{\nu} \rangle_{\Gamma_C} + \frac{r}{2} t \| (\mathbf{w} - \mathbf{u}) \cdot \boldsymbol{\nu} \|^2_{L^2(\Gamma_C)} \geq 0 \;\; \forall \mathbf{w} \in \mathbf{V}.$$

By letting $t \to 0^+$ in above inequality we obtain

$$a(\mathbf{u}, \mathbf{w} - \mathbf{u}) + \langle \lambda, (\mathbf{w} - \mathbf{u}) \cdot \boldsymbol{\nu} \rangle_{\Gamma_C} \geq L(\mathbf{w} - \mathbf{u}) \;\; \forall \mathbf{w} \in \mathbf{V},$$

which yields clearly

$$a(\mathbf{u}, \mathbf{v}) + \langle \lambda, \mathbf{v} \cdot \boldsymbol{\nu} \rangle_{\Gamma_C} = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}. \tag{2.14}$$

Now since $\mathbf{u} \in \mathbf{K}$ is the solution Problem (P), by applying Green formula in (2.14), (2.3)-(2.4), (2.7) and the assumption $\sigma_n(\mathbf{u}) \in L^2(\Gamma_C)$ we have that

$$\int_{\Gamma_C} (\sigma_{ij}(\mathbf{u})\nu_i\nu_j + \lambda)(\mathbf{v} \cdot \boldsymbol{\nu})ds = 0 \quad \forall \mathbf{v} \in \mathbf{V}.$$

This implies $\lambda = -\sigma_n(\mathbf{u})$ by the density of $H^{1/2}(\Gamma_C)$ in $L^2(\Gamma_C)$ and the surjectivity of the trace operator $\gamma^0_{\Sigma n} : \mathbf{V} \to H^{1/2}_{00}(\Sigma)$. Finally, $\lambda \geq 0$ on Γ_C follows from (2.5).

2. We claim that Problem (Q) has a unique solution.
The uniqueness is a direct consequence of (2.12) since the solution of Problem (P) is unique. To show the existence, we just let $(\mathbf{u}, p; \lambda) \in \mathbf{V} \times H \times H$ be defined according to (2.12) with $\mathbf{u} \in \mathbf{K}$ be the solution of Problem (P). Now the left inequality in (2.11) is obvious since $p = \mathbf{u} \cdot \boldsymbol{\nu} - g$. In fact we have $\mathcal{L}_r(\mathbf{u}, p; \mu) = \mathcal{L}_r(\mathbf{u}, p; \lambda)$, $\forall \mu \in H$. On the other hand, from $\lambda = -\sigma_n(\mathbf{u})$ we know that (2.14) holds. Thus

$$G(\mathbf{u}) \leq G(\mathbf{v}) + \langle \lambda, (\mathbf{v} - \mathbf{u}) \cdot \boldsymbol{\nu} \rangle_{\Gamma_C} \quad \forall \mathbf{v} \in \mathbf{V}.$$

But $\lambda \geq 0$ a.e. on Γ_C and $\langle \lambda, \mathbf{u} \cdot \boldsymbol{\nu} - g \rangle_{\Gamma_C} = 0$ by (2.5)-(2.6), we conclude that

$$G(\mathbf{u}) \leq G(\mathbf{v}) + \langle \lambda, \mathbf{v} \cdot \boldsymbol{\nu} - g - q \rangle_{\Gamma_C} \quad \forall \mathbf{v} \in \mathbf{V}, q \in H \text{ s.t. } q \leq 0 \text{ a.e. on } \Gamma_C.$$

This implies (2.13) and completes the proof. □

3 Discrete Saddle Point Problem

Let Ω_h be a polyhedral domain in $\mathbf{R}^3$ which approximates the domain Ω and $\{\mathcal{M}_h\}_{h>0}$ be a family of regular and quasiuniform triangulations over Ω_h. We assume that all the vertices lying on $\Gamma_h = \partial\Omega_h$ also lie on Γ, and Γ_h decomposed into subsets Γ_D^h, Γ_F^h and Γ_C^h in such a way that all the vertices of Γ_D^h, Γ_F^h and Γ_C^h lie on Γ_D, Γ_F and Γ_C respectively, and those on $\partial\Gamma_D^h$, $\partial\Gamma_F^h$ and $\partial\Gamma_C^h$ lie on $\partial\Gamma_D, \partial\Gamma_F$ and $\partial\Gamma_C$ respectively. Denote by $\mathcal{M}_h^C$ the collection of all contact elements, that is, all the elements in $\mathcal{M}_h$ having one face included in $\bar{\Gamma}_C^h$. We also let $\mathcal{B}_h$ be the reduced triangulation over Γ_C^h, that is, the collection of all faces $e = \partial S \cap \bar{\Gamma}_C^h$ for some $S \in \mathcal{M}_h^C$.

Let W_h be the standard conforming linear finite element space over $\mathcal{M}_h$ and $I_h : C(\bar{\Omega}_h) \to W_h$ the standard linear Lagrange interpolant. We define the finite element subspaces

$$\begin{aligned}
\mathbf{V}_h &= \{\mathbf{v}_h \in C(\bar{\Omega}_h)^3 : \ \mathbf{v}_h \text{ is linear on all } S \in \mathcal{M}_h \text{ and } \mathbf{v}_h = 0 \text{ on } \Gamma_D^h\}, \\
H_h &= \{\mu_h \in C(\bar{\Gamma}_C^h) : \mu_h \text{ is linear on all } e \in \mathcal{B}_h\}.
\end{aligned}$$

