# Analytic and Algebraic Properties of Canal Surfaces ⋆


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**Abstract**

The envelope of a one-parameter set of spheres with radii $r(t)$ and centers $\mathbf{m}(t)$ is a canal surface with $\mathbf{m}(t)$ as the spine curve and $r(t)$ as the radii function. This concept is a generalization of the classical notion of an offset of a plane curve. In this paper, we firstly survey the principle geometric features of canal surfaces. In particular, a sufficient condition of canal surfaces without local self-intersection is presented. Moreover,the Gaussian curvature and a simple expression for the area of canal surfaces are given. We also consider the implicit equation $f(x, y, z) = 0$ of canal surfaces. In particular, the degree of $f(x, y, z)$ is presented. By using the degree of $f(x, y, z)$, a low boundary of the degree of parametrizations representations of canal surfaces is presented. We also prove the low boundary can be reached in some cases.



*Key words:* Canal surfaces, self-intersections, implicit equations


## 1. Introduction

A canal surface is defined as envelope of a non-parameter set of spheres, centered at a spine curve $\mathbf{m}(t)$ with radius $r(t)$. When $r(t)$ is a constant function, the canal surface is the envelope of a moving sphere and is called a pipe surface. The defining equations for canal surfaces are(cf.[10])


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$$\sum(t) : (x - \mathbf{m}(t))^2 - r(t)^2 = 0, \tag{1}$$

$$\sum{}'(t) : (x - \mathbf{m}(t)) \cdot \mathbf{m}'(t) + r(t)r'(t) = 0. \tag{2}$$

To guarantee that the canal surface is a smooth, non-degenerate surface, we assume that $\mathbf{m}(t)$ and $r(t)$ are smooth and satisfy the following non-degeneracy conditions: $|\mathbf{m}'(t)|^2 > r'(t)^2$(cf.[10]).

Canal surfaces have wide applications in CAGD, such as construction of blending surfaces, shape reconstruction,transition surfaces between pipes, robotic path planning or robotic path planning etc(cf.[6][11][12]).

Most of the literature on canal surfaces within the CAGD context has been motivated by the observation that canal surfaces with rational spine curve and rational radius function is rational, and it is therefore natural to ask for methods which allow one to construct a rational parametrization of canal surfaces from its spine curve and radius function. In this paper we shall not be concerned with parametrization but rather with the certain fundamental geometric and algebraic characteristics of canal surfaces.

The canal surface can be considered as a generalization of the classical notion of an offset of a plane curve. In [4] and [5], the analysis and algebraic properties of offset curves are discussed in detail. In [3], do Carmo discussed some geometrical features of pipe surfaces. Moreover, by using pipe surfaces, do Carmo proved two very important theorems in Differential Geometry concerning the total curvature of space curves, namely Fenchel's theorem and the Fary-Milnor theorem. To the best of our knowledge, there is no literature concerned with the geometric or algebraic features of canal surfaces. In this paper, we shall discuss the fundamental geometric and algebraic properties for the canal surfaces. Some results are also useful for finding the lowest degree rational parametrization of canal surfaces.

The paper is organized as follows. In section 2, we shall discuss geometry properties of canal surfaces. The first and second fundamental form of canal surfaces is presented. By using the first fundamental form, a sufficient condition of canal surfaces without local self-intersection is presented. By the second fundamental form, the Gaussian curvature of canal surfaces is given. Moreover, a simple expression for the area of the canal surface is also discussed. In section 3, algebraic properties of canal surfaces are discussed. In particular, the degree of the implicit equation of canal surfaces is presented. In section 4, by using the results in section 3, a low boundary of the degree of parametrization form of canal surfaces is presented. Moreover, we prove the low boundary can be reached in some cases.

