A Companion Technical Report of "Efficient CI-Based One-Bit Precoding for Multiuser Downlink Massive MIMO Systems with PSK Modulation"

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I. Proof of Theorem 1 in [1]

In this section, we give the proof of Theorem 1 in [1]. We first state the theorem as follows.

Theorem. *The CI-based one-bit precoding problem (P_0) is NP-hard in the single-user case, i.e.,* $K = 1$.

Notice that when $K = 1$, (P_0) reduces to the following problem:

$$\begin{aligned} \max_{\mathbf{x}_T} \ & \min \ \left\{\alpha^A, \alpha^B\right\} \\ \text{s.t.} \ & \mathbf{h}^\mathsf{T}\mathbf{x}_T = \alpha^A s^A + \alpha^B s^B, \\ & \mathbf{x}_T(i) \in \{\pm 1 \pm j\}, \quad i = 1, 2, \ldots, N_t. \end{aligned} \tag{1}$$

Next we shall build a polynomial-time transformation from the partition problem [2] to problem (1). The partition problem is to determine whether a given set of N positive integers $\{a_1, a_2, \ldots, a_N\}$ can be partitioned into two subsets such that the sum of elements in each subset is the same.

Now we construct an instance of problem (1) based on the given instance of the partition problem. Let the number of antennas at the BS be N and the transmitted data symbol be $s = 1$, which is drawn from the QPSK constellation set. In this case, $s^A = \frac{\sqrt{2}}{2}(1-j)$ and $s^B = \frac{\sqrt{2}}{2}(1+j)$.

Moreover, set the channel vector $\mathbf{h}$ to be $\mathbf{h} = \sqrt{2}\mathbf{a}$ with $\mathbf{a} = [a_1, a_2, \ldots, a_N]^\mathsf{T}$. With the above constructed parameters, problem (1) can be expressed as

$$\begin{aligned} \max_{\mathbf{x}_T} \quad & \min\left\{\alpha^A, \alpha^B\right\} \\ \text{s.t.} \quad & \begin{bmatrix}\alpha^A \\ \alpha^B\end{bmatrix} = \begin{bmatrix}\mathbf{a}^\mathsf{T} & -\mathbf{a}^\mathsf{T} \\ \mathbf{a}^\mathsf{T} & \mathbf{a}^\mathsf{T}\end{bmatrix} \begin{bmatrix}\mathcal{R}(\mathbf{x}_T) \\ \mathcal{I}(\mathbf{x}_T)\end{bmatrix}, \\ & \mathbf{x}_T(i) \in \{\pm 1 \pm j\}, \; i = 1, 2, \ldots, N. \end{aligned} \tag{2}$$

Let the optimal solution of problem (2) be $\mathbf{x}_T^*$. Since $\mathbf{a} > \mathbf{0}$, it is easy to argue that $\mathcal{R}(\mathbf{x}_T^*) = \mathbf{1}$. By defining $\mathcal{S} = \{i \in \{1, 2, \ldots, N\} \mid \mathcal{I}(\mathbf{x}_T^*(i)) = 1\}$, it then follows that

$$\alpha^A = 2\sum_{i \notin S} a_i, \quad \alpha^B = 2\sum_{i \in S} a_i.$$

Now, it is straightforward to argue that the optimal value of our constructed problem (2) is $\sum_{i=1}^N a_i$ if and only if the partition problem has a "yes" answer. Finally, the above transformation can be done in polynomial time. Since the partition problem is NP-complete, problem (1) is NP-hard.

II. Proof of Theorem 2 in [1]

In this section, we give the proof of Theorem 2 in [1]. We first state the theorem as follows.

Theorem. *The CI-based one-bit precoding problem (P_0) is strongly NP-hard. Moreover, there is no polynomial-time constant approximation algorithm for (P_0), unless P=NP.*

Proof. The proof is based on a polynomial-time transformation from the 3-SAT problem [2] to problem (P_0). The 3-SAT problem is to determine whether a given set of disjunctive clauses, each consisting of 3 Boolean variables, is satisfiable. Given any instance of the 3-SAT problem consisting of m disjunctive clauses $c_1, c_2, \ldots, c_m$ defined on n Boolean variables $x_1, x_2, \ldots, x_n$, we construct below a problem instance of (P_0) with $K = m$ and $N_t = n + 1$.

