Numerical Methods for Moving Interface Problems

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Outline

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3. DLM Fictitious Domain Method
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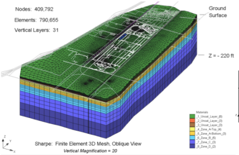

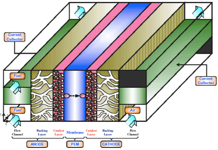

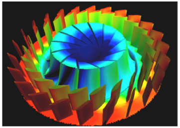

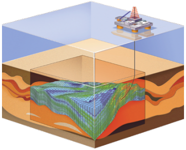

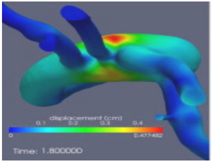

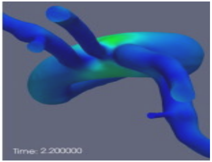

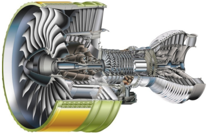

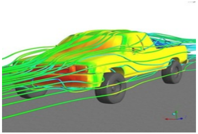

Our Team & Collaborators

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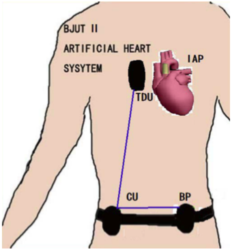
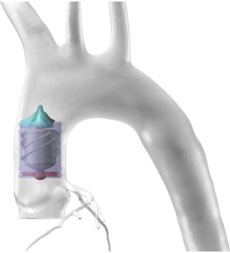
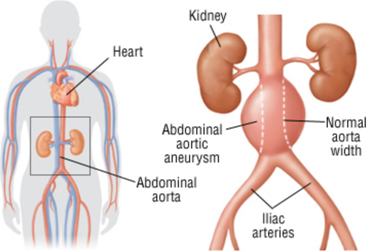
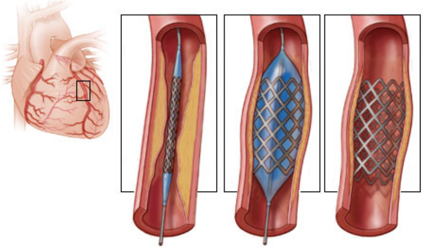
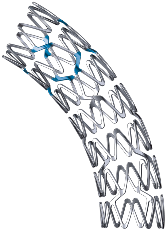
Applications in CVD Treatment







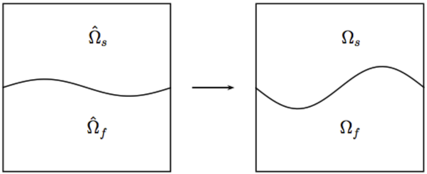




Artificial heart pump, abdominal aortic aneurysm, artery stenosis and dissection, ...

Mathematical Model for FSI

A multiphysics problem which studies one or more solid structures (rigid or flexible) interact with an internal or surrounding fluid flow!



$$\rho D_t v - \nabla \cdot \sigma = 0 \qquad \star \qquad v_s = v_f \qquad \star \qquad \sigma_s n = \sigma_f n$$

- Coupling: Weak vs Strong
- Formulation: Partitioned vs Monolithic
- Coordinates: Eulerian vs Lagrangian or ALE
- Interface: Body-fitted (tracking) vs Non-body-fitted (capturing)
- Meshing: Conforming vs Non-conforming
- Model: Macroscale vs Mesoscale vs Microscale

[Richter 2010; Hou, Wang, Layton 2012; Bazilevs, Takizawa, Tezduyar 2013; ...]

Partitioned Methods

1. Algorithm:
   - Predict and update interface position
   - Regenerate mesh for the fluid domain
   - Solve the fluid equation ($S \rightarrow F$)
   - Compute interface force and solve the solid equation ($F \rightarrow S$)
2. Advantages:
   - Well-established methods and legacy code available for F and S
   - Explicit information communication for interfacial conditions
   - Available in almost all commercial software
3. Challenges:
   - ☞ Stability: Difficult to achieve convergence, stability, and accuracy at once
   - Added mass: Need to account for the added mass effect
   - Accuracy: How to exchange interfacial conditions accurately
   - Reusability: Coordinate the disciplinary code with minimal modification

If partitioned methods do not work, how about monolithic methods?

Eulerian–Eulerian Methods

1. Algorithm:
   - Formulate both fluid and solid in the Eulerian coordinate
   - Most Eulerian methods are of interface-capturing type
   - Some typical examples

     > Phase field method; Volume of fluid method; Level-set method;
     > Initial point set method

2. Advantages:
   - Very large deformation and topology changes can be handled
   - Standard description of the flow problem in Eulerian coordinates
   - No artificial domain mapping is used

3. Challenges:
   - ☞ Accuracy: loss of accuracy near the interface
   - Overhead: capturing the moving interface is needed
   - Efficiency: potentially more expensive (to achieve same accuracy)?

