Reverse Time Migration for Reconstructing Extended Obstacles in Planar Acoustic Waveguides

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Abstract   We propose a new reverse time migration method for reconstructing extended obstacles in the planar waveguide using acoustic waves at a fixed frequency. We prove the resolution of the reconstruction method in terms of the aperture and the thickness of the waveguide. The resolution analysis implies that the imaginary part of the cross-correlation imaging function is always positive and thus may have better stability properties. Numerical experiments are included to illustrate the powerful imaging quality and to confirm our resolution results.

Keywords   Reverse time migration, planar waveguide, resolution analysis, extended obstacle

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1   Introduction

We propose a reverse time migration (RTM) method to find the support of an unknown obstacle embedded in a planar acoustic waveguide from the measurement of the wave field on part of the boundary of the waveguide which is far away from the obstacle (see Figure 1). Let $\mathbb{R}_h^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \in (0, h)\}$ be the waveguide of thickness $h > 0$. Denote by $\Gamma_0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$ and $\Gamma_h = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = h\}$ the boundaries of $\mathbb{R}_h^2$. Let the obstacle occupy a bounded Lipschitz domain D included in $B_R = (-R, R) \times (0, h)$, $R > 0$, with ν the unit outer normal to its boundary Γ_D. We assume the incident wave is a point source excited at $x_s \in \Gamma_h$. The measured wave field satisfies the following equations:

$$\Delta u + k^2 u = -\delta_{x_s}(x) \qquad \text{in } \mathbb{R}_h^2 \backslash \bar{D}, \tag{1.1}$$
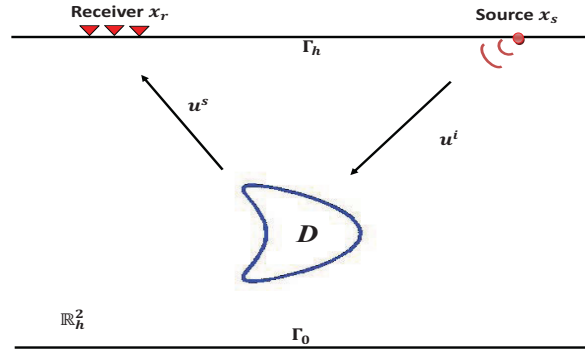
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Figure 1   The geometric setting of the inverse problem in planar waveguides.

$$\frac{\partial u}{\partial \nu} + \mathbf{i}k\eta(x)u = 0 \qquad \text{on } \Gamma_D, \tag{1.2}$$

$$u = 0 \ \text{ on } \Gamma_0, \qquad \frac{\partial u}{\partial x_2} = 0 \ \text{ on } \Gamma_h. \tag{1.3}$$

Here $k > 0$ is the wave number and $\eta(x) > 0$ is a bounded function on Γ_D. The equation (1.1) is understood as the limit when $x_s \in \mathbb{R}_h^2 \backslash \bar{D}$ tends to Γ_h whose precise meaning will be given below after we introduce the definition of the radiation condition and the waveguide Green function. The impedance boundary condition in (1.2) is assumed only for the convenience of the analysis of this paper. The RTM method studied in this paper does not require any a priori information of the physical properties of the obstacle such as penetrable and non-penetrable, and for the non-penetrable obstacles, the type of boundary conditions on the boundary of the obstacle (see section 6 below).

Now we introduce the radiation condition for the planar waveguide problem [27]. Since $D \subset B_R$, we have by separation of variables the following mode expansion:

$$u(x_1, x_2) = \sum_{n=1}^{\infty} u_n(x_1)\sin(\mu_n x_2), \qquad \forall\, |x_1| > R, \tag{1.4}$$

where $\mu_n = \frac{2n-1}{2h}\pi$, $n = 1, 2, \cdots$, are called *cut-off* frequencies. In this paper we will always assume

$$k \neq \frac{2n-1}{2h}\pi, \quad n = 1, 2, \cdots. \tag{1.5}$$

The mode expansion coefficients $u_n(x_1)$, $n = 1, 2, \cdots$, satisfy the 1D Helmholtz equation:

$$u_n'' + \xi_n^2 u_n = 0, \qquad \forall\, |x_1| > R, \quad n = 1, 2, ..., \tag{1.6}$$

where $\xi_n = \sqrt{k^2 - \mu_n^2}$ if $k > \mu_n$ and $\xi_n = \mathbf{i}\sqrt{\mu_n^2 - k^2}$ if $k < \mu_n$. The radiation condition for the planar waveguide problem is then to impose the mode expansion coefficient $u_n(x_1)$ to satisfy

$$\lim_{|x_1| \to \infty} \left(\frac{\partial u_n}{\partial |x_1|} - \mathbf{i}\xi_n u_n \right) = 0, \quad n = 1, 2, \cdots, \tag{1.7}$$

which guarantees the uniqueness of the solution of the 1D Helmholtz equation (1.6). The existence and uniqueness of the waveguide scattering problem (1.1)-(1.3) with the radiation condition (1.7) is an intensively studied subject in the literature, see e.g. [2,20–22,27]. The difficulty is the possible existence of the so-called embedded trapped modes which destroys the uniqueness of the solution [19]. In this paper we will show that the impedance boundary condition on the scatterer guarantees the uniqueness of the scattering solution. We also prove the existence of the solution by the limiting absorption principle.

It is well known that imaging a scatterer in a waveguide is much more challenging than in the free space. Indeed, because of the presence of two parallel infinite boundaries of the waveguide, only a finite number of modes can propagate at long distance, while the other modes decay exponentially [27]. We refer to [1] for MUSIC type algorithm to locate small inclusions, [28] for the generalized dual space method, [3, 6, 24] for the linear sampling method, [25] for a selective imaging method based on Kirchhoff migration, and the inversion method in [23] for reconstructing obstacles in waveguides.

The RTM method, which consists of back-propagating the complex conjugated data into the background medium and computing the cross-correlation between the incident wave field and the backpropagation field to output the final imaging profile, is nowadays widely used in exploration geophysics [4, 5, 13]. In [10, 11], the RTM method for reconstructing extended targets using acoustic and electromagnetic waves at a fixed frequency in the free space is proposed and studied. The resolution analysis in [10, 11] is achieved without using the small inclusion or geometrical optics assumption previously made in the literature.

The purpose of this paper is to extend the RTM method in [10, 11] to find extended targets in the planar acoustic waveguide. Our new RTM algorithm is motivated by a generalized Helmholtz-Kirchhoff identity for the waveguide scattering problems. We show our new imaging function enjoys the nice feature that it is always positive and thus may have better stability properties. The key ingredient in the analysis is a decay estimate of the difference of the Green function for the waveguide problem and the half space Green function. We also refer to [17] for the study of the resolution of time-reversal experiments.

The rest of this paper is organized as follows. In section 2 we introduce some necessary results concerning the direct scattering problem. In section 3 we prove the generalized Helmholtz-Kirchhoff identity and introduce our RTM algorithm. In section 4 we study the resolution of the finite aperture Helmholtz-Kirchhoff function which plays a key role in the resolution analysis of RTM algorithm in section 5. In section 6 we consider the extension of the resolution results for reconstructing penetrable obstacles or non-penetrable obstacles with sound soft or sound hard boundary conditions. In section 7 we report extensive numerical experiments to show the competitive performance of the RTM algorithm. In section 8 we include some concluding remarks. The appendix is devoted to the proof of the existence of the solution of the direct scattering problem by the limiting absorption principle.

2   Direct scattering problem

We start by introducing the Green function $N(x, y)$, where $y \in \mathbb{R}_h^2$, which is the radiation solution satisfying the equations:

$$\Delta N(x, y) + k^2 N(x, y) = -\delta_y(x) \qquad \text{in } \mathbb{R}_h^2,$$

$$N(x, y) = 0 \text{ on } \Gamma_0, \qquad \frac{\partial N(x, y)}{\partial x_2} = 0 \text{ on } \Gamma_h.$$

Let $\hat{N}_y(\xi, x_2) = \int_{-\infty}^{\infty} N(x, y) e^{-\mathbf{i}(x_1 - y_1)\xi} dx_1$ be the Fourier transform in the first variable. It is easy to find by the assumption that $N(x, y)$ is a radiation solution that

$$\hat{N}_y(\xi, x_2) = \frac{\mathbf{i}}{2\mu} \left(e^{\mathbf{i}\mu|x_2 - y_2|} - e^{\mathbf{i}\mu(x_2 + y_2)} - \frac{2 \sin(\mu x_2)}{\cos(\mu h)} \sin(\mu y_2) e^{\mathbf{i}\mu h} \right), \tag{2.1}$$

where $\mu = \sqrt{k^2 - \xi^2}$ and we choose the branch cut of $\sqrt{z}$ such that $\text{Re}\,(\sqrt{z}) \geqslant 0$ throughout the paper. By using the limiting absorption principle one can obtain the following formula for the Green function by taking the inverse Fourier transform on the Sommerfeld Integral Path (SIP) (see Figure 2):

$$N(x, y) = \frac{1}{2\pi} \int_{\text{SIP}} \hat{N}_y(\xi, x_2) e^{\mathbf{i}\xi(x_1 - y_1)} d\xi. \tag{2.2}$$

We refer to [12, Chapter 2] for more discussion on the SIPs. We will also use the following well-known *normal mode expression* for the Green function $N(x, y)$, see e.g. [27]:

$$N(x, y) = \sum_{n=1}^{\infty} \frac{\mathbf{i}}{h\xi_n} \sin(\mu_n x_2) \sin(\mu_n y_2) e^{\mathbf{i}\xi_n|x_1 - y_1|}, \tag{2.3}$$
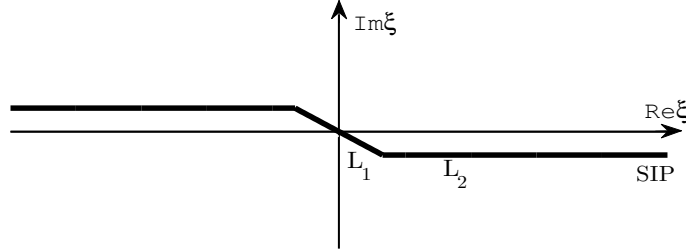
Figure 2   The Sommerfeld Integral Path (SIP).

where $\mu_n = \frac{2n-1}{2h}\pi$ and $\xi_n = \sqrt{k^2 - \mu_n^2}$ if $k > \mu_n$, $\xi_n = \mathbf{i}\sqrt{\mu_n^2 - k^2}$ if $k < \mu_n$, $n = 1, 2, \cdots$. It is obvious that the series in the normal mode expression is absolutely convergent if $x_1 \neq y_1$. If $x_1 = y_1$ but $x_2 \neq y_2$, the series in (2.3) is also convergent by using the method of Dirichlet's test [26, §8.B.13-15].

Lemma 2.1. *If $|x_1 - y_1| \geqslant \alpha h$ for some constant $\alpha > 0$, then $N(x, y)$ and $\nabla_x N(x, y)$ are uniformly bounded.*

Proof. We only prove $N(x, y)$ is uniformly bounded. The proof for $\nabla_x N(x, y)$ is similar. Let M be the integer such that $\mu_M < k < \mu_{M+1}$. Since $\frac{e^{-\alpha k h \sqrt{t^2-1}}}{\sqrt{t^2-1}}$ is a decreasing function in $(1, \infty)$, we know that

$$\sum_{n=M+1}^{\infty} \frac{1}{h|\xi_n|} e^{-|\xi_n||x_1-y_1|} \leqslant \frac{1}{h|\xi_{M+1}|} + \frac{1}{\pi}\int_1^{\infty} \frac{e^{-\alpha k h\sqrt{t^2-1}}}{\sqrt{t^2-1}} dt \leqslant \frac{1}{h|\xi_{M+1}|} + \frac{1}{\alpha k h\pi}.$$

On the other hand, note that $\left|\sum_{n=1}^{M} \frac{\mathbf{i}}{h\xi_n}\sin(\mu_n x_2)\sin(\mu_n y_2)e^{\mathbf{i}\xi_n|x_1-y_1|}\right| < \sum_{n=1}^{M} \frac{1}{h\xi_n}$, we obtain

$$\sum_{n=1}^{M} \frac{1}{h\xi_n} \leqslant \frac{1}{h|\xi_M|} + \sum_{n=1}^{M-1} \frac{1}{h\xi_n} \leqslant \frac{1}{h|\xi_M|} + \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{1-t^2}}dt = \frac{1}{h|\xi_M|} + \frac{1}{2},$$

where we have used the fact that $\frac{1}{\sqrt{1-t^2}}$ is an increasing function in $(0, 1)$. This completes the proof. $\square$

Now we consider the existence and uniqueness of the radiation solution of the following waveguide problem:

$$\Delta\psi + k^2\psi = 0 \qquad \text{in } \mathbb{R}_h^2 \backslash \bar{D}, \tag{2.4}$$

$$\frac{\partial\psi}{\partial\nu} + \mathbf{i}k\eta(x)\psi = g \qquad \text{on } \Gamma_D, \tag{2.5}$$

$$\psi = 0 \ \text{ on } \Gamma_0, \qquad \frac{\partial\psi}{\partial x_2} = 0 \ \text{ on } \Gamma_h, \tag{2.6}$$

where $g \in H^{-1/2}(\Gamma_D)$. We first show the uniqueness of the solution.