We first express c_k as $c_k = \alpha_{\pi(k)} \vee \beta_{\rho(k)} \vee \gamma_{\tau(k)}$ and define the channel vectors as $\mathbf{h}_k = e^{-\frac{j\pi}{4}}\mathbf{g}_k = e^{-\frac{j\pi}{4}}[g_{k1}, g_{k2}, \ldots, g_{kn}, 1]^\mathsf{T} \in \mathbb{R}^{n+1}, \; k = 1, 2, \ldots, m,$ with

$$g_{ki} = \begin{cases} 1, & \text{if } \alpha_{\pi(k)} = x_i \text{ or } \beta_{\rho(k)} = x_i \text{ or } \gamma_{\tau(k)} = x_i; \\ -1, & \text{if } \alpha_{\pi(k)} = \bar{x}_i \text{ or } \beta_{\rho(k)} = \bar{x}_i \text{ or } \gamma_{\tau(k)} = \bar{x}_i; \\ 0, & \text{otherwise,} \end{cases} \quad i = 1, 2, \ldots, n.$$

For example, if $c_k = x_1 \vee \bar{x}_3 \vee x_4$, then $\mathbf{g}_k = [1, 0, -1, 1, 0, \dots, 0, 1]$. Moreover, we set the modulation scheme to be QPSK and the data symbols for all users to be

$$s_k = 1, \quad k = 1, 2, \dots, m.$$

It follows immediately that $s_k^A = e^{-\frac{j\pi}{4}}$ and $s_k^B = e^{\frac{j\pi}{4}}$ for all $k = 1, 2, \dots, m$. With the above constructed parameters and by multiplying $e^{\frac{j\pi}{4}}$ on both sides of the first constraint of (P_0), problem (P_0) becomes

$$\begin{aligned} \max_{\mathbf{x}_T} \quad & \min_{k \in \{1,2,\dots,m\}} \ \{\alpha_k^A, \alpha_k^B\} \\ \text{s.t.} \quad & \mathbf{g}_k^{\mathsf{T}} \mathcal{R}(\mathbf{x}_T) = \alpha_k^A, \quad k = 1, 2, \dots, m, \\ & \mathbf{g}_k^{\mathsf{T}} \mathcal{I}(\mathbf{x}_T) = \alpha_k^B, \quad k = 1, 2, \dots, m, \\ & \mathbf{x}_T(i) \in \{\pm 1 \pm j\}, \quad i = 1, 2, \dots, n+1, \end{aligned} \tag{3}$$

which is equivalent to

$$\begin{aligned} \max_{\mathbf{y}, t} \quad & t \\ \text{s.t.} \quad & \mathbf{g}_k^{\mathsf{T}} \mathbf{y} \geq t, \quad k = 1, 2, \dots, m, \\ & y_i \in \{-1, 1\}, \quad i = 1, 2, \dots n+1. \end{aligned} \tag{4}$$

Let the optimal solution of (4) be $\mathbf{y}^*$. Since the last entries of all $\{\mathbf{g}_k\}_{k=1}^m$ are 1, it is easy to argue that $y_{n+1}^* = 1$. Based on this, and by further defining $\hat{\mathbf{g}}_k = \mathbf{g}_k[1:n]$ for all $k = 1, 2, \dots, m$ and $z_i = (y_i + 1)/2$ for all $i = 1, 2, \dots, n$, problem (4) can be equivalently expressed as

$$\begin{aligned} \max_{\mathbf{z}, t} \quad & t \\ \text{s.t.} \quad & 2\hat{\mathbf{g}}_k^{\mathsf{T}} \mathbf{z} - \mathbf{1}^{\mathsf{T}} \hat{\mathbf{g}}_k + 1 \geq t, \quad k = 1, 2, \dots, m, \\ & z_i \in \{0, 1\}, \quad i = 1, 2, \dots n. \end{aligned} \tag{5}$$

Now we claim that the 3-SAT problem is satisfied if and only if the optimal value of our constructed problem is greater than or equal to zero, or equivalently, there exists $\mathbf{z} \in \{0, 1\}^n$ such that

$$2\hat{\mathbf{g}}_k^{\mathsf{T}} \mathbf{z} - \mathbf{1}^{\mathsf{T}} \hat{\mathbf{g}}_k + 1 \geq 0, \quad k = 1, 2, \dots, m. \tag{6}$$

From the choice of $\mathbf{g}_k$, we know that c_k is satisfied if and only if

$$g_{k\pi(k)} x_{\pi(k)} + \frac{1 - g_{k\pi(k)}}{2} + g_{k\rho(k)} x_{\rho(k)} + \frac{1 - g_{k\rho(k)}}{2} + g_{k\tau(k)} x_{\tau(k)} + \frac{1 - g_{k\tau(k)}}{2} \geq 1,$$

which is equivalent to

$$2\hat{\mathbf{g}}_k^{\mathsf{T}} \mathbf{x} - \mathbf{1}^{\mathsf{T}} \hat{\mathbf{g}}_k + 1 \geq 0.$$