ALE–Lagrangian Methods

1. Algorithm:
   - Using interface-tracking type methods for the F/S interface
   - Fluid mesh deforms according to the F/S interface movement
   - Some examples

     > Arbitrary Lagrangian–Eulerian method; Deforming spatial domain / stablized space-time method

2. Advantages:
   - Explicitly representation of moving interface
   - Interface conditions can be easily embedded in variational form
   - Widely used and tested in many engineering applications

3. Challenges:
   - ☞ Meshing: mesh smoothing/re-meshing is necessary from time to time
   - Applicability: large deformation/displacement, nearly contact structures
   - Efficiency: need efficient solvers for coupled nonlinear/linear systems

Eulerian–Lagrangian Methods

1. Algorithm:
   - Eulerian mesh for F (physical and fictitious) and Lagrangian mesh for S
   - Solve the fluid equation with an artificial force or Lagrange multiplier
   - Some typical examples

     > Immersed domain/boundary method; Immersed interface method;
     > Distributed Lagrange multiplier method; Direct forcing method;
     > Volume of fluid method; eXtended finite element method

2. Advantages:
   - Non-conforming meshes: fluid equation solved on whole domain
   - Allow different coordinate systems for F/S, easier to implement
   - Able to simulate large solid deformation and displacement
3. Challenges:
   - ☞ Accuracy: leakage, need to adjust F/S mesh sizes
   - Stability: may need small time stepsizes
   - Applicability: closed boundary, volume-free, incompressible structure

An FSI Model Problem

Incompressible viscous fluid + elastic solid:

$$\begin{cases} \hat{\rho}_s \partial_{tt} \hat{u}_s - \nabla \cdot \hat{\sigma}_s = 0 & \text{in } \hat{\Omega}_s \\ \rho_f D_t v_f - \nabla \cdot \sigma_f = 0 & \text{in } \Omega_f \\ \nabla \cdot v_f = 0 & \text{in } \Omega_f \\ v_f \circ x_s = \partial_t \hat{u}_s, \quad (\sigma_f n_f) \circ x_s = -\hat{\sigma}_s n_s & \text{on } \hat{\Gamma} \end{cases}$$

Some notations:

- Flow map $F(\hat{x}, t) := \frac{\partial x}{\partial \hat{x}}$ $\quad J(\hat{x}, t) := |F|$
- Linear elasticity model $\hat{\sigma}_s \approx 2\mu_s \varepsilon(\hat{u}_s) + \lambda_s \nabla \cdot \hat{u}_s I =: \tilde{P}_s$
- Material derivative $D_t v := \partial_t v + v \cdot \nabla v$
- Newtonian fluid model $\sigma_f := 2\mu_f \varepsilon(v_f) - p I$

Difficulties: Rotating structure, turbulence flow, complex fluid

Part I. Modeling Rotational Structure

St. Venant–Krichhoff material:

- First Piola–Kirchhoff stress tensor $\hat{P}_s = \hat{\sigma}_s = J\sigma_s F^{-T}$
- Second Piola–Kirchhoff stress tensor $\hat{S}_s := F^{-1}\hat{P}_s = JF^{-1}\sigma_s F^{-T}$
- Green–Lagrangian finite strain tensor $E := \frac{1}{2}(F^T F - I)$
- StVK constitutive law $\hat{S}_s = 2\mu_s E + \lambda_s tr(E)I$

Rotational elastic structure:

- Given structure centroid in static X_0
- Given or unknown rotator is assumed to be R
- Divide the motion into two parts: deformation and rotation:

$$x - X_0 = R(\hat{x} + \hat{u}_d - X_0)$$

- Structure displacement: $\hat{u}_s := x - \hat{x} = R\hat{u}_d + (R - I)(\hat{x} - X_0)$
- Rotating StVK material:

$$\hat{P}_s = R\big(2\mu_s\varepsilon(\hat{u}_d) + \lambda_s tr(\varepsilon(\hat{u}_d))I\big)$$