Lemma 2.2. *Let $\eta > 0$ be bounded on Γ_D. The scattering problem (2.4)-(2.6) has at most one radiation solution.*

Proof. We include a proof here for the sake of completeness. Let $g = 0$ in (2.5). We multiply (2.4) by $\bar\psi$ and integrate over $B_R\backslash\bar{D}$ to obtain by integration by parts that

$$-\text{Im}\int_{\Gamma_D} \bar\psi\frac{\partial\psi}{\partial\nu}ds + \text{Im}\int_{\partial B_R} \bar\psi\frac{\partial\psi}{\partial\nu}ds = 0, \tag{2.7}$$

where ν is the unit outer normal to ∂B_R on ∂B_R and to Γ_D on Γ_D. By the boundary condition satisfied by ψ, $\int_{(\Gamma_0\cup\Gamma_h)\cap\partial B_R} \bar\psi\frac{\partial\psi}{\partial\nu}ds = 0$. On the other hand, for $|x_1| > R$, similar to (1.4) we have the mode expansion

$\psi(x) = \sum_{n=1}^{\infty} \psi_n(x_1) \sin(\mu_n x_2)$ with $\psi_n(x_1)$ satisfying (1.6)-(1.7). Thus there exist constants $\psi_n^{\pm}$ such that $\psi_n(x_1) = \psi_n^{\pm} e^{\mathbf{i}\xi_n|x_1|}$ for $\pm x_1 > R$. By the Parseval identity, we have then

$$\int_{\Gamma_R^+ \cup \Gamma_R^-} \bar{\psi} \frac{\partial \psi}{\partial \nu} ds = \frac{h}{2} \sum_{n=1}^{M} \mathbf{i}\xi_n(|\psi_n^+|^2 + |\psi_n^-|^2) - \frac{h}{2} \sum_{n=M+1}^{+\infty} |\xi_n|(|\psi_n^+|^2 + |\psi_n^-|^2)e^{-2|\xi_n|R},$$

where $\Gamma_R^{\pm} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \pm R, x_2 \in (0, h)\}$. Thus by taking the imaginary part of the above identity and inserting it into (2.7) we have

$$-\mathrm{Im} \int_{\Gamma_D} \bar{\psi} \frac{\partial \psi}{\partial \nu} ds + \frac{h}{2} \sum_{n=1}^{M} \xi_n(|\psi_n^+|^2 + |\psi_n^-|^2) = 0. \tag{2.8}$$

By using the impedance condition and the assumption $\eta > 0$ on Γ_D we have $\psi = 0$ on Γ_D and $\psi_n^{\pm} = 0, n = 1, 2, ..., M$. This implies that $\frac{\partial \psi}{\partial \nu} = 0$ on Γ_D. By the unique continuation principle we conclude $\psi = 0$ in $\mathbb{R}_h^2 \backslash \bar{D}$. This completes the proof. □

In this paper, we call $\psi_n^{\pm}, n = 1, 2, \cdots, M$, which are the coefficients of the propagating modes, the *far-field pattern* of the radiation solution ψ of the planar waveguide problem (2.4)-(2.6).

We remark that under some assumption on the geometry of the obstacle, the uniqueness of the solution to the acoustic waveguide scattering problem for the sound soft obstacle was first proved in [20] based on the Rellich type identity. The proof was refined in [22] and was also used in Arens [2] for 3D scattering problems. For general geometry of the obstacle, the embedded trapped mode may appear which makes the uniqueness fail [19].

The following theorem which is useful in our resolution analysis for the RTM algorithm will be proved in the Appendix by using the method of limiting absorption principle.

Theorem 2.3. *Let* $g \in H^{-1/2}(\Gamma_D)$ *and* $\eta(x) > 0$ *be bounded on* Γ_D. *Then the problem (2.4)-(2.6) admits a unique radiation solution* $\psi \in H_{\mathrm{loc}}^1(\mathbb{R}_h^2 \backslash \bar{D})$. *Moreover, for any bounded open set* $\mathcal{O} \subset \mathbb{R}^2 \backslash \bar{D}$, *there exists a constant* C *such that* $\|\psi\|_{H^1(\mathcal{O})} \leqslant C\|g\|_{H^{-1/2}(\Gamma_D)}$.

To conclude this section we remark that the solution u in (1.1)-(1.3) is defined as $u(x) = N(x, y) + u^s(x)$ with $u^s(x)$ being the radiation solution of (2.4)-(2.6) with $g = -\frac{\partial N(x,y)}{\partial \nu} - \mathbf{i}k\eta(x)N(x, y)$.

3 The reverse time migration algorithm

In this section we develop the reverse time migration type algorithm for inverse scattering problems in the planar acoustic waveguide. Let $G(x, y)$ be the half-space Green function, where $y \in \mathbb{R}_+^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$, which satisfies the Sommerfeld radiation condition and the following equations:

$$\begin{aligned} \Delta G(x, y) + k^2 G(x, y) &= -\delta_y(x) & \text{in } \mathbb{R}_+^2, \\ G(x, y) &= 0 & \text{on } \Gamma_0. \end{aligned}$$

It is well known by the image method that

$$G(x, y) = \frac{\mathbf{i}}{4} H_0^{(1)}(k|x - y|) - \frac{\mathbf{i}}{4} H_0^{(1)}(k|x - y'|), \tag{3.1}$$

where $H_0^{(1)}(z)$ is the first Hankel function of zeroth order and $y' = (y_1, -y_2)$ is the image point of $y = (y_1, y_2)$ with respect to $y_2 = 0$.

We start by proving the generalized Helmholtz-Kirchhoff identity which plays a key role in this paper.

Lemma 3.1. *Let* $S(x, y) = N(x, y) - G(x, y)$. *Then we have*

$$\int_{\Gamma_h} \frac{\partial G(x, \zeta)}{\partial \zeta_2} \overline{N(\zeta, y)} ds(\zeta) = 2\mathbf{i} \, \mathrm{Im} \, N(x, y) - S(x, y), \quad \forall x, y \in \mathbb{R}_h^2. \tag{3.2}$$

Proof. Let $x, y \in B_R = (-R, R) \times (0, h)$ for some $R > 0$. Since $\operatorname{Im} G(x, \cdot)$ satisfies the Helmholtz equation, by the integral representation formula we obtain

$$\operatorname{Im} G(x, y) = \int_{\partial B_R} \left(\frac{\partial \operatorname{Im} G(x, \zeta)}{\partial \nu(\zeta)} N(\zeta, y) - \frac{\partial N(\zeta, y)}{\partial \nu(\zeta)} \operatorname{Im} G(x, \zeta) \right) ds(\zeta).$$

Again by the integral representation formula we have

$$\int_{\partial B_R} \left(\frac{\partial G(x, \zeta)}{\partial \nu(\zeta)} N(\zeta, y) - \frac{\partial N(\zeta, y)}{\partial \nu(\zeta)} G(x, \zeta) \right) ds(\zeta) = -N(x, y) + G(x, y) = -S(x, y).$$

Thus, since $\operatorname{Im} G(x, y) = \frac{1}{2\mathbf{i}} (G(x, y) - \overline{G(x, y)})$, we have

$$
\begin{aligned}
2\mathbf{i} \operatorname{Im} G(x, y) + S(x, y) \;=\; & -\int_{\partial B_R} \left(\frac{\partial \overline{G(x, \zeta)}}{\partial \nu(\zeta)} N(\zeta, y) - \frac{\partial N(\zeta, y)}{\partial \nu(\zeta)} \overline{G(x, \zeta)} \right) ds(\zeta) \\
\;=\; & -\int_{\Gamma_h \cap \partial B_R} \frac{\partial \overline{G(x, \zeta)}}{\partial \nu(\zeta)} N(\zeta, y) ds(\zeta) \\
& - \int_{\Gamma_R^\pm} \left(\frac{\partial \overline{G(x, \zeta)}}{\partial \nu(\zeta)} N(\zeta, y) - \frac{\partial N(\zeta, y)}{\partial \nu(\zeta)} \overline{G(x, \zeta)} \right) ds(\zeta), \quad\quad (3.3)
\end{aligned}
$$

where we have used $\frac{\partial N(\zeta, y)}{\partial \nu(\zeta)} = 0$ on Γ_h and $G(x, \zeta) = N(\zeta, y) = 0$ on Γ_0. By (3.1) we know that $|G(x, \zeta)| = O(|x - \zeta|^{-1/2})$ and $|\frac{\partial G(x, \zeta)}{\partial \zeta_1}| = O(|x - \zeta|^{-1/2})$ as $|x - \zeta| \to \infty$. Therefore, by using Lemma 2.1 we conclude that the integral on $\Gamma_R^\pm$ in (3.3) vanishes as $R \to \infty$. This shows by letting $R \to \infty$ that

$$2\mathbf{i} \operatorname{Im} G(x, y) + S(x, y) = -\int_{\Gamma_h} \frac{\partial \overline{G(x, \zeta)}}{\partial \nu(\zeta)} N(\zeta, y) ds(\zeta).$$

This completes the proof by taking the complex conjugate and noticing $2\mathbf{i} \operatorname{Im} G(x, y) + S(x, y) = 2\mathbf{i} \operatorname{Im} N(x, y) + \overline{S(x, y)}$. $\qquad\qquad\square$

Now assume that there are N_s sources and N_r receivers uniformly distributed on Γ_h^d, where $\Gamma_h^d = \{(x_1, x_2) \in \Gamma_h : x_1 \in (-d, d)\}$, $d > 0$ is the aperture. We denote by $\Omega \subset B_d = (-d, d) \times (0, h)$ the sampling domain in which the obstacle is sought. Let $u^i(x, x_s) = N(x, x_s)$ be the incident wave and $u^s(x_r, x_s) = u(x_r, x_s) - u^i(x_r, x_s)$ be the scattered field measured at x_r, where $u(x, x_s)$ is the solution of the problem (1.1)-(1.3) and (1.7). Our RTM algorithm consists of two steps. The first step is the back-propagation in which we back-propagate the complex conjugated data $\overline{u^s(x_r, x_s)}$ into the domain using the half space Green function $G(x, y)$. The second step is the cross-correlation in which we compute the imaginary part of the cross-correlation of $\frac{\partial G(x, y)}{\partial y_2}$ and the back-propagated field.

Algorithm 3.1. (REVERSE TIME MIGRATION)
Given the data $u^s(x_r, x_s)$ which is the measurement of the scattered field at $x_r = (x_1(x_r), x_2(x_r))$ when the source is emitted at $x_s = (x_1(x_s), x_2(x_s))$, $s = 1, \dots, N_s$, $r = 1, \dots, N_r$.
1° Back-propagation: For $s = 1, \dots, N_s$, compute the back-propagation field

$$v_b(z, x_s) = \frac{|\Gamma_h^d|}{N_r} \sum_{r=1}^{N_r} \frac{\partial G(z, x_r)}{\partial x_2(x_r)} \overline{u^s(x_r, x_s)}, \quad \forall z \in \Omega. \quad\quad (3.4)$$

2° Cross-correlation: For $z \in \Omega$, compute

$$I_d(z) = \operatorname{Im} \left\{ \frac{|\Gamma_h^d|}{N_s} \sum_{s=1}^{N_s} \frac{\partial G(z, x_s)}{\partial x_2(x_s)} v_b(z, x_s) \right\}. \quad\quad (3.5)$$

The back-propagation field v_b can be viewed as the solution which satisfies the Sommerfeld radiation condition and the following equations:

$$\Delta v_b(x, x_s) + k^2 v_b(x, x_s) = \frac{|\Gamma_h^d|}{N_r} \sum_{r=1}^{N_r} \overline{u^s(x_r, x_s)} \frac{\partial}{\partial x_2} \delta_{x_r}(x) \quad \text{in } \mathbb{R}_+^2,$$

$$v_b(x, x_s) = 0 \quad \text{on } \Gamma_0.$$

Taking the imaginary part of the cross-correlation of the incident field and the back-propagated field in (3.5) is motivated by the resolution analysis in the next section. It is easy to see that

$$I_d(z) = \operatorname{Im} \left\{ \frac{|\Gamma_h^d||\Gamma_h^d|}{N_s N_r} \sum_{s=1}^{N_s} \sum_{r=1}^{N_r} \frac{\partial G(z, x_r)}{\partial x_2(x_r)} \frac{\partial G(z, x_s)}{\partial x_2(x_s)} \overline{u^s(x_r, x_s)} \right\}, \quad \forall z \in \Omega. \tag{3.6}$$

This formula is used in all our numerical experiments in section 7. By letting $N_s, N_r \to \infty$, we know that (3.6) can be viewed as an approximation of the following continuous integral:

$$\hat{I}_d(z) = \operatorname{Im} \int_{\Gamma_h^d} \int_{\Gamma_h^d} \frac{\partial G(z, x_r)}{\partial x_2(x_r)} \frac{\partial G(z, x_s)}{\partial x_2(x_s)} \overline{u^s(x_r, x_s)} ds(x_s) ds(x_r), \quad \forall z \in \Omega. \tag{3.7}$$

We will study the resolution of the function $\hat{I}_d(z)$ in the section 5. To this end we will first consider the resolution of the finite aperture Helmholtz-Kirchhoff function in the next section.

4 Resolution of the finite aperture Helmholtz-Kirchhoff function

By the Helmholtz-Kirchhoff identity (3.2) we know that for any $x, y \in \mathbb{R}_h$,

$$\int_{\Gamma_h^d} \frac{\partial G(x, \zeta)}{\partial \zeta_2} \overline{N(\zeta, y)} ds(\zeta) = 2\mathbf{i} \operatorname{Im} N(x, y) - S(x, y) - S_d(x, y), \tag{4.1}$$

where

$$S_d(x, y) := \int_{\Gamma_h \setminus \bar{\Gamma}_h^d} \frac{\partial G(x, \zeta)}{\partial \zeta_2} \overline{N(\zeta, y)} ds(\zeta), \quad \forall x, y \in \mathbb{R}_h. \tag{4.2}$$

The integral on the left-hand side of (4.1), $H_d(x, y) = \int_{\Gamma_h^d} \frac{\partial G(x, \zeta)}{\partial \zeta_2} \overline{N(\zeta, y)} ds(\zeta)$, will be called the finite aperture Helmholtz-Kirchhoff function in the following. In this section we will estimate $S(x, y)$ and $S_d(x, y)$ in (4.1) which provides the resolution of $H_d(x, y)$.

We assume the obstacle $D \subset \Omega$ and there exist positive constants c_0, c_1, c_2, where $c_0, c_1 \in (0, 1)$, such that

$$|y_1| \leqslant c_0 d, \quad |y_2| \leqslant c_1 h, \quad k|y_1 - z_1| \leqslant c_2 \sqrt{kh}, \quad \forall y, z \in \Omega. \tag{4.3}$$

The first condition means that the search domain should not be close to the boundary of the aperture. The second condition is rather mild in practical applications as we are interested in finding obstacles far away from the surface of the waveguide where the data is collected. The third condition indicates that the horizontal width of the search domain should not be very large comparing with the thickness of the waveguide. This is reasonable since we are interested in the case when the size of the scatterer is smaller than or comparable with the probe wavelength and the thickness h is large compared with the probe wavelength, i.e., $kh \gg 1$.