Therefore, if there exists a truth assignment $x_1, x_2, \ldots, x_n$ for the 3-SAT problem, we can simply set $z_i = x_i,\ i = 1, 2, \ldots, n$, to obtain a solution of (5) with the objective value greater than or equal to 0. On the other hand, if the optimal value of the constructed problem is greater than or equal to 0, i.e., there exists $\mathbf{z} \in \{0,1\}^n$ satisfying (6), we can simply assign $x_i = z_i,\ i = 1, 2, \ldots, n$, to obtain a truth assignment. Since the transformation is in polynomial time and the 3-SAT problem is strongly NP-hard, we can conclude that problem (P_0) is strongly NP-hard.

It follows immediately from the above proof that there is no polynomial-time constant approximation algorithm for solving (P_0). Otherwise, we can check whether the optimal value of the constructed problem (3) is nonnegative in polynomial time, which in turn solves the corresponding 3-SAT problem in polynomial time. This contradicts with the strong NP-hardness of the 3-SAT problem. □

III. PROOF OF THEOREM 3 IN [1]

In this section, we prove Theorem 3 in [1]. The theorem is stated as follows.

Theorem. *If the penalty parameter λ in (P_λ) satisfies $\lambda > \max_l \|\mathbf{a}_l\|_\infty$, then the following results hold:*

(1) *Any optimal solution of (P_λ) is also an optimal solution of (P), and vice versa.*

(2) *Any local minimizer of (P_λ) is a feasible point of (P); on the other hand, any feasible point of (P) is a (strict) local minimizer of (P_λ).*

Proof. Notice that (1) is a direct corollary of (2), thus it is sufficient to prove (2). Given a penalty parameter $\lambda > \max_l \|\mathbf{a}_l\|_\infty$ and a local minimizer $\bar{\mathbf{x}}$ of (P_λ), we will show that $\bar{\mathbf{x}}$ is a feasible point of (P). Suppose for contradiction that there exists $s \in \{1, 2, \ldots n\}$ such that $|\bar{x}_s| < 1$. We claim that for any $\delta > 0$, there exists $\mathbf{z} = [z_1, z_2, \ldots, z_n]^\mathsf{T} \in B(\bar{\mathbf{x}}, \delta) \cap [-1, 1]^n$ such that

$$\max_l \mathbf{a}_l^\mathsf{T} \mathbf{z} - \lambda \|\mathbf{z}\|_1 < \max_l \mathbf{a}_l^\mathsf{T} \bar{\mathbf{x}} - \lambda \|\bar{\mathbf{x}}\|_1,$$

which contradicts with the fact that $\bar{\mathbf{x}}$ is a local minimizer. Specifically, let

$$z_i = \begin{cases} \mathrm{sgn}(\bar{x}_i) \min\{|\bar{x}_i| + \delta, 1\}, & \text{if } i = s; \\ \bar{x}_i, & \text{otherwise}, \end{cases} \qquad i = 1, 2, \ldots, n.$$

It is easy to check that $\mathbf{z} \in B(\bar{\mathbf{x}}, \delta) \cap [-1, 1]^n$ and

$$\|\mathbf{z}\|_1 - \|\bar{\mathbf{x}}\|_1 = \|\mathbf{z} - \bar{\mathbf{x}}\|_1 = \min\{1 - |\bar{x}_s|, \delta\},$$

which, together with $\lambda > \max_l \|\mathbf{a}_l\|_\infty$, implies that

$$\begin{aligned}&\max_l \mathbf{a}_l^\mathsf{T}\mathbf{z} - \max_l \mathbf{a}_l^\mathsf{T}\bar{\mathbf{x}} - \lambda(\|\mathbf{z}\|_1 - \|\bar{\mathbf{x}}\|_1)\\ \le& \max_l \|\mathbf{a}_l\|_\infty \|\mathbf{z} - \bar{\mathbf{x}}\|_1 - \lambda(\|\mathbf{z}\|_1 - \|\bar{\mathbf{x}}\|_1)\\ =&(\max_l \|\mathbf{a}_l\|_\infty - \lambda)\min\{1 - |\bar{x}_s|, \delta\} < 0.\end{aligned}$$

Therefore, we can conclude that any local minimizer of (P$_\lambda$) is a feasible point of (P) when $\lambda > \max_l \|\mathbf{a}_l\|_\infty$.