## 2. Differential Properties

Throughout this section, we assume the spine curve $\mathbf{m}(t)$ is regular, i.e. $|\mathbf{m}'(t)| \neq 0, \forall t$. To simplify our exposition, we shall restrict ourselves to the spine curve parametrized by arc length $s$. The radii function is also considered as a function about $s$. By the non-degenerated condition , $r'(s)^2 < |\mathbf{m}'(s)|^2 = 1$. The canal surface can be parametrized using the Frenet trihedron $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ as follows:

$$C(s,v) = m(s) + r(s)(\sqrt{(1-r'(s)^2)}\cos(v)\mathbf{n} + \sqrt{(1-r'(s)^2)}\sin(v)\mathbf{b} - r'(s)\mathbf{t}),$$

$$s \in [0,l], v \in [0, 2\pi),$$

where $l$ is the total length of the spine curve. When $C_s(s_0,v_0) \times C_v(s_0,v_0) = 0$, the point $(s_0, v_0)$ is called a singular point of $C$. Similar with [9], we refer to this singularity as local self-intersection. A canal surface without local self-intersection is a regular canal surface. The condition for local self-intersection of a pipe surface has been discussed widely (cf.[7],[3],[9]). In [11], Shani and Ballard described a method to prevent local self-intersection of a generalized cylinder. However, to the best of our knowledge, there is no literature which deals with the problem of local self-intersection of a canal surface. In this section,by using the first fundamental form, a necessary condition of canal surfaces without local self-intersection is presented.

Recall the Frenet equations for a space curve:

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}, \frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} + \tau\mathbf{b}, \frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}, \tag{3}$$

where, $\mathbf{t}$ is the unit tangent vector, $\mathbf{n}$ is the unit normal vector, $\mathbf{b}$ is the unit binormal vector, $\tau$ is the torsion, $\kappa$ is the curvature.

For convenience,let $g := r\sqrt{1-r'^2}, h := rr'$. By using Frenet equation, we can obtain the first fundamental form of canal surfaces:

$$E = C_s \cdot C_s = (1 - \kappa g \cos v - h')^2 + (g\tau + h\kappa \sin v)^2 + (g' - h\kappa \cos v)^2, \tag{4}$$
$$F = C_s \cdot C_v = g^2\tau + gh\kappa \sin v, \tag{5}$$
$$G = C_v \cdot C_v = g^2. \tag{6}$$

Hence,

$$|C_s \times C_v|^2 = EG - F^2 = g^2((1 - \kappa g \cos v - h')^2 + (h\kappa \cos v - g')^2). \tag{7}$$

**Lemma 1** *When* $1-\kappa(s_0)g(s_0)\cos v_0-h'(s_0)=0$, $h(s_0)\kappa(s_0)\cos v_0-g'(s_0)=0, s_0\in[0,l], v_0\in[0,2\pi)$.

*proof:* Since $r'(s)^2<|\mathbf{m}'(s)|^2=1, g(s)=r(s)\sqrt{1-r'^2(s)}\neq 0, \forall s\in[0,l]$. By $1-\kappa(s_0)g(s_0)\cos v_0-h'(s_0)=0, \kappa(s_0)\cos v_0=\frac{1-h'(s_0)}{g(s_0)}$. Moreover, it is not difficult for proving $h(s)-h(s)h'(s)-g(s)g'(s)\equiv 0, \forall s\in[0,l]$. Hence, when $1-\kappa(s_0)g(s_0)\cos v_0-h'(s_0)=0$, $h(s_0)\kappa(s_0)\cos v_0-g'(s_0)=h(s_0)\frac{1-h'(s_0)}{g(s_0)}-g'(s_0)=(h(s_0)-h(s_0)h'(s_0)-g(s_0)g'(s_0))/g(s_0)=0$. The theorem holds. □

Moreover, we have the following theorem

**Theorem 1** *When* $r(s)<\frac{2-(r^2(s))''}{2|\kappa|}$, *the canal surface has no local self-intersection.*

*proof:* By the first fundamental form, the canal surface has local self-intersection if and only if $1-\kappa g\cos v-h'=0, h\kappa\cos v-g'=0$. By Lemma 1, we only need to consider $1-\kappa g\cos v-h'=0$. Since $\cos v$ varies between -1 and 1, there will be no local self-intersection if $|\kappa|g+h'<1$. By $\sqrt{1-r'^2}\leq 1$, $|\kappa|g+h'=|\kappa|r\sqrt{1-r'^2}+\frac{1}{2}(r^2)''\leq|\kappa|r+\frac{1}{2}(r^2)''$. Hence, when $|\kappa|r+\frac{1}{2}(r^2)''<1$, $|\kappa|g+h'<1$, i.e. when $r(s)<\frac{2-(r^2(s))''}{2|\kappa|}$, $|\kappa|g+h'<1$. The theorem holds. □

In fact, a sufficient condition for canal surfaces without local self-intersection is presented in Theorem 1. Moreover, we have the following interesting corollaries.