We start with the following formula for $S(x, y)$.

Lemma 4.1. *Let $S(x, y) = N(x, y) - G(x, y)$. Then we have*

$$S(x, y) = \frac{1}{2\pi} \int_{\text{SIP}} \hat{S}_y(\xi, x_2) e^{\mathbf{i}\xi|x_1 - y_1|} d\xi, \quad \hat{S}_y(\xi, x_2) = -\frac{2\mathbf{i}}{\mu} \frac{\sin(\mu x_2)}{e^{2\mathbf{i}\mu h} + 1} \sin(\mu y_2) e^{2\mathbf{i}\mu h}.$$

Proof. Let

$$\hat{G}_y(\xi, x_2) = \int_{-\infty}^{\infty} G(x, y) e^{-\mathbf{i}(x_1-y_1)\xi} dx_1, \quad \hat{S}_y(\xi, x_2) = \int_{-\infty}^{\infty} S(x, y) e^{-\mathbf{i}(x_1-y_1)\xi} dx_1,$$

be the Fourier transform of $G(x, y)$ and $S(x, y)$ in the first variable, respectively. It is easy to find that

$$\hat{G}_y(\xi, x_2) = \frac{\mathbf{i}}{2\mu}\left(e^{\mathbf{i}\mu|x_2-y_2|} - e^{\mathbf{i}\mu(x_2+y_2)}\right).$$

Thus by (2.1) we know that

$$\hat{S}_y(\xi, x_2) = -\frac{\mathbf{i}}{\mu}\frac{\sin(\mu x_2)}{\cos(\mu h)}\sin(\mu y_2)e^{\mathbf{i}\mu h}.$$

This completes the proof by taking the inverse Fourier transform along SIP.   □

Theorem 4.2. *Let $kh > \pi/2$ and (4.3) be satisfied. We have*

$$|S(x, y)| \leqslant \frac{C}{|\cos(kh)|}\frac{1}{\sqrt{kh}}, \quad |\nabla_x S(x, y)| \leqslant \frac{Ck}{|\cos(kh)|}\frac{1}{\sqrt{kh}}, \quad \forall x, y \in \Omega, \tag{4.4}$$

where C is a constant independent of k, h but may depend on c_1, c_2.

We remark that since $\mu_1 = \pi/(2h)$, the condition $kh > \pi/2$ means that there exists at least one propagating mode in the received scattering field on Γ_h, which is the minimum requirement that any imaging method could work. We also remark that the decay estimate (4.4) can not hold uniformly for $x, y \in \mathbb{R}_h^2$ since $N(x, y)$ keeps oscillatory and bounded as $|x_1 - y_1| \to \infty$ but $G(x, y)$ decays to 0 as $|x_1 - y_1| \to \infty$.

Proof. Denote by $\gamma = 1/\sqrt{2kh}$ and by the assumption $kh \geqslant \pi/2$, $\gamma \leqslant 1/\sqrt{\pi}$. Write $\xi = \xi_1 + \mathbf{i}\xi_2$, $\xi_1, \xi_2 \in \mathbb{R}$. Let SIP$^+$ be the part of the SIP in the fourth quadrant. By taking the coordinate transform $\xi \to -\xi$ in the second quadrant we know from Lemma 4.1 that

$$S(x, y) = \frac{1}{2\pi}\int_{\text{SIP}^+} \hat{S}_y(\xi, x_2)(e^{\mathbf{i}\xi(x_1-y_1)} + e^{-\mathbf{i}\xi(x_1-y_1)})d\xi := S_1(x, y) + S_2(x, y),$$

where $S_j(x, y) = \frac{1}{2\pi}\int_{L_j} \hat{S}_y(\xi, x_2)(e^{\mathbf{i}\xi(x_1-y_1)} + e^{-\mathbf{i}\xi(x_1-y_1)})d\xi$, $j = 1, 2$, and L_1, L_2 are the sections of SIP$^+$ (see Figure 2):

$$L_1 = \{\xi \in \mathbb{C} : \xi_1 \in (0, k\gamma), \xi_2 = -\xi_1\}, \quad L_2 = \{\xi \in \mathbb{C} : \xi_1 \in (k\gamma, \infty), \xi_2 = -k\gamma\}.$$

Let $\mu = \sqrt{k^2 - \xi^2} = \mu_1 + \mathbf{i}\mu_2$, $\mu_1, \mu_2 \in \mathbb{R}$. It is easy to see that

$$|\mu|^2 = \sqrt{(k^2 - \xi_1^2 + \xi_2^2)^2 + 4\xi_1^2\xi_2^2}, \quad \mu_2 = \frac{\sqrt{2}\,\xi_1|\xi_2|}{\sqrt{|\mu|^2 + (k^2 - \xi_1^2 + \xi_2^2)}}. \tag{4.5}$$

It is clear by using (4.3) that

$$|e^{\mathbf{i}\xi(x_1-y_1)} + e^{-\mathbf{i}\xi(x_1-y_1)}| \leqslant 2e^{-\xi_2|x_1-y_1|} \leqslant 2e^{c_2/\sqrt{2}}, \quad \forall \xi \in L_1 \cup L_2.$$

Thus, since $|\sin(\mu x_2)\sin(\mu y_2)e^{\mathbf{i}\mu h}| \leqslant e^{-\mu_2(2h-x_2-y_2)}$, we have

$$|S_j(x, y)| \leqslant C\left|\int_{L_j} \frac{1}{|\mu|}\frac{1}{|1 + e^{2\mathbf{i}\mu h}|}e^{-\mu_2(2h-x_2-y_2)}d\xi\right|. \tag{4.6}$$

We first estimate $S_1(x, y)$ and thus assume $\xi \in L_1$. By (4.5) it is clear that $|\mu| \geqslant k$. Next

$$\frac{1}{|1 + e^{2\mathbf{i}\mu h}|} \quad \leqslant \quad \left|\frac{1}{1 + e^{2\mathbf{i}\mu h}} - \frac{1}{1 + e^{2\mathbf{i}kh}}\right| + \frac{1}{|1 + e^{2\mathbf{i}kh}|}$$

$$= \frac{1}{2|\cos(kh)|}\left(1 + \frac{|e^{2\mathbf{i}\mu h} - e^{2\mathbf{i}kh}|}{|1 + e^{2\mathbf{i}\mu h}|}\right). \tag{4.7}$$

By using the elementary inequality $1 - e^{-t} \geqslant t - t^2/2$ for $t \geqslant 0$, we have $|1 + e^{2\mathbf{i}\mu h}| \geqslant 1 - e^{-2\mu_2 h} \geqslant 2\mu_2 h(1 - \mu_2 h)$. By (4.5) we have $\mu_2 = \frac{\sqrt{2}\xi_1^2}{\sqrt{|\mu|^2 + k^2}}$ which implies $\mu_2 h \leqslant \frac{\xi_1^2 h}{k} \leqslant \frac{1}{2}$ for $\xi \in L_1$. Therefore $|1 + e^{2\mathbf{i}\mu h}| \geqslant \frac{\sqrt{2}\xi_1^2 h}{\sqrt{|\mu|^2 + k^2}}$. On the other hand, since $\mu = \sqrt{k^2 + 2\mathbf{i}\xi_1^2}$ on L_1, $\mu(0) = k$, by using elementary calculus one obtains

$$
\begin{aligned}
|e^{2\mathbf{i}\mu h} - e^{2\mathbf{i}kh}| &\leqslant \sqrt{2}\max_{t \in (0,\xi_1)}\left|\frac{de^{2\mathbf{i}\mu h}}{d\xi_1}\bigg|_{\xi_1 = t}\right| \times \xi_1 \\
&= \sqrt{2}\max_{t \in (0,\xi_1)}\left(\frac{4hte^{-2\mu_2(t)h}}{|\mu(t)|}\right)\xi_1 \leqslant 4\sqrt{2}\frac{\xi_1^2 h}{k}.
\end{aligned}
$$

Thus by (4.7)

$$\frac{1}{|1 + e^{2\mathbf{i}\mu h}|} \leqslant \frac{C}{|\cos(kh)|}\frac{\sqrt{|\mu|^2 + k^2}}{k} \leqslant \frac{C}{|\cos(kh)|},$$

where we have used the fact that $|\mu| = (k^4 + 4\xi_1^4)^{1/4} \leqslant k(1 + 4\gamma^4)^{1/4} \leqslant Ck$ for $\xi \in L_1$. Now it follows from (4.6) that

$$|S_1(x,y)| \leqslant \frac{C}{|\cos(kh)|}\gamma \leqslant \frac{C}{|\cos(kh)|}\frac{1}{\sqrt{kh}}. \tag{4.8}$$

Now we estimate $S_2(x,y)$ and thus let $\xi \in L_2$. By (4.5) we have $|\mu|^2 \leqslant k^2 + \xi_1^2 + \xi_2^2$ and thus

$$\mu_2 \geqslant \frac{\xi_1|\xi_2|}{\sqrt{k^2 + \xi_2^2}} = \frac{\xi_1\gamma}{\sqrt{1 + \gamma^2}} \geqslant \frac{\xi_1\gamma}{\sqrt{2}}, \tag{4.9}$$

which implies $\mu_2 h \geqslant kh\gamma^2/\sqrt{2} = 1/(2\sqrt{2})$ and consequently $|1 + e^{2\mathbf{i}\mu h}| \geqslant 1 - e^{-1/\sqrt{2}}$ for $\xi \in L_2$. Now by (4.6) we have

$$|S_2(x,y)| \leqslant C\int_{k\gamma}^{+\infty}\frac{1}{|\mu|}e^{-\mu_2(2h - x_2 - y_2)}d\xi_1. \tag{4.10}$$

For $\xi_1 \in (k\gamma, k/\sqrt{\pi})$, we know from (4.5) that $|\mu| \geqslant \sqrt{k^2 - \xi_1^2} \geqslant k\sqrt{1 - 1/\pi}$. This implies by using (4.9) that

$$
\begin{aligned}
\int_{k\gamma}^{\frac{k}{\sqrt{\pi}}}\frac{1}{|\mu|}e^{-\mu_2(2h - x_2 - y_2)}d\xi_1 &\leqslant \frac{C}{k}\int_{k\gamma}^{\frac{k}{\sqrt{\pi}}}e^{-(1-c_1)\frac{\xi_1 h}{\sqrt{kh}}}d\xi_1 \\
&= -\frac{C}{\sqrt{kh}}\left[e^{-(1-c_1)\frac{\xi_1 h}{\sqrt{kh}}}\right]\Big|_{k\gamma}^{\frac{k}{\sqrt{\pi}}} \leqslant C\frac{1}{\sqrt{kh}}.
\end{aligned}
$$

For $\xi_1 > \frac{k}{\sqrt{\pi}}$, by (4.5), we have $|\mu| \geqslant \sqrt{2\xi_1|\xi_2|} \geqslant \frac{\sqrt{2}}{\sqrt[4]{\pi}}k\gamma^{1/2}$. Thus

$$
\begin{aligned}
\int_{\frac{k}{\sqrt{\pi}}}^{+\infty}\frac{1}{|\mu|}e^{-\mu_2(2h - x_2 - y_2)}d\xi_1 &\leqslant \frac{C}{k\gamma^{1/2}}\int_{\frac{k}{\sqrt{\pi}}}^{+\infty}e^{-(1-c_1)\frac{\xi_1 h}{\sqrt{kh}}}d\xi_1 \\
&= -\frac{C}{\sqrt[4]{kh}}\left[e^{-(1-c_1)\frac{\xi_1 h}{\sqrt{kh}}}\right]\Big|_{\frac{k}{\sqrt{\pi}}}^{+\infty} \leqslant C\frac{1}{\sqrt{kh}}.
\end{aligned}
$$

This shows the first estimate in (4.4) upon substituting into (4.10) and noticing (4.8). The estimate for $\nabla_x S(x,y)$ can be proved in a similar way by noticing that

$$\frac{\partial S(x,y)}{\partial x_1} = \frac{1}{2\pi}\int_{\mathrm{SIP+}}\frac{\xi}{\mu}\frac{\sin(\mu x_2)}{\cos(\mu h)}\sin(\mu y_2)e^{\mathbf{i}\mu h}(e^{\mathbf{i}\xi(x_1 - y_1)} - e^{-\mathbf{i}\xi(x_1 - y_1)})d\xi,$$

$$\frac{\partial S(x,y)}{\partial x_2} = \frac{1}{2\pi}\int_{\mathrm{SIP+}}(-\mathbf{i})\frac{\cos(\mu x_2)}{\cos(\mu h)}\sin(\mu y_2)e^{\mathbf{i}\mu h}(e^{\mathbf{i}\xi(x_1 - y_1)} + e^{-\mathbf{i}\xi(x_1 - y_1)})d\xi.$$

Here we omit the details. This completes the proof. $\qquad\square$

Now we consider the effect of the finite aperture by estimating $S_d(x, y)$ in (4.2). We first recall the following estimate for the first Hankel function in [29, P.197].

Lemma 4.3. *For any $t > 0$, we have*

$$H_0^{(1)}(t) = \left(\frac{2}{\pi t}\right)^{1/2} e^{\mathbf{i}(t - \pi/4)} + R_0(t), \quad H_1^{(1)}(t) = \left(\frac{2}{\pi t}\right)^{1/2} e^{\mathbf{i}(t - 3\pi/4)} + R_1(t),$$

where $|R_j(t)| \leqslant Ct^{-3/2}$, $j = 0, 1$, for some constant $C > 0$ independent of t.