The energy functional $G : \mathbf{V} \to \mathbf{R}$ is approximated by

$$G_h(\mathbf{v}) = \frac{1}{2} a_h(\mathbf{v}, \mathbf{v}) - L_h(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h,$$

where the bilinear form $a_h : \mathbf{V}_h \times \mathbf{V}_h \to \mathbf{R}$ and the linear form $L_h : \mathbf{V}_h \to \mathbf{R}$ are given by

$$a_h(\mathbf{u}, \mathbf{v}) = \int_{\Omega_h} I_h(E_{ijkl}) u_{k,l} v_{i,j} dx \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}_h,$$
$$L_h(\mathbf{v}) = \int_{\Omega_h} I_h(\mathbf{f} \cdot \mathbf{v}) dx + \int_{\Gamma_F^h} I_h(\mathbf{t} \cdot \mathbf{v}) ds \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

We also need the discrete inner product and norm

$$\langle \varphi, \psi \rangle_{\Gamma_C^h}^h = \int_{\Gamma_C^h} I_h(\varphi\psi) ds, \quad \| \varphi \|_{h,\Gamma_C^h}^2 = \langle \varphi, \varphi \rangle_{\Gamma_C^h}^h$$

for any piecewise uniformly continuous function φ and ψ.
Let $g_h = I_h g$ and $\boldsymbol{\nu}_h = I_h \boldsymbol{\nu}$, now we introduce the discrete augmented Lagrangian functional $L_r : \mathbf{V}_h \times H_h \times H_h \to \bar{\mathbf{R}}$ given by

$$L_r(\mathbf{v}_h, q_h; \mu_u) = F(q_h) + G_h(\mathbf{v}_h) + \langle \mu_h, \mathbf{v}_h \cdot \boldsymbol{\nu}_h - g_h - q_h \rangle_{\Gamma_C^h}^h + \frac{r}{2} \| \mathbf{v}_h \cdot \boldsymbol{\nu}_h - g_h - q_h \|_{h,\Gamma_C^h}^2$$

and the associate saddle point problem

Problem ($\mathbf{Q}_h$). Find $(\mathbf{u}_h, p_h; \lambda_h) \in \mathbf{V}_h \times H_h \times H_h$ such that

$$L_r(\mathbf{u}_h, p_h; \mu_h) \le L_r(\mathbf{u}_h, p_h; \lambda_h) \le L_r(\mathbf{v}_h, q_h; \lambda_h) \quad \forall (\mathbf{v}_h, q_h; \mu_u) \in \mathbf{V}_h \times H_h \times H_h. \tag{3.1}$$

The following theorem is the main result of this section.

Theorem 3.1 *There exists a unique saddle point of the augmented Lagrangian functional $L_r : \mathbf{V}_h \times H_h \times H_h \to \bar{\mathbf{R}}$.*

The proof of this theorem depends on the following lemmas.

Lemma 3.1 *There exists a constant $\alpha > 0$ independent of h such that*

$$\sup_{0 \ne \mathbf{v}_h \in \mathbf{V}_h} \frac{\langle \mu_h, \mathbf{v}_h \cdot \boldsymbol{\nu}_h \rangle_{\Gamma_C^h}^h}{\| \mathbf{v}_h \|_{H^1(\Omega_h)}} \ge \alpha h^{1/2} \| \mu_h \|_{L^2(\Gamma_C^h)}, \quad \forall \mu_h \in H_h.$$

Proof. Let $\{x_j\}_{j=1}^K$ be the nodes of $\mathcal{M}_h$ and $\{\psi_j\}_{j=1}^K$ the usual nodal basis of W_h. Denote by $\{x_j\}_{j=1}^{K_c}$ the nodes on $\bar{\Gamma}_C^h$. For any $\mu_h \in H_h$, define

$$\mathbf{v}_h = \sum_{j=1}^{K_c} \left[\mu_h(x_j) \boldsymbol{\nu}_h(x_j) \right] \psi_j(x).$$

Then $\mathbf{v}_h \in \mathbf{V}_h$. Simple scaling arguments yield that

$$\| \mathbf{v}_h \|_{H^1(\Omega_h)} \le C h^{-1/2} \| \mu_h \|_{L^2(\Gamma_C^h)}.$$

On the other hand, by the definition of quadrature rule, we have

$$\langle \mu_h, \mathbf{v}_h \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} = \langle \mu_h, \mu_h \rangle^h_{\Gamma^h_C} \geq C \| \mu_h \|^2_{L^2(\Gamma^h_C)}.$$

This completes the proof. □

Lemma 3.2 *Let $(\mathbf{u}_h, p_h; \lambda_h) \in \mathbf{V}_h \times H_h \times H_h$ be the saddle point of $L_r : \mathbf{V}_h \times H_h \times H_h \to \bar{\mathbf{R}}$, then the following relations hold*

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + \langle \lambda_h, \mathbf{v}_h \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} = L_h(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.2}$$

$$p_h = I_h(\mathbf{u}_h \cdot \boldsymbol{\nu}_h) - g_h, \qquad p_h \leq 0, \tag{3.3}$$

$$\langle \lambda_h, p_h \rangle^h_{\Gamma^h_C} = 0, \qquad \lambda_h \geq 0. \tag{3.4}$$