Part Ⅱ. Modeling Turbulence Flow

Momentum equation

$$\rho_{\mathrm{f}}\Big(\frac{\partial v_{\mathrm{f}}}{\partial t} + v_{\mathrm{f}}\cdot\nabla v_{\mathrm{f}}\Big) = \nabla\cdot(\sigma_{\mathrm{f}} + \sigma_{\mathrm{R}}), \quad \nabla\cdot v_{\mathrm{f}} = 0, \quad \text{in } \Omega_{\mathrm{f}}$$

where the classical viscous stress and the Reynolds stress are defined as

$$\sigma_{\mathrm{f}} := -p_{\mathrm{f}} I + \mu_{\mathrm{f}}\big(\nabla v_{\mathrm{f}} + (\nabla v_{\mathrm{f}})^T\big)$$

$$\sigma_{\mathrm{R}} := -\frac{2}{3}\rho_{\mathrm{f}} k I + \mu_{\mathrm{t}}\big(\nabla v_{\mathrm{f}} + (\nabla v_{\mathrm{f}})^T\big) \qquad \mu_{\mathrm{t}} := \rho_{\mathrm{f}}\frac{k}{\omega}$$

k–ω turbulence model (Shear Stress Transport)

$$\frac{\partial}{\partial t}(\rho_{\mathrm{f}} k) + \nabla\cdot(\rho_{\mathrm{f}} v_{\mathrm{f}} k) = \nabla\cdot(\mu_{\mathrm{eff},k}\nabla k) + P_k - \beta^*\rho_{\mathrm{f}} k\omega$$

$$\frac{\partial}{\partial t}(\rho_{\mathrm{f}}\omega) + \nabla\cdot(\rho_{\mathrm{f}} v_{\mathrm{f}}\omega) = \nabla\cdot(\mu_{\mathrm{eff},\omega}\nabla\omega) + \tilde{\alpha}\frac{\omega}{k}P_k - \tilde{\beta}\rho_{\mathrm{f}}\omega^2 + 2(1-F_1)\sigma_{\omega 2}\frac{\rho_{\mathrm{f}}}{\omega}\nabla k\cdot\nabla\omega$$

SST k–ω Turbulence Model Parameters

Blending function $\tilde{\sigma} := F_1\sigma_1 + (1-F_1)\sigma_2$, where σ_1 and σ_2 are some parameters:

$$F_1 := \tanh\left(\left(\min\left\{\max\left\{\frac{\sqrt{k}}{\beta^\star d_\perp \omega}, \frac{500\mu_\mathrm{f}}{d_\perp^2 \rho_\mathrm{f}\omega}\right\}, \frac{4\sigma_{\omega 2}k}{CD_{k\omega}d_\perp^2}\right\}\right)^4\right),$$

$$CD_{k\omega} := \max\left\{2\rho_\mathrm{f}\sigma_{\omega 2}\frac{1}{\omega}\nabla k\cdot\nabla\omega,\ 10^{-10}\right\},$$

where $d_\perp$ is the distance from the nearest wall. We apply $\beta^\star = 0.09$ and

$$\begin{aligned} &\alpha_1 = 0.5532, && \beta_1 = 0.0750, && \sigma_{k1} = 0.850, && \sigma_{\omega 1} = 0.500,\\ &\alpha_2 = 0.4403, && \beta_2 = 0.0828, && \sigma_{k2} = 1.000, && \sigma_{\omega 2} = 0.856. \end{aligned}$$

In the near-wall region, we define the width of the viscous sublayer as: $d^+_{\mathrm{lim}} = 11.06$, $d^+ = \frac{C_\mu^{1/4}k_{\mathrm{avg}}^{1/2}d_C}{\nu}$, and $C_\mu = 0.09$. If the grid point closest to a wall is in the viscous sublayer ($d^+ < d^+_{\mathrm{lim}}$), we define the boundary behavior:

$$k_{\mathrm{wall}} := \max\left\{\frac{2400C_\mu^{1/2}k_{\mathrm{avg}}}{C_{\epsilon 2}^2\left(\frac{1}{(d^+ + C)^2} + 2\frac{d^+}{C^3} - \frac{1}{C^2}\right)}, 10^{-15}\right\},\ \omega_{\mathrm{wall}} = \frac{6\mu}{\beta_1\rho_\mathrm{f}(d^+)^2},\ C = 11,\ C_{\epsilon 2} = 1.9.$$

If the grid point nearest to a wall is in the inertial sublayer ($d^+ > d^+_{\mathrm{lim}}$), we define $B_k = 8.366$, $\kappa = 0.41$

$$k_{\mathrm{wall}} := \max\left\{C_\mu^{1/2}k_{\mathrm{avg}}\left(\frac{C_k}{\kappa}\ln(d^+) + B_k\right), 10^{-15}\right\},\ \omega_{\mathrm{wall}} := \frac{\sqrt{k}}{C_\mu^{1/4}\kappa d^+},\ C_k = -0.416.$$

How to Handle Moving Domains? ALE

A widely used approach in engineering:

