ON THE ITERATIVE SOLUTION OF SADDLE POINT PROBLEMS

Philippe Devloo
Giovane Avancini
Jeferson Fernandes
Sonia Gomes

Outline

- Two different preconditioners are presented
- 1. A spectrally efficient preconditioner for the Darcy equation
- 2. A preconditioner for incompressible elasticity or the Stokes equation

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Part 1
An efficient preconditioner for mixed finite element approximations

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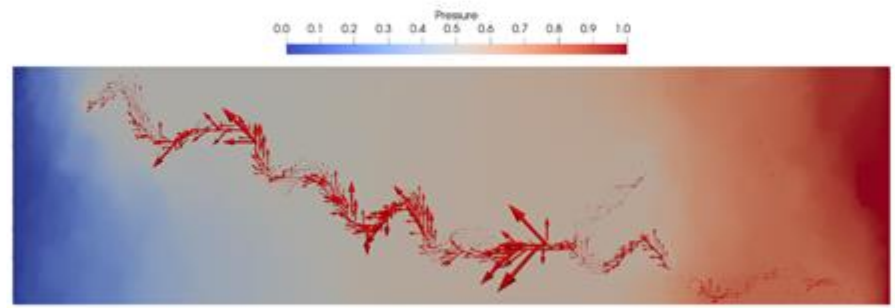
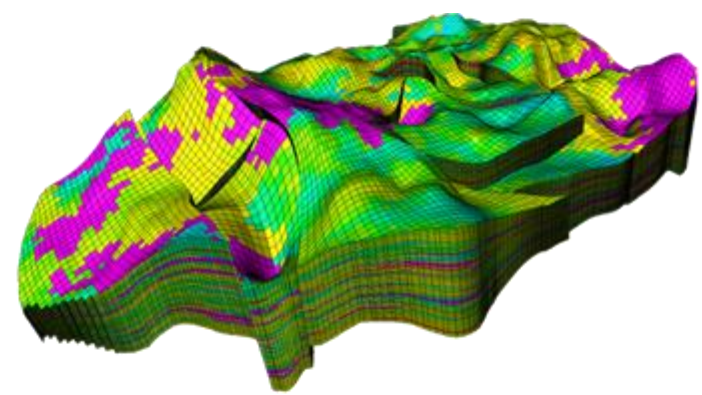
Motivation

Mixed finite element simulations are applicable to reservoir simulation: they allow using unstructured meshes with a higher precision when compared to FV.

The goals of this work are:

- Develop a fast prototype solver;
- Derive low cost and efficient preconditioners for high order mixed finite element computations and reservoir simulations;
- **Obtain local conservation for the approximate solutions at each iteration**;





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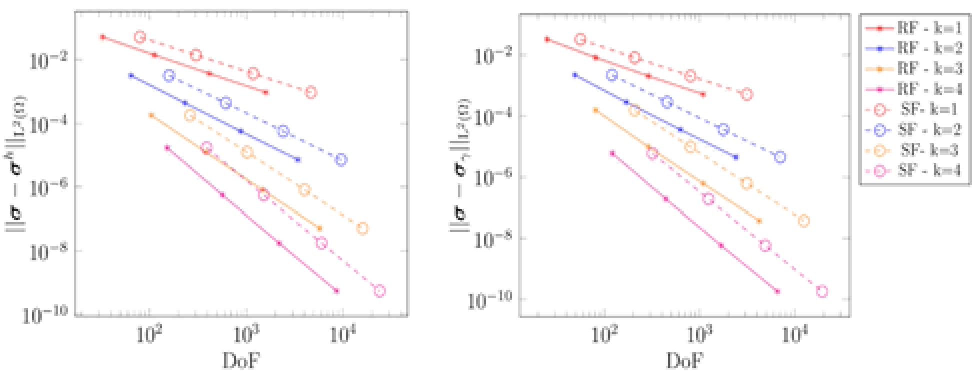
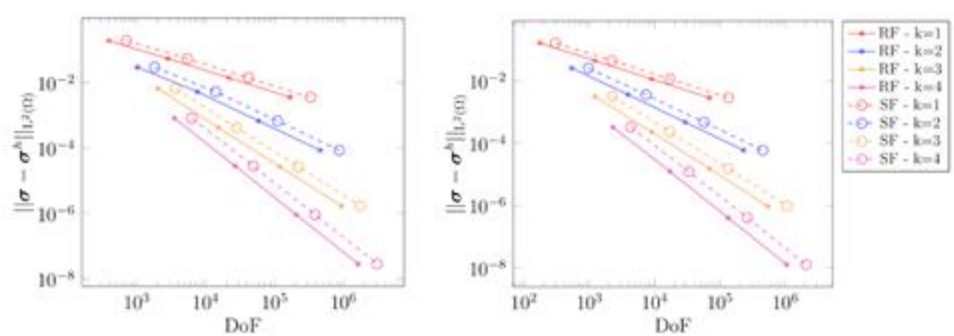
Methodology

First step: development of low cost div-free and div-constant H(div) spaces



(a) Triangle (b) Quadrilateral

Reduction of **55%-65%** in # DoF for the same result.



(c) Tetrahedra (d) Hexahedra

Reduction of **45%-50%** in # DoF for the same result.

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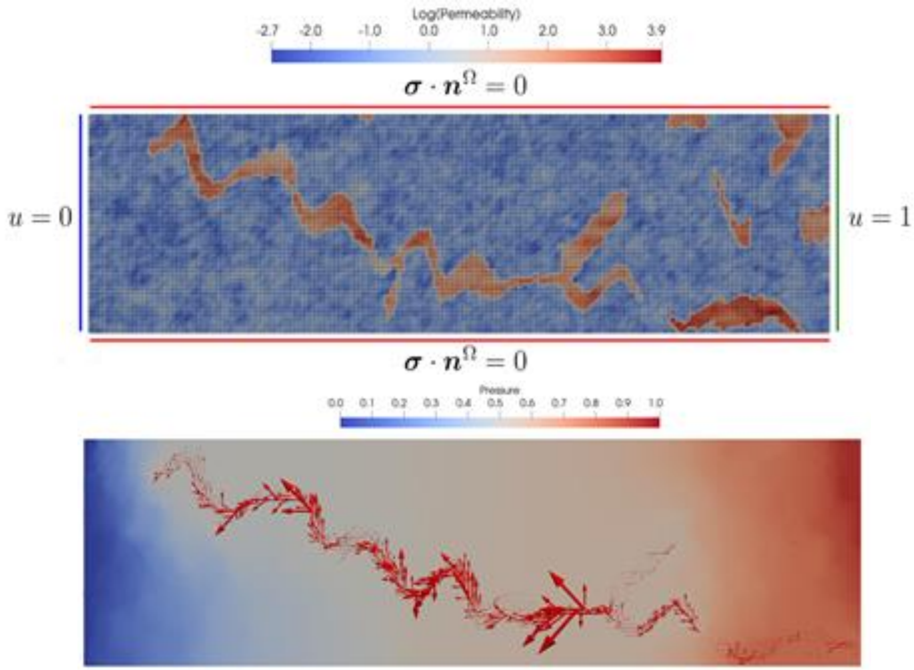
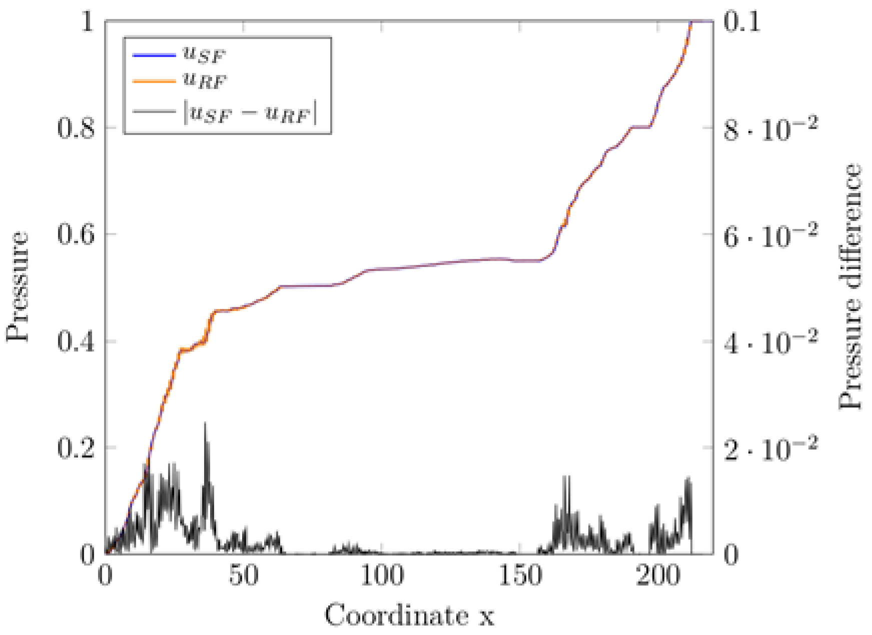
Methodology