On the other hand, let $\bar{\mathbf{x}}$ be a feasible point of (P), and we will show that for any $\lambda > \max_l \|\mathbf{a}_l\|_\infty$, it is also a (strict) local minimizer of (P$_\lambda$). The main idea of the proof is similar to that in [3, Theorem 3.5]. We first define $\mathbf{x}^s,\ s = 1, 2, \ldots, 2^n - 1$, to be all the remaining feasible points of (P) except $\bar{\mathbf{x}}$ and set $\mathbf{d}^s = \mathbf{x}^s - \bar{\mathbf{x}}$ for all s. Given $t \in (0, \frac{1}{2})$, let $\mathbf{x}_s = \bar{\mathbf{x}} + t\mathbf{d}^s, s = 1, 2, \ldots, 2^n - 1$, and let $\mathbf{x}_N = \bar{\mathbf{x}}$, where $N = 2^n$. We claim that $\mathbf{x}_N$ is the local minimizer in conv$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$, where conv$(\cdot)$ denotes the convex hull of the corresponding set.

Note that any given $\mathbf{x} \in$ conv$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ can be expressed as $\mathbf{x} = \sum_{s=1}^N \mu_s \mathbf{x}_s$ with some $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_N] \in \mathbb{R}^N$ satisfying $\sum_{s=1}^N \mu_s = 1$ and $\boldsymbol{\mu} \ge \mathbf{0}$. It follows that

$$\begin{aligned}\max_l \mathbf{a}_l^\mathsf{T}\mathbf{x} - \max_l \mathbf{a}_l^\mathsf{T}\mathbf{x}_N =& \max_l \mathbf{a}_l^\mathsf{T}\left(\sum_{s=1}^N \mu_s \mathbf{x}_s\right) - \max_l \mathbf{a}_l^\mathsf{T}\mathbf{x}_N\\ \ge& \mathbf{a}_{l_0}^\mathsf{T}\left(\sum_{s=1}^N \mu_s \mathbf{x}_s - \mathbf{x}_N\right)\\ \ge& -\|\mathbf{a}_{l_0}\|_\infty \sum_{s=1}^N \mu_s \|\mathbf{x}_s - \mathbf{x}_N\|_1\\ \ge& -\max_l \|\mathbf{a}_l\|_\infty \sum_{s=1}^{N-1} \mu_s t \|\mathbf{d}^s\|_1,\end{aligned} \tag{7}$$

where $l_0 \in \arg\max_l \mathbf{a}_l^\mathsf{T}\mathbf{x}_N$. Furthermore,

$$\begin{aligned}\|\mathbf{x}_N\|_1 - \|\mathbf{x}\|_1 =& \|\mathbf{x}_N\|_1 - \left\|\sum_{s=1}^N \mu_s \mathbf{x}_s\right\|_1\\ \ge& \|\mathbf{x}_N\|_1 - \sum_{s=1}^N \mu_s \|\mathbf{x}_s\|_1\\ =& \sum_{s=1}^{N-1} \mu_s \left(\|\mathbf{x}_N\|_1 - \|\mathbf{x}_s\|_1\right).\end{aligned} \tag{8}$$

Combining (7) and (8) gives

$$\begin{aligned}&\max_l \mathbf{a}_l^\mathsf{T}\mathbf{x} - \lambda\|\mathbf{x}\|_1 - \left(\max_l \mathbf{a}_l^\mathsf{T}\mathbf{x}_N - \lambda\|\mathbf{x}_N\|_1\right)\\ \geq &\sum_{s=1}^{N-1}\mu_s\left(\lambda\|\mathbf{x}_N\|_1 - \lambda\|\mathbf{x}_s\|_1 - \max_l \|\mathbf{a}_l\|_\infty t\|\mathbf{d}^s\|_1\right).\end{aligned} \tag{9}$$

For all $s \in \{1, 2, \ldots, N-1\}$, we denote

$$\begin{aligned}\Gamma_1^s &= \{i \mid \bar{\mathbf{x}}(i) = 1, \mathbf{x}^s(i) = 1\}, \quad \Gamma_2^s = \{i \mid \bar{\mathbf{x}}(i) = 1, \mathbf{x}^s(i) = -1\},\\ \Gamma_3^s &= \{i \mid \bar{\mathbf{x}}(i) = -1, \mathbf{x}^s(i) = 1\}, \; \Gamma_4^s = \{i \mid \bar{\mathbf{x}}(i) = -1, \mathbf{x}^s(i) = -1\}.\end{aligned}$$