Interface-tracking, mesh conforming, suitable for small deformation

Given solid trajectory x_s^n on $\hat{\Gamma}$, the moving grid can be described by $\mathcal{A}^n : \hat{\Omega}_f \mapsto \Omega_f$

$$\text{ALE mapping} \qquad \begin{cases} \mathcal{A}^n(\hat{x}) = \hat{x} & \text{on } \partial\hat{\Omega}_f \cap \partial\hat{\Omega} \\ \mathcal{A}^n(\hat{x}) = x_s^n(\hat{x}, t) & \text{on } \hat{\Gamma} \end{cases}$$

We obtain an approximation of material derivatives as follows: For $x = \mathcal{A}(\hat{x}, t^{n+1})$

$$D_t v|_{t^{n+1}} \approx (D_{t,k} v)^{n+1} := \partial_{t,k}^{\mathcal{A}} v(x, t^{n+1}) + \big((v - \partial_{t,k}\mathcal{A}) \cdot \nabla\big) v(x, t^{n+1}),$$

where

$$\partial_{t,k}^{\mathcal{A}} v|_{(\mathcal{A}(\hat{x}, t^{n+1}), t^{n+1})} := \frac{1}{k}\Big[v(\mathcal{A}(\hat{x}, t^{n+1}), t^{n+1}) - v(\mathcal{A}(\hat{x}, t^n), t^n)\Big],$$

$$(\partial_{t,k}\mathcal{A})(\hat{x}, t) := \frac{1}{k}\Big[\mathcal{A}(\hat{x}, t^{n+1}) - \mathcal{A}(\hat{x}, t^n)\Big].$$

How to determine $\mathcal{A}^n$? Solve $-\Delta \mathcal{A}^n = 0$ or $\mathcal{L}\mathcal{A}^n = 0$, in $\hat{\Omega}_f$

Development and Analysis of ALE

Development of ALE methods

- FDM + ALE [Noh 1963; Hirt, Amsden, Cook 1974; ...]
- FEM + ALE [Hughes, Liu, Zimmermann 1981; Donea, Giuliani, Halleux 1982; ...]
- FVM + ALE [Farhat, Lesoninne, Maman 1994; Lesoinne, Farhat 1995; ...]
- Many practical applications [Hu, Joseph, Crochet 1992; Hu, Patankar, Zhu 2001; Tallec, Mouro 2001; Bazilevs, Takizawa, Tezduyar 2013; ...]
  - Fluid problems on a moving domain
  - Fluid–particle interactions
  - Fluid–structure interactions
  - ......

A priori error estimates of ALE/FEM

- Geometric Conservation Law [Lesoinne, Farhat 1995; Formaggia, Nobile 1999]
- Linear advection-diffusion problem [Gastaldi 2001]
- Stokes equation: $H^1(\Omega)$-error [Martín, Smarand, Takahashi 2009]
- Fluid-structure interaction $H^1(\Omega)$-error [Lee and Xu 2016a, 2016b]
- Stokes-parabolic problem $H^1(\Omega)$-error and $L^2(\Omega)$-error [Sun et al., in preparation]

Monolithic ALE Formulation

Function spaces:

$$
\begin{aligned}
\hat{\mathbb{V}}_{\mathrm{s}} &:= \left\{\hat{v}_{\mathrm{s}} \in (H^1(\hat{\Omega}_{\mathrm{s}}))^d \mid \hat{v}_{\mathrm{s}} = v_{\mathrm{f}} \circ \mathcal{A} \ \text{ on } \hat{\Gamma}\right\},\\
\mathbb{V}_{\mathrm{f}} &:= \left\{v_{\mathrm{f}} \in (H^1(\Omega_{\mathrm{f}}))^d \mid v_{\mathrm{f}} = v_{\mathrm{B}} \ \text{ on } \partial\Omega\right\},\\
\mathbb{W}_{\mathrm{f}} &:= L^2(\Omega_{\mathrm{f}}),\\
\hat{\mathbb{Q}}_{\mathrm{f}} &:= \left\{\mathcal{A} \in (H^1(\hat{\Omega}_{\mathrm{f}}))^d \mid \mathcal{A} = 0 \ \text{ on } \partial\hat{\Omega}_{\mathrm{f}} \cap \partial\hat{\Omega},\ \mathcal{A} = \hat{u}_{\mathrm{s}} \ \text{ on } \hat{\Gamma}\right\}.
\end{aligned}
$$

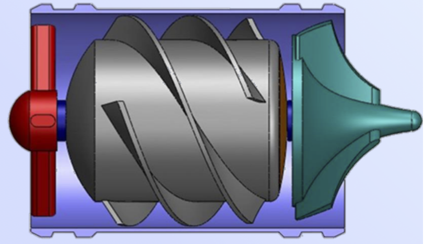
Weak formulation:

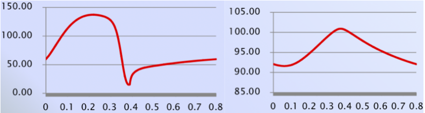
Find $(\hat{v}_{\mathrm{s}}, v_{\mathrm{f}}, p, \mathcal{A}) \in L^\infty(0,T;\hat{\mathbb{V}}_{\mathrm{s}} \times \mathbb{V}_{\mathrm{f}} \times \mathbb{W}_{\mathrm{f}} \times \hat{\mathbb{Q}}_{\mathrm{f}})$ such that

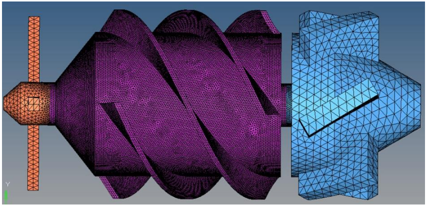
$$
\left\{
\begin{aligned}
&\left(\rho_{\mathrm{f}}\partial_t^{\mathcal{A}} v_{\mathrm{f}}, \psi\right)_{\Omega_{\mathrm{f}}} + \left(\rho_{\mathrm{f}}\left(v_{\mathrm{f}} - w\right)\cdot\nabla v_{\mathrm{f}}, \psi\right)_{\Omega_{\mathrm{f}}} + \left(\sigma_{\mathrm{f}} + \sigma_{\mathrm{R}}, \varepsilon(\psi)\right)_{\Omega_{\mathrm{f}}}\\
&\qquad + \left(\rho_{\mathrm{s}}\frac{\partial\hat{v}_{\mathrm{s}}}{\partial t}, \phi\right)_{\hat{\Omega}_{\mathrm{s}}} + \left(\hat{\sigma}_{\mathrm{s}}\left(\hat{u}_{\mathrm{s}}^0 + \int_0^t \hat{v}_{\mathrm{s}}(\tau)\,d\tau\right), \varepsilon\left(\phi\right)\right)_{\hat{\Omega}_{\mathrm{s}}} = 0,\\
&\qquad (\nabla\cdot v_{\mathrm{f}}, q)_{\Omega_{\mathrm{f}}} = 0,\\
&\qquad (\nabla\mathcal{A}, \nabla\xi)_{\hat{\Omega}_{\mathrm{f}}} = 0,\\
&\qquad \forall\phi \in \hat{\mathbb{V}}_{\mathrm{s}},\ \psi \in \mathbb{V}_{\mathrm{f}},\ q \in \mathbb{W}_{\mathrm{f}},\ \xi \in \hat{\mathbb{Q}}_{\mathrm{f}}.
\end{aligned}
\right.
$$

[Sun, Leng, Z. et al. 2018; Leng, Z., Sun, et al. 2019]

Simulating Artificial Heart







- BJUT-II LVAD
  - Purple: Enclosure
  - Red: Head
  - Green: Tail
  - Gray: Rotor
- Design optimization
  - Smaller still / backward flow zone
  - Reduce turbulence flow
- Major difficulties in simulation
  - Fluid, solid, and coupling
  - High-speed rotation ($\approx$ 7000rpm)
  - Turbulence flow
  - Meshing (blood vessel wall)
  - Large problem size

Partitioning and Meshing

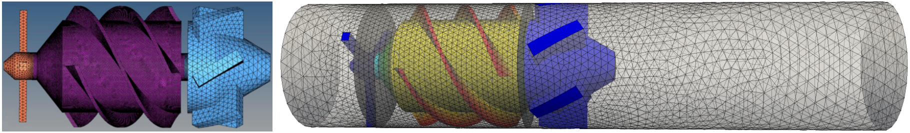


Figure: Left: the artificial heart pump (head guide, rotor and tail guide); Right: the blood flow mesh in a vascular lumen (the rotational part is separated from the stationary part by two discs).

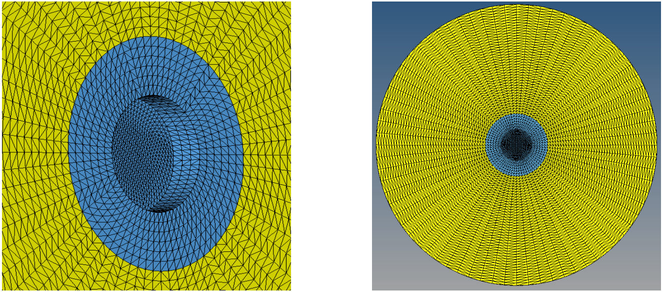
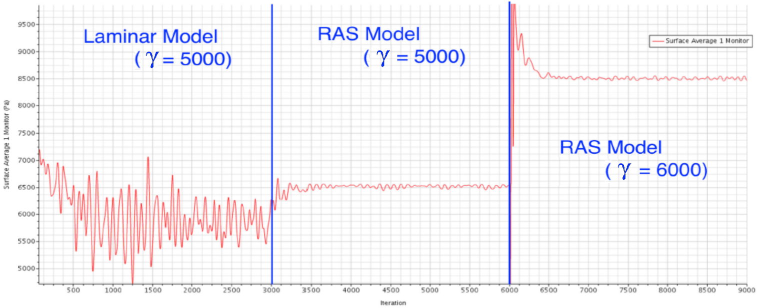
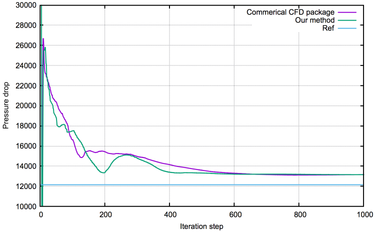
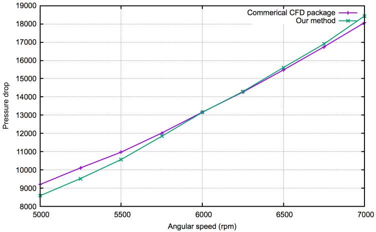