**Corollary 1** *When* $r(s)=as+b$, *if* $r<\frac{1-a^2}{|\kappa|}$, *the canal surface has no local self-intersection.*

**Corollary 2** *When* $r(s)=\sqrt{as+b}$, *if* $r<\frac{1}{|\kappa|}$, *the canal surface has no local self-intersection.*

By Theorem 1, it is easy for proving Corollary 1 and Corollary 2. Hence,we omit its proofs.

We turn to an examination of certain global characteristics of a canal surface, deriving the Gaussian curvature and a simple expressions for the area of a canal surface.

The developable surface plays a important role in CAGD. A national question is when the canal surface is developable. It is well known that, at regular points, the Gaussian curvature of a developable surface is identically zero. Hence, to answer the question, we have to compute the Gaussian curvature of canal surfaces. To compute the Gaussian curvature of a canal surface is really tough work. Firstly, we compute the second fundamental form. Let $\mathbf{N}$ denote the normal vector on the canal surface, then

$$\mathbf{N} = \frac{(g' - \kappa h \cos v) \cdot \mathbf{t} + (\kappa g \cos v + h' - 1) \sin v \cdot \mathbf{b} + (\kappa g \cos v + h' - 1) \cos v \cdot \mathbf{n}}{\sqrt{(g' - \kappa h \cos v)^2 + (\kappa g \cos v + h' - 1)^2}}.$$

Hence,we can obtain the second fundamental form of canal surfaces

$$\begin{aligned}
L = C_{ss} \cdot \mathbf{N} = & \frac{1}{\sqrt{(g' - kh \cos v)^2 + (kg \cos v + h' - 1)^2}} \\
& \times (-(2h' - 1)k(\kappa g \cos v + h' - 1) \cos v - k^2 g(\kappa g \cos v + h' - 1) \cos^2 v \\
& - 2(g' - kh \cos v)kg' \cos v + k(g' - kh \cos v)g\tau \sin v \\
& + (g' - kh \cos v)(k^2 h - h'') - (\kappa g \cos v + h' - 1)(\tau^2 g - g'') \\
& - \tau(\kappa g \cos v + h' - 1)\kappa h), \\
M = C_{sv} \cdot \mathbf{N} = & \frac{1}{\sqrt{(g' - kh \cos v)^2 + (\kappa g \cos v + h' - 1)^2}} \\
& \times ((g' - kh \cos v)kg \sin v - \tau g(\kappa g \cos v + h' - 1)), \\
N = C_{vv} \cdot \mathbf{N} = & -\frac{g(\kappa g \cos v + h' - 1)}{\sqrt{(g' - kh \cos v)^2 + (\kappa g \cos v + h' - 1)^2}}.
\end{aligned}$$

It is well known that the Gaussian curvature $K = \frac{LN - M^2}{EG - F^2}$. Hence,

$$\begin{aligned}
K = \frac{LN - M^2}{EG - F^2} = & \frac{g(\kappa g cos v + h' - 1)}{g^2((\kappa g cos v + h' - 1)^2 + (g' - \kappa h \cos v)^2)^2} \\
& \times (\kappa(\kappa g cos v + h' - 1)(\kappa g cos v + 2h' - 1) \cos v + 2(g' - \kappa h \cos v)\kappa g' \cos v \\
& + \tau\kappa(g' - \kappa h \cos v)g \sin v - (g' - \kappa h \cos v)(\kappa^2 h - h'') - g''(\kappa g cos v + h' - 1) \\
& + (\kappa g cos v + h' - 1)\tau\kappa h - (g' - \kappa h \cos v)^2 \kappa^2 g sin v^2/(\kappa g cos v + h' - 1)). \quad (8)
\end{aligned}$$

Hence, we have

**Theorem 2** *The regular canal surface is developable if and only if the canal surface is a cylinder or cone.*

*proof:* Obviously, when it is a cylinder or cone, the canal surface is developable.