For any $x, y \in \Omega$, by the normal mode expression of the Green function $N(x, y)$ in (2.3) we know that

$$S_d(x, y) = \sum_{n=1}^{\infty} \frac{-\mathbf{i}}{h\bar{\xi}_n} \sin(\mu_n h) \sin(\mu_n y_2) \int_{\Gamma_h \setminus \bar{\Gamma}_h^d} \frac{\partial G(x, \zeta)}{\partial \zeta_2} e^{-\mathbf{i}\bar{\xi}_n |\zeta_1 - y_1|} d\zeta_1. \tag{4.11}$$

For $\zeta = (\zeta_1, h) \in \Gamma_h$, $G(x, \zeta) = \frac{\mathbf{i}}{4} H_0^{(1)}(k|x - \zeta|) - \frac{\mathbf{i}}{4} H_0^{(1)}(k|x - \zeta'|)$, where $\zeta' = (\zeta_1, -h)$. Thus, since $H_0^{(1)'}(\xi) = -H_1^{(1)}(\xi)$ for any $\xi \in \mathbb{C}$,

$$\frac{\partial G(x, \zeta)}{\partial \zeta_2} = f(x_1, \zeta_1, h + x_2) - f(x_1, \zeta_1, h - x_2), \quad \forall \zeta \in \Gamma_h, \tag{4.12}$$

where

$$f(x_1, \zeta_1, t) = \frac{\mathbf{i}}{4} H_1^{(1)}(k\Theta) \frac{kt}{\Theta}, \quad \Theta = \sqrt{(\zeta_1 - x_1)^2 + t^2}, \quad t \in [h - x_2, h + x_2].$$

By Lemma 4.3 we have

$$f(x_1, \zeta_1, t) = \frac{\mathbf{i}}{4} e^{-\mathbf{i}\frac{3\pi}{4}} \left(\frac{2}{\pi k\Theta}\right)^{1/2} \frac{kt}{\Theta} e^{\mathbf{i}k\Theta} + \frac{\mathbf{i}}{4} R_1(k\Theta) \frac{kt}{\Theta}.$$

Inserting the above equation into (4.11) we obtain

$$
\begin{aligned}
|S_d(x, y)| \quad \leqslant \quad & \max_{t \in [h - x_2, h + x_2]} \sum_{n=1}^{\infty} \frac{k^{1/2} t}{h|\xi_n|} \left| \int_{\Gamma_h \setminus \bar{\Gamma}_h^d} \Theta^{-3/2} e^{\mathbf{i} f_n(x_1, \zeta_1, t)} d\zeta_1 \right| \\
& + \max_{t \in [h - x_2, h + x_2]} \sum_{n=1}^{\infty} \frac{1}{h|\xi_n|} \left| \int_{\Gamma_h \setminus \bar{\Gamma}_h^d} R_1(k\Theta) \frac{kt}{\Theta} e^{-\mathbf{i}\bar{\xi}_n |\zeta_1 - y_1|} d\zeta_1 \right|,
\end{aligned} \tag{4.13}
$$

where $f_n(x_1, \zeta_1, t) = k\Theta - \bar{\xi}_n |\zeta_1 - y_1|$. We note that for $1 \leqslant n \leqslant M$, in which case $\xi_n = \sqrt{k^2 - \mu_n^2}$ is real, $f_n(x_1, \zeta_1, t)$ has a critical point at $\zeta_1 = x_1 + p_n$:

$$\frac{\partial f_n}{\partial \zeta_1}(x_1, x_1 + p_n, t) = 0, \quad \text{where } p_n = t\frac{\xi_n}{\mu_n}, \quad 1 \leqslant n \leqslant M.$$

Lemma 4.4. *Let $1 \leqslant n \leqslant M$ and (4.3) be satisfied. Then there exists a constant $C > 0$ independent of k, h, d such that for any $x, y \in \Omega$ and $t \in [h - x_2, h + x_2]$, we have,*

$$\left| \int_{\eta}^{\infty} \Theta^{-3/2} e^{\mathbf{i} f_n(x_1, \zeta_1, t)} d\zeta_1 \right| \leqslant C \left(\frac{kt^2}{p_n^2}\right)^{-1} (\eta - x_1)^{-3/2}, \quad \forall \eta \geqslant x_1 + 2p_n, \tag{4.14}$$

and if $x_1 + p_n/2 \geqslant d$,

$$\left| \int_d^{x_1 + p_n/2} \Theta^{-3/2} e^{\mathbf{i} f_n(x_1, \zeta_1, t)} d\zeta_1 \right| \leqslant C \left(\frac{kt^2}{p_n^2}\right)^{-1} (d - x_1)^{-3/2}. \tag{4.15}$$

Proof. It is clear that for any $\zeta_1 \geqslant \eta \geqslant x_1 + 2p_n$,

$$\frac{\partial f_n}{\partial \zeta_1}(x_1, \zeta_1, t) = k\frac{\zeta_1 - x_1}{\Theta} - \xi_n \geqslant k \left(\frac{2p_n}{\sqrt{4p_n^2 + t^2}} - \frac{p_n}{\sqrt{p_n^2 + t^2}}\right) \geqslant C\frac{kt^2}{p_n^2}. \tag{4.16}$$

Thus $\frac{\Theta(x_1,\zeta_1,t)^{-3/2}}{\mathrm{i}\partial_{\zeta_1} f_n(x_1,\zeta_1,t)} \to 0$ as $\zeta_1 \to \infty$. By integration by parts we have then

$$
\int_\eta^\infty \Theta^{-3/2} e^{\mathrm{i} f_n(x_1,\zeta_1,t)} d\zeta_1 = -\frac{\Theta(x_1,\eta,t)^{-3/2}}{\mathrm{i}\partial_{\zeta_1} f_n(x_1,\eta,t)} e^{\mathrm{i} f_n(x_1,\eta,t)}
$$
$$
- \int_\eta^\infty \frac{\partial}{\partial \zeta_1}\left(\frac{\Theta(x_1,\zeta_1,t)^{-3/2}}{\mathrm{i}\partial_{\zeta_1} f_n(x_1,\zeta_1,t)}\right) e^{\mathrm{i} f_n(x_1,\zeta_1,t)} d\zeta_1. \tag{4.17}
$$

Since $\frac{\partial^2 f_n(x_1,\zeta_1,t)}{\partial \zeta_1^2} = \frac{kt^2}{\Theta^3}$, $\frac{\partial}{\partial \zeta_1}\left(\frac{\Theta(x_1,\zeta_1,t)^{-3/2}}{\partial_{\zeta_1} f_n(x_1,\zeta_1,t)}\right) \leqslant 0$. Thus

$$
\left|\int_\eta^\infty \frac{\partial}{\partial \zeta_1}\left(\frac{\Theta(x_1,\zeta_1,t)^{-3/2}}{\mathrm{i}\partial_{\zeta_1} f_n(x_1,\zeta_1,t)}\right) e^{\mathrm{i} f_n(x_1,\zeta_1,t)} d\zeta_1\right| \leqslant \int_\eta^\infty \left|\frac{\partial}{\partial \zeta_1}\left(\frac{\Theta(x_1,\zeta_1,t)^{-3/2}}{\partial_{\zeta_1} f_n(x_1,\zeta_1,t)}\right)\right| d\zeta_1
$$
$$
= \left|\int_\eta^\infty \frac{\partial}{\partial \zeta_1}\left(\frac{\Theta(x_1,\zeta_1,t)^{-3/2}}{\partial_{\zeta_1} f_n(x_1,\zeta_1,t)}\right) d\zeta_1\right|
$$
$$
\leqslant [\partial_{\zeta_1} f_n(x_1,\eta,t)]^{-1}(\eta - x_1)^{-3/2}.
$$

This shows (4.14) by using (4.17) and (4.16). The estimate (4.15) can be proved similarly since for $d \leqslant \zeta_1 \leqslant x_1 + p_n/2$,

$$
\left|\frac{\partial f_n}{\partial \zeta_1}(x_1,\zeta_1,t)\right| = \xi_n - k\frac{\zeta_1 - x_1}{\Theta} \geqslant k\left(\frac{p_n}{\sqrt{p_n^2 + t^2}} - \frac{p_n}{\sqrt{p_n^2 + 4t^2}}\right) \geqslant C\frac{kt^2}{p_n^2}.
$$

This completes the proof. $\qquad\square$

We will use the following Van der Corput lemma, see e.g. in [15, Corollary 2.6.8], to estimate the oscillatory integral around the critical point.

Lemma 4.5. *There is a constant $C > 0$ such that for any $-\infty < a < b < \infty$, for every real-valued C^2 function u that satisfies $u''(t) \geqslant 1$ for $t \in (a,b)$, for any function ψ defined on (a,b) with an integrable derivative, and for any $\lambda > 0$,*

$$
\left|\int_a^b e^{\mathrm{i}\lambda u(t)}\psi(t) dt\right| \leqslant C\lambda^{-1/2}\left[|\psi(b)| + \int_a^b |\psi'(t)| dt\right],
$$

where the constant C is independent of the constants a, b, λ and the functions u, ψ.

We remark that if the function ψ in Lemma 4.5 is monotonic decreasing and non-negative in (a,b), then we have

$$
|\psi(b)| + \int_a^b |\psi'(t)| dt = |\psi(b)| + \left|\int_a^b \psi'(t) dt\right| = |\psi(a)|. \tag{4.18}
$$

Lemma 4.6. *Let $1 \leqslant n \leqslant M$ and (4.3) be satisfied. Then there exists a constant $C > 0$ independent of k, h, d such that for any $x, y \in \Omega$ and $t \in [h - x_2, h + x_2]$, we have*

$$
\left|\int_\eta^{x_1 + 2p_n} \Theta^{-3/2} e^{\mathrm{i} f_n(x_1,\zeta_1,t)} d\zeta_1\right| \leqslant Ckt^{-1}\xi_n^{-3/2}, \quad \forall \eta \geqslant x_1 + p_n/2. \tag{4.19}
$$

Proof. It is easy to see that for any $x_1 + p_n/2 \leqslant \zeta_1 \leqslant x_1 + 2p_n$,

$$
\frac{\partial_{\zeta_1}^2 f_n(x_1,\zeta_1,t)}{\partial_{\zeta_1}^2 f_n(x_1,x_1+p_n,t)} = \frac{\Theta(x_1,x_1+p_n,t)^3}{\Theta(x_1,\zeta_1,t)^3} \geqslant \left(\frac{p_n^2 + t^2}{4p_n^2 + t^2}\right)^{3/2} \geqslant \frac{1}{8}.
$$

We can use Lemma 4.5 and (4.18) for $\lambda = \partial_{\zeta_1}^2 f_n(x_1, x_1 + p_n, t)/8 = \mu_n^3/(8k^2 t)$ and $u(\zeta_1) = \frac{8\partial_{\zeta_1}^2 f_n(x_1,\zeta_1,t)}{\partial_{\zeta_1}^2 f_n(x_1,x_1+p_n,t)}$ to obtain, for any $\eta \geqslant x_1 + p_n/2$,

$$
\left|\int_\eta^{x_1 + 2p_n} \Theta^{-3/2} e^{\mathrm{i} f_n(x_1,\zeta_1,t)} d\zeta_1\right| \leqslant C\left(\frac{\mu_n^3}{8k^2 t}\right)^{-1/2} p_n^{-3/2},
$$

This completes the proof by using $p_n = t\xi_n/\mu_n$. $\qquad\square$

Lemma 4.7. *Let* $n \geqslant M + 1$ *and* (4.3) *be satisfied. Let the aperture* $d \geqslant c_3 h$ *for some constant* $c_3 > 0$ *independent of* k, h, d. *Then there exists a constant* $C > 0$ *independent of* k, h, d *such that for any* $x, y \in \Omega$ *and* $t \in [h - x_2, h + x_2]$, *we have*

$$\left| \int_d^\infty \Theta^{-3/2} e^{\mathbf{i} f_n(x_1, \zeta_1, t)} d\zeta_1 \right| \leqslant C k^{-1} d^{-3/2} \chi_n,$$

where $\chi_n = e^{-(1-c_0)|\xi_n|d}$ *for* $n \geqslant M + 1$ *and* $\chi_n = 1$ *for* $n \leqslant M$.

Proof. Since $\xi_n = \mathbf{i}\sqrt{\mu_n^2 - k^2}$ for $n \geqslant M + 1$, by (4.3) we know that $|e^{-\mathbf{i}\bar{\xi}_n|\zeta_1 - y_1|}| \leqslant \chi_n$ for any $\zeta_1 \geqslant d$. The proof of this lemma is essentially the same as that of Lemma 4.4 by using (4.17) and noticing that now we have $|\partial_{\zeta_1} f_n(x_1, \zeta_1, t)| \geqslant k \frac{|\zeta - x_1|}{\Theta} \geqslant Ck$ since $|\zeta_1 - x_1| \geqslant (1 - c_0)d \geqslant (1 - c_0)c_3 h$ and $|t| \leqslant (1 + c_1)h$. We omit the details. $\qquad\square$

Theorem 4.8. *Let* $kh > \frac{\pi}{2}$ *and* (4.3) *be satisfied. Let* $d \geqslant c_3 h$ *for some constant* $c_3 > 0$ *independent of* k, h, d. *Then there exists a constant* $C > 0$ *independent of* k, h, d *such that for any* $x, y \in \Omega$,

$$|S_d(x, y)| \leqslant C \left(\frac{1}{\sqrt{kd}} + \frac{h}{d} \right), \quad |\nabla_x S_d(x, y)| \leqslant Ck \left(\frac{1}{\sqrt{kd}} + \frac{h}{d} \right).$$