Proof. From the left inequality of (3.1) we have

$$\langle \mu_h - \lambda_h, \mathbf{u}_h \cdot \boldsymbol{\nu}_h - g_h - p_h \rangle^h_{\Gamma^h_C} \leq 0 \quad \forall \mu_h \in H_h$$

which implies $p_h = I_h(\mathbf{u}_h \cdot \boldsymbol{\nu}_h) - g_h$. From the right inequality of (3.1) and the definition of the functional $F : H \to \bar{\mathbf{R}}$ we know that $p_h \leq 0$ on Γ^h_C. This proves (3.3). Then the right inequality of (3.1) reduces to

$$G_h(\mathbf{u}_h) \leq G_h(\mathbf{v}_h) + \langle \lambda_h, (\mathbf{v}_h - \mathbf{u}_h) \cdot \boldsymbol{\nu}_h + p_h - q_h \rangle^h_{\Gamma^h_C} + \frac{r}{2} \| (\mathbf{v}_h - \mathbf{u}_h) \cdot \boldsymbol{\nu}_h + (p_h - q_h) \|^2_{h,\Gamma^h_C} \tag{3.5}$$

for any $\mathbf{v}_h \in \mathbf{V}_h, q_h \in H_h$ satisfying $q_h \leq 0$. For any $t \in (0,1)$ and $\mathbf{w}_h \in \mathbf{V}_h$, we select $\mathbf{v}_h = \mathbf{u}_h + t(\mathbf{w}_h - \mathbf{u}_h) \in \mathbf{V}_h$ and $q_h = p_h \leq 0$ in (3.5) to obtain

$$G_h(\mathbf{u}_h) - G_h(\mathbf{u}_h + t(\mathbf{w}_h - \mathbf{u}_h)) \leq t \langle \lambda_h, \mathbf{w}_h - \mathbf{u}_h \rangle^h_{\Gamma^h_C} + \frac{r}{2} t^2 \| (\mathbf{v}_h - \mathbf{u}_h) \cdot \boldsymbol{\nu}_h \|^2_{h,\Gamma^h_C}.$$

Dividing above inequality by t and then letting $t \to 0^+$, we get (3.2).
For any $t \in (0,1)$ and $\theta_h \in H_h$ such that $\theta_h \leq 0$, we select $\mathbf{v}_h = \mathbf{u}_h$ and $q_h = p_h + t(\theta_h - p_h) \leq 0$ in (3.5) to get

$$t \langle \lambda_h, p_h - \theta_h \rangle^h_{\Gamma^h_C} + \frac{r}{2} t^2 \| p_h - \theta_h \|^2_{h,\Gamma^h_C} \geq 0, \qquad \forall \theta_h \in H_h, \theta_h \leq 0.$$

Dividing above inequality by t and then letting $t \to 0^+$, we have

$$\langle \lambda_h, p_h - \theta_h \rangle^h_{\Gamma^h_C} \geq 0, \qquad \forall \theta_h \in H_h, \theta_h \leq 0.$$

This gives (3.4). □

Remark 3.1. Denote by

$$\mathbf{K}_h = \{ \mathbf{v}_h \in \mathbf{V}_h : \ I_h(\mathbf{v}_h \cdot \boldsymbol{\nu}_h) - g_h \leq 0 \text{ on } \Gamma^h_C \}$$

the closed convex subset of $\mathbf{V}_h$. From (3.3) we know that $\mathbf{u}_h \in \mathbf{K}_h$ and from (3.4) we have, for any $\mathbf{v}_h \in \mathbf{K}_h$,

$$\langle \lambda_h, (\mathbf{v}_h - \mathbf{u}_h) \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} = \langle \lambda_h, \mathbf{v}_h \cdot \boldsymbol{\nu}_h - g_h \rangle^h_{\Gamma^h_C} \le 0.$$

Thus, from (3.2) we know that $\mathbf{u}_h \in \mathbf{K}_h$ is indeed the solution of the following discrete variational inequality

$$a_h(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \ge L_h(\mathbf{v}_h - \mathbf{u}_h) \quad \forall \mathbf{v}_h \in \mathbf{K}_h. \tag{3.6}$$

Proof of Theorem 3.1. 1. We first prove the uniqueness. Let $(\mathbf{u}_h, p_h; \lambda_h)$ and $(\bar{\mathbf{u}}_h, \bar{p}_h; \bar{\lambda}_h)$ be two saddle points of $L_r : \mathbf{V}_h \times H_h \times H_h \to \bar{\mathbf{R}}$. Then we know that $\mathbf{u}_h \in \mathbf{K}_h$ and $\bar{\mathbf{u}}_h \in \mathbf{K}_h$ both are the solutions of the variational inequality (3.6). We deduce easily from the coerciveness of $a_h : \mathbf{V}_h \times \mathbf{V}_h \to \mathbf{R}$ that $\mathbf{u}_h = \bar{\mathbf{u}}_h$. Now (3.3) in Lemma 3.2 implies that $p_h = \bar{p}_h$. Finally from (3.2) in Lemma 3.2 we know that

$$\langle \lambda_h - \bar{\lambda}_h, \mathbf{v}_h \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} = 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

This implies $\lambda_h = \bar{\lambda}_h$ by the inf-sup condition in Lemma 3.1.

2. Now we turn to the proof of existence. To do that, let $\mathbf{u}_h \in \mathbf{K}_h$ be the unique solution of (3.6). We define $p_h = I_h(\mathbf{u}_h \cdot \boldsymbol{\nu}_h) - g_h$ and $\lambda_h \in H_h$ according to the prescription

$$\langle \lambda_h, \mathbf{v}_h \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} = L_h(\mathbf{v}_h) - a_h(\mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.7}$$

The existence of unique such λ_h is guaranteed by Lemma 3.1. Since $\mathbf{u}_h \in \mathbf{K}_h$ satisfies (3.6), we have $p_h \le 0$ on Γ^h_C and

$$\langle \lambda_h, (\mathbf{v}_h - \mathbf{u}_h) \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} \le 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.8}$$