Application: flow in heterogeneous porous media.





P. R. B. Devloo et al. “An efficient construction of divergence-free spaces in the context of exact finite element de Rham sequences”. *Computer Methods in Applied Mechanics and Engineering* (2022).

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Methodology

Second step: move forward and solve the reduced problem in a more efficient way.

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \Rightarrow \mathbf{Ku} = \mathbf{F}$$

Direct solvers

Involves the inversion (or decomposition)

of matrix **K**;

In practice, its efficiency is limited to the matrix dimensions and sparsity.

Iterative solvers

Do not require the inversion of matrix **K** but uses projections to correct an initial guess solution.

Matrix **K** needs to be properly conditioned;

Suitable for large scale problems.

Model problem - Darcy flow
Semi-hybrid form

Now the formulation is stated as: find $\tilde{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^0 \in \hat{\mathbf{V}}_{\eta^0} \oplus \mathbf{V}_{\eta}^0$, with $\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}|_{\partial\Omega_N} = \bar{g}$, $\boldsymbol{\sigma}^0 \cdot \mathbf{n}|_{\partial\Omega_N} = g^{\perp}$ $\tilde{u} \in W_{\eta^0}$ and $\tilde{\lambda} \in \Lambda_{\eta^0}$ such that

$$\int_\Omega \mathbb{K}^{-1}\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{v}}\, d\mathbf{x} + \int_\Omega \mathbb{K}^{-1}\boldsymbol{\sigma}^0 \cdot \hat{\mathbf{v}}\, d\mathbf{x} - \sum_{K\in\mathcal{T}_h} \int_K \tilde{u}\nabla\cdot\hat{\mathbf{v}}\, d\mathbf{x} + \sum_{K\in\mathcal{T}_h} \int_{\partial K\setminus\partial\Omega} \tilde{\lambda}\,\hat{\mathbf{v}}\cdot\mathbf{n}^K ds = -\int_{\partial\Omega_D} u_D(\hat{\mathbf{v}}\cdot\mathbf{n})ds,$$

$$\int_\Omega \mathbb{K}^{-1}\hat{\boldsymbol{\sigma}} \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbb{K}^{-1}\boldsymbol{\sigma}^0 \cdot \mathbf{v}\, d\mathbf{x} = -\int_{\partial\Omega_D} u_D(\mathbf{v}\cdot\mathbf{n})ds,$$

$$\sum_{K\in\mathcal{T}_h} \int_K \nabla\cdot\hat{\boldsymbol{\sigma}}\, w d\mathbf{x} = \int_\Omega f w d\mathbf{x},$$

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K\setminus\partial\Omega} \mu\,\hat{\boldsymbol{\sigma}}\cdot\mathbf{n}^K ds = 0,$$

Introduce a constant pressure Lagrange multiplier between elements to enforce continuity in a weak sense

Model problem - Darcy flow
Iterative method

Static condensation can be applied to the problem:

$$\begin{bmatrix} \mathbf{M}_{\mu\mu} & \mathbf{M}_{\mu\mathbf{v}^e} \\ \mathbf{M}_{\mathbf{v}^e\mu} & \mathbf{M}_{\mathbf{v}^e\mathbf{v}^e} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{S}^e \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{\mu} \\ \mathbf{G}_{\mathbf{v}^e} \end{bmatrix} \Rightarrow$$

$$\mathbf{M}_{\mu\mu}\mathbf{\Lambda} = \mathbf{G}_{\mu} - \mathbf{M}_{\mu\mathbf{v}^e}\mathbf{S}^e. \quad (1)$$

$$\left(\mathbf{M}_{\mathbf{v}^e\mathbf{v}^e} - \mathbf{M}_{\mathbf{v}^e\mu}\mathbf{M}_{\mu\mu}^{-1}\mathbf{M}_{\mu\mathbf{v}^e}\right)\mathbf{S}^e = \mathbf{G}_{\mathbf{v}^e} - \mathbf{M}_{\mathbf{v}^e\mu}\mathbf{M}_{\mu\mu}^{-1}\mathbf{G}_{\mu}. \quad (2)$$

ALGORITHM

- Decompose matrix $\mathbf{M}_{\mu\mu}$, which has **dimension independent of the polynomial degree**, as it corresponds to the constant flux approximations, and is equal to **the corresponding FV problem**.
- **Solve iteratively problem (2) applying the conjugate gradient method** and a block diagonal preconditioner;
- Once problem (1)-(2) reaches convergence, the condensed variables can be explicitly recovered.
- **Remark:** A conservative solution can be obtained at each iteration;

Numerical Results
Problem setup

- A tolerance of $\varepsilon = 10^{-10}$ (Euclidean norm) in the iterative process was considerd;
- The permeability tensor is equal to the identity, namely K = I
- Numerical tests were performed with quadrilateral (in 2D) and hexahedral (in 3D) meshes for several configurations, with both *h* and *p* refinement.
- The following exact solutions were adopted:

2D problem

$$u = e^{\pi x} \sin(\pi y)$$

3D problem

$$u = \frac{\sin(\pi x)\sin(\pi y)\sinh(\sqrt{2}\pi z)}{\sinh\left(\sqrt{2}\pi\right)}$$

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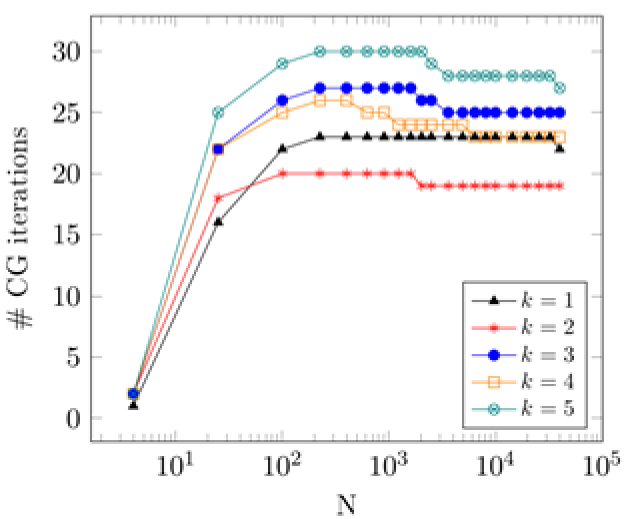
2D case - *h* refinement

? How many iterations do the method need to achieve convergence?