Since $t \in (0, \frac{1}{2})$, we have

$$\begin{aligned}\|\mathbf{x}_N\|_1 - \|\mathbf{x}_s\|_1 &= |\Gamma_2^s|(1 - |1-2t|) + |\Gamma_3^s|(1 - |-1+2t|)\\ &= (|\Gamma_2^s| + |\Gamma_3^s|)2t\end{aligned}$$

and

$$\|\mathbf{d}^s\|_1 = 2(|\Gamma_2^s| + |\Gamma_3^s|), \quad s = 1, 2, \ldots, N-1,$$

which, together with (9), implies

$$\begin{aligned}&\max_l \mathbf{a}_l^\mathsf{T}\mathbf{x} - \lambda\|\mathbf{x}\|_1 - (\max_l \mathbf{a}_l^\mathsf{T}\mathbf{x}_N - \lambda\|\mathbf{x}_N\|_1)\\ \geq\; &2\sum_{s=1}^{N-1} t\mu_s(|\Gamma_2^s| + |\Gamma_3^s|)(\lambda - \max_l \|\mathbf{a}_l\|_\infty) \geq 0,\end{aligned}$$

where the last inequality holds strictly if $\mu_1, \mu_2, \cdots, \mu_{N-1}$ are not all 0, i.e., $\mathbf{x} \neq \mathbf{x}_N$. Therefore, for any $t \in (0, \frac{1}{2})$ and any $\lambda > \max \|\mathbf{a}_l\|_\infty$, it holds that for all $\mathbf{x} \in \mathrm{conv}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ and $\mathbf{x} \neq \mathbf{x}_N$,

$$\max_l \mathbf{a}_l^\mathsf{T}\mathbf{x} - \lambda\|\mathbf{x}\|_1 > \max_l \mathbf{a}_l^\mathsf{T}\mathbf{x}_N - \lambda\|\mathbf{x}_N\|_1,$$

which proves our claim. Moreover, we can always choose a sufficiently small but fixed $\epsilon > 0$ such that $B(\mathbf{x}_N, \epsilon) \cap [-1, 1]^n \subset \mathrm{conv}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$. Consequently, $\bar{\mathbf{x}} = \mathbf{x}_N$ is a (strict) local minimizer of (P_λ), which completes our proof. □

IV. PROOF OF THEOREM 4 IN [1]

In this section, we prove Theorem 4 in [1]. To begin, let us state the theorem.

Theorem. *Suppose that Assumptions 1 and 2 in [1] hold. Let* $\{(\mathbf{x}_k, \mathbf{y}_k)\}$ *be the sequence generated by Algorithm 2 in [1] with* $\rho_k = \rho$. *If* $0 < \rho \leq \frac{2}{L_y + 2\beta_1}$, $c_k = \frac{\beta_1}{k^\gamma}$ *with* $0 < \gamma \leq 0.5$,

$\beta_1 > 0$, and $\tau_k = \frac{16\beta_2 L_{12}^2}{\rho c_k^2} + \beta_3$ with $\beta_2 > 1$ and $\beta_3 \geq \rho L_{12}^2 + L_x$, then any limit point of $\{(\mathbf{x}_k, \mathbf{y}_k)\}$ is a stationary point of the corresponding min-max problem.

To prove the theorem, we need the following lemma.

Lemma. *Suppose that Assumption 1 in [1] holds and assume that $\{c_k\}$ is a nonnegative monotonically decreasing sequence. Let $\{(\mathbf{x}_k, \mathbf{y}_k)\}$ be the sequence generated by Algorithm 2 with $\rho_k = \rho$. Also denote $F_{k+1} = F(\mathbf{x}_{k+1}, \mathbf{y}_{k+1})$,*

$$S_{k+1} = \frac{4}{\rho^2 c_{k+1}} \|\mathbf{y}_{k+1} - \mathbf{y}_k\|_2^2 - \frac{4}{\rho} \left(\frac{c_{k-1}}{c_k} - 1 \right) \|\mathbf{y}_{k+1}\|_2^2, \tag{10}$$

$$\Phi_{k+1} = F_{k+1} + S_{k+1} - \frac{7}{2\rho} \|\mathbf{y}_{k+1} - \mathbf{y}_k\|_2^2 - \frac{c_k}{2} \|\mathbf{y}_{k+1}\|_2^2. \tag{11}$$

If

$$\frac{1}{c_{k+1}} - \frac{1}{c_k} \leq \frac{\rho}{10}, \quad \rho \leq \frac{2}{L_y + 2c_1}, \tag{12}$$

then for all $k \geq 1$, it holds that

$$\begin{aligned} \Phi_{k+1} - \Phi_k \leq & - \left(\frac{\tau_k - L_x}{2} - \frac{\rho L_{12}^2}{2} - \frac{8 L_{12}^2}{\rho c_k^2} \right) \|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2^2 \\ & - \frac{1}{10\rho} \|\mathbf{y}_{k+1} - \mathbf{y}_k\|_2^2 + \frac{c_{k-1} - c_k}{2} \|\mathbf{y}_{k+1}\|_2^2 \\ & + \frac{4}{\rho} \left(\frac{c_{k-2}}{c_{k-1}} - \frac{c_{k-1}}{c_k} \right) \|\mathbf{y}_k\|_2^2 . \end{aligned}$$