By taking

$$\mathbf{v}_h = \sum_{j=1}^{K_c} \Big[(\mathbf{u}_h(x_j) - 1) \boldsymbol{\nu}_h(x_j) \Big] \psi_j(x) \in \mathbf{K}_h$$

in (3.8) we obtain $\lambda_h \ge 0$ on Γ^h_C, where we recall that $\{x_j\}_{j=1}^{K_c}$ are the nodes of $\mathcal{M}^C_h$ on $\bar{\Gamma}^h_C$. Next we take

$$\mathbf{v}_h = \sum_{j=1}^{K_c} \Big[g_h(x_j) \boldsymbol{\nu}(x_j) \Big] \psi_j(x) \in \mathbf{K}_h$$

in (3.8) to get

$$\langle \lambda_h, g_h - \mathbf{u}_h \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} \le 0.$$

But, we know from $\lambda_h \geq 0$ and $g_h - \mathbf{u}_h \cdot \boldsymbol{\nu}_h = -p_h \geq 0$ on Γ_C^h that

$$\langle \lambda_h, g_h - \mathbf{u}_h \cdot \boldsymbol{\nu}_h \rangle_{\Gamma_C^h}^h \geq 0.$$

Thus, we conclude that

$$\langle \lambda_h, p_h \rangle_{\Gamma_C^h}^h = 0. \tag{3.9}$$

The remainder of the argument is similar to that in the proof of Theorem 2.1. From (3.7) we have

$$G_h(\mathbf{u}_h) \leq G_h(\mathbf{v}_h) + \langle \lambda_h, (\mathbf{v}_h - \mathbf{u}_h) \cdot \boldsymbol{\nu}_h \rangle_{\Gamma_C^h}^h, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h$$

which yields, after using (3.9) and $\lambda_h \geq 0$,

$$G_h(\mathbf{u}_h) \leq G_h(\mathbf{v}_h) + \langle \lambda_h, \mathbf{v}_h \cdot \boldsymbol{\nu}_h - g_h - q_h \rangle_{\Gamma_C^h}^h, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, q_h \in H_h \text{ s.t. } q_h \leq 0 \text{ on } \Gamma_C^h.$$

This implies

$$L_r(\mathbf{u}_h, p_h; \lambda_h) \leq L_r(\mathbf{v}_h, q_h; \lambda_h) \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times H_h.$$

Finally, we know from the definition of p_h that

$$L_r(\mathbf{u}_h, p_h; \mu_h) \leq L_r(\mathbf{u}_h, p_h; \lambda_h) \quad \forall \mu_h \in H_h.$$

This completes the proof. □

Remark 3.2. We observe that in order to obtain the existence and uniqueness of saddle points to L_r and the equivalence of the discrete saddle formulation to the discrete variational inequality (3.6), the inf-sup condition in Lemma 3.1 is crucial. If we use piecewise constant finite element, which is used in [12, Chapter 7] for 2D case, to approximate the multiplier instead of piecewise linear finite element here, the inf-sup condition in Lemma 3.1 is in general *not* valid. More precisely, let $\tilde{H}_h$ be the piecewise constant finite element space over Γ_C^h, we claim that the following inf-sup relation is in general not true

$$\inf_{0 \neq \mu_h \in \tilde{H}_h} \sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{\int_{\Gamma_C^h} \mu_h (\mathbf{v}_h \cdot \boldsymbol{\nu}_h) ds}{\| \mu_h \|_{L^2(\Gamma_C^h)} \| \mathbf{v}_h \|_{H^1(\Omega)}} \geq \alpha_h > 0. \tag{3.10}$$

In fact (3.10) implies that the number of nodes on Γ_C^h must be greater than the number of faces in $\mathcal{B}_h$. It is easy to construct counterexamples violating this restriction. Finally we remark that (3.10) is indeed valid in 2D case, however, in this situation, the saddle point formulation is not equivalent to the discrete variational inequality in the form (3.6).

Remark 3.3. The saddle point problem (Q_h) can be solved by Uzawa's algorithm or the decomposition Uzawa's algorithms studied in [11], [12]. For example, the so-called algorithm ALG2 reads as follows:

- Given $(p_h^0, \lambda_h^1) \in H_h^- \times H_h$, where $H_h^- = \{q_h \in H_h : \; q_h \le 0 \;\text{ on } \Gamma_C^h\}$;
- For any $n \ge 1$, (p_h^{n-1}, λ_h^n) known, find $(\mathbf{u}_h^n, p_h^n, \lambda_h^{n+1}) \in \mathbf{V}_h \times H_h^- \times H_h$ such that

$$a_h(\mathbf{u}_h^n, \mathbf{v}_h) + r\langle \mathbf{u}_h^n \cdot \boldsymbol{\nu}_h - g_h - p_h^{n-1}, \mathbf{v}_h \cdot \boldsymbol{\nu}_h \rangle_{\Gamma_C^h}^h = L_h(\mathbf{v}_h) - \langle \lambda_h^n, \mathbf{v}_h \cdot \boldsymbol{\nu}_h \rangle_{\Gamma_C^h}^h, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h;$$

$$p_h^n = \min(0, I_h(\mathbf{u}_h^n \cdot \boldsymbol{\nu}_h) - g_h + \frac{1}{r}\lambda_h^n);$$

$$\lambda_h^{n+1} = \lambda_h^n + \rho(I_h(\mathbf{u}_h^n \cdot \boldsymbol{\nu}_h) - g_h - p_h^n), \quad 0 < \rho < \frac{1+\sqrt{5}}{2}\, r.$$

4 Finite Element Error Analysis

We begin by stating the hypotheses concerning the data and solution.

(H1) $E_{ijkl} \in W^{1,\infty}(\Omega), \mathbf{f} \in W^{1,\beta}(\Omega)^3, \mathbf{t} \in C^{0,1}(\bar{\Gamma}_F)^3$ and $g \in H^{3/2}(\Omega)$, where $\beta > 3$;

(H2) The solution of Problem (P) satisfies $\mathbf{u} \in H^2(\Omega)^3$.