Proof. The starting point is (4.13). We first estimate the second term. Since $|R_1(k\Theta)| \leqslant C(k\Theta)^{-3/2}$ by Lemma 4.3, we have

$$\left| \int_{\Gamma_h \backslash \bar{\Gamma}_h^d} R_1(k\Theta) \frac{kt}{\Theta} e^{\mathbf{i}\xi_n|\zeta_1 - y_1|} d\zeta_1 \right| \leqslant C\chi_n k^{-1/2} h \max_{|x_1| \leqslant c_0 d} \int_d^\infty \Theta^{-5/2} d\zeta_1 \leqslant \frac{C\chi_n}{\sqrt{kd}},$$

where χ_n is defined in Lemma 4.7 and we have used $d - x_1 \geqslant (1 - c_0)d \geqslant (1 - c_0)c_3 h$. This implies

$$\sum_{n=1}^\infty \frac{1}{h|\xi_n|} \left| \int_{\Gamma_h \backslash \bar{\Gamma}_h^d} R_1(k\Theta) \frac{kt}{\Theta} e^{\mathbf{i}\xi_n|\zeta_1 - y_1|} d\zeta_1 \right| \leqslant \frac{C}{\sqrt{kd}}, \tag{4.20}$$

where have used the fact that $\sum_{n=1}^\infty \frac{\chi_n}{h|\xi_n|} \leqslant C$ by the argument in the proof of Lemma 2.1. For estimating the first term in (4.13), we first use Lemma 4.7 to obtain that

$$\sum_{n=M+1}^\infty \frac{k^{1/2}t}{h|\xi_n|} \left| \int_{\Gamma_n \backslash \bar{\Gamma}_h^d} \Theta^{-3/2} e^{\mathbf{i} f_n(x_1, \zeta_1, t)} d\zeta_1 \right| \leqslant \sum_{n=M+1}^\infty \frac{k^{1/2}t}{h|\xi_n|} \frac{C\chi_n}{kd^{3/2}} \leqslant \frac{C}{\sqrt{kd}}. \tag{4.21}$$

It remains to estimate

$$\sum_{n=1}^M \frac{k^{1/2}t}{h|\xi_n|} \left| \int_{\Gamma_n \backslash \bar{\Gamma}_h^d} \Theta^{-3/2} e^{\mathbf{i} f_n(x_1, \zeta_1, t)} d\zeta_1 \right|$$
$$\leqslant \sum_{n=1}^M \frac{2}{h|\xi_n|} \max_{|x_1| \leqslant c_0 d} \left| \int_d^\infty k^{1/2}t\Theta^{-3/2} e^{\mathbf{i} f_n(x_1, \zeta_1, t)} d\zeta_1 \right|. \tag{4.22}$$

Let $n_0 \geqslant 1$ be such that $x_1 + 2p_{n_0} > d$ and $x_1 + 2p_{n_0+1} \leqslant d$, which is equivalent to

$$k^{-1}\mu_{n_0} < \frac{2t}{\sqrt{d_1^2 + 4t^2}}, \quad k^{-1}\mu_{n_0+1} \geqslant \frac{2t}{\sqrt{d_1^2 + 4t^2}}, \quad d_1 := d - x_1. \tag{4.23}$$

Clearly $n_0 \leqslant M$ since $k^{-1}\mu_{n_0} < 1$. For $n \geqslant n_0 + 1$, we have $x_1 + 2p_n \leqslant d$ and thus by (4.14) with $\eta = d$ we obtain

$$\sum_{n=n_0+1}^M \frac{k^{1/2}t}{h\xi_n} \left| \int_d^\infty \Theta^{-3/2} e^{\mathbf{i} f_n(x_1, \zeta_1, t)} d\zeta_1 \right| \leqslant C \sum_{n=n_0+1}^M \frac{k^{1/2}t}{h\xi_n} \left(\frac{kt^2}{p_n^2} \right)^{-1} d^{-3/2}$$
$$\leqslant C \frac{k^{1/2}t}{d^{3/2}} \sum_{n=n_0+1}^M \frac{\mu_n^{-2}}{h} \leqslant \frac{C}{\sqrt{kd}}, \tag{4.24}$$

where we have used $\xi_n \leqslant k$ for $n \leqslant M$ and the fact that by (4.23), $\sum_{n=n_0+1}^{M} h^{-1}\mu_n^{-2} \leqslant h^{-1}\mu_{n_0+1}^{-2} + \pi^{-1}\mu_{n_0+1}^{-1} \leqslant Cd/(kh)$.

For $n \leqslant n_0$, by (4.23), we have $\xi_n \geqslant \frac{kd_1}{\sqrt{d_1^2+4t^2}} \geqslant Ck$. By (4.14) with $\eta = x_1 + 2p_n$ we obtain

$$\sum_{n=1}^{n_0} \frac{k^{1/2}t}{h\xi_n} \left| \int_{x_1+2p_n}^{\infty} \Theta^{-3/2} e^{\mathbf{i}f_n(x_1,\zeta_1,t)} d\zeta_1 \right| \leqslant C \sum_{n=1}^{n_0} \frac{k^{1/2}t}{h\xi_n} \frac{p_n^{1/2}}{kt^2}$$

$$\leqslant C \sum_{n=1}^{n_0} \frac{1}{kt^{1/2}} \frac{\mu_n^{-1/2}}{h} \leqslant \frac{C}{\sqrt{kd}}, \tag{4.25}$$

where we have used the fact that $\sum_{n=1}^{n_0} h^{-1}\mu_n^{-1/2} \leqslant \frac{2}{\pi}\mu_{n_0}^{1/2} \leqslant C(kh)^{1/2}/d^{1/2}$ by (4.23).

To proceed, let $n_1 \geqslant 1$ be such that $x_1 + p_{n_1}/2 > d$ and $x_1 + p_{n_1+1}/2 \leqslant d$, which is equivalent to

$$k^{-1}\mu_{n_1} < \frac{t}{\sqrt{4d_1^2+t^2}}, \quad k^{-1}\mu_{n_1+1} \geqslant \frac{t}{\sqrt{4d_1^2+t^2}}, \quad d_1 := d - x_1. \tag{4.26}$$

Clearly $n_1 \leqslant n_0$. We write

$$\sum_{n=1}^{n_0} \frac{k^{1/2}t}{h\xi_n} \left| \int_{d}^{x_1+2p_n} \Theta^{-3/2} e^{\mathbf{i}f_n(x_1,\zeta_1,t)} d\zeta_1 \right|$$

$$\leqslant \sum_{n=1}^{n_1} \frac{k^{1/2}t}{h\xi_n} \left| \int_{d}^{x_1+p_n/2} \Theta^{-3/2} e^{\mathbf{i}f_n(x_1,\zeta_1,t)} d\zeta_1 \right|$$

$$+ \sum_{n=1}^{n_1} \frac{k^{1/2}t}{h\xi_n} \left| \int_{x_1+p_n/2}^{x_1+2p_n} \Theta^{-3/2} e^{\mathbf{i}f_n(x_1,\zeta_1,t)} d\zeta_1 \right|$$

$$+ \sum_{n=n_1+1}^{n_0} \frac{k^{1/2}t}{h\xi_n} \left| \int_{d}^{x_1+2p_n} \Theta^{-3/2} e^{\mathbf{i}f_n(x_1,\zeta_1,t)} d\zeta_1 \right| := \text{I} + \text{II} + \text{III}. \tag{4.27}$$

Let $n_2 = \frac{kh}{\sqrt{kd}}$. If $n_1 \leqslant n_2$, then since $\xi_n \geqslant Ck$ for $n \leqslant n_0$,

$$\text{I} \leqslant C \sum_{n=1}^{n_2} \frac{k^{1/2}t}{h\xi_n} d^{-1/2} \leqslant C \frac{n_2}{\sqrt{kd}} \leqslant C \frac{h}{d}.$$

Otherwise, if $n_1 \geqslant n_2 + 1$, we split the sum and use (4.15) to have

$$\text{I} \leqslant C\frac{h}{d} + \sum_{n=n_2+1}^{n_1} \frac{k^{1/2}t}{h\xi_n} \left| \int_{d}^{x_1+p_n/2} \Theta^{-3/2} e^{\mathbf{i}f_n(x_1,\zeta_1,t)} d\zeta_1 \right|$$

$$\leqslant C\frac{h}{d} + C \sum_{n=n_2+1}^{n_1} \frac{k^{1/2}t}{h\xi_n} \left(\frac{kt^2}{p_n^2} \right)^{-1} d^{-3/2} \leqslant C\frac{h}{d},$$

where we have used the fact that

$$\sum_{n=n_2+1}^{n_1} h^{-1}\mu_n^{-2} \leqslant h^{-1}\mu_{n_2+1}^{-2} + \pi^{-1}\mu_{n_2+1}^{-1} \leqslant Chn_2^{-1} = Ck^{-1/2}d^{1/2}.$$

Therefore, we have $\text{I} \leqslant Ch/d$. By using (4.19) with $\eta = x_1 + p_n/2$ for the term II and with $\eta = d \geqslant x_1 + p_n/2$ for the term III, we have

$$\text{II} + \text{III} \leqslant C \sum_{n=1}^{n_0} \frac{k^{1/2}t}{h\xi_n} kt^{-1} \xi_n^{-3/2} \leqslant C \sum_{n=1}^{n_0} \frac{1}{h\xi_n} \leqslant C\frac{h}{d},$$

where we have used $n_0 \leqslant Ckh^2/d$ by (4.23). Therefore, by (4.27),

$$\sum_{n=1}^{n_0} \frac{k^{1/2}t}{h\xi_n} \left| \int_{d}^{x_1+2p_n} \Theta^{-3/2} e^{\mathbf{i}f_n(x_1,\zeta_1,t)} d\zeta_1 \right| \leqslant C\frac{h}{d} + \frac{C}{\sqrt{kd}}.$$

This shows the estimate for $|S_d(x,y)|$ by (4.13), (4.20), (4.21)-(4.22), (4.24)-(4.25), and the above estimate.
The estimate for $\nabla_x S_d(x,y)$ can be proved similarly by noticing that

$$\frac{\partial^2 G(x,\zeta)}{\partial x_2 \partial \zeta_2} = \frac{\partial f}{\partial t}(x_1,\zeta_1,h+x_2) + \frac{\partial f}{\partial t}(x_1,\zeta_1,h-x_2),$$

$$\frac{\partial^2 G(x,\zeta)}{\partial x_1 \partial \zeta_2} = \frac{\partial f}{\partial x_1}(x_1,\zeta_1,h+x_2) - \frac{\partial f}{\partial x_1}(x_1,\zeta_1,h-x_2),$$

where after using the identity $H_1^{(1)'}(\xi) = H_0^{(1)}(\xi) - \frac{1}{\xi}H_1^{(1)}(\xi)$ for any $\xi \in \mathbb{C}$,

$$\frac{\partial f}{\partial t}(x_1,\zeta_1,t) = \frac{\mathbf{i}}{4}H_0^{(1)}(k\Theta)\frac{k^2 t^2}{\Theta^2} + \frac{\mathbf{i}}{4}H_1^{(1)}(k\Theta)\left(\frac{k}{\Theta} - \frac{2kt^2}{\Theta^3}\right),$$

$$\frac{\partial f}{\partial x_1}(x_1,\zeta_1,t) = \frac{\mathbf{i}}{4}H_0^{(1)}(k\Theta)\frac{k^2(x_1-\zeta_1)t}{\Theta^2} - \frac{\mathbf{i}}{4}H_1^{(1)}(k\Theta)\frac{2k(x_1-\zeta_1)t}{\Theta^3}.$$

We omit the details. This completes the proof. $\square$

We remark that by Theorem 4.2 and Theorem 4.8, the resolution of the finite aperture Helmholtz-Kirchhoff
function $H_d(x,y)$ is the same as the resolution of $\mathrm{Im}\,N(x,y)$ for $x,y \in \Omega$ when $kh \gg 1$ and $kd/(kh) \gg 1$.

5   The resolution analysis of the RTM algorithm

In this section we study the resolution of the imaging function in (3.6). We first notice that since $S(\cdot,z)$
satisfies the Helmholtz equation in $\mathbb{R}_h^2$, it follows from Theorem 4.2 that

$$\|S(\cdot,z)\|_{H^{1/2}(\Gamma_D)} + \left\|\frac{\partial S(\cdot,z)}{\partial \nu}\right\|_{H^{-1/2}(\Gamma_D)} \leqslant C\|S(\cdot,z)\|_{H^1(D)} + \|\Delta S(\cdot,z)\|_{L^2(D)}$$

$$\leqslant \frac{C}{|\cos(kh)|}\frac{1}{\sqrt{kh}}, \quad \forall z \in \Omega, \tag{5.1}$$

for some constant C independent of h. Similarly, since $S_d(\cdot,z)$ also satisfies the Helmholtz equation, by
Theorem 4.8, for any $z \in \Omega$,

$$\|S_d(\cdot,z)\|_{H^{1/2}(\Gamma_D)} + \left\|\frac{\partial S_d(\cdot,z)}{\partial \nu}\right\|_{H^{-1/2}(\Gamma_D)} \leqslant C\left(\frac{1}{\sqrt{kh}} + \frac{h}{d}\right), \tag{5.2}$$

for some constant C independent of h,d. Here we have used the assumption $d \geqslant c_3 h$.