Figure: Interface meshes on $\partial\Omega_{\mathrm{rs}}$ between the stationary fluid and the rotational fluid regions.

Numerical Validation

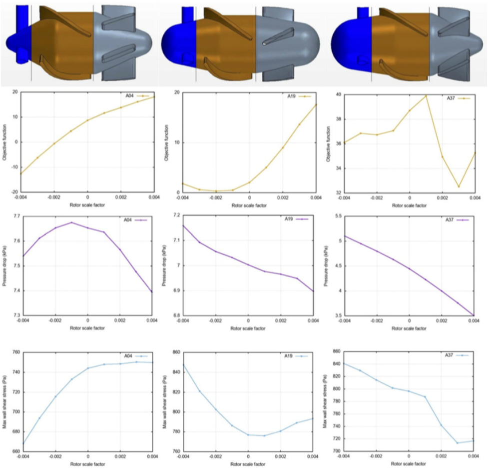






L: Convergence for rotation speed = 7000rpm; R: Comparison with commercial software.

Shape Optimization and Animal Tests



When applied to solve the artificial heart problem on the LSSC-IV cluster (LSEC, AMSS), the whole simulation costs about 2 hours (using 128 processing cores).

An Alternative Approach: DLM/FD

Difficulties when applying body-fitted methods: meshing for evolving domain

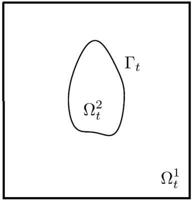
- Moving, drafting, kissing, topological changes, ...
- Mesh generation (rotation + boundary layers) and re-meshing
- Example: Fluid–particle interactions [Glowinski, Pan, Hesla, Joseph 1999]

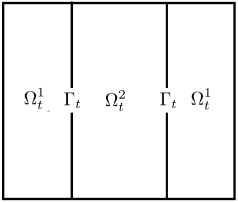
Fictitious domain methods / Domain-embedding methods

- Extend a problem on a geometrically complex (time-dependent) domain to a larger (but simpler) domain
  - Simple geometry $\Longrightarrow$ simple (or even regular) mesh $\Longrightarrow$ fast solvers
  - ☞ Moving domain $\Longrightarrow$ fixed domain $\Longrightarrow$ no re-meshing needed
- Need to find a way to enforce boundary conditions on the original domain
  - As a constraint using a boundary-supported Lagrange multiplier
  - ☞ As a constraint using a distributed Lagrange multiplier
  - Using least-square method
- Many examples: [Hyman 1952; Saulev 1963; Buzbee, et al. 1971; Glowinski, et al. 1994, 1995, 1999; Boffi, Gastaldi 2017; Lundberg, Sun, Wang 2019; ...]

Stokes Interface Problem

$$
\begin{cases}
\rho_1 \partial_t v_1 - \nabla \cdot (\mu_1 \nabla v_1) + \nabla p_1 = f_1, & \Omega_t^1 \times (0,T] \\
\nabla \cdot v_1 = 0, & \Omega_t^1 \times (0,T] \\
\rho_2 \partial_t v_2 - \nabla \cdot (\mu_2 \nabla v_2) + \nabla p_2 = f_2, & \Omega_t^2 \times (0,T] \\
\nabla \cdot v_2 = 0, & \Omega_t^2 \times (0,T] \\
v_1 - v_2 = 0, & \Gamma_t \times (0,T] \\
(\mu_1 \nabla v_1 - p_1 I) n_1 - (\mu_2 \nabla v_2 - p_2 I) n_2 = g, & \Gamma_t \times (0,T] \\
v_1 = 0, & \partial\Omega_t^1 \backslash \Gamma_t \times (0,T] \\
v_2 = 0, & \partial\Omega_t^2 \backslash \Gamma_t \times (0,T] \\
v_1(\cdot,0) = v_1^0(\cdot), & \Omega_0^1 \\
v_2(\cdot,0) = v_2^0(\cdot), & \Omega_0^2
\end{cases}
$$