Let us denote by Ω_h^{ext} the union of Ω_h and those elements which are reflections across Γ_h of elements with one face on Γ_h (boundary elements). Since $\Gamma \in C^{2,1}$, the property $\Omega \subset \Omega_h^{\text{ext}}$ holds for all h less than some $h_0 > 0$. For any $\chi_h \in W_h$ we denote by χ_h^{ext} the extension of χ_h to Ω_h^{ext} such that $\chi_h^{\text{ext}} = \chi_h$ on Ω_h and χ_h^{ext} is linear in the union of any boundary element and its reflected element. It is proved in [19, Lemma 1] that for any $\chi_h \in W_h$, we have

$$C_1 \| \varphi \|_{L^2(\Omega)} \le \| \varphi \|_{L^2(\Omega_h)} \le C_2 \| \varphi \|_{L^2(\Omega)} \tag{4.1}$$

for either $\varphi = \chi_h^{\text{ext}}$ or $\varphi = \nabla \chi_h^{\text{ext}}$. Whenever no confusion is possible, we shall use from now on the same symbol χ_h to denote both the discrete function in W_h and its extension χ_h^{ext}.

The following theorem is the main result of this section.

Theorem 4.1 *Under the assumptions* (H1)-(H2), *there exists a constant $h_0 > 0$ such that for all $h \le h_0$, we have*

$$\| \mathbf{u} - \mathbf{u}_h \|_{H^1(\Omega)} \le C h^{3/4}.$$

To prove this theorem, we denote by $\mathbf{e}_h = \mathbf{u}_h - I_h \mathbf{u} \in \mathbf{V}_h$ and note that by using Green formula and (2.3)-(2.7),

$$\begin{aligned} a(\mathbf{u}, \mathbf{e}_h) &= \int_\Omega \sigma_{ij}(\mathbf{u})(\mathbf{e}_h)_{i,j} dx \\ &= \int_\Omega \mathbf{f} \cdot \mathbf{e}_h dx + \int_{\Gamma_D \cup \Gamma_F \cup \Gamma_C} (\sigma(\mathbf{u})\boldsymbol{\nu}) \cdot \mathbf{e}_h ds \\ &= L(\mathbf{e}_h) - \langle \lambda, \mathbf{e}_h \cdot \boldsymbol{\nu} \rangle_{\Gamma_C} + \int_{\Gamma_D} (\sigma(\mathbf{u})\boldsymbol{\nu}) \cdot \mathbf{e}_h ds. \end{aligned}$$

Thus, upon using (3.2), we obtain easily

$$\begin{aligned}
a(\mathbf{e}_h, \mathbf{e}_h) &= a(\mathbf{u} - I_h\mathbf{u}, \mathbf{e}_h) + a(\mathbf{u}_h, \mathbf{e}_h) - a(\mathbf{u}, \mathbf{e}_h) \\
&= a(\mathbf{u} - I_h\mathbf{u}, \mathbf{e}_h) + \Big[a(\mathbf{u}_h, \mathbf{e}_h) - a_h(\mathbf{u}_h, \mathbf{e}_h)\Big] \\
&+ \Big[L_h(\mathbf{e}_h) - L(\mathbf{e}_h)\Big] + \Big[\langle \lambda, \mathbf{e}_h \cdot \boldsymbol{\nu}\rangle_{\Gamma_C} - \langle \lambda_h, \mathbf{e}_h \cdot \boldsymbol{\nu}_h\rangle^h_{\Gamma^h_C}\Big] \\
&- \int_{\Gamma_D} (\sigma(\mathbf{u})\boldsymbol{\nu}) \cdot \mathbf{e}_h ds =: \text{(I)} + \cdots + \text{(V)}.
\end{aligned} \tag{4.2}$$

Now we are going to estimate the various terms in (4.2).

Lemma 4.1 *There exists a constant $m^* > 0$ independent of $h \le h_0$ such that*

$$a(\mathbf{v}_h, \mathbf{v}_h) \ge m^* \| \mathbf{v}_h \|^2_{H^1(\Omega)} \qquad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

Proof. By Korn's inequality and the the assumption (2.2) we know that, for any $\mathbf{v} \in H^1(\Omega)^3$,

$$\begin{aligned}
\| \mathbf{v} \|^2_{H^1(\Omega)} &\le C \int_\Omega \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) dx + C\| \mathbf{v} \|^2_{L^2(\Gamma_D)} \\
&\le Ca(\mathbf{v}, \mathbf{v}) + C\| \mathbf{v} \|^2_{L^2(\Gamma_D)}
\end{aligned}$$

where $\varepsilon_{ij}(\mathbf{v}) = (1/2)(v_{i,j} + v_{j,i})$. But since $\mathbf{v}_h = 0$ on Γ^h_D for any $\mathbf{v}_h \in \mathbf{V}_h$, from [19, Lemma 3] we know that

$$\| \mathbf{v}_h \|_{L^2(\Gamma_D)} \le Ch^{3/2} \| \mathbf{v}_h \|_{H^1(\Omega)} \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{4.3}$$

This completes the proof for appropriately small $h_0 > 0$. $\Box$

Lemma 4.2 *We have*

$$|L(\mathbf{v}_h) - L_h(\mathbf{v}_h)| \le Ch \| \mathbf{v}_h \|_{H^1(\Omega)}, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h \tag{4.4}$$

$$|a(\mathbf{v}_h, \mathbf{w}_h) - a_h(\mathbf{v}_h, \mathbf{w}_h)| \le Ch \| \mathbf{v}_h \|_{H^1(\Omega)} \| \mathbf{w}_h \|_{H^1(\Omega)}, \qquad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h. \tag{4.5}$$