Theorem 5.1. *Let $kh > \pi/2$, $d \geqslant c_3 h$, and (4.3) be satisfied. For any $z \in \Omega$, let $\psi(x,z)$ be the radiation
solution of the problem*

$$\Delta\psi(x,z) + k^2\psi(x,z) = 0 \quad in \; \mathbb{R}_h^2 \backslash \bar{D}, \tag{5.3}$$

$$\psi = 0 \; on \; \Gamma_0, \quad \frac{\partial\psi}{\partial x_2} = 0 \; on \; \Gamma_h, \tag{5.4}$$

$$\frac{\partial\psi}{\partial \nu} + \mathbf{i}k\eta\psi = -\left(\frac{\partial\,\mathrm{Im}\,N(x,z)}{\partial \nu} + \mathbf{i}k\eta\,\mathrm{Im}\,N(x,z)\right) \quad on \; \Gamma_D. \tag{5.5}$$

Then we have, for any $z \in \Omega$,

$$\hat{I}_d(z) = 2h\sum_{n=1}^M \xi_n(|\psi_n^+|^2 + |\psi_n^-|^2) + 4k\int_{\Gamma_D} \eta(\zeta)\,|\psi(\xi,z) + \mathrm{Im}\,N(\zeta,z)|^2\,ds(\xi) + w_{\hat{I}}(z),$$

*where $\psi_n^\pm$, $n = 1,2,\cdots,M$, are the far-field pattern of the radiation solution of $\psi(\cdot,z)$ and $\|w_{\hat{I}}\|_{L^\infty(\Omega)} \leqslant
\frac{C}{|\cos(kh)|}\frac{1}{\sqrt{kh}} + C\frac{h}{d}$.*

Proof. By the integral representation formula we know that

$$u^s(x_r, x_s) = \int_{\Gamma_D} \left(u^s(\zeta, x_s) \frac{\partial N(x_r, \zeta)}{\partial \nu(\zeta)} - \frac{\partial u^s(\zeta, x_s)}{\partial \nu(\zeta)} N(x_r, \zeta) \right) ds(\zeta).$$

By using Lemma 4.2 we obtain that, for any $z \in \Omega$,

$$
\begin{aligned}
\hat{v}_b(z, x_s) &= \int_{\Gamma_h^d} \frac{\partial G(z, x_r)}{\partial x_2(x_r)} \overline{u^s(x_r, x_s)} ds(x_r) \\
&= \int_{\Gamma_D} \overline{u^s(\zeta, x_s)} \frac{\partial}{\partial \nu(\zeta)} \big(2\mathrm{i}\mathrm{Im}\, N(\zeta, z) - S(\zeta, z) - S_d(\zeta, z) \big) ds(\zeta) \\
&\quad - \int_{\Gamma_D} \frac{\partial \overline{u^s(\zeta, x_s)}}{\partial \nu(\zeta)} \big(2\mathrm{i}\mathrm{Im}\, N(\zeta, z) - S(\zeta, z) - S_d(\zeta, z) \big) ds(\zeta),
\end{aligned}
$$

where we have used the reciprocity relation $N(\zeta, z) = N(z, \zeta), G(\zeta, z) = G(z, \zeta)$. By (3.6) we obtain then

$$
\begin{aligned}
\hat{I}_d(z) &= \mathrm{Im} \int_{\Gamma_D} v_s(\zeta, z) \frac{\partial}{\partial \nu(\zeta)} \big(2\mathrm{i}\,\mathrm{Im}\, N(\zeta, z) - S(\zeta, z) - S_d(\zeta, z) \big) ds(\zeta) \\
&\quad - \mathrm{Im} \int_{\Gamma_D} \frac{\partial v_s(\zeta, z)}{\partial \nu(\zeta)} \big(2\mathrm{i}\mathrm{Im}\, N(\zeta, z) - S(\zeta, z) - S_d(\zeta, z) \big) ds(\zeta), \quad (5.6)
\end{aligned}
$$

where $v_s(\zeta, z) = \int_{\Gamma_h^d} \frac{\partial G(z, x_s)}{\partial x_2(x_s)} \overline{u^s(\zeta, x_s)} ds(x_s)$. By taking the complex conjugate, we have

$$\overline{v_s(\zeta, z)} = \int_{\Gamma_h^d} \frac{\partial \overline{G(z, x_s)}}{\partial x_2(x_s)} u^s(\zeta, x_s) ds(x_s).$$

Thus $\overline{v_s(\zeta, z)}$ is a weighted superposition of the scattered waves $u^s(\zeta, x_s)$. Therefore, $\overline{v_s(\zeta, z)}$ is the radiation solution of the Helmholtz equation

$$\Delta_\zeta \overline{v_s(\zeta, z)} + k^2 \overline{v_s(\zeta, z)} = 0 \quad \text{in } \mathbb{R}_h^2 \backslash \bar{D},$$

$$\overline{v_s(\zeta, z)} = 0 \text{ on } \Gamma_0, \qquad \frac{\partial \overline{v_s(\zeta, z)}}{\partial \zeta_2} = 0 \text{ on } \Gamma_h,$$

satisfying the impedance boundary condition

$$
\begin{aligned}
&\left(\frac{\partial}{\partial \nu(\zeta)} + \mathrm{i}k\eta(\zeta) \right) \overline{v_s(\zeta, z)} \\
&= \int_{\Gamma_h^d} \frac{\partial \overline{G(z, x_s)}}{\partial x_2(x_s)} \left(\frac{\partial}{\partial \nu(\zeta)} + \mathrm{i}k\eta(\zeta) \right) u^s(\zeta, x_s) ds(x_s) \\
&= \int_{\Gamma_h^d} \frac{\partial \overline{G(z, x_s)}}{\partial x_2(x_s)} \left(\frac{\partial}{\partial \nu(\zeta)} + \mathrm{i}k\eta(\zeta) \right) (-N(\zeta, x_s)) ds(x_s) \\
&= \left(\frac{\partial}{\partial \nu(\zeta)} + \mathrm{i}k\eta(\zeta) \right) (2\mathrm{i}\,\mathrm{Im}\, N(\zeta, z) + \overline{S(\zeta, z)} + \overline{S_d(\zeta, z)}) \quad \text{on } \Gamma_D,
\end{aligned}
$$

where we have used (4.1) in the last equality. This implies by using (5.3)-(5.5) that $\overline{v_s(\zeta, z)} = -2\mathrm{i}\psi(\zeta, z) + w(\zeta, z)$, where $w(\cdot, z)$ satisfies the impedance scattering problem in Theorem 2.3 with $g(\cdot) = \left(\frac{\partial}{\partial \nu} + \mathrm{i}k\eta(\cdot) \right) (\overline{S(\cdot, z)} + \overline{S_d(\cdot, z)})$.

By Theorem 2.3, (5.1)-(5.2), and the boundary condition satisfied by w on Γ_D, we know that w satisfies

$$\|w(\cdot, z)\|_{H^{1/2}(\Gamma_D)} + \left\| \frac{\partial w(\cdot, z)}{\partial \nu} \right\|_{H^{-1/2}(\Gamma_D)} \leqslant \frac{C}{|\cos(kh)|} \frac{1}{\sqrt{kh}} + C\frac{h}{d}.$$

Moreover, since $N(\cdot, z) = G(\cdot, z) + S(\cdot, z)$, we also have

$$\|N(\cdot, z)\|_{H^{1/2}(\Gamma_D)} + \left\| \frac{\partial N(\cdot, z)}{\partial \nu} \right\|_{H^{-1/2}(\Gamma_D)} \leqslant C, \quad \forall z \in \Omega.$$

Now substituting $v_s(\zeta, z) = 2\mathbf{i}\overline{\psi(\zeta, z)} + \overline{w(\zeta, z)}$ into (5.6) we obtain

$$
\begin{aligned}
\hat{I}(z) &= -4\operatorname{Im}\int_{\Gamma_D}\left(\overline{\psi(\zeta, z)}\frac{\partial\operatorname{Im}N(\zeta, z)}{\partial\nu(\zeta)} - \frac{\partial\overline{\psi(\zeta, z)}}{\partial\nu(\zeta)}\operatorname{Im}N(\zeta, z)\right)ds(\zeta) + w_{\hat{I}}(z) \\
&= 4\operatorname{Im}\int_{\Gamma_D}\left(\psi(\zeta, z)\frac{\partial\operatorname{Im}N(\zeta, z)}{\partial\nu(\zeta)} - \frac{\partial\psi(\zeta, z)}{\partial\nu(\zeta)}\operatorname{Im}N(\zeta, z)\right)ds(\zeta) + w_{\hat{I}}(z),
\end{aligned}
$$

where

$$
\begin{aligned}
|w_{\hat{I}}(z)| &\leqslant 2\|\psi(\cdot, z)\|_{H^{1/2}(\Gamma_D)}\left\|\frac{\partial(S(\cdot, z) + S_d(\cdot, z))}{\partial\nu}\right\|_{H^{-1/2}(\Gamma_D)} \\
&\quad + 2\|S(\cdot, z) + S_d(\cdot, z)\|_{H^{1/2}(\Gamma_D)}\left\|\frac{\partial\psi(\cdot, z)}{\partial\nu}\right\|_{H^{-1/2}(\Gamma_D)} \\
&\quad + \|w(\cdot, z)\|_{H^{1/2}(\Gamma_D)}\left\|\frac{\partial(2\mathbf{i}\operatorname{Im}N(\cdot, z) - S(\cdot, z) - S_d(\cdot, z))}{\partial\nu}\right\|_{H^{-1/2}(\Gamma_D)} \\
&\quad + \|2\mathbf{i}\operatorname{Im}N(\cdot, z) - S(\cdot, z) - S_d(\cdot, z)\|_{H^{1/2}(\Gamma_D)}\left\|\frac{\partial w(\cdot, z)}{\partial\nu}\right\|_{H^{-1/2}(\Gamma_D)} \\
&\leqslant \frac{C}{|\cos(kh)|}\frac{1}{\sqrt{kh}} + C\frac{h}{d}.
\end{aligned}
$$

By (5.4) we have

$$
\begin{aligned}
&\operatorname{Im}\int_{\Gamma_D}\left(\psi(\zeta, z)\frac{\partial\operatorname{Im}N(\zeta, z)}{\partial\nu(\zeta)} - \frac{\partial\psi(\zeta, z)}{\partial\nu(\zeta)}\operatorname{Im}N(\zeta, z)\right)ds(\zeta) \\
&= \operatorname{Im}\int_{\Gamma_D}\left[\psi(\zeta, z)\left(\frac{\partial\operatorname{Im}N(\zeta, z)}{\partial\nu(\zeta)} - \mathbf{i}k\eta(\zeta)\operatorname{Im}N(\zeta, z)\right)\right. \\
&\qquad\left. -\left(\frac{\partial\psi(\zeta, z)}{\partial\nu(\zeta)} + \mathbf{i}k\eta(\zeta)\psi(\zeta, z)\right)\operatorname{Im}N(\zeta, z) + 2\mathbf{i}k\eta(\zeta)\operatorname{Im}N(\zeta, z)\psi(\zeta, z)\right]ds(\zeta) \\
&= \operatorname{Im}\int_{\Gamma_D}\left[-\psi(\zeta, z)\cdot\left(\frac{\partial\overline{\psi(\zeta, z)}}{\partial\nu(\zeta)} - \mathbf{i}k\eta(\zeta)\overline{\psi(\zeta, z)}\right)\right. \\
&\qquad\left. +\left(\frac{\partial\operatorname{Im}N(\zeta, z)}{\partial\nu(\zeta)} + \mathbf{i}k\eta(\zeta)\operatorname{Im}N(\zeta, z)\right)\operatorname{Im}N(\zeta, z) + 2\mathbf{i}k\eta(\zeta)\operatorname{Im}N(\zeta, z)\psi(\zeta, z)\right]ds(\zeta) \\
&= -\operatorname{Im}\int_{\Gamma_D}\psi(\zeta, z)\frac{\partial\overline{\psi(\zeta, z)}}{\partial\nu(\zeta)}ds(\zeta) + k\int_{\Gamma_D}\eta(\zeta)\left|\psi(\zeta, z) + \operatorname{Im}N(\zeta, z)\right|^2 ds(\zeta).
\end{aligned}
$$

By (2.8) we know that the far-field pattern $\psi_n^{\pm}$, $n = 1, 2, \cdots, M$, satisfy

$$
-\operatorname{Im}\int_{\Gamma_D}\psi\frac{\partial\overline{\psi}}{\partial\nu}ds = \frac{h}{2}\sum_{n=1}^{M}\xi_n(|\psi_n^+|^2 + |\psi_n^-|^2).
$$

This completes the proof. $\qquad\square$

We remark that $\psi(x, z)$ is the scattering solution of the Helmholtz equation in the waveguide with the incoming field $\operatorname{Im}N(x, z)$. Since

$$
\begin{aligned}
\operatorname{Im}N(x, z) &= \operatorname{Im}G(x, z) + \operatorname{Im}S(x, z) \\
&= \frac{1}{4}J_0(k|x - z|) - \frac{1}{4}J_0(k|x - z'|) + \operatorname{Im}S(x, z),
\end{aligned}
$$

where $J_0(t)$ is the first kind Bessel function of zeroth order and $z' = (z_1, -z_2)$ is the imagine point of $z = (z_1, z_2)$. It is well-known that $J_0(t)$ peaks at $t = 0$ and decays like $t^{-1/2}$ away from the origin. By Theorem 4.2, $S(x, z)$ is small when $kh \gg 1$ which implies $\operatorname{Im}N(x, z)$ of the problem (5.3)-(5.5) will peak at the boundary of the scatterer D and becomes small when z moves away from ∂D. Thus we expect that the imaging function $\hat{I}_d(z)$ will have contrast at the boundary of the scatterer D and decay outside the boundary ∂D if $kh \gg 1$ and $kd/(kh) \gg 1$. This is indeed confirmed in our numerical experiments.

6   Extensions

In this section we consider the reconstruction of the sound soft and penetrable obstacles in the planar waveguide by our RTM algorithm. For the sound soft obstacle, the measured data $u^s(x_r, x_s) = u(x_r, x_s) - u^i(x_r, x_s)$, where $u(x, x_s)$ is the radiation solution of the following problem

$$\Delta u + k^2 u = -\delta_{x_s}(x) \qquad \text{in } \mathbb{R}_h^2 \backslash \bar{D}, \tag{6.1}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{6.2}$$

$$u = 0 \text{ on } \Gamma_0, \qquad \frac{\partial u}{\partial x_2} = 0 \text{ on } \Gamma_h. \tag{6.3}$$

The well-posedness of the problem under some geometric condition of the obstacle D is known [20, 22]. Here we assume that the scattering problem (6.1)-(6.3) has a unique solution. By modifying the argument in Theorem 5.1 we can show the following result whose proof is omitted.

Theorem 6.1. *Let $kh > \pi/2$, $d \geqslant c_3 h$, and (4.3) be satisfied. For any $z \in \Omega$, let $\psi(x, z)$ be the radiation solution of the problem*

$$\Delta \psi(x, z) + k^2 \psi(x, z) = 0 \qquad in \ \mathbb{R}_h^2 \backslash \bar{D},$$

$$\psi = 0 \ \ on \ \Gamma_0, \qquad \frac{\partial \psi}{\partial x_2} = 0 \ \ on \ \Gamma_h,$$

$$\psi(x, z) = -\operatorname{Im} N(x, z) \qquad on \ \Gamma_D.$$

Then we have, for any $z \in \Omega$,

$$\hat{I}_d(z) = 2h \sum_{n=1}^{M} \xi_n (|\psi_n^+|^2 + |\psi_n^-|^2) + w_{\hat{I}}(z),$$

where $\psi_n^\pm$, $n = 1, 2, \cdots, M$, are the far-field pattern of the radiation solution of $\psi(\cdot, z)$ and $\|w_{\hat{I}}\|_{L^\infty(\Omega)} \leqslant \frac{C}{|\cos(kh)|} \frac{1}{\sqrt{kh}} + C \frac{h}{d}$.