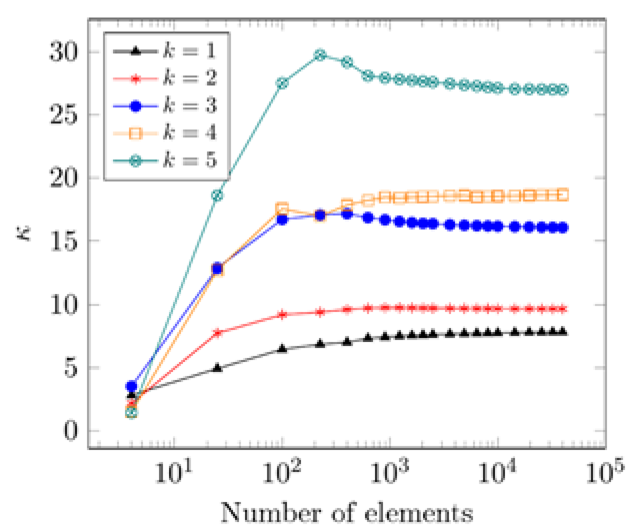
? What about the matrix conditioning?

? How big is the problem size reduction?

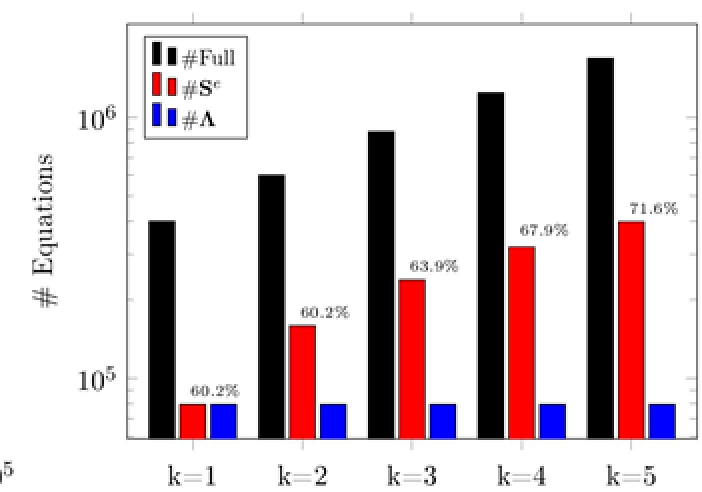
Number of CG iterations



Condition number



Problem size reduction



k: polynomial order

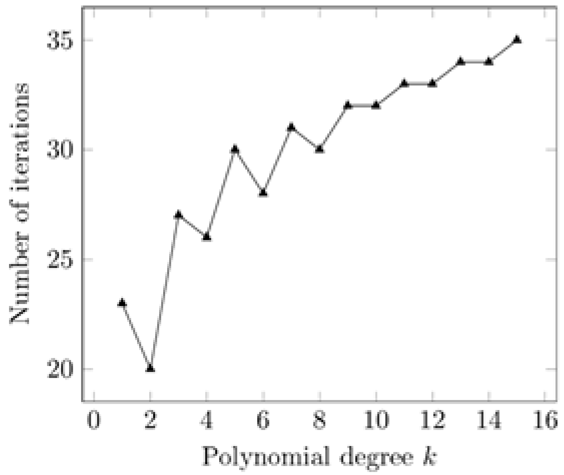
blue : size of the constant pressure matrix

2D case - *p* refinement

? How the method works with higher order approximations?

? Will the method work any unstructured mesh?

Number of CG iterations



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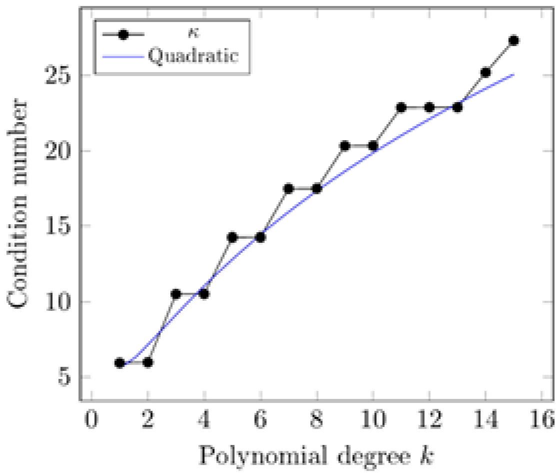

	a_0	a_1	a_2	a_3	$\|error\|_{L^\infty}$	$\|error\|_{L^2}$
Linear	1.742	8.315			4.18769	7.09641
Quadratic	5.573	0.535	2.640		1.44206	3.32993
Cubic	5.735	-0.687	3.785	-0.273	1.39989	3.29934

Theoretical results verified:

$$\kappa = \mathcal{O}(1 + \log k)^2$$

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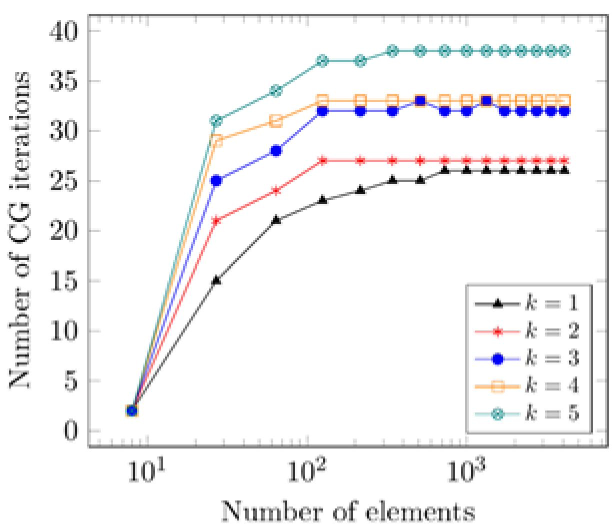
3D case - *h* refinement

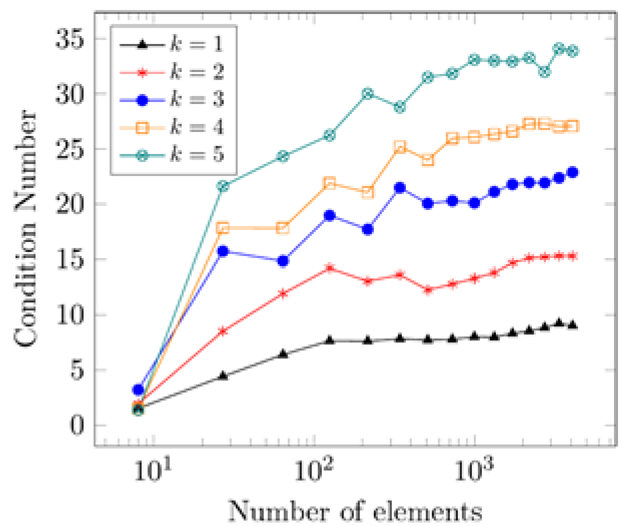
? The conclusions of the 2D analysis are still valid?