Proof. Since $\Gamma \in C^{2,1}$, we can extend $E_{ijkl}, \mathbf{f}$ and $\mathbf{t}$ to $\mathbf{R}^3$ such that $E_{ijkl} \in W^{1,\infty}(\mathbf{R}^3), \mathbf{f} \in W^{1,\beta}(\mathbf{R}^3)^3$ and $\mathbf{t} \in C^{0,1}_0(\mathbf{R}^3)^3$ for some $\beta > 3$. To show (4.4) we note first that

$$\begin{aligned}
L(\mathbf{v}_h) - L_h(\mathbf{v}_h) &= \int_\Omega (\mathbf{f} - I_h f) \cdot \mathbf{v}_h dx + \Big[\int_\Omega (I_h \mathbf{f}) \cdot \mathbf{v}_h dx - \int_{\Omega_h} I_h(\mathbf{f} \cdot \mathbf{v}_h) dx\Big] \\
&+ \int_{\Gamma_F} (\mathbf{t} - I_h \mathbf{t}) \cdot \mathbf{v}_h ds + \Big[\int_{\Gamma_F} (I_h \mathbf{t}) \cdot \mathbf{v}_h ds - \int_{\Gamma^h_F} I_h(\mathbf{t} \cdot \mathbf{v}_h) ds\Big] \\
&=: \text{(VI)}_1 + \cdots + \text{(VI)}_4.
\end{aligned} \tag{4.6}$$

By standard finite element interpolation theory [6], we know that, for $\beta > 3$,

$$\begin{aligned}
|(\mathrm{VI})_1| &\leq Ch\|\,\mathbf{f}\,\|_{W^{1,\beta}(\Omega)}\|\,\mathbf{v}_h\,\|_{L^2(\Omega)} \leq Ch\|\,\mathbf{v}_h\,\|_{H^1(\Omega)}, \\
|(\mathrm{VI})_3| &\leq Ch\|\,\mathbf{t}\,\|_{C^{0,1}(\Gamma_F)}\|\,\mathbf{v}_h\,\|_{L^2(\Gamma_F)} \leq Ch\|\,\mathbf{v}_h\,\|_{H^1(\Omega)}.
\end{aligned}$$

On the other hand, arguing as in [19, Lemma 2] we obtain easily that

$$\begin{aligned}
|(\mathrm{VI})_2| &\leq Ch^2\|\,I_h\mathbf{f}\,\|_{H^1(\Omega)}\|\,\mathbf{v}_h\,\|_{H^1(\Omega)} \leq Ch^2\|\,\mathbf{v}_h\,\|_{H^1(\Omega)}, \\
|(\mathrm{VI})_3| &\leq Ch\|\,I_h\mathbf{t}\,\|_{H^1(\Omega)}\|\,\mathbf{v}_h\,\|_{H^1(\Omega)} \leq Ch\|\,\mathbf{v}_h\,\|_{H^1(\Omega)}.
\end{aligned}$$

Substituting these estimates into (4.6) we get (4.4). The estimate (4.5) can be proved similarly. □

Lemma 4.3 *We have*

$$\langle \lambda_h, \mathbf{e}_h \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} \geq 0, \tag{4.7}$$

$$\langle \lambda, \mathbf{e}_h \cdot \boldsymbol{\nu} \rangle_{\Gamma_C} \leq Ch^{3/2} + Ch\|\,\mathbf{e}_h\,\|_{H^1(\Omega)}. \tag{4.8}$$

Proof. 1. We first note that since $\mathbf{u} \cdot \boldsymbol{\nu} - g \leq 0$ on Γ_C, we have $I_h(\mathbf{u} \cdot \boldsymbol{\nu}) - g_h \leq 0$ on Γ^h_C. Thus from the definition of the quadrature formula and (3.4) we obtain

$$\begin{aligned}
\langle \lambda_h, \mathbf{e}_h \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} = \langle \lambda_h, (\mathbf{u}_h - I_h\mathbf{u}) \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} &= \langle \lambda_h, g_h - (I_h\mathbf{u}) \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} \\
&\geq \langle \lambda_h, I_h(\mathbf{u} \cdot \boldsymbol{\nu}) - (I_h\mathbf{u}) \cdot \boldsymbol{\nu}_h \rangle^h_{\Gamma^h_C} = 0.
\end{aligned}$$

This proves (4.7).

2. To show (4.8) we introduce the local coordinates along Γ^h_C. For any $S \in \mathcal{M}^C_h$, let $\tilde{S}$ denote the corresponding reflected element across Γ^h_C. We set $\Gamma_S := \Gamma_C \cap (S \cup \tilde{S})$ and $\Gamma^h_S = \Gamma^h_C \cap \partial S$. Let $x^S_3 = \Phi(x'_S)$, where $x'_S = (x^S_1, x^S_2)$, locally describes the manifold Γ_C, and let $x^S_3 = 0$ locally represents Γ^h_C. Then since $\Gamma \in C^{2,1}$, we have

$$|\Phi| \leq Ch^2_S, \quad |\nabla\Phi| \leq Ch_S, \quad |D^2\Phi| \leq C, \tag{4.9}$$

where h_S is the diameter of S.

