A companion technical report of "Joint Power and Admission Control: Non-Convex L_q Approximation and An Effective Polynomial Time Deflation Approach"

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I. Proof of Lemma 1 in [1]

In this part, we prove Lemma 1 in [1], i.e., for any $q \in [0, 1]$, problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_q^q + \alpha \bar{\mathbf{p}}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{0} \le \mathbf{x} \le \mathbf{e} \end{aligned} \tag{1}$$

is equivalent to

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_q^q + \alpha \bar{\mathbf{p}}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b}, \\ & \mathbf{0} \le \mathbf{x} \le \mathbf{e}. \end{aligned} \tag{2}$$

The lemma is a consequence of the special structure of $\mathbf{A}$ and $\mathbf{b}$.

Before showing the lemma, we first introduce some notations which will be used later. For any subset $\mathcal{I} \subseteq \mathcal{K}$, we use $\mathbf{A}_{\mathcal{I}}$ to denote the matrix formed by the rows of $\mathbf{A}$ indexed by $\mathcal{I}$. For a vector $\mathbf{x}$, the notation $\mathbf{x}_{\mathcal{I}}$ is similarly defined. Moreover, for any $\mathcal{J} \subseteq \mathcal{K}$, the notation $\mathbf{A}_{\mathcal{I},\mathcal{J}}$ will denote the submatrix of $\mathbf{A}$ obtained by taking the rows and columns of $\mathbf{A}$ indexed by $\mathcal{I}$ and $\mathcal{J}$ respectively.

Notice that problem (2) is equivalent to

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & \|\mathbf{y}\|_q^q + \alpha\, \bar{\mathbf{p}}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{y} = \mathbf{b} - \mathbf{A}\mathbf{x}, \\ & \mathbf{0} \le \mathbf{x} \le \mathbf{e},\ \mathbf{y} \ge \mathbf{0}. \end{aligned}$$

Thus, to show the equivalence of (1) and (2), it suffices to show that any optimal solution $\tilde{\mathbf{x}}$ of (1) always satisfies $\mathbf{b}-\mathbf{A}\tilde{\mathbf{x}}\geq\mathbf{0}$. For simplicity of notations, let $\tilde{\mathbf{z}}=\mathbf{A}\tilde{\mathbf{x}}-\mathbf{b}$. Next, we show

$$\tilde{\mathbf{z}}=(\tilde{z}_1,\tilde{z}_2,\ldots,\tilde{z}_K)^T\leq\mathbf{0}.$$

Denote $\mathcal{K}^{+}=\{\,k\mid\tilde{z}_k>0\,\}$, $\mathcal{K}^{=}=\{\,k\mid\tilde{z}_k=0\,\}$, and $\mathcal{K}^{-}=\{\,k\mid\tilde{z}_k<0\,\}$. We claim $|\mathcal{K}^{+}|=0$. Assume the contrary so that $|\mathcal{K}^{+}|\geq 1$. We will derive a contradiction. Let $\mathcal{I}=\mathcal{K}^{+}\cup\mathcal{K}^{=}$. Then $\tilde{\mathbf{z}}_{\mathcal{I}}=\mathbf{A}_{\mathcal{I}}\tilde{\mathbf{x}}-\mathbf{b}_{\mathcal{I}}\geq\mathbf{0}$, which further implies

$$\mathbf{A}_{\mathcal{I},\mathcal{I}}\tilde{\mathbf{x}}_{\mathcal{I}}-(\mathbf{b}_{\mathcal{I}}-\mathbf{A}_{\mathcal{I},\mathcal{I}^c}\tilde{\mathbf{x}}_{\mathcal{I}^c})\geq\mathbf{0}.$$