For the penetrable obstacle, the measured data $u^s(x_r, x_s) = u(x_r, x_s) - u^i(x_r, x_s)$, where $u(x, x_s)$ is the radiation solution of the following problem

$$\Delta u + k^2 n(x) u = -\delta_{x_s}(x) \qquad \text{in } \mathbb{R}_h^2, \tag{6.4}$$

$$u = 0 \text{ on } \Gamma_0, \qquad \frac{\partial u}{\partial x_2} = 0 \text{ on } \Gamma_h, \tag{6.5}$$

where $n(x) \in L^\infty(\mathbb{R}_h^2)$ is a positive function which is equal to 1 outside the scatterer D. The well-posedness of the problem under some condition on $n(x)$ is known [8]. Here we assume that the scattering problem (6.4)-(6.5) has a unique solution. By modifying the argument in Theorem 5.1, the following theorem can be proved. We refer to [10, Theorem 3.1] for a similar result. Here we omit the details.

Theorem 6.2. *Let $kh > \pi/2$, $d \geqslant c_3 h$, and (4.3) be satisfied. For any $z \in \Omega$, let $\psi(x, z)$ be the radiation solution of the problem*

$$\Delta \psi(x, z) + k^2 n(x) \psi(x, z) = -k^2(n(x) - 1)\operatorname{Im} N(x, z) \qquad in \ \mathbb{R}_h^2,$$

$$\psi = 0 \ \ on \ \Gamma_0, \qquad \frac{\partial \psi}{\partial x_2} = 0 \ \ on \ \Gamma_h.$$

Then we have, for any $z \in \Omega$,

$$\hat{I}_d(z) = 2h \sum_{n=1}^{M} \xi_n (|\psi_n^+|^2 + |\psi_n^-|^2) + w_{\hat{I}}(z),$$

where $\psi_n^\pm$, $n = 1, 2, \cdots, M$, are the far-field pattern of the radiation solution of $\psi(\cdot, z)$ and $\|w_{\hat{I}}\|_{L^\infty(\Omega)} \leqslant \frac{C}{|\cos(kh)|} \frac{1}{\sqrt{kh}} + C \frac{h}{d}$.

We remark that for the penetrable scatterers, $\psi(x, z)$ is again the scattering solution with the incoming field $\operatorname{Im} N(x, z)$. Therefore we again expect the imaging function $\hat{I}_d(z)$ will have contrast on the boundary of the scatterer and decay outside the scatterer if $kh \gg 1$ and $kd/(kh) \gg 1$.
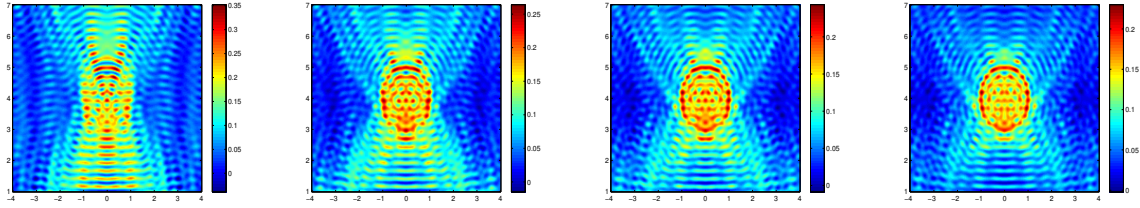
Figure 3   The imaging results with the aperture $d = 10, 20, 30, 50$ from left to right, respectively.

7   Numerical experiments

In this section we present several numerical examples to demonstrate the effectiveness of our RTM method for planar acoustic waveguide. To synthesize the scattering data we compute the solution $u^s(x_r, x_s)$ of the scattering problem by representing the ansatz solution as the double layer potential with the Green function $N(x, y)$ as the kernel and discretizing the integral equation by standard Nyström methods [14]. The boundary integral equations on Γ_D are solved on a uniform mesh over the boundary with ten points per probe wavelength. The sources and receivers are both placed on the surface Γ_h^d with equal-distribution, where d is the aperture. The boundaries of the obstacles used in our numerical experiments are parameterized as follows:

$$
\begin{aligned}
\text{Circle:} \qquad & x_1 = \rho\cos(\theta), \;\; x_2 = \rho\sin(\theta), \;\; \theta \in (0, 2\pi], \\
\text{Kite:} \qquad & x_1 = \cos(\theta) + 0.65\cos(2\theta) - 0.65, \;\; x_2 = 1.5\sin(\theta), \;\; \theta \in (0, 2\pi], \\
\text{Rounded Square:} \qquad & x_1 = 0.5(\cos^3(\theta) + \cos(\theta)), \; x_2 = 0.5(\sin^3(\theta) + \sin(\theta)), \; \theta \in (0, 2\pi].
\end{aligned}
$$

Example 7.1.   In this example we consider the imaging of a sound soft circle of radius $\rho = 1$ for different values of the aperture d. We take the probe wavelength $\lambda = 0.5$, where $\lambda = 2\pi/k$, the thickness $h = 10$, and $N_s = N_r = 401$. We choose the aperture $d = 10, 20, 30, 50$ for the tests. The imaging results are shown in Figure 3. We observe that our RTM imaging function peaks at the boundary of the obstacle and the imaging quality improves with the increase of the aperture d. This conforms with our theoretical results in Theorem 6.1.

Example 7.2.   In this example we first consider the imaging of a circle of radius $\rho = 1$, a kite, and a rounded square with the impedance boundary condition with $\eta = 1$ or $\eta = 1000$ on Γ_D. Let $\Omega = (-4, 4) \times (1, 7)$ be the search region. The imaging function is computed at the nodal points of a uniform 201×201 mesh with the probed wavelength $\lambda = 0.5$. The imaging results on the top and bottom row shown in Figure 4 correspond to the surface impedance $\eta = 1$ and $\eta = 1000$, respectively. We observe our imaging algorithm is quite robust with respect to the magnitude of the surface impedance η.

We then consider to find a penetrable obstacle with the refraction index $n(x) = 0.25$, a non-penetrable obstacle with homogeneous Neumann, homogeneous Dirichlet, and partially coated impedance boundary condition ($\eta = 1000$ on the upper half boundary and $\eta = 1$ on the lower half boundary), respectively. The results are shown in Figure 5 which indicates clearly that our RTM method can reconstruct the boundary of the obstacle without a priori information on penetrable or non-penetrable obstacles, and in the case of non-penetrable obstacles, the type of the boundary conditions on the boundary of the obstacle.

Example 7.3.   In this example we consider the stability of the imaging function with respect to the complex additive Gaussian random noise. We introduce the additive Gaussian noise as follows (see e.g. [10]):

$$
u_{noise} = u_s + \nu_{\text{noise}},
$$

where u_s is the synthesized data and ν_{noise} is the complex Gaussian noise with mean zero and standard deviation μ times the maximum of the data $|u_s|$, i.e. $\nu_{\text{noise}} = \frac{\mu \max |u_s|}{\sqrt{2}}(\varepsilon_1 + \mathbf{i}\varepsilon_2)$, and $\varepsilon_j \sim \mathcal{N}(0, 1)$ for the real ($j = 1$) and imaginary part ($j = 2$).
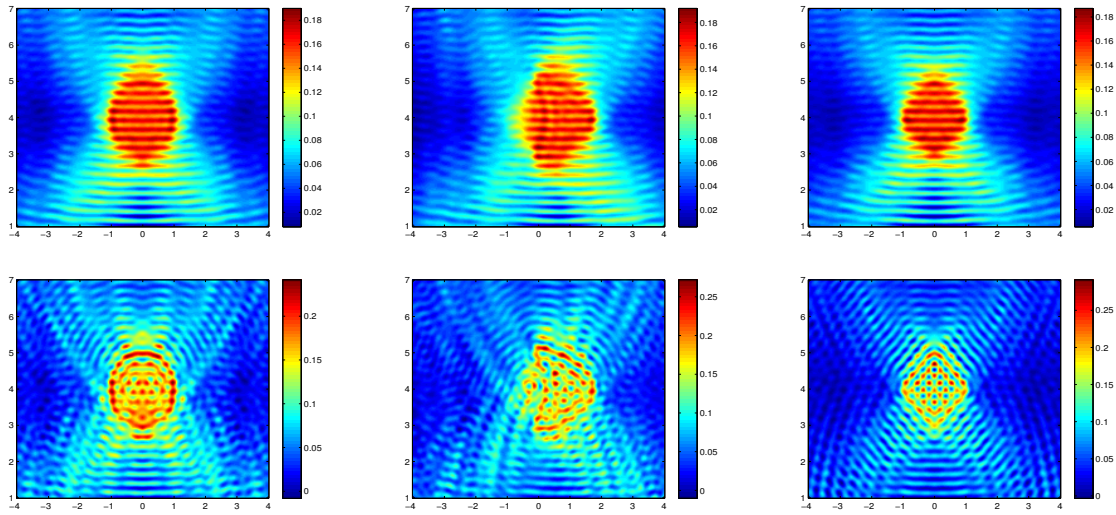
Figure 4.   The imaging results with the probe wavelength $\lambda = 0.5$, the thickness $h = 10$, the aperture $d = 30$, and $N_s = N_r = 401$ for a circle, a kite, and a rounded square, respectively, The top and bottom row show the imaging results of the RTM method for the surface impedance $\eta = 1$ and $\eta = 1000$, respectively.
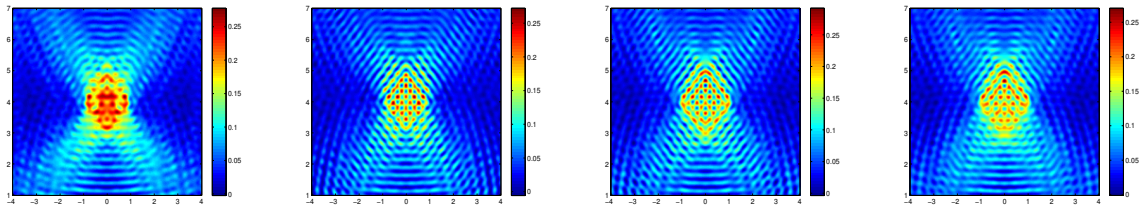

Figure 5.   The imaging results, from left to right, for the penetrable obstacle with $n(x) = 0.25$, a non-penetrable obstacle with homogeneous Neumann condition, homogeneous Dirichlet condition, and partially coated impedance boundary condition ($\eta = 1000$ on the upper half boundary and $\eta = 1$ on the lower half boundary), respectively. The probe wavelength $\lambda = 0.5$, the thickness $h = 10$, the aperture $d = 30$, and $N_s = N_r = 401$.
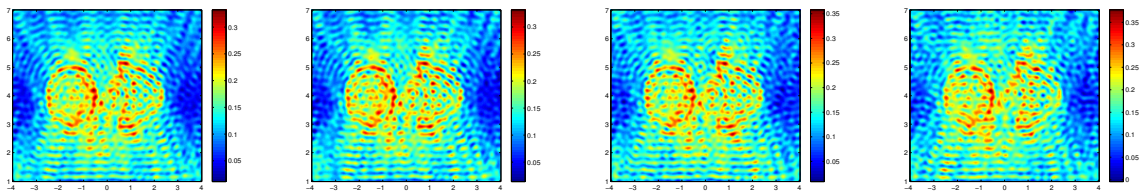

Figure 6.   The imaging results using data added with additive Gaussian noise and $\mu = 10\%, 20\%, 30\%, 40\%$ from left to right, respectively. The probe wavelength $\lambda = 0.5$, the thickness $h = 10$, the aperture $d = 30$, and $N_s = N_r = 401$.

Table 1. The signal level and noise level in the case of single frequency data (left) and multi-frequency data (right).

μ	σ	$\|u_s\|_{\ell^2}$	$\|\nu_{\text{noise}}\|_{\ell^2}$	μ	σ	$\|u_s\|_{\ell^2}$	$\|\nu_{\text{noise}}\|_{\ell^2}$
0.1	0.0360	0.1033	0.0293	0.1	0.0355	0.1033	0.0290
0.2	0.0720	0.1033	0.0589	0.2	0.0710	0.1033	0.0580
0.3	0.1079	0.1033	0.0876	0.3	0.1064	0.1033	0.0869
0.4	0.1439	0.1033	0.1178	0.4	0.1419	0.1033	0.1159

For the fixed probe wavelength $\lambda = 0.5$, we choose one kite and one circle in our test. The search domain is $\Omega = (-4, 4) \times (1, 7)$ with a sampling 201×201 mesh. Figure 6 shows the imaging results with the noise level $\mu = 10\%, 20\%, 30\%, 40\%$ in the single frequency scattered data, respectively. The left table in Table 1 shows the noise level in this case, where $\sigma = \mu \max_{x_r, x_s} |u^s(x_s, x_r)|$, $\|u_s\|_{\ell^2}^2 = \frac{1}{N_s N_r} \sum_{s,r=1}^{N_s, N_r} |u^s(x_s.x_r)|^2$, $\|\nu_{\text{noise}}\|_{\ell^2}^2 = \frac{1}{N_s N_r} \sum_{s,r=1}^{N_s, N_r} |\nu_{\text{noise}}(x_s.x_r)|^2$.