? How is the problem size reduction in the 3D case?

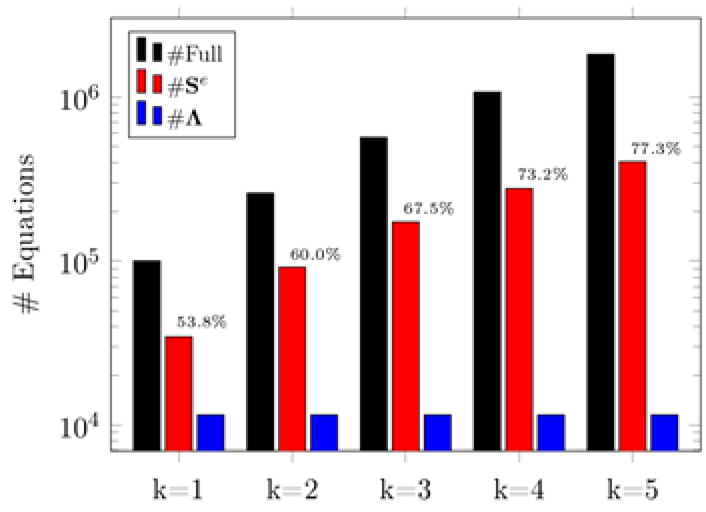
Number of CG iterations



Condition number



Problem size reduction



k: polynomial order

blue : size of the constant pressure matrix

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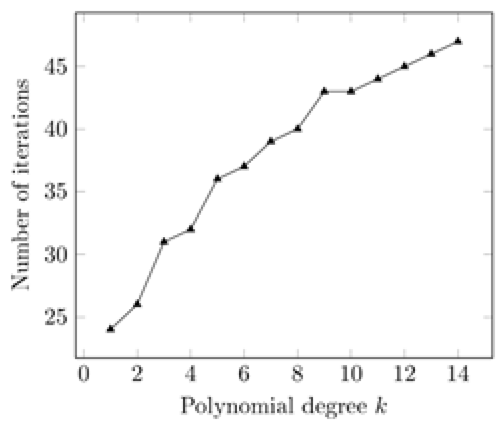
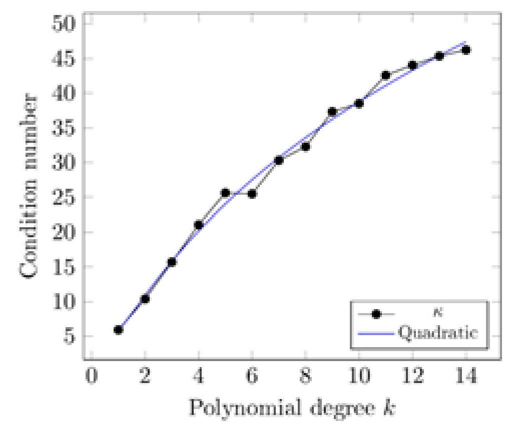
3D case - *p* refinement

The conclusions of the 2D analysis are still valid?

Number of CG iterations



Condition number



	a_0	a_1	a_2	a_3	$\|\text{error}\|_{L^\infty}$	$\|\text{error}\|_{L^2}$
Linear	0.0315	16.684			5.89807	10.2099
Quadratic	5.774	4.455	4.286		2.04349	3.83837
Cubic	5.859	3.759	4.960	-0.165	2.09517	3.83047

Theoretical results verified:

$$\kappa = \mathcal{O}(1+\log k)^2$$

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- Efficient preconditioner obtained, with low number of iterations;
- The number of iterations doesn't increase mesh refinement;
- The number of iterations increases in a log scale with polynomial degree;
- Theoretical results for the matrix condition number were obtained with optimal rate;
- Good results for both 2D and 3D approaches;
- High reduction in the number of equations of the global system.
- A conservative solution can be obtained at each iteration;

Conclusions

Part 2
An efficient preconditioner for mixed finite element approximations of Stokes equations

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Outline

- Challenges in inverting saddle point problems
- Problem statement
- A preconditioner introducing a small compressibility
- A positive definite preconditioner
- Numerical results
- Conclusions

Motivation

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- A linear system of equations modeling a saddle point problem has a structure as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix}$$

- Depending on the ordering of the equations, its decomposition will require pivoting equations
- If small numerical perturbations occur during decomposition, the linear solver may not pivot where necessary and give wrong results

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Applications

- The iterative scheme is tested for two kind of boundary value problems:

- 1) Darcy equations

- 2) Stokes or incompressible elasticity equations

Darcy problem statement

$$\begin{aligned} \boldsymbol{\sigma} = -\mathcal{K}\nabla p, &\quad \text{in} \quad \Omega, \\ \nabla \cdot \boldsymbol{\sigma} = f, &\quad \text{in} \quad \Omega, \\ p = p_D, &\quad \text{on} \quad \partial\Omega_D, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = g, &\quad \text{on} \quad \partial\Omega_N, \end{aligned}$$

- The finite element approximation using De Rham compatible flux/pressure spaces is

find $\boldsymbol{\sigma}^\gamma \in \mathbf{V}^\gamma$ and $p^\gamma \in W^\gamma$ such that for all $\mathbf{w}_{\boldsymbol{\sigma}}^\gamma \in \mathbf{V}^\gamma$ and $w_p^\gamma \in W^\gamma$:

$$\sum_{\Omega_e \in \mathcal{T}} \left(\int_{\Omega_e} \mathcal{K}^{-1} \boldsymbol{\sigma}^\gamma \cdot \mathbf{w}_{\boldsymbol{\sigma}}^\gamma \, d\Omega - \int_{\Omega_e} p^\gamma \, \nabla \cdot \mathbf{w}_{\boldsymbol{\sigma}}^\gamma \, d\Omega \right) = - \sum_{\Omega_e \in \partial\Omega_D} \int_{\partial\Omega_e} p_D \left(\mathbf{w}_{\boldsymbol{\sigma}}^\gamma \cdot \mathbf{n} \right) d\partial\Omega,$$

$$\sum_{\Omega_e \in \mathcal{T}} \int_{\Omega_e} \nabla \cdot \boldsymbol{\sigma}^\gamma \, w_p^\gamma \, d\Omega = \sum_{\Omega_e \in \mathcal{T}} \int_{\Omega_e} f w_p^\gamma \, d\Omega.$$