Hence, $\tilde{\mathbf{x}}_{\mathcal{I}}$ is a feasible solution to the following linear subsystem in $\mathbf{x}_{\mathcal{I}}$:

$$\mathbf{A}_{\mathcal{I},\mathcal{I}}\mathbf{x}_{\mathcal{I}}-(\mathbf{b}_{\mathcal{I}}-\mathbf{A}_{\mathcal{I},\mathcal{I}^c}\tilde{\mathbf{x}}_{\mathcal{I}^c})\geq\mathbf{0}. \tag{3}$$

Since $\mathbf{A}_{\mathcal{I},\mathcal{I}^c}\leq\mathbf{0}$ (the off-diagonals of $\mathbf{A}$ are nonpositive), it follows that $\mathbf{b}_{\mathcal{I}}-\mathbf{A}_{\mathcal{I},\mathcal{I}^c}\tilde{\mathbf{x}}_{\mathcal{I}^c}\geq\mathbf{b}_{\mathcal{I}}>\mathbf{0}$. It can be checked that the other assumptions of Lemma **3.1** in [2] all hold for (3) so that there exists a vector $\bar{\mathbf{x}}_{\mathcal{I}}$ such that $\mathbf{b}_{\mathcal{I}}\leq\bar{\mathbf{x}}_{\mathcal{I}}\leq\tilde{\mathbf{x}}_{\mathcal{I}}$ and $\mathbf{A}_{\mathcal{I},\mathcal{I}}\bar{\mathbf{x}}_{\mathcal{I}}-(\mathbf{b}_{\mathcal{I}}-\mathbf{A}_{\mathcal{I},\mathcal{I}^c}\tilde{\mathbf{x}}_{\mathcal{I}^c})=\mathbf{0}$. Define $\bar{\mathbf{x}}_{\mathcal{I}^c}=\tilde{\mathbf{x}}_{\mathcal{I}^c}$. Then we have

$$\mathbf{A}_{\mathcal{I}}\bar{\mathbf{x}}-\mathbf{b}_{\mathcal{I}}=\mathbf{A}_{\mathcal{I},\mathcal{I}}\bar{\mathbf{x}}_{\mathcal{I}}-(\mathbf{b}_{\mathcal{I}}-\mathbf{A}_{\mathcal{I},\mathcal{I}^c}\tilde{\mathbf{x}}_{\mathcal{I}^c})=\mathbf{0}. \tag{4}$$

Moreover, we have $\mathbf{0}\leq\bar{\mathbf{x}}\leq\tilde{\mathbf{x}}\leq\mathbf{e}$, so $\bar{\mathbf{x}}$ is a feasible power allocation. With this new power allocation $\bar{\mathbf{x}}$, there holds

$$\begin{aligned}\mathbf{A}_{\mathcal{I}^c}\bar{\mathbf{x}}-\mathbf{b}_{\mathcal{I}^c} &= \mathbf{A}_{\mathcal{I}^c,\mathcal{I}^c}\tilde{\mathbf{x}}_{\mathcal{I}^c}+\mathbf{A}_{\mathcal{I}^c,\mathcal{I}}\bar{\mathbf{x}}_{\mathcal{I}}-\mathbf{b}_{\mathcal{I}^c}\\ &\geq \mathbf{A}_{\mathcal{I}^c,\mathcal{I}^c}\tilde{\mathbf{x}}_{\mathcal{I}^c}+\mathbf{A}_{\mathcal{I}^c,\mathcal{I}}\tilde{\mathbf{x}}_{\mathcal{I}}-\mathbf{b}_{\mathcal{I}^c}\\ &= \mathbf{A}_{\mathcal{I}^c}\tilde{\mathbf{x}}-\mathbf{b}_{\mathcal{I}^c},\end{aligned} \tag{5}$$

where the inequality follows from $\mathbf{A}_{\mathcal{I}^c,\mathcal{I}}\leq\mathbf{0}$ and $\bar{\mathbf{x}}\leq\tilde{\mathbf{x}}$. This further implies