The regular canal surface is developable iff Gaussian curvature $K \equiv 0, i.e. LN - M^2 \equiv 0$. By (8), when $LN - M^2 \equiv 0$,

$$\tau\kappa(\kappa g \cos v + h' - 1)(g' - \kappa h \cos v)g \sin v \equiv (\kappa g \cos v + h' - 1)$$

$$
\begin{aligned}
&(\kappa(\kappa gcosv + h' - 1)(\kappa g\cos v + 2h' - 1)\cos v + 2(g' - \kappa h\cos v)\kappa g'\cos v \\
&- (g' - \kappa h\cos v)(\kappa^2 h - h'') - g''(\kappa gcosv + h' - 1) + (\kappa gcosv + h' - 1)\tau\kappa h \\
&- (g' - \kappa h\cos v)^2\kappa^2 gsinv^2/(\kappa gcosv + h' - 1)). \qquad (9)
\end{aligned}
$$

Square both sides of (9). By $\sin^2 v = 1 - \cos^2 v$, the left and right side of (9) are polynomial about $\cos v$ with degree 6 and 8 respectively. By properties of polynomial, the coefficient of the highest term of right sided of (9) is identical to 0. Hence, $\kappa^4 g^2(g^2 + h^2) \equiv 0$. Since $g \neq 0, \kappa \equiv 0$.

Substitute $\kappa \equiv 0$ into (8), we have

$$
(h' - 1)(g'h'' - g''(h' - 1)) \equiv 0. \qquad (10)
$$

It is easy for proving when $\kappa \equiv 0$, if $h'(s_0) - 1 = 0, s_0 \in [0, l]$, then $EG - F^2 = 0$ at $(s_0, v), v \in [0, 2\pi)$. Hence, when $h'(s_0) - 1 = 0$, the canal surface is not regular. According to the assumption of canal surfaces being regular, we have

$$
g'h'' - g''(h' - 1) \equiv 0. \qquad (11)
$$

It is easy to verify that $g'' = (\frac{3r'r'' + rr'''}{h' - 1} + \frac{r''}{r'(1 - r'^2)})g'$. Using these results, equation (11) is simplified as

$$
-(h' - 1)\frac{r''}{r'(1 - r'^2)}g' \equiv 0. \qquad (12)
$$

From above analysis, we know that $h' - 1 \not\equiv 0$. By $g' = r'(1 - h')/\sqrt{1 - r'^2}$, if $g' \equiv 0$, then $r' \equiv 0$. Hence, when $g' \equiv 0$, $r$ is constant. If $r'' \equiv 0$, $r$ is linear about $s$ or constant. Hence, from (12), $r$ is constant or a linear function about $s$. Since $\kappa \equiv 0$, the spine curve is a line. When $r$ is constant and linear function about $s$, the canal surface is a cylinder and cone respectively. Hence, the theorem holds. □

In the following theorem, a simple expressions for the area of a canal surface is given.

**Theorem 3** *If the canal surface $C(s, v)$ is regular on $s \in [0, l], v \in [0, 2\pi)$, the total area $A$ is given by :*

$$
A = \pi|\int_0^l r(r^2)'' - 2rds|. \qquad (13)
$$

*proof:* It is well known that $A = \int_0^l \int_0^{2\pi} \sqrt{EG - F^2} dvds$. By (7), $EG - F^2 = g^2((1 - \kappa g \cos v - h')^2 + (h\kappa \cos v - g')^2)$. Substituting $g = r\sqrt{1 - r'^2}, h = rr'$ into $EG - F^2$, a simple expressions of $EG - F^2$ is presented, i.e. $EG - F^2 = (r^2\sqrt{1 - r^2}\kappa \cos v + rr'' + r'^2 - 1)^2$. Hence,

$$A = \int_0^l \int_0^{2\pi} |r^2\sqrt{1 - r^2}\kappa \cos v + rr'' + r'^2 - 1| dvds. \tag{14}$$