Darcy formulation properties

- For each element, the flux shape functions can be partitioned
  - Internal fluxes
  - Boundary fluxes
- The pressure functions can be partitioned
  - Pressures with zero average
  - Constant pressure
- The internal fluxes and pressures with zero average value can be statically condensed at the element level

$$[K]_{el} = \begin{bmatrix} K_{\sigma_n \sigma_n} & K_{\sigma_n \bar{p}} \\ K_{\bar{p} \sigma_n} & 0 \end{bmatrix}$$

Stokes problem statement

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f} &\quad \text{in} \quad \Omega,\\
-\nabla \cdot \mathbf{u} = 0 &\quad \text{in} \quad \Omega,\\
\mathbf{u} = \mathbf{u}_D &\quad \text{on} \quad \partial\Omega_D,\\
\boldsymbol{\sigma}\mathbf{n} = \boldsymbol{\sigma}_N &\quad \text{on} \quad \partial\Omega_N,
\end{aligned}
$$

- A double-hybrid finite element formulation with compatible De Rham spaces is developed, resulting in

Find $\mathbf{u}$, p, $\boldsymbol{\lambda}^t$, $\mathbf{u}^t \in \vec{\mathbb{V}}_D \times \mathbb{Q}_d \times \boldsymbol{\Lambda}^t_* \times \boldsymbol{\mathcal{L}}^t_D$ such that for all $\mathbf{v}, q, \boldsymbol{\eta}^t, \mathbf{v}^t \in \vec{\mathbb{V}}_0 \times \mathbb{Q}_d \times \boldsymbol{\Lambda}^t_* \times \boldsymbol{\mathcal{L}}^t_0$, the following equations are satisfied

$$
\sum_{\Omega_e \in \mathcal{T}} \int_{\Omega_e} 2\mu\boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{u}) d\Omega_e - \sum_{\Omega_e \in \mathcal{T}} \int_{\Omega_e} p(\nabla \cdot \mathbf{v}) d\Omega_e - \sum_{\partial\Omega_e \in \partial\mathcal{T}} \int_{\partial\Omega_e} \boldsymbol{\lambda}^t \cdot \mathbf{v} d\partial\Omega_e = \sum_{\Omega_e \in \mathcal{T}} \int_{\Omega_e} \mathbf{v} \cdot \mathbf{f} d\Omega_e + \sum_{E \in \partial\Omega_N} \int_E \boldsymbol{\sigma}^n_N (\mathbf{v} \cdot \mathbf{n}) d\partial\Omega_e
$$

$$
\sum_{\Omega_e \in \mathcal{T}} - \int_{\Omega_e} q(\nabla \cdot \vec{u}) d\Omega_e = 0
$$

$$
\sum_{\partial\Omega_e \in \partial\mathcal{T}} - \int_{\partial\Omega_e} \mathbf{u} \cdot \boldsymbol{\eta}^t d\partial\Omega_e + \sum_{E \in \varepsilon_0} \int_E [\![\boldsymbol{\eta}^t]\!] \cdot \mathbf{u}^t d\partial\Omega_e = 0
$$

$$
\sum_{E} \int_E [\![\boldsymbol{\lambda}^t]\!] \cdot \mathbf{v}^t d\partial\Omega_e = \sum_{E \in \partial\Omega_N} \int_E \mathbf{v}^t \cdot (\boldsymbol{\lambda}^t - \boldsymbol{\sigma}^t_N)\, d\partial\Omega_e
$$

Stokes formulation properties

- For each element, the velocity shape functions can be partitioned
  - Internal velocities
  - Boundary normal velocities
- The pressure functions can be partitioned
  - Pressures with zero average
  - Constant pressure
- The internal velocities pressures with zero average value and tangential stresses can be statically condensed at the element level

$$[K]_{el} = \begin{bmatrix} K_{\mathbf{uu}^t} & K_{\mathbf{u}p} \\ K_{\mathbf{u}p}^T & 0 \end{bmatrix}$$

Global system of equations

- Grouping the fluxes first and pressures second

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\sigma} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{\boldsymbol{\sigma}} \\ \mathbf{f}_p \end{Bmatrix}$$

- The matrix A is positive definite
- Each element contributes to one equation in B

Preconditioner

- A preconditioner is proposed adding a small compressibility to the global system of eqs

$$[\mathbf{G}] = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \Rightarrow \tilde{\mathbf{G}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{C} \end{bmatrix}$$

- Where C is a positive definite matrix representing an "artificial" compressibility
- The pressure equations can now be statically condensed

$$\bar{\mathbf{G}} = \mathbf{A} + \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$$
$$\bar{\mathbf{f}} = \mathbf{f}_{\boldsymbol{\sigma}} + \mathbf{B}\mathbf{C}^{-1}\mathbf{f}_p$$

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Iterative method

$$\bar{\mathbf{G}} = \mathbf{A} + \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T \qquad \bar{\mathbf{f}} = \mathbf{f}_{\boldsymbol{\sigma}} + \mathbf{B}\mathbf{C}^{-1}\mathbf{f}_p$$

- Compute an initial solution for flux and pressure

$$\boldsymbol{\sigma}^0 = \bar{\mathbf{G}}^{-1}\bar{\mathbf{f}} \qquad \mathbf{p}^0 = C^{-1}\left(\mathbf{B}^T\boldsymbol{\sigma}^0 - \mathbf{f}_p\right)$$

- Compute the residual

$$\mathbf{r}^k = -\mathbf{B}\mathbf{C}^{-1}\left(\mathbf{B}^T\boldsymbol{\sigma}^k - \mathbf{f}_p\right)$$

The original constraint

- Solve the system to obtain the flux correction

$$\Delta\boldsymbol{\sigma}^k = \bar{\mathbf{G}}^{-1}\mathbf{r}^k \qquad \boldsymbol{\sigma}^{k+1} = \boldsymbol{\sigma}^k + \Delta\boldsymbol{\sigma}^k$$

- Update the pressure explicitly

$$\Delta\mathbf{p}^k = C^{-1}\mathbf{B}^T\Delta\boldsymbol{\sigma}^k \qquad \mathbf{p}^{k+1} = \mathbf{p}^k + \Delta\mathbf{p}^k$$

- **Notice that the flux update does not depend on the pressure!**

Convergence of the method

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$$[\mathbf{G}] = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix}$$

$$[\bar{\mathbf{G}}] = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{C} \end{bmatrix}$$

$$[\mathbf{G}]\,[\mathbf{U}] = [\mathbf{F}]$$

$$[\mathbf{U}]^{k+1} - [\mathbf{U}]^{k} = [\bar{\mathbf{G}}]^{-1}\left([\mathbf{F}] - [\mathbf{G}]\,[\mathbf{U}]^{k}\right)$$

$$[\mathbf{U}]^{k+2} - [\mathbf{U}]^{k+1} = [\bar{\mathbf{G}}]^{-1}\left([\mathbf{F}] - [\mathbf{G}]\,[\mathbf{U}]^{k+1}\right)$$

$$\Delta\,[\mathbf{U}]^{k+1} - \Delta\,[\mathbf{U}]^{k} = [\bar{\mathbf{G}}]^{-1}\left(-[\mathbf{G}]\,\Delta\,[\mathbf{U}]^{k}\right)$$

$$\Delta\,[\mathbf{U}]^{k+1} = \left([I] - [\bar{\mathbf{G}}]^{-1}[\mathbf{G}]\right)\Delta\,[\mathbf{U}]^{k}$$