$$(\mathbf{b}_{\mathcal{I}^c}-\mathbf{A}_{\mathcal{I}^c}\bar{\mathbf{x}})_+\leq(\mathbf{b}_{\mathcal{I}^c}-\mathbf{A}_{\mathcal{I}^c}\tilde{\mathbf{x}})_+$$

where $(\cdot)_+$ denotes the projection to the nonnegative orthant. Since $\mathbf{b}_{\mathcal{I}}-\mathbf{A}_{\mathcal{I}}\bar{\mathbf{x}}=\mathbf{0}$ (cf. (4)), it follows that

$$(\mathbf{b}-\mathbf{A}\bar{\mathbf{x}})_+\leq(\mathbf{b}-\mathbf{A}\tilde{\mathbf{x}})_+ \tag{6}$$

with the inequality holds true *strictly* for entries indexed by $\mathcal{K}^{+}$. Define the function

$$\phi(\mathbf{x})=\|(\mathbf{b}-\mathbf{A}\mathbf{x})_+\|_q^q+\alpha\,\bar{\mathbf{p}}^T\mathbf{x}.$$

Clearly, we have

$$\phi(\mathbf{x})\leq\|\mathbf{b}-\mathbf{A}\mathbf{x}\|_q^q+\alpha\,\bar{\mathbf{p}}^T\mathbf{x},\quad\forall\,\mathbf{x}\geq\mathbf{0},$$

where the equality holds whenever $\mathbf{A}\mathbf{x} - \mathbf{b} \le \mathbf{0}$. Since $\mathbf{0} \le \bar{\mathbf{x}} \le \tilde{\mathbf{x}}$, we can use (6) to obtain

$$\phi(\bar{\mathbf{x}}) = \|(\mathbf{b} - \mathbf{A}\bar{\mathbf{x}})_+\|_q^q + \alpha\, \bar{\mathbf{p}}^T\bar{\mathbf{x}} < \|(\mathbf{b} - \mathbf{A}\tilde{\mathbf{x}})_+\|_q^q + \alpha\, \bar{\mathbf{p}}^T\tilde{\mathbf{x}} = \phi(\tilde{\mathbf{x}}).$$

Moreover, it follows from $|\mathcal{K}^+| \ge 1$ that

$$\|\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}\|_0 = K - |\mathcal{I}| = K - |\mathcal{K}^=| - |\mathcal{K}^+| < K - |\mathcal{K}^=| = \|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_0.$$

We claim that, without loss of generality, we can assume $\mathbf{A}_{\mathcal{I}^c}\bar{\mathbf{x}} - \mathbf{b}_{\mathcal{I}^c} \le \mathbf{0}$ so that $\mathbf{A}\bar{\mathbf{x}} - \mathbf{b} \le \mathbf{0}$. This is because otherwise we can repeat the above steps by replacing the vector $\tilde{\mathbf{x}}$ with $\bar{\mathbf{x}}$ to find a new vector $\mathbf{0} \le \hat{\mathbf{x}} \le \bar{\mathbf{x}}$ such that

$$\phi(\hat{\mathbf{x}}) < \phi(\bar{\mathbf{x}}) < \phi(\tilde{\mathbf{x}}) \quad \text{and} \quad \|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_0 < \|\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}\|_0 < \|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_0.$$

Since the ℓ_0-norm can be reduced at most finitely many times, it follows that by repeatedly applying the above steps we will eventually obtain a power allocation (still denote by $\bar{\mathbf{x}}$) such that

$$\mathbf{0} \le \bar{\mathbf{x}} \le \tilde{\mathbf{x}} \le \mathbf{e}, \quad \phi(\bar{\mathbf{x}}) < \phi(\tilde{\mathbf{x}}), \quad \mathbf{A}\bar{\mathbf{x}} - \mathbf{b} \le \mathbf{0}.$$

Notice that when $\mathbf{A}\bar{\mathbf{x}} - \mathbf{b} \le \mathbf{0}$ we have

$$\|\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}\|_q^q + \alpha\, \bar{\mathbf{p}}^T\bar{\mathbf{x}} = \phi(\bar{\mathbf{x}}) < \phi(\tilde{\mathbf{x}}) \le \|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_q^q + \alpha\, \bar{\mathbf{p}}^T\tilde{\mathbf{x}}.$$

This contradicts the optimality of $\tilde{\mathbf{x}}$. This completes the proof.