Since $I_h(\mathbf{u}_h \cdot \boldsymbol{\nu}_h) - g_h \leq 0$ on Γ^h_C, we have

$$\begin{aligned}
\mathbf{e}_h \cdot \boldsymbol{\nu}_h &= (\mathbf{u}_h - I_h\mathbf{u}) \cdot \boldsymbol{\nu}_h \\
&\leq g_h - I_h\mathbf{u} \cdot \boldsymbol{\nu}_h + \mathbf{u}_h \cdot \boldsymbol{\nu}_h - I_h(\mathbf{u}_h \cdot \boldsymbol{\nu}_h) \\
&= -I_hp + I_h(\mathbf{u} \cdot \boldsymbol{\nu}) - I_h\mathbf{u} \cdot \boldsymbol{\nu}_h + \mathbf{u}_h \cdot \boldsymbol{\nu}_h - I_h(\mathbf{u}_h \cdot \boldsymbol{\nu}_h) \\
&= -I_hp + \mathbf{e}_h \cdot \boldsymbol{\nu}_h - I_h(\mathbf{e}_h \cdot \boldsymbol{\nu}_h), \quad \text{on } \Gamma^h_C
\end{aligned}$$

where $p = \mathbf{u} \cdot \boldsymbol{\nu} - g$. Using this relation and $\langle \lambda, p \rangle_{\Gamma_C} = 0$, we have

$$
\begin{aligned}
& \langle \lambda, \mathbf{e}_h \cdot \boldsymbol{\nu} \rangle_{\Gamma_C} \\
= & \sum_{S \in \mathcal{M}_h^C} \int_{\Gamma_S^h} \lambda(x_S', \Phi(x_S')) \Big[\mathbf{e}_h(x_S', \Phi(x_S')) - \mathbf{e}_h(x_S', 0) \Big] \cdot \boldsymbol{\nu} \sqrt{1 + |\nabla \Phi|^2} \, dx_S' \\
+ & \sum_{S \in \mathcal{M}_h^C} \int_{\Gamma_S^h} \lambda(x_S', \Phi(x_S')) \mathbf{e}_h(x_S', 0) \cdot (\boldsymbol{\nu} - \boldsymbol{\nu}_h) \sqrt{1 + |\nabla \Phi|^2} \, dx_S' \\
+ & \sum_{S \in \mathcal{M}_h^C} \int_{\Gamma_S^h} \lambda(x_S', \Phi(x_S')) \Big[(I_h p)(x_S', \Phi(x_S')) - (I_h p)(x_S', 0) \Big] \sqrt{1 + |\nabla \Phi|^2} \, dx_S' \\
+ & \sum_{S \in \mathcal{M}_h^C} \int_{\Gamma_S^h} \lambda(x_S', \Phi(x_S')) (p - I_h p)(x_S', \Phi(x_S')) \sqrt{1 + |\nabla \Phi|^2} \, dx_S' \\
+ & \sum_{S \in \mathcal{M}_h^C} \int_{\Gamma_S^h} \lambda(x_S', \Phi(x_S')) (\mathbf{e}_h \cdot \boldsymbol{\nu}_h - I_h(\mathbf{e}_h \cdot \boldsymbol{\nu}_h))(x_S', 0) \sqrt{1 + |\nabla \Phi|^2} \, dx_S' \\
=: & \; (\mathrm{VII})_1 + \cdots + (\mathrm{VII})_5.
\end{aligned}
\tag{4.10}
$$

Since $\nabla \mathbf{e}_h$ is constant over S, from (4.9) we know that

$$
\begin{aligned}
|\mathbf{e}_h(x_S', \Phi(x_S')) - \mathbf{e}_h(x_S', 0)| & \leq \int_0^{\Phi(x_S')} \Big| \frac{\partial \mathbf{e}_h}{\partial x_3^S}(x_S', \xi) \Big| \, d\xi \\
& \leq C h_S^2 \| \nabla \mathbf{e}_h \|_{L^\infty(S)} \\
& \leq C h^{1/2} \| \nabla \mathbf{e}_h \|_{L^2(S)},
\end{aligned}
\tag{4.11}
$$

Therefore, we obtain

$$
(\mathrm{VII})_1 \leq C h^{3/2} \| \lambda \|_{L^2(\Gamma_C)} \| \mathbf{e}_h \|_{H^1(\Omega_h)} \leq C h^{3/2} \| \mathbf{e}_h \|_{H^1(\Omega)}.
$$

Next since $|\boldsymbol{\nu} - \boldsymbol{\nu}_h| \leq C h_S$ on Γ_S^h by (4.9), we have upon using (4.1)

$$
(\mathrm{VII})_2 \leq C h \| \lambda \|_{L^2(\Gamma_C)} \| \mathbf{e}_h \|_{L^2(\Gamma_C^h)} \leq C h \| \mathbf{e}_h \|_{H^1(\Omega)}.
$$

Similar to the estimation of $(\mathrm{VII})_1$ we have

$$
(\mathrm{VII})_3 \leq C h^{3/2} \| \lambda \|_{L^2(\Gamma_C)} \| I_h p \|_{H^1(\Omega_h)} \leq C h^{3/2},
$$

where since $\Gamma \in C^{2,1}$, we have extended $p \in H^{3/2}(\Gamma_C)$ to a function in $H^2(\mathbf{R}^3)$ which, by finite element stability estimates, yields $\| I_h p \|_{H^1(\Omega_h)} \leq C$.
By finite element interpolation estimates and (4.9) we get

$$
\begin{aligned}
(\mathrm{VII})_4 & \leq \sum_{S \in \mathcal{M}_h^C} C h_S^{3/2} \| \lambda \|_{L^2(\Gamma_S)} \| p(\cdot, \Phi(\cdot)) \|_{H^{3/2}(\Gamma_S^h)} \\
& \leq C h^{3/2} \| \lambda \|_{L^2(\Gamma_C)} \| p \|_{H^{3/2}(\Gamma_C)} \leq C h^{3/2}.
\end{aligned}
$$