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Convergence

$$\bar{\mathbf{G}} = \mathbf{G} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C} \end{bmatrix}$$

Taylor series expansion

$$\bar{\mathbf{G}}^{-1} = \mathbf{G}^{-1} - \mathbf{G}^{-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C} \end{bmatrix} \mathbf{G}^{-1} + O(\mathbf{C})$$

$$(\mathbf{I} - \bar{\mathbf{G}}^{-1}\mathbf{G}) \simeq \mathbf{G}^{-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C} \end{bmatrix}$$

$$\Delta\left[\mathbf{U}\right]^{k+1} = \left([I] - \left[\bar{\mathbf{G}}\right]^{-1}\left[\mathbf{G}\right]\right)\Delta\left[\mathbf{U}\right]^{k}$$

- **If C is suficiently small, the method will converge**
- **QUESTION: How to define matrix C?**

$$[C] = \alpha\, h_e [I]$$

Artificial compressibility parameter

Where are we different from Mardal/Winter?

CONSTRUCTION OF PRECONDITIONERS BY MAPPING PROPERTIES FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

KENT-ANDRE MARDAL[1] AND RAGNAR WINTHER[2]

$$\mathcal{A}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} -\Delta & -\operatorname{grad} \\ \operatorname{div} & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

$$\mathcal{B} = \begin{pmatrix} -\Delta^{-1} & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix}\begin{Bmatrix} \boldsymbol{\sigma} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{\boldsymbol{\sigma}} \\ \mathbf{f}_p \end{Bmatrix}$$

$$\tilde{\mathbf{G}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{C} \end{bmatrix}$$

We just found a “clever” way of efficiently inverting $\tilde{G}$

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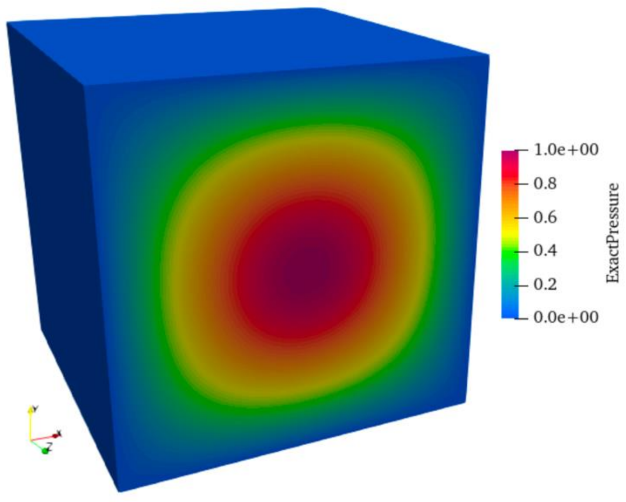
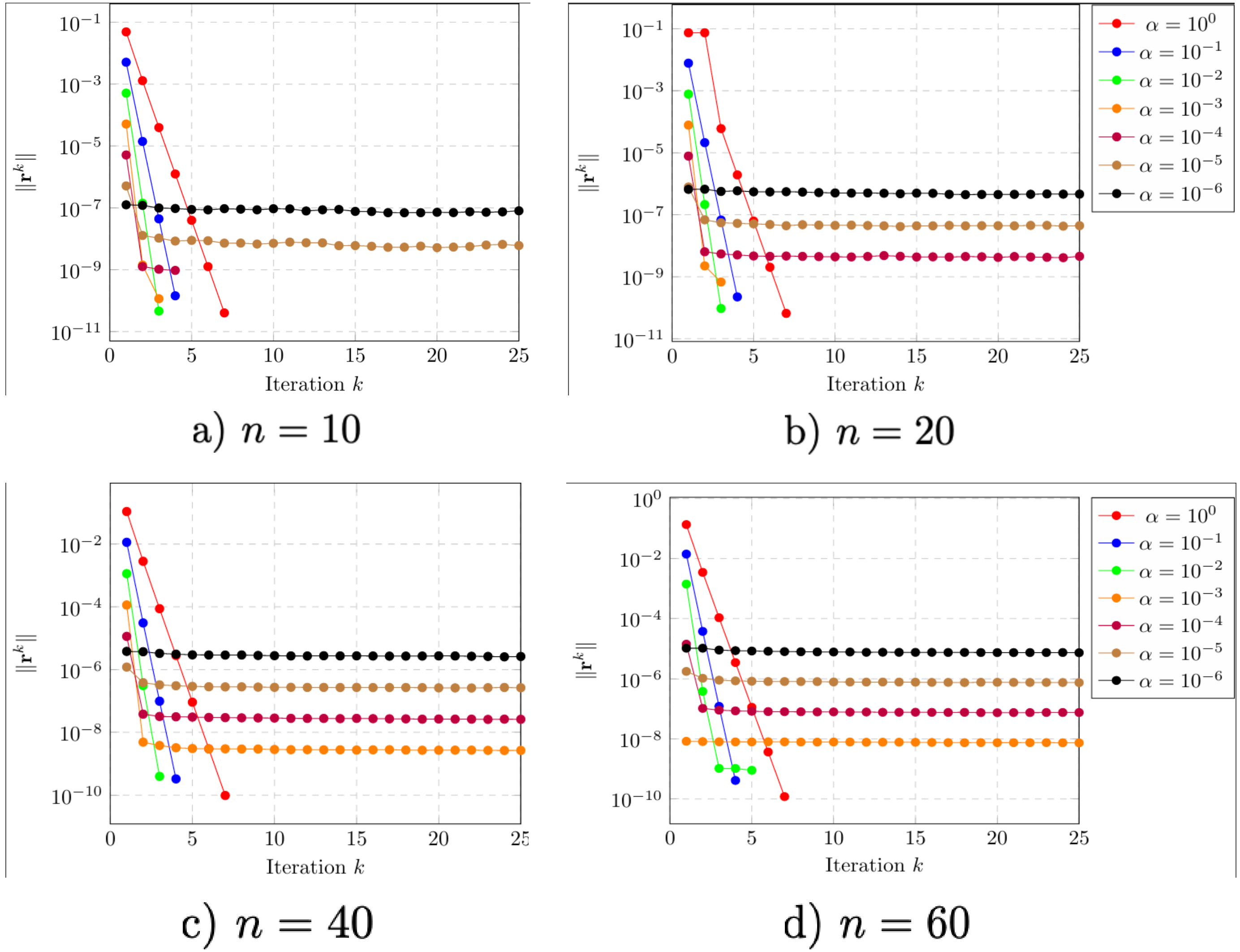
Darcy Numerical results

- Given a 3D Darcy problem on a unit cube

$$\mathcal{K} = \mathbf{I}_3.$$

$$p(x,y,z) = \frac{\sin(\pi x)\sin(\pi y)\sinh(\sqrt{2}\pi z)}{\sinh(\sqrt{2}\pi)}.$$