II. Global Minimizer of Problem (12) in [1]

In this part, we show that, for any $q \in (0,1)$, $\mathbf{x}^* = (0.5, 0.5, 0)^T$ is the unique global minimizer of the non-convex optimization problem

$$\begin{aligned} \min \quad & h_q(\mathbf{x}) = (0.5 - x_1 + x_3)^q + (0.5 - x_2 + x_3)^q + (0.5 + x_1 + x_2 - x_3)^q + \alpha \sum_{i=1}^{3} x_i \\ \text{s.t.} \quad & 0.5 - x_1 + x_3 \ge 0, \\ & 0.5 - x_2 + x_3 \ge 0, \\ & 0.5 + x_1 + x_2 - x_3 \ge 0, \\ & 0 \le x_1, x_2, x_3 \le 1, \end{aligned} \tag{7}$$

where

$$0 < \alpha \le \bar{\alpha}_q := \min\left\{1 + (0.5)^q, 2^q\right\} - (1.5)^q. \tag{8}$$

To show the desired result, let us first introduce the following useful facts.

Fact 1. Suppose $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are concave functions, so is $f(\mathbf{x}) := \min\left\{f_1(\mathbf{x}), f_2(\mathbf{x})\right\}.$

Fact 2. Suppose $f(x)$ is an univariate strictly concave function in $x \in [a, b]$, then

$$f(x) > \min\{f(a), f(b)\}, \ \forall\, x \in (a, b).$$

Fact 3. For any $q \in (0, 1)$, $\|\mathbf{A}\mathbf{x} + \mathbf{b}\|_q^q$ is concave in $\{\mathbf{x} \,|\, \mathbf{A}\mathbf{x} + \mathbf{b} \geq \mathbf{0}\}$. Furthermore, it is strictly concave if $\mathbf{A}$ is column full rank.

Fact 4. For any $q \in (0, 1), x_1 \geq 0, x_2 \geq 0$, we have $x_1^q + x_2^q \geq (x_1 + x_2)^q$, where the inequality holds with "=" if and only if at least one of x_1 and x_2 is zero.

From **Fact 3,** we know $h_q(\mathbf{x})$ defined in (7) is strictly concave in its feasible region, and thus is also strictly concave in each of its components. **Fact 4** implies that $\bar{\alpha}_q$ defined in (8) is positive.

We are now ready to show the desired result. It suffices to show that for any feasible set $\mathcal{S} \subseteq \{1, 2, 3\}$, the optimal value of problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & h_{q,\mathcal{S}}(\mathbf{x}) = \sum_{i \notin \mathcal{S}} [\mathbf{b} - \mathbf{A}\mathbf{x}]_i^q + \alpha \mathbf{e}^T \mathbf{x} \\ \text{s.t.} \quad & [\mathbf{b} - \mathbf{A}\mathbf{x}]_i = 0, \ i \in \mathcal{S}, \\ & [\mathbf{b} - \mathbf{A}\mathbf{x}]_i > 0, \ i \notin \mathcal{S}, \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{e}, \end{aligned} \tag{9}$$

is strictly greater than $h_q(\mathbf{x}^*) = (1.5)^q + \alpha$ except at the point $\mathbf{x} = \mathbf{x}^*$.