Proof. Since $\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y})$ is Lipschitz continuous for fixed $\mathbf{y}$, we have

$$f(\mathbf{x}_{k+1}, \mathbf{y}_k) - f(\mathbf{x}_k, \mathbf{y}_k) \leq \langle \nabla_{\mathbf{x}} f(\mathbf{x}_k, \mathbf{y}_k), \mathbf{x}_{k+1} - \mathbf{x}_k \rangle + \frac{L_x}{2} \|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2^2.$$

By the update rule of $\mathbf{x}$, we further have

$$\langle \nabla_{\mathbf{x}} f(\mathbf{x}_k, \mathbf{y}_k), \mathbf{x}_{k+1} - \mathbf{x}_k \rangle - g(\mathbf{x}_{k+1}) + g(\mathbf{x}_k) \leq -\frac{\tau_k}{2} \|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2^2.$$

Combining the above two inequalities yields

$$F(\mathbf{x}_{k+1}, \mathbf{y}_k) - F(\mathbf{x}_k, \mathbf{y}_k) \leq \frac{L_x - \tau_k}{2} \|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2^2.$$

Then the rest of the proof is the same as in [4, Lemma 3.6]. □

Now we are ready to prove the theorem.

Proof. With the selected parameters in the theorem, it is easy to check that $\rho \leq \frac{2}{L_y + 2c_1}$ and $\lim_{k\to\infty} \left(\frac{1}{c_{k+1}} - \frac{1}{c_k} \right) = 0$, and thus there exists k_0 such that condition (12) holds for all $k \geq k_0$.

In the following, we shall first prove that

$$\tau_k\|\mathbf{x}_{k+1}-\mathbf{x}_k\|_2 \to 0 \text{ and } \|\mathbf{y}_{k+1}-\mathbf{y}_k\|_2 \to 0.$$

Let $\alpha_k = \frac{8(\beta_2-1)L_{12}^2}{\rho c_k^2}$, and then τ_k can be expressed as $\tau_k = \frac{16L_{12}^2}{\rho c_k^2}+2\alpha_k+\beta_3$. Since $\beta_3 \geq L_x+\rho L_{12}^2$, it follows from the above lemma that for all $k \geq k_0$,

$$\begin{aligned}&\alpha_k\|\mathbf{x}_{k+1}-\mathbf{x}_k\|_2^2+\frac{1}{10\rho}\|\mathbf{y}_{k+1}-\mathbf{y}_k\|_2^2\\ &\leq \Phi_k-\Phi_{k+1}+\frac{4}{\rho}\left(\frac{c_{k-2}}{c_{k-1}}-\frac{c_{k-1}}{c_k}\right)\|\mathbf{y}_k\|_2^2+\frac{c_{k-1}-c_k}{2}\|\mathbf{y}_{k+1}\|_2^2.\end{aligned} \tag{13}$$

For all $K > k_0$, summing both sides of (13) from $k=k_0$ to $k=K$ gives

$$\begin{aligned}&\sum_{k=k_0}^{K}\alpha_k\|\mathbf{x}_{k+1}-\mathbf{x}_k\|_2^2+\frac{1}{10\rho}\|\mathbf{y}_{k+1}-\mathbf{y}_k\|_2^2\\ &\leq \Phi_{k_0}-\Phi_{K+1}+\frac{4}{\rho}\left(\frac{c_{k_0-2}}{c_{k_0-1}}-\frac{c_{K-1}}{c_K}\right)\sigma_y^2+\frac{c_{k_0-1}-c_K}{2}\sigma_y^2,\end{aligned}$$

where $\sigma_y = \max\{\|\mathbf{y}\|_2 \mid \mathbf{y}\in\mathcal{Y}\}$. Furthermore, it follows from the definitions (10) and (11) that Φ_k is bounded from below, and thus we have

$$\sum_{k=1}^{\infty}\alpha_k\|\mathbf{x}_{k+1}-\mathbf{x}_k\|_2^2 < +\infty \tag{14}$$