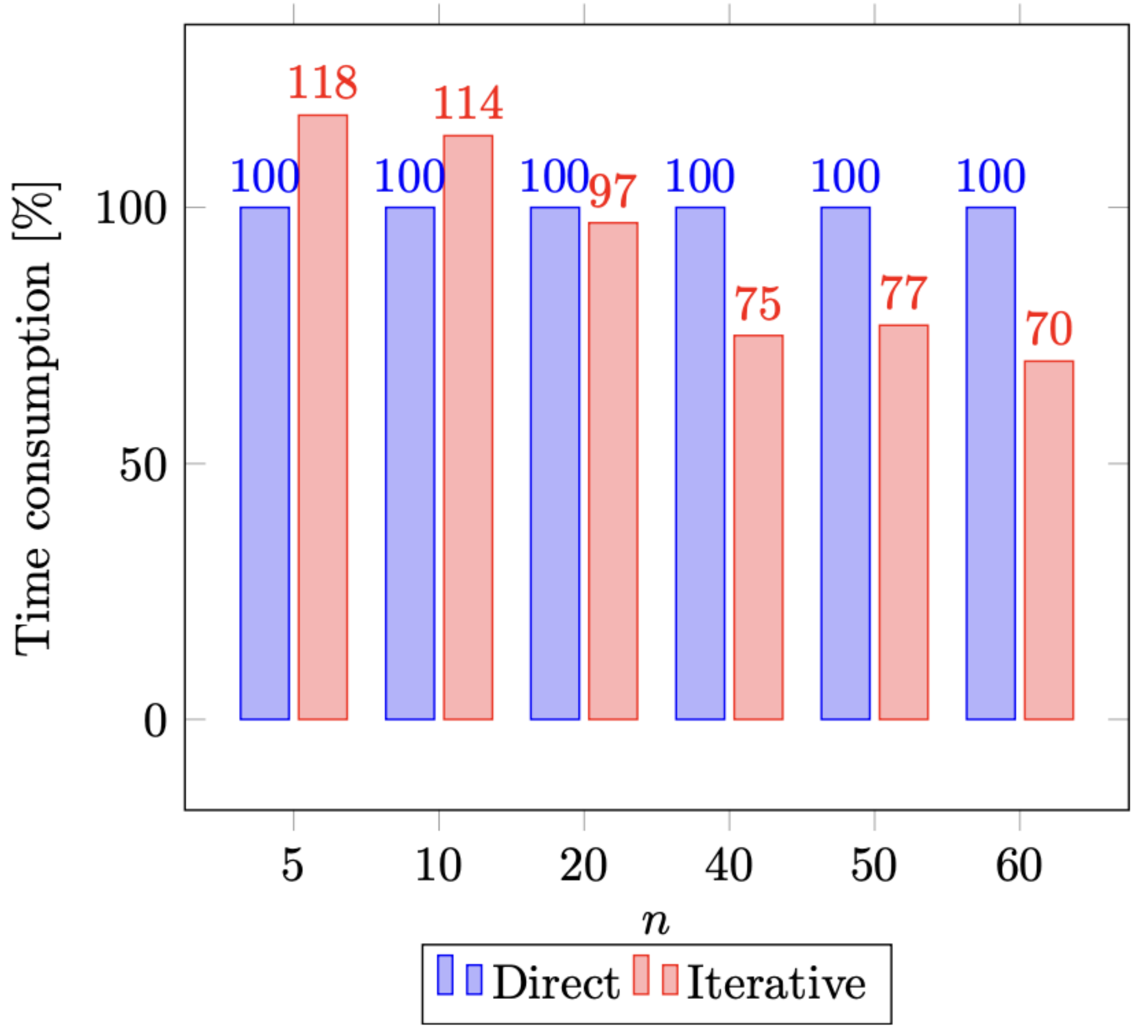


Convergence behavior

If alpha is too small roundoff error will limit the convergence

Convergence is very smooth

Optimum Alpha between 10^{-3} and 10^{-2}



Acceleration

Compare direct solver with iterative solver

We gain in cpu time

We gain in stability

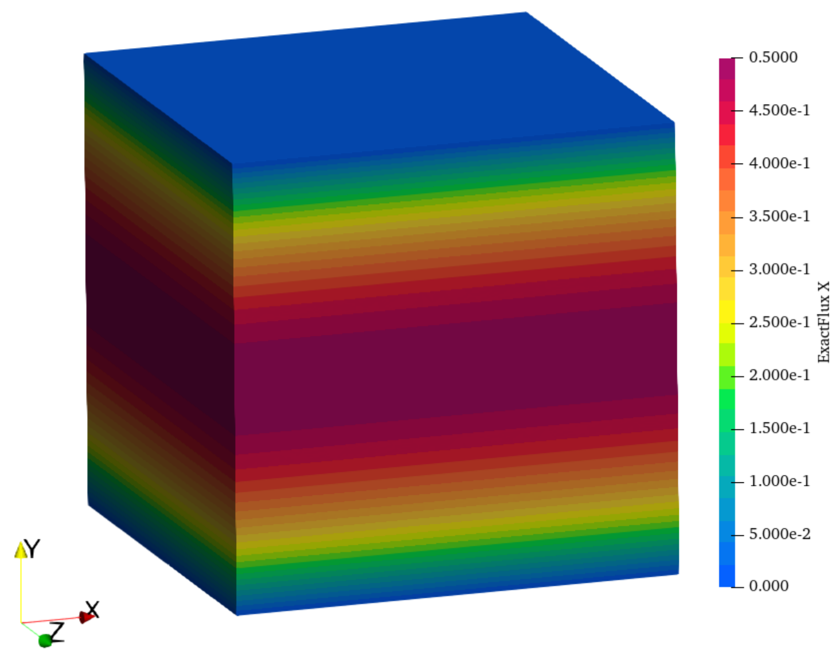
Stokes Numerical results

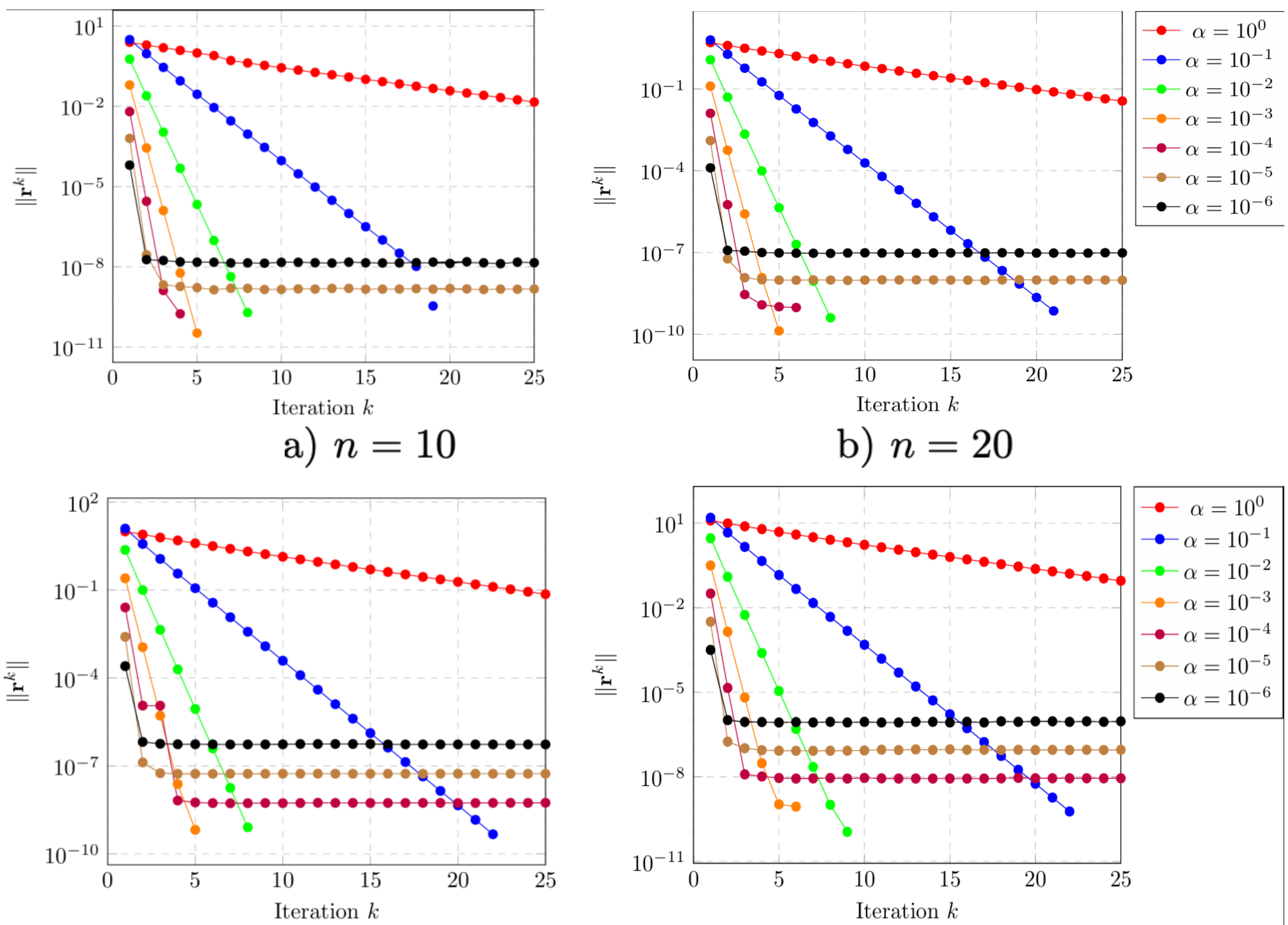
- Poiseuille flow in a rectangular domain

[0,2]x[-1,1]x[-1,1]

$$\mu = 1.0$$

$$u_x(y) = \frac{\nabla p}{2\mu}(1 - y^2)$$





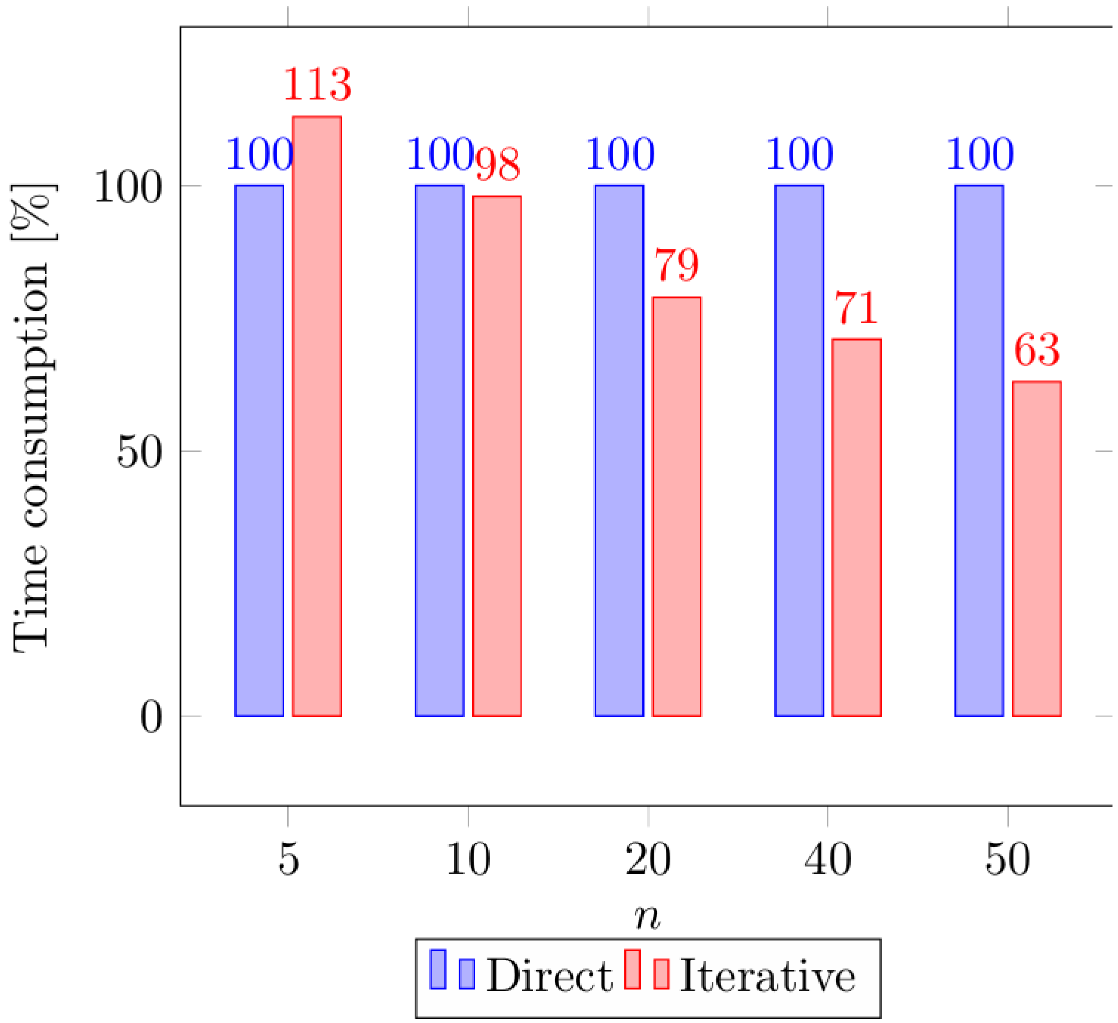
a) $n = 10$ b) $n = 20$

c) $n = 40$ d) $n = 50$

Convergence behavior

Convergence is very smooth

Optimum Alpha between 10^{-4} and 10^{-3}



Acceleration

Compare direct solver with iterative solver

We gain in cpu time

We gain in stability

Conclusion Stokes/Darcy

- An iterative method was presented for solving saddle point problems using a positive definite preconditioner
- The system of the preconditioner is smaller than the global problem
- The iterative method was up to 30% and 40% faster than the direct method for Darcy and Stokes, respectively

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