Finally, by the standard finite element interpolation theory and the fact that $\mathbf{e}_h$ is linear over S, we have

$$\begin{aligned}
\|\mathbf{e}_h\cdot\boldsymbol{\nu}_h - I_h(\mathbf{e}_h\cdot\boldsymbol{\nu}_h)\|_{L^\infty(\Gamma_S^h)} &\le Ch_S^2\|\,\mathbf{e}_h\cdot\boldsymbol{\nu}_h\,\|_{W^{2,\infty}(\Gamma_S^h)}\\
&\le Ch_S^2\|\,\mathbf{e}_h\,\|_{W^{1,\infty}(\Gamma_S^h)}\\
&\le Ch_S^{1/2}\|\,\mathbf{e}_h\,\|_{H^1(S)}. \qquad (4.12)
\end{aligned}$$

This implies

$$(\mathrm{VII})_5 \le Ch^{3/2}\|\,\lambda\,\|_{L^2(\Gamma_C)}\|\,\mathbf{e}_h\,\|_{H^1(\Omega)} \le Ch^{3/2}\|\,\mathbf{e}_h\,\|_{H^1(\Omega)}.$$

Now the desired estimate (4.8) follows from above estimates and (4.10). $\square$

Proof of Theorem 4.1. The starting point is (4.2). By finite element interpolation theory we have

$$(\mathrm{I}) \le Ch\|\,\mathbf{u}\,\|_{H^2(\Omega)}\|\,\mathbf{e}_h\,\|_{H^1(\Omega)} \le Ch\|\,\mathbf{e}_h\,\|_{H^1(\Omega)}.$$

By Lemmas 4.2-4.3, we obtain

$$(\mathrm{II}) + \cdots + (\mathrm{IV}) \le Ch\|\,\mathbf{e}_h\,\|_{H^1(\Omega)} + Ch^{3/2}.$$

For the last term, we use (4.3) and get

$$(\mathrm{V}) \le C\|\,\sigma(\mathbf{u})\boldsymbol{\nu}\,\|_{L^2(\Gamma_D)}\|\,\mathbf{e}_h\,\|_{L^2(\Gamma_D)} \le Ch^{3/2}\|\,\mathbf{e}_h\,\|_{H^1(\Omega)}.$$

Now the desired estimate follows from substituting these estimates into (4.2) and using Lemma 4.1. $\square$

Remark 4.1. Under additional assumptions that $\mathbf{u}\in C^{0,1}(\bar\Omega)^3$ and the boundary of the coincidence set $\mathcal{C} = \{x\in\Gamma_C:\ \mathbf{u}(x)\cdot\boldsymbol{\nu}(x) - g(x) = 0\}$ has finite length, we can improve the error estimate in Theorem 4.1 to $O(h)$, as observed in [3] for unilateral problem for Laplacian operator $-\Delta$. In fact, in this case, the only troublesome terms $(\mathrm{VII})_3, (\mathrm{VII})_4$ in (4.10) can instead be estimated as follows. First we have $\lambda\in C(\bar\Gamma_C)$ by Theorem 2.1 and $\mathbf{u}\in C^{0,1}(\bar\Omega)^3$. Since $\lambda = 0$ on $\Gamma_C\backslash\mathcal{C}$, we have

$$(\mathrm{VII})_3 = \sum_{S\in\mathcal{M}_h^C,(S\cup\tilde S)\cap\mathcal{C}\neq\emptyset}\int_{\Gamma_S^h}\lambda(x_S',\Phi(x_S'))\Big[(I_hp)(x_S',\Phi(x_S')) - (I_hp)(x_S',0)\Big]\sqrt{1+|\nabla\Phi|^2}\,dx_S'.$$

By the argument in deriving (4.11) we get

$$|(I_hp)(x_S',\Phi(x_S')) - (I_hp)(x_S',0)| \le Ch_S^2\|\,\nabla I_hp\,\|_{L^\infty(S)} \le Ch_S.$$

Thus, after noticing that the number of the elements $S \in \mathcal{M}_h^C$ satisfying $(S \cup \tilde{S}) \cap \mathcal{C} \neq \emptyset$ is of order $O(h)$ by the quasi-uniformness of the mesh $\mathcal{M}_h$, we obtain

$$(\mathrm{VII})_3 \leq \sum_{S \in \mathcal{M}_h^C, (S \cup \tilde{S}) \cap \mathcal{C} \neq \emptyset} C h_S^3 \leq C h^2.$$

Similarly, we have $(\mathrm{VII})_4 \leq C h^2$. Using these new estimates, we note that (4.8) can be improved to

$$\langle \lambda, \mathbf{e}_h \cdot \boldsymbol{\nu} \rangle_{\Gamma_C} \leq C h^2 + C h \| \mathbf{e}_h \|_{H^1(\Omega)}.$$

This proves that $\| \mathbf{u} - \mathbf{u}_h \|_{H^1(\Omega)} \leq C h$ by the argument in the proof of Theorem 4.1.

5 Conclusions

In this paper we first transformed the Signorini elastic contact problem into a saddle point problem of some augmented Lagrangian functional. The saddle point problem then was solved by finite element methods. In particular, the unique existence of the discrete saddle point problem and the equivalence of saddle point formulation to the discrete variational inequality were established via a discrete inf-sup condition. Optimal error estimates for the finite element approximation were derived for general smooth domains which are not necessarily convex.

The results obtained in this paper can be used to solve more general elastic contact problems with Coulomb's friction law by incremental methods and can be extended directly to solve the frictionless contact problems of two elastic bodies. The finite element analysis developed in this paper for Signorini problem is also useful to analyze the augmented Lagrangian methods solving other more complicated nonlinear elastic problems [12]. Finally, we remark that the numerical results in [12, Chapter 7] and [1] indicated clearly the effectiveness of the augmented Lagrangian methods solving Signorini problems.

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