It is simple to check that the sets $\{1, 3\}$, $\{2, 3\}$, and (thus) $\{1, 2, 3\}$ are infeasible for problem (9). In the sequential, due to the symmetry of x_1 and x_2, it is sufficient to consider the following feasible sets (including the empty set):

A: When $\mathcal{S} = \{1, 2\}$, we have $0.5 - x_1 + x_3 = 0$ and $0.5 - x_2 + x_3 = 0$. Therefore,

$$\begin{aligned} h_{q,\mathcal{S}}(\mathbf{x}) &= (0.5 + x_1 + x_2 - x_3)^q + \alpha(x_1 + x_2 + x_3) \\ &= (1.5 + x_3)^q + \alpha + 3\alpha x_3 \\ &\geq h_q(\mathbf{x}^*), \end{aligned}$$

where "=" holds if and only if $\mathbf{x} = \mathbf{x}^*$.

B: When $\mathcal{S} = \{1\}$, we have $x_3 = x_1 - 0.5 \geq 0$ and $x_1 \geq 0.5$. Therefore,

$$\begin{aligned} h_{q,\mathcal{S}}(\mathbf{x}) &= (0.5 - x_2 + x_3)^q + (0.5 + x_1 + x_2 - x_3)^q + \alpha(x_1 + x_2 + x_3) \\ &= (x_1 - x_2)^q + (1 + x_2)^q + 2\alpha x_1 + \alpha x_2 - 0.5\alpha \quad (10) \\ &\geq \min\left\{(1 + x_1)^q + 3\alpha x_1 - 0.5\alpha, x_1^q + 1 + 2\alpha x_1 - 0.5\alpha\right\} \quad (11) \\ &\geq \min\left\{(1.5)^q + \alpha, (0.5)^q + 1 + 0.5\alpha\right\} \quad (12) \\ &= h_q(\mathbf{x}^*), \quad (13) \end{aligned}$$

where (10) is due to $x_3 = x_1 - 0.5$, (11) comes from the fact that the function in the right hand of (10) is strictly concave in $x_2 \in [0, x_1]$ and **Fact 2**, (12) is due to the fact that the function in the right hand of (11) is increasing in $x_1 \in [0.5, 1]$, and (13) is due to $\alpha \leq \bar{\alpha}_q$ (cf. (8)). Concluding the above analysis, we have $h_{q,\mathcal{S}}(\mathbf{x}) \geq h_q(\mathbf{x}^*)$ and the equality holds true if and only if $\mathbf{x} = \mathbf{x}^*$.

C: When $\mathcal{S} = \{3\}$, we have $x_3 = x_1 + x_2 + 0.5$. Then,

$$\begin{aligned} h_{q,\mathcal{S}}(\mathbf{x}) &\geq (0.5 - x_1 + x_3)^q + (0.5 - x_2 + x_3)^q \\ &= (1 + x_2)^q + (1 + x_1)^q \\ &\geq 2 > h_q(\mathbf{x}^*), \end{aligned}$$

where the last strict inequality is due to the fact $\alpha \leq \bar{\alpha}_q \leq 2^q - (1.5)^q < 2 - (1.5)^q$ (cf. (8)).

D: When $\mathcal{S} = \emptyset$, we further consider the following cases separately.

D1: When $x_1 = 0, x_2 \in [0, 1], x_3 \in [0, 0.5]$, we have

$$\begin{aligned} h_{q,\emptyset}(\mathbf{x}) &= (0.5 + x_3)^q + (0.5 - x_2 + x_3)^q + (0.5 + x_2 - x_3)^q + \alpha(x_2 + x_3) \\ &\geq (0.5 + x_3)^q + \alpha x_3 + \min\left\{(0.5 + x_3)^q + (0.5 - x_3)^q, 1 + \alpha(0.5 + x_3)\right\} \quad (14) \\ &\geq \min\left\{(0.5)^q + 1 + 0.5\alpha, 3(0.5)^q, 2 + 0.5\alpha, 2 + 1.5\alpha\right\} \quad (15) \\ &> h_q(\mathbf{x}^*), \quad (16) \end{aligned}$$

where (14) is due to the fact that $h_{q,\emptyset}(\mathbf{x})$ is strictly concave in $x_2 \in [0, 0.5 + x_3]$ and **Fact 2**, (15) is due to **Fact 1** (the function in the right hand side of (14) is strictly concave in $x_3 \in [0, 0.5]$) and **Fact 2**, and the last strict inequality (16) is due to (8).