and

$$\sum_{k=1}^{\infty}\|\mathbf{y}_{k+1}-\mathbf{y}_k\|_2^2 < +\infty,$$

which immediately shows that

$$\|\mathbf{y}_{k+1}-\mathbf{y}_k\|_2 \to 0.$$

On the other hand, since $\lim_{k\to\infty} c_k = 0$ and $\beta_2 > 1$, we have

$$\lim_{k\to\infty}\frac{\tau_k}{\alpha_k} = \lim_{k\to\infty}\frac{\frac{16\beta_2 L_{12}^2}{\rho c_k^2}+\beta_3}{\frac{8(\beta_2-1)L_{12}^2}{\rho c_k^2}} = \frac{2\beta_2}{\beta_2-1},$$

which implies that the sequence $\left\{\frac{\tau_k}{\alpha_k}\right\}$ is bounded, i.e., there exists d_1 such that

$$\frac{\tau_k}{\alpha_k} \leq d_1, \quad \forall k \geq 1.$$

Therefore, the following inequality holds for all $k \geq 1$,

$$\frac{1}{d_1\tau_k}\tau_k^2\|\mathbf{x}_{k+1}-\mathbf{x}_k\|_2^2 \leq \alpha_k\|\mathbf{x}_{k+1}-\mathbf{x}_k\|_2^2,$$

which, together with (14), gives

$$\sum_{k=1}^{\infty} \frac{1}{d_1 \tau_k} \tau_k^2 \|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2^2 < +\infty.$$

Note that $d_1 \tau_k \leq d_1^2 \alpha_k = \mathcal{O}(k^{2\gamma})$ with $0 < \gamma \leq 0.5$. Then we know from the above assertion that

$$\tau_k \|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2 \to 0. \tag{15}$$

Next we shall show that any limit point of $\{(\mathbf{x}_k, \mathbf{y}_k)\}$ is a stationary point of the corresponding min-max problem. Given a limit point $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ of $\{(\mathbf{x}_k, \mathbf{y}_k)\}$, there exists $\{k_j\}$ such that

$$\lim_{j \to \infty} (\mathbf{x}_{k_j}, \mathbf{y}_{k_j}) = (\hat{\mathbf{x}}, \hat{\mathbf{y}}).$$

By the update rules of $\mathbf{x}$ and $\mathbf{y}$ in Algorithm 2, it holds that

$$\mathbf{0} \in \nabla_{\mathbf{x}} f(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}) - \partial g(\mathbf{x}_{k_j+1}) + \tau_{k_j}(\mathbf{x}_{k_j+1} - \mathbf{x}_{k_j}) + \partial \mathbb{I}_{\mathcal{X}}(\mathbf{x}_{k_j+1}), \tag{16a}$$

$$\mathbf{0} \in -\nabla_{\mathbf{y}} f(\mathbf{x}_{k_j+1}, \mathbf{y}_{k_j}) + \frac{1}{\rho}(\mathbf{y}_{k_j+1} - \mathbf{y}_{k_j}) + c_{k_j} \mathbf{y}_{k_j} + \partial \mathbb{I}_{\mathcal{Y}}(\mathbf{y}_{k_j+1}). \tag{16b}$$

The above (16a) can be equivalently expressed as

$$\langle -\nabla_{\mathbf{x}} f(\mathbf{x}_{k_j}, \mathbf{y}_{k_j}) + \mathbf{s}_{k_j+1} - \tau_{k_j}(\mathbf{x}_{k_j+1} - \mathbf{x}_{k_j}), \mathbf{x} - \mathbf{x}_{k_j+1} \rangle \leq 0, \quad \forall \mathbf{x} \in \mathcal{X}, \tag{17}$$

where $\mathbf{s}_{k_j+1}$ is an element in $\partial g(\mathbf{x}_{k_j+1})$ that guarantees (16a) holds. By Assumption 2 in [1], we know that the sequence $\{\mathbf{s}_{k_j+1}\}$ is bounded. Without loss of generality, we assume

$$\lim_{j \to \infty} \mathbf{s}_{k_j+1} = \hat{\mathbf{s}},$$

otherwise we can extract a convergent subsequence. With (15) and notice that $\tau_k \to \infty$, we have $\mathbf{x}_{k_j+1} \to \hat{\mathbf{x}}$. Taking limits of the left-hand side of inequality (17) gives

$$\langle -\nabla_{\mathbf{x}} f(\hat{\mathbf{x}}, \hat{\mathbf{y}}) + \hat{\mathbf{s}}, \mathbf{x} - \hat{\mathbf{x}} \rangle \leq 0, \quad \forall \mathbf{x} \in \mathcal{X},$$

which further implies that

$$\mathbf{0} \in \nabla_{\mathbf{x}} f(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \hat{\mathbf{s}} + \partial \mathbb{I}_{\mathcal{X}}(\hat{\mathbf{x}}) \subset \nabla_{\mathbf{x}} f(\hat{\mathbf{x}}, \hat{\mathbf{y}}) - \partial g(\hat{\mathbf{x}}) + \partial \mathbb{I}_{\mathcal{X}}(\hat{\mathbf{x}}). \tag{18}$$