The imaging quality can be improved by using multi-frequency data as illustrated in Figure 7, in which we show the imaging results added with the noise level $\mu = 10\%, 20\%, 30\%, 40\%$ Gaussian noise by summing the imaging functions for the probed wavelengths $\lambda = 1/1.8, 1/1.9, 1/2.0, 1/2.1, 1/2.2$. The right table in Table 1 shows the noise level in the case of multi-frequency data, where σ, $\|u_s\|_{\ell^2}$, and $\|\nu_{\text{noise}}\|_{\ell^2}$ are the arithmetic mean of the corresponding values for different frequencies, respectively. The imaging result is visually much more better than the single frequency imaging result, and the noise is greatly suppressed after the summation over individual frequency imaging results.
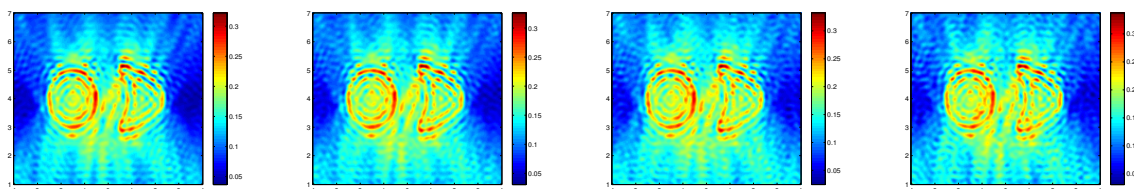

Figure 7. The imaging results using multi-frequency data added with additive Gaussian noise and $\mu = 10\%, 20\%, 30\%, 40\%$ from left to right, respectively. The probe wavelengths $\lambda = 1/1.8, 1/1.9, 1/2.0, 1/2.1, 1/2.2$, the thickness $h = 10$, the aperture $d = 30$, and $N_s = N_r = 401$.

8   Concluding remarks

In this paper we have developed a novel reverse time migration algorithm based on the generalized Helmholtz-Kirchhoff identity for the obstacle shape reconstruction in planar acoustic waveguide. The algorithm consists of using the half space Green function instead of the waveguide Green function in both the back-propagation and cross-correlation processes. The algorithm is quite robust with respect to the random noise. Our numerical experiments indicate that the RTM algorithm based on multiple frequency superposition can effectively suppress the random noise. Extending the results in this paper to the electromagnetic and elastic waveguide imaging problem is of considerable practical interests and will be pursued in our future works.

9   Appendix: Proof of Theorem 2.3

We will prove the existence of the radiation solution of the problem (2.4)-(2.6) by the method of limiting absorption principle. The argument is standard and generalizes that for Helmholtz scattering problem in the free space, see e.g. [18]. Here we only outline the main steps.

For any $z = 1 + \mathbf{i}\varepsilon$, $\varepsilon > 0$, $f \in L^2(\mathbb{R}_h^2)$ with compact support in $B_R = (-R, R) \times (0, h)$, where $R > 0$, we consider the problem

$$\Delta u_z + zk^2 u_z = -f \qquad \text{in } \mathbb{R}_h^2, \tag{9.1}$$

$$u_z = 0 \ \text{ on } \Gamma_0, \qquad \frac{\partial u_z}{\partial x_2} = 0 \ \text{ on } \Gamma_h. \tag{9.2}$$

By Lax-Milgram lemma we know that (9.1)-(9.2) has a unique solution $u_z \in H^1(\mathbb{R}_h^2)$. For any domain $\mathcal{D} \subset \mathbb{R}_h^2$, we define the weighted space $L^{2,s}(\mathcal{D}), s \in \mathbb{R}$, by

$$L^{2,s}(\mathcal{D}) = \{ v \in L^2_{\mathrm{loc}}(\mathcal{D}) : (1 + |x_1|^2)^{s/2} v \in L^2(\mathcal{D}) \}$$

with the norm $\|v\|_{L^{2,s}(\mathcal{D})} = (\int_{\mathcal{D}} (1 + |x_1|^2)^s |v|^2 dx)^{1/2}$. The weighted Sobolev space $H^{1,s}(\mathcal{D}), s \in \mathbb{R}$, is defined as the set of functions in $L^{2,s}(\mathcal{D})$ whose first derivative is also in $L^{2,s}(\mathcal{D})$. The norm $\|v\|_{H^{1,s}(\mathcal{D})} = (\|v\|_{L^{2,s}(\mathcal{D})}^2 + \|\nabla v\|_{L^{2,s}(\mathcal{D})}^2)^{1/2}$.

Lemma 9.1. *Let $f \in L^2(\mathbb{R}_h^2)$ with compact support in B_R. For any $z = 1 + \mathbf{i}\varepsilon$, $0 < \varepsilon < 1$, we have, for any $s > 1/2$, $\|u_z\|_{H^{1,-s}(\mathbb{R}_h^2)} \leqslant C\|f\|_{L^2(\mathbb{R}_h^2)}$ for some constant independent of ε, u_z, and f.*

Proof. We first note that by testing (9.1) by $(1 + |x_1|^2)^{-s}\bar{u}_z$, $s > 1/2$, one can obtain $\|u_z\|_{H^{1,-s}(\mathbb{R}_h^2)} \leqslant C\|u_z\|_{L^{2,s}(\mathbb{R}_h^2)} + C\|f\|_{L^2(\mathbb{R}_h^2)}$ by standard argument. It remains to show $\|u_z\|_{L^{2,s}(\mathbb{R}_h^2)} \leqslant C\|f\|_{L^2(\mathbb{R}_h^2)}$. It is obvious that we only need to prove the estimate for $f \in C_0^\infty(\mathbb{R}_h^2)$. We start with the following integral representation formula

$$u_z(x) = \int_{\mathbb{R}_h^2} N^z(x, y) f(y) dy, \qquad x \in \mathbb{R}_h^2. \tag{9.3}$$

Here $N^z(x, y)$ is the Green function of the problem (9.1)-(9.2) with the complex wave number $kz^{1/2}$, where $\mathrm{Im}\,(z^{1/2}) > 0$ for $\varepsilon > 0$. Similar to (2.3), it is easy to check that

$$N^z(x, y) = \sum_{n=1}^{\infty} \frac{\mathbf{i}}{h\xi_n^z} \sin(\mu_n x_2) \sin(\mu_n y_2) e^{\mathbf{i}\xi_n^z |x_1 - y_1|}, \tag{9.4}$$

where $\xi_n^z = \sqrt{zk^2 - \mu_n^2}$ whose imaginary part $\mathrm{Im}\,\xi_n^z \geqslant 0$. It follows from (9.3)-(9.4) that u_z has the mode expansion

$$u_z(x) = \sum_{n=1}^{\infty} u_n^z(x_1) \sin(\mu_n x_2), \tag{9.5}$$

where, since f is supported in B_R,

$$u_n^z(x_1) = \frac{h}{2} \int_{-R}^{R} \frac{\mathbf{i}}{h\xi_n^z} e^{\mathbf{i}\xi_n^z |x_1 - y_1|} f_n(y_1) dy_1, \qquad f_n(x_1) = \frac{2}{h} \int_0^h f(x) \sin(\mu_n x_2) dx_2.$$

Since $\mathrm{Im}\,\xi_n^z \geqslant 0$ and $|\xi_n^z|^2 = \sqrt{(\mu_n^2 - k^2)^2 + (k^2\varepsilon)^2} \geqslant |\mu_n^2 - k^2|$, we have

$$|u_n^z(x_1)| \leqslant \frac{h}{2} \frac{\sqrt{2R}}{h|\mu_n^2 - k^2|^{1/2}} \|f_n\|_{L^2(\mathbb{R})}.$$

Therefore

$$\|u\|_{L^{2,-s}(\mathbb{R}_h^2)}^2 \quad = \quad \frac{h}{2} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} (1 + |x_1|^2)^{-s} |u_n^z(x_1)|^2 dx_1$$

$$\leqslant \left(\frac{h}{2}\right)^2 \|f\|^2_{L^2(\mathbb{R}^2_h)} \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \frac{2R}{h^2|\mu_n^2 - k^2|}(1+|x_1|^2)^{-s} dx_1$$

$$\leqslant C\|f\|^2_{L^2(\mathbb{R}^2_h)}.$$

where we have used $\|f\|^2_{L^2(\mathbb{R}^2_h)} = \frac{h}{2}\sum_{n=1}^{\infty}\|f_n\|^2_{L^2(\mathbb{R})}$. This completes the proof.   □

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. For any $0 < \varepsilon < 1$, we consider the problem

$$\Delta u_\varepsilon + (1 + \mathbf{i}\varepsilon)k^2 u_\varepsilon = 0 \qquad \text{in } \mathbb{R}^2_h\backslash\bar{D}, \tag{9.6}$$

$$u_\varepsilon = 0 \text{ on } \Gamma_0, \qquad \frac{\partial u_\varepsilon}{\partial x_2} = 0 \text{ on } \Gamma_h, \tag{9.7}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} + \mathbf{i}k\eta u_\varepsilon = g \quad \text{on } \Gamma_D. \tag{9.8}$$

We know that the above problem has a unique solution $u_\varepsilon \in H^1(\mathbb{R}^2_h\backslash\bar{D})$ by the Lax-Milgram Lemma.

Let $\chi \in C^\infty(\mathbb{R}^2_h)$ be the cut-off function such that $0 \leqslant \chi \leqslant 1$, $\chi = 0$ in B_R, and $\chi = 1$ outside of B_{R+1}. Let $v_\varepsilon = \chi u_\varepsilon$, then v_ε satisfies the equation (9.1) with $z = 1 + \mathbf{i}\varepsilon$ and $f = u_\varepsilon \Delta\chi + 2\nabla u_\varepsilon \cdot \nabla\chi$. Obviously, f is supported in B_{R+1}. By Lemma 9.1, we have $\|v_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_h\backslash\bar{D})} \leqslant C\|u_\varepsilon\|_{H^1(B_{R+1}\backslash\bar{D})}$. Since $\chi = 1$ outside B_{R+1}, we have then

$$\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_h\backslash\bar{D})} \leqslant C\|u_\varepsilon\|_{H^1(B_{R+1}\backslash\bar{D})}. \tag{9.9}$$

Next let $\chi_1 \in C_0^\infty(\mathbb{R}^2_h)$ be the cut-off function with that $0 \leqslant \chi_1 \leqslant 1$, $\chi_1 = 1$ in B_{R+1}, and $\chi_1 = 0$ outside of B_{R+2}. For $g \in H^{-1/2}(\Gamma_D)$, let $u_g \in H^1(\mathbb{R}^2_h\backslash\bar{D})$ be the lifting function such that $\frac{\partial u_g}{\partial \nu} + \mathbf{i}k\eta u_g = g$ on Γ_D and $\|u_g\|_{H^1(\mathbb{R}^2_h\backslash\bar{D})} \leqslant C\|g\|_{H^{-1/2}(\Gamma_D)}$ hold. By testing (9.6) with $\chi_1^2(\overline{u_\varepsilon - u_g})$, we have by the standard argument

$$\|u_\varepsilon\|_{H^1(B_{R+1}\backslash\bar{D})} \leqslant C(\|u_\varepsilon\|_{L^2(B_{R+2}\backslash\bar{D})} + \|g\|_{H^{-1/2}(\Gamma_D)}). \tag{9.10}$$

Now we claim

$$\|u_\varepsilon\|_{L^2(B_{R+2}\backslash\bar{D})} \leqslant C\|g\|_{H^{-1/2}(\Gamma_D)}, \tag{9.11}$$

for any $g \in H^{-1/2}(\Gamma_D)$ and $\varepsilon > 0$. If it were false, there would exist sequences $\{g_m\} \subset H^{-1/2}(\Gamma_D)$ and $\{\varepsilon_m\} \subset (0,1)$, and $\{u_{\varepsilon_m}\}$ be the corresponding solution of (9.6)-(9.8) such that

$$\|u_{\varepsilon_m}\|_{L^2(B_{R+2}\backslash\bar{D})} = 1 \text{ and } \|g_m\|_{H^{-1/2}(\Gamma_D)} \leqslant \frac{1}{m}. \tag{9.12}$$

Then $\|u_{\varepsilon_m}\|_{H^{1,-s}(\mathbb{R}^2_h\backslash\bar{D})} \leqslant C$, and thus there is a subsequence of $\{\varepsilon_m\}$, which is still denoted by $\{\varepsilon_m\}$, such that $\varepsilon_m \to \varepsilon' \in [0,1]$, and a subsequence of $\{u_{\varepsilon_m}\}$, which is still denoted by $\{u_{\varepsilon_m}\}$, such that it converges weakly to some $u_{\varepsilon'} \in H^{1,-s}(\mathbb{R}^2_h\backslash\bar{D})$. The function $u_{\varepsilon'}$ satisfies (9.6)-(9.8) with $g = 0$ and $\varepsilon = \varepsilon'$. By the integral representation formula, we have, for $x \in \mathbb{R}^2_h\backslash\bar{D}$,

$$u_{\varepsilon'}(x) = -\int_{\Gamma_D}\left(\frac{\partial N^{1+\mathbf{i}\varepsilon'}(x,y)}{\partial\nu(y)}u_{\varepsilon'}(y) - N^{1+\mathbf{i}\varepsilon'}(x,y)\frac{\partial u_{\varepsilon'}(y)}{\partial\nu(y)}\right)ds(y). \tag{9.13}$$

If $\varepsilon' > 0$, we deduce from (9.13) that $u_{\varepsilon'}$ decays exponentially and thus $u_{\varepsilon'} \in H^1(\mathbb{R}^2_h\backslash\bar{D})$, then $u_{\varepsilon'} = 0$ by the uniqueness of the solution in $H^1(\mathbb{R}^2_h\backslash\bar{D})$ with positive absorption. If $\varepsilon' = 0$, (9.13) implies that $u_{\varepsilon'}$ satisfies the mode radiation condition (1.7), and then $u_{\varepsilon'} = 0$ by the uniqueness Lemma 2.2. Therefore, in any case, $u_{\varepsilon'} = 0$, which, however, contradicts to (9.12).

This shows (9.11). Consequently, by (9.9) and (9.10),

$$\|u_\varepsilon\|_{H^{1,-s}(\mathbb{R}^2_h\backslash\bar{D})} \leqslant C\|g\|_{H^{-1/2}(\Gamma_D)}. \tag{9.14}$$

Now, it is easy to see that u_ε has a convergent subsequence which converges weakly to some $u \in H^{1,-s}(\mathbb{R}^2_h\backslash\bar{D})$ and satisfies (2.4)-(2.6). The desired estimate follows from (9.14). This completes the proof.   □

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