D2: Similar as case D1, when $x_1 = 0, x_2 \in [0,1], x_3 \in [0.5,1]$, we have

$$\begin{aligned} h_{q,\emptyset}(\mathbf{x}) &= (0.5+x_3)^q + (0.5-x_2+x_3)^q + (0.5+x_2-x_3)^q + \alpha(x_2+x_3) \\ &\geq (0.5+x_3)^q + \alpha x_3 + \min\left\{(1.5-x_3)^q + (x_3-0.5)^q + \alpha, 1+\alpha(x_3-0.5)\right\} \quad (17) \\ &\geq \min\left\{(1.5)^q + 2(0.5)^q + 2\alpha, (1.5)^q + 1.5\alpha + 1, 2+1.5\alpha, 2+0.5\alpha\right\} \quad (18) \\ &> h_q(\mathbf{x}^*), \quad (19) \end{aligned}$$

where (17) is due to the fact that $h_{q,\emptyset}(\mathbf{x})$ is strictly concave in $x_2 \in [x_3 - 0.5, 1]$ and **Fact 2**, (18) is due to **Fact 1** (the function in the right hand side of (17) is strictly concave in $x_3 \in [0.5, 1]$) and **Fact 2**, and the last strictly inequality (19) is due to (8).

D3: When $x_1 \in (0,1), x_2 \in [0,1], x_3 = 0$, we have

$$\begin{aligned} h_{q,\emptyset}(\mathbf{x}) &= (0.5-x_1)^q + (0.5-x_2)^q + (0.5+x_1+x_2)^q + \alpha(x_1+x_2) \\ &\geq (1-(x_1+x_2))^q + (0.5+(x_1+x_2))^q + \alpha(x_1+x_2) \quad (20) \\ &\geq \min\left\{(1.5)^q + \alpha, 1+(0.5)^q\right\} \quad (21) \\ &= h_q(\mathbf{x}^*), \quad (22) \end{aligned}$$

where (20) is due to **Fact 4**, (21) is due to **Fact 2** and the fact that $x_1 \leq 0.5$ and $x_2 \leq 0.5$, and (22) is due to (8). Hence, $h_{q,\emptyset}(\mathbf{x}) \geq h_q(\mathbf{x}^*)$ for any $x_1 \in (0,1), x_2 \in [0,1], x_3 = 0$, and the inequality holds with "=" if and only if $\mathbf{x} = \mathbf{x}^*$.

D4: In a similar fashion as the case D3, when $x_1 \in (0,1), x_2 \in [0,1], x_3 = 1$, we can show

$$\begin{aligned} h_{q,\emptyset}(\mathbf{x}) &\geq (1.5-x_1)^q + (1.5-x_2)^q + (x_1+x_2-0.5)^q \\ &\geq (3-x_1-x_2)^q + (x_1+x_2-0.5)^q \\ &\geq \min\left\{1+(1.5)^q, (2.5)^q\right\} \\ &= (2.5)^q > h_q(\mathbf{x}^*), \end{aligned}$$

where the third inequality comes from the fact that $(3-(x_1+x_2))^q + (x_1+x_2-0.5)^q$ is strictly concave in $x_1 + x_2 \in [0.5, 2]$ and **Fact 2**.

D5: The remaining case is $x_1 \in (0,1), x_2 \in (0,1), x_3 \in (0,1)$. Recall that $h_q(\mathbf{x})$ in problem (7) is strictly concave in $\mathbf{x}$. Combining this, the fact $\mathcal{S} = \emptyset$, and **Fact 2**, we know that the minimum of problem (9) will not lie in this region.

From the above analysis, we conclude that $\mathbf{x}^*$ is the unique global minimizer of problem (7).

REFERENCES

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