The last inclusion holds since g is a proper closed convex function, and thus the graph of $\partial g(\mathbf{x})$ is closed [5, Theorem 24.4], i.e., $\mathbf{s}_{k_j+1} \in \partial g(\mathbf{x}_{k_j+1})$ with $\mathbf{s}_{k_j+1} \to \hat{\mathbf{s}}$ and $\mathbf{x}_{k_j+1} \to \hat{\mathbf{x}}$ can imply $\hat{\mathbf{s}} \in \partial g(\hat{\mathbf{x}})$.

Since $c_k \to 0$ and $\|\mathbf{y}_{k+1} - \mathbf{y}_k\|_2 \to 0$, similarly we can show that

$$\mathbf{0} \in -\nabla_{\mathbf{y}} f(\hat{\mathbf{x}}, \hat{\mathbf{y}}) + \partial \mathbb{I}_{\mathcal{Y}}(\hat{\mathbf{y}}). \tag{19}$$

Combining (18) and (19), we can conclude that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a stationary point.

□

V. Proof of Lemma 1 in [1]

In this section, we give a proof of Lemma 1 in [1], which is stated as follows.

Lemma. *If* $\lambda > \max_l \|\mathbf{a}_l\|_\infty$, *all stationary points* $\hat{\mathbf{x}}$ *of (*P_λ*) must satisfy*

$$\hat{x}_i \in \{-1, 1, 0\}, \; i = 1, 2, \ldots, n.$$

Given a stationary point $\hat{\mathbf{x}}$ of (P_λ), there exists $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_+$ such that

$$\mathbf{0} \in \partial(\max_l \mathbf{a}_l^\mathsf{T} \hat{\mathbf{x}}) - \lambda \partial \|\hat{\mathbf{x}}\|_1 - \mathbf{u} + \mathbf{v}, \tag{20a}$$

$$u_i(\hat{x}_i + 1) = 0, \; v_i(\hat{x}_i - 1) = 0, \; i = 1, 2, \ldots, n. \tag{20b}$$

Next we shall show that if $\lambda > \max_l \|\mathbf{a}_l\|_\infty$, $\hat{\mathbf{x}}$ must satisfy $|\hat{x}_i| \in \{0, 1\}$ for all $i = 1, 2, \ldots, n$. Suppose for contradiction that there exists s, such that $0 < |\hat{x}_s| < 1$. It follows immediately from (20b) that $u_s = v_s = 0$. Moreover, from the calculation rule of the subdifferential [5], we know that $(\partial \|\hat{\mathbf{x}}\|_1)_s = \mathrm{sgn}(\hat{x}_s)$ and

$$\partial(\max_l \mathbf{a}_l^\mathsf{T} \hat{\mathbf{x}}) = \left\{ \mathbf{A}^\mathsf{T} \mathbf{t} \mid \mathbf{t} \in \Delta, t_i = 0 \text{ if } i \notin \mathcal{I} \right\},$$

where $\Delta = \{\mathbf{t} \in \mathbb{R}^m \mid \mathbf{1}^\mathsf{T} \mathbf{t} = 1, \; \mathbf{t} \geq \mathbf{0}\}$ is and $\mathcal{I}$ is defined as

$$\mathcal{I} = \left\{ i \in \{1, 2, \ldots, m\} \mid \mathbf{a}_i^\mathsf{T} \hat{\mathbf{x}} = \max_l \mathbf{a}_l^\mathsf{T} \hat{\mathbf{x}} \right\}.$$

Therefore, for all $\mathbf{s} \in \partial(\max_l \mathbf{a}_l^\mathsf{T} \hat{\mathbf{x}})$, we have

$$\begin{aligned} \|\mathbf{s}\|_\infty = \|\mathbf{A}^\mathsf{T} \mathbf{t}\|_\infty &= \max_l \left| \mathbf{a}_l^\mathsf{T} \mathbf{t} \right| \\ &\leq \max_l \|\mathbf{a}_l\|_\infty \|\mathbf{t}\|_1 \\ &= \max_l \|\mathbf{a}_l\|_\infty < \lambda, \end{aligned}$$

which implies that the condition (20a) cannot be satisfied for the s-th component. As a result, $\hat{\mathbf{x}}$ must have all its elements being either ± 1 or 0.

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