DLM Fictitious Domain Method

Introduce a fictitious problem

$$\begin{cases} \tilde{\rho}_2\partial_t\tilde{v}_2 - \nabla\cdot(\tilde{\mu}_2\nabla\tilde{v}_2) + \nabla\tilde{p}_2 = \tilde{f}_2, & \Omega_t^2\times(0,T] \\ \nabla\cdot\tilde{v}_2 = 0, & \Omega_t^2\times(0,T] \\ \tilde{v}_2 = v_2, & \Gamma_t\times(0,T] \\ \tilde{v}_2 = 0, & \partial\Omega_t^2\backslash\Gamma_t\times(0,T] \\ \tilde{v}_2(\cdot,0) = v_2^0(\cdot), & \Omega_0^2 \end{cases}$$

DLM/FD formulation: Find a solution $(\tilde{v}, v_2, \tilde{p}, \lambda)$ in $(H^1\cap L^\infty)(0,T;\mathbb{V})\times(H^1\cap L^\infty)(0,T;\mathbb{V}_2(\cdot))\times L^2(0,T;\mathbb{Q})\times L^2(0,T;\mathbb{V}_2(\cdot))$, s.t.

$$\begin{cases} \left(\tilde{\rho}\partial_t\tilde{v},\psi\right)_\Omega + \left(\tilde{\mu}\nabla\tilde{v},\nabla\psi\right)_\Omega - \left(\tilde{p},\nabla\cdot\psi\right)_\Omega + (\lambda,\psi)_{\mathbb{V}_2(\cdot)} = (\tilde{f},\psi)_\Omega, \\ (\nabla\cdot\tilde{u},q)_\Omega = 0, \\ \left((\rho_2-\tilde{\rho}_2)\partial_t^{\mathcal{A}}v_2,\psi_2\right)_{\Omega_t^2} + \left((\rho_2-\tilde{\rho}_2)\,(v_2-w)\cdot\nabla v_2,\psi\right)_{\Omega_t^2} \\ +\left((\mu_2-\tilde{\mu}_2)\nabla v_2,\nabla\psi_2\right)_{\Omega_t^2} - (\lambda,\psi_2)_{\mathbb{V}_2(t)} = (f_2-\tilde{f}_2,\psi_2)_{\Omega_t^2} + (g,\psi_2)_{\Gamma_t}, \\ (\phi_2,\tilde{v}_2-v_2)_{\mathbb{V}_2(t)} = 0, \\ \forall\,\psi\in\mathbb{V},\ \psi_2\in\mathbb{V}_2(\cdot),\ q\in\mathbb{Q},\ \phi_2\in\mathbb{V}_2(\cdot). \end{cases}$$

Problem Setting and Assumptions

Steady-state Stokes interface case

$$\begin{cases} -\nabla\cdot(\mu_i\nabla v_i)+\nabla p_i=f_i, & \Omega_t^i\times(0,T] \\ \nabla\cdot v_i=0, & \Omega_t^i\times(0,T] \\ (\mu_1\nabla v_1-p_1I)n_1-(\mu_2\nabla v_2-p_2I)n_2=g, & \Gamma_t\times(0,T] \end{cases}$$

We have [Shibata, Shimizu 2003; Abels, Liu 2018]: If Γ_t is smooth enough, then

$$\|v\|_{H^1(\Omega)}+\sum_i\left(\|v\|_{H^2(\Omega_t^i)}+\|p\|_{H^1(\Omega_t^i)}\right)\lesssim\|f\|_{L^2(\Omega)}+\|g\|_{H^{\frac{1}{2}}(\Gamma_t)}$$

Assumptions

- On regularity: Γ_t is Lipschitz, $v(t)\in H^\alpha(\Omega_t^1\cup\Omega_t^2)$ with $\frac{3}{2}<\alpha\leq 2$
- On the immersed domain: $\max\left\{\|w\|_{1,\infty,\Omega_t^2},\|\partial_t w\|_{1,\infty,\Omega_t^2}\right\}\leq C$
- On the numerical method: $\mu_2>\tilde{\mu}_2$ and $\rho_2>\tilde{\rho}_2$

Function spaces

$$\mathbb{V}:=\left(H_0^1(\Omega)\right)^d,\quad \mathbb{W}:=L^2(\Omega),\quad \mathbb{V}_2(t):=\left(H_0^1(\Omega_t^2)\right)^d$$

$$(\lambda,v)_{\mathbb{V}_2(t)}:=(\lambda,v)_{\Omega_t^2}+(\nabla\lambda,\nabla v)_{\Omega_t^2}$$