Since by assumption the canal surface is regular, $EG - F^2$ must be of constant sigh on $s \in [0, l], v \in [0, 2\pi)$, and the right-hand side of (14) may thus be re-written in the form

$$\begin{aligned} A &= \int_0^l \int_0^{2\pi} |r^2\sqrt{1 - r^2}\kappa \cos v + rr'' + r'^2 - 1| dvds \\ &= |\int_0^l \int_0^{2\pi} r^2\sqrt{1 - r^2}\kappa \cos v + rr'' + r'^2 - 1 dvds| \\ &= |2\pi \int_0^l rr'' + r'^2 - 1 dvds| = \pi |\int_0^l r(r^2)'' - 2r ds|. \end{aligned} \tag{15}$$

□

## 3. Algebraic Properties

In Section 2, we have analyzed the geometric features of canal surfaces with spine curve $\mathbf{m}(t)$ and radii function $r(t)$. The analysis was uninhibited by assumption regarding the functional form of the spine curve $\mathbf{m}(t)$ and radii function $r(t)$ and was thus equally valid for any analytic curve. However, in most practical applications, only spine curve and radii function parameterized by polynomial or rational functions are employed. The canal surface with such spine curves and radii functions admit further analysis based on purely algebraic considerations. In this paper, we shall only discuss the canal surface with the polynomial spine curve and radii function. In fact, the algebraic properties of canal surfaces with the rational spine curve and radii function can be discussed by using similar method.

Suppose $\mathbf{m}(t) = (m_1(t), m_2(t), m_3(t))$ and $m_i(t)$ and $r(t)$ is polynomial function with coefficients in $\mathbf{R}$. In general, the defining equations of canal surfaces is

$$\sum(t) : (x - m_1)^2 + (y - m_2)^2 + (z - m_3)^2 - r^2 = 0, \tag{16}$$
$$\sum{}'(t) : (x - m_1)m_1' + (y - m_2)m_2' + (z - m_3)m_3' + rr' = 0. \tag{17}$$

To guarantee the canal surface is non-degeneracy, we assume $m_1'(t)^2+m_2'(t)^2+m_3'(t)^2 > r'(t)^2, t \in \mathbf{R}$. Obviously, the plane $\sum{}'(t)$ is perpendicular to the derivative vector $m'$. But, if $\tau$ is a value of the parameter such that $m_1'(\tau) = m_2'(\tau) = m_3'(\tau) = r(\tau)r'(\tau) = 0$, equation (17) will degenerate to an identity at this value, being then satisfy for any $(x, y, z)$. This defect can be remedied as follows. Let $\phi(t)$ be the greatest common divisor of $m_1'(t), m_2'(t), m_3'(t)$ and $r(t)r'(t)$, i.e.

$$\phi(t) = GCD(m_1'(t), m_2'(t), m_3'(t), r(t)r'(t)). \tag{18}$$

If $\phi(t)$ is not a constant, we can compute $p_1(t), p_2(t), p_3(t)$ and $q(t)$ through polynomial divisions:

$$q(t) = \frac{r(t)r'(t)}{\phi(t)}, p_i(t) = \frac{m_i'(t)}{\phi(t)}, i = 1, 2, 3. \tag{19}$$

We now substitute in place of (17), and obtain

$$p_1(t)(x - m_1(t)) + p_2(t)(y - m_2(t)) + p_3(t)(z - m_3(t)) + q(t) = 0. \tag{20}$$

Let $\tau$ denote a root of $\phi(t)$. In fact, the equation (20) defines the planar at $(m_1(\tau), m_2(\tau), m_3(\tau))$ as the proper limit of the planar $\sum{}'(t)$ at regular curve points $(m_1(t), m_2(t), m_3(t))$ as $t \to \tau$.

Let $\widehat{\phi}(t) = GCD(m_1'(t), m_2'(t), m_3'(t))$. Obviously, $\phi(t)$ is a factor of $\widehat{\phi}(t)$. If $\tau$ is a value of the parameter such that $\phi(\tau) \neq 0$ and $\widehat{\phi}(\tau) = 0$, then equation (17) will become inconsistency. Hence, we can not find a solution of (16) and (17) in Euclid space. But in complex project space, the solution of (16) and (17) corresponding to the circular point at infinity. The homogeneous coordinates of solutions is $(x_0, y_0, z_0, 0)$, where $x_0^2 + y_0^2 + z_0^2 = 0$.

In fact, when $\tau \in \mathbf{R}$, by $r'(\tau)^2 < m_1'(\tau)^2 + m_2'(\tau)^2 + m_3'(\tau)^2$, if $\widehat{\phi}(\tau) = 0$ then $\phi(\tau) = 0$. Hence, if only consider the real parameter value, we do not worry about the inconsistency equation appear.

Now for a prescribed canal surface, we form the following polynomials in $t$ :

$$P(t, x, y, z) = (x - m_1)^2 + (y - m_2)^2 + (z - m_3)^2 - r(t)^2 = 0, \tag{21}$$
$$Q(t, x, y, z) = (x - m_1)p_1(t) + (y - m_2)p_2(t) + (z - m_3)p_3(t) + q(t) = 0. \tag{22}$$

Obviously, the implicit equation of the canal surface with spine curve $m(t)$ and radii function $r(t)$ is given by the resultant

$$f(x,y,z) = Res_t(P(t,x,y,z), Q(t,x,y,z)), \tag{23}$$

where $Res_t(P,Q)$ denotes the sylvester resultant of $P$ and $Q$ with respect to $t$.

Suppose the degree of $\mathbf{m}(t)$, $r(t)$ and $\phi(t)$ is $n$, $m$ and $v$ respectively. Then we have

**Theorem 4** *The canal surface $f(x,y,z) = 0$ given by (23) is of degree $4\max(m,n) - 2 - 2v$, where $n, m$ and $v$ is the degree of $\mathbf{m}(t), r(t)$ and $\phi(t)$ respectively.*

*proof:* Suppose $m \geq n$. Obviously, $P$ and $Q$ are of degree $2m$ and $2m-1-v$ in $t$. Hence, the Sylvester determinant for (23) is of dimension $4m-1-v$. Let $\delta_{i,j}(x,y,z)$ be a typical entry in the Sylvester determinant for the resultant (23). By the definition of Sylvester determinant and (21) (22) we see that,

when $1 \leq i \leq 2m-1-v$ and $j \leq i$ , $1 \leq i \leq 2m-1-v$ and $j > i+2m$, $2m-v \leq i \leq 4m-1-v$ and $j < i+1-(2m-v)$ ,$2m-v \leq i \leq 4m-1-v$ and $j > i$, $\delta_{ij}(x,y,z) = 0$ ,

when $1 \leq i \leq 2m-1-v$ and $i \leq j \leq i+n-1$ ,$2m-v \leq i \leq 4m-1-v$ and $i-(2m-v)+1 \leq j \leq i+n-(2m-v)$ , $\delta_{ij}(x,y,z)$ is a constant,

when $1 \leq i \leq 2m-1-v$ and $j = i+2m$ $deg(\delta_{ij}) = 2$,

otherwise, $deg(\delta_{ij}) = 1$.

The expansion of the Sylvester determinant for (22) and (21) may be written as

$$f(x,y,z) = \sum_{\sigma \in S} (-1)^{sign(\sigma)} \Pi_{i=1}^{4m-1-v} \delta_{i\sigma_i}(x,y), \tag{24}$$

where $S$ is the set of all permutations of the sequence of integer $\{1, \cdots, 4m-1-v\}$, $\sigma$ is a typical member of this set which maps $i$ to $\sigma_i$, and $(-1)^{sign(\sigma)}$ is just $+1$ for even permutations, and $-1$ for odd permutations.

Consider a permutation $\sigma$ which contributes a non-zero product to the sum (24). This contribution may be expressed in terms of the five sub-products:

$$\pi_1(\sigma) = \prod_{1 \leq i \leq 2m-1-v, \sigma_i = i+2m} \delta_{i\sigma_i}(x,y,z),$$

$$\pi_2(\sigma) = \prod_{1\le i\le 2m-1-v, i+n\le\sigma_i\le i+2m-1} \delta_{i\sigma_i}(x,y,z),$$
$$\pi_3(\sigma) = \prod_{2m-v\le i\le 4m-1-v, i-(2m-v)+n+1\le\sigma_i\le i} \delta_{i\sigma_i}(x,y,z),$$
$$\pi_4(\sigma) = \prod_{1\le i\le 2m-1-v, i\le\sigma_i\le i+n-1} \delta_{i\sigma_i}(x,y,z),$$
$$\pi_5(\sigma) = \prod_{2m-v\le i\le 4m-1-v, i-(2m-v)+1\le\sigma_i\le i-(2m-v)+n} \delta_{i\sigma_i}(x,y,z).$$