Convergence of DLM/FD

Theorem (Stability)

Let $(\tilde{v}_h^m, v_{2,h_2}^m, p_h^m, \lambda_{h_2}^m) \in \mathbb{V}_h \times \mathbb{V}_{2,h_2}^m \times \mathbb{W}_h \times \mathbb{V}_{2,h_2}^m$. Then we have

$$\|\tilde{v}_h^m\|_{0,\Omega} + \|v_{2,h_2}^m\|_{0,\Omega_{t_m}^2} + \left(k \sum_{m=0}^{M} \left(\|\tilde{v}_h^m\|_{\mathbb{V}}^2 + \|v_{2,h_2}^m\|_{\mathbb{V}_2(t_m)}^2 \right) \right)^{\frac{1}{2}}$$
$$\lesssim \|\tilde{v}_h^0\|_{0,\Omega} + \|v_{2,h_2}^0\|_{0,\Omega_0^2} + k \sum_{m=0}^{M} \left(\|\tilde{f}^m\|_{0,\Omega} + \|f_2^m\|_{0,\Omega_m^2} + \|g^m\|_{0,\Gamma_m} \right)$$

Theorem (Error Estimate)

Let $(\tilde{v}_h^m, v_{2,h_2}^m, p_h^m, \lambda_{h_2}^m) \in \mathbb{V}_h \times \mathbb{V}_{2,h_2}^m \times \mathbb{W}_h \times \mathbb{V}_{2,h_2}^m$. Then we have

$$\|\tilde{v}^m - \tilde{v}_h^m\|_{0,\Omega} + \|v_2^m - v_{2,h_2}^m\|_{0,\Omega_{t_m}^2} + \left(k \sum_{m=0}^{M} \left(\|\tilde{v}^m - \tilde{v}_h^m\|_{\mathbb{V}}^2 + \|v_2^m - v_{2,h_2}^m\|_{\mathbb{V}_2(t_m)}^2 \right) \right)^{\frac{1}{2}}$$
$$\lesssim h^{\alpha-1} + h_2^{\alpha-1} + k$$

Finite element space used is P^2-P^2-P^1-P^2 [Lundberg, Sun, Wang, Z., 2019].

Numerical Experiments

h	h_2	$\|v - v_h\|_1$	$\|v - v_h\|_0$	$\|p - p_h\|_0$
1/10	1/40	5.8308e-3	2.3015e-4	6.9973e-3
1/16	1/64	5.2881e-3	1.8555e-4	4.5674e-3
1/24	1/96	3.6631e-3	9.5663e-5	1.9919e-3
1/28	1/112	2.9270e-3	7.6531e-5	2.5130e-3
		0.67	1.07	0.99

Table: Spatial convergence (rough case: $\mu_2/\mu_1 = 10^4$, $\rho_2/\rho_1 = 10^3$, $k = 1/128$)

k	$\|v - v_h\|_1$	$\|p - p_h\|_0$
1/64	8.4674e-3	3.1714e-2
1/128	4.3102e-3	1.6299e-2
1/256	2.1778e-3	8.3803e-3
	0.99	0.96

Table: Temporal convergence (smooth case: $\mu_2/\mu_1 = 2$, $\rho_2/\rho_1 = 2$, $h = 1/32$, $h_2 = 1/128$)

Finite element space used is P^2-P^2-P^1-P^2 [Lundberg, Sun, Wang, Z., 2019].

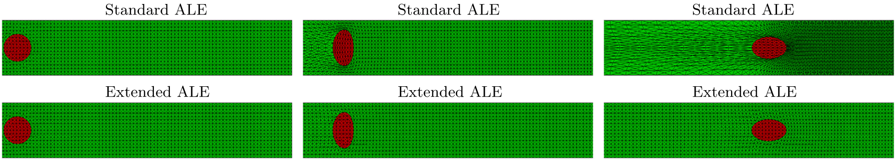
Summay

Pros and cons

- ALE methods are mature in many engineering applications
- ALE methods rely on good linear solvers, meshing, etc
- Standard ALE methods not good for problems with large displacement or deformation
- FD methods can take advantages of fixed meshes; can be applied in many cases
- Constraints are enforced via Lagrange multipliers on two meshes in the fictitious domain

Extended ALE method: [Basting, Quaini, Čanić, Glowinski 2017]

- Keep mesh connectivity while maintain body-fitted with moving structures
- Employ a constrained optimization approach to enforce mesh aligning with structures
- Provably optimal mesh quality and non-degenerate (especially for large displacement)



(a) $t = 0$s (b) $t = 5$s (c) $t = 15$s

How to solve FSI problems with large deformation / displacement?

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