Let $n_k(\sigma)$ be the number of terms in the $k-$th sub-product $\pi_k(\sigma)$. We have

$$n_1(\sigma) + n_2(\sigma) + n_3(\sigma) \le 2m - 1 - v. \quad (25)$$

The degree of the product corresponding to the permutation $\sigma$ is of degree $2n_1(\sigma) + n_2(\sigma) + n_3(\sigma)$.

Obviously, $2n_1(\sigma)+n_2(\sigma)+n_3(\sigma) \le 2(n_1(\sigma)+n_2(\sigma)+n_3(\sigma)) \le 4m-2-2v$. The maximum degree can be achieved when $n_1(\sigma) = 2m-1-v, n_2(\sigma) = n_3(\sigma) = 0$. Indeed, we can find such $\sigma$ reach the maximum degree. Hence, the degree of $f(x,y,z)$ is precisely $4m - 2 - 2v$.

When $m < n$, by using similar method, we can prove $degf(x,y,z) = 4n - 2 - 2v$. Hence, the theorem holds.□

**4. A low boundary of degree of parametrization of canal surfaces**

A rational surface is defined by $\Phi(t,u) = (a(t,u)/d(t,u), b(t,u)/d(t,u), c(t,u)/d(t,u))$, where $a, b, c$ and $d$ are polynomial about $t, u$ with degree $(n_t, n_u)$, i.e. $n_t$ and $n_u$ are the highest degree of the parametric equation with respect to $t$ and $u$ respectively. It may happen that there values of $t, u$ for which $a(t,u) = b(t,u) = c(t,u) = d(t,u) = 0$. These are known as base points in contemporary algebraic geometry. By [2], the implicit degree of $\Phi(t,u)$ is $2n_tn_u - \rho$, where $\rho$ is the number of base points.

It is well known that canal surfaces with rational spine curve and rational radius function are rational ([10]). To be precise, they admit real rational parametrization of their real components. There are plenty of literature to discuss the parametrization of canal surfaces(cf.[9][1][10][8]). Denote the rational parametric form of canal surfaces by $\Phi(t,u)$. The highest degree of $\Phi(t,u)$ with respect to $t$ and $u$ is denoted by $n_t$ and $n_u$ respectively. Since the low degree representations are important for practical use, in [10], degree reductions are discussed. Hence, a natural question is raised: what is the lowest degree of parametrization of canal surfaces? The question is also described by the following : given a canal surface with a rational spine curve and radii function, what are the minimum $n_t$ and $n_u$?

**Theorem 5** *Given a canal surface with a polynomial spine curve* $m(t) = (m_1(t), m_2(t), m_3(t))$ *and radii function* $r(t)$*. Suppose the degree of* $m(t)$ *and* $r(t)$ *is* $n$ *and* $m$ *respectively. Let* $\Phi(t,u)$ *be rational parametrization form of the canal surface. Then we have* $n_t n_u \geq 2\max(m,n) - 1 - v$*, where* $v$ *is the degree of* $GCD(m_1'(t), m_2'(t), m_3'(t), r(t)r'(t))$*.*

*proof:* By Theorem 4, the implicit degree of the canal surface is $4\max(m,n) - 2 - 2v$. By [2], we have $2n_t n_u - \rho = 4\max(m,n) - 2 - 2v$, where $\rho$ is the number of base points. Since $\rho \geq 0$, $2n_t n_u \geq 4\max(m,n) - 2 - 2v$. Hence, $n_t n_u \geq 2\max(m,n) - 1 - v$. □

In general, the degree of $u$ in $\Phi(t,u)$ is 2. Obviously, the minimum $n_u$ is also 2. Then we have

**Corollary 3** *When* $n_u$ *is selected as 2, under the condition of Theorem 5,* $n_t \geq \lceil \max(m,n) - (1+v)/2 \rceil$.

Moreover, we can prove the low boundary in Theorem 5 can be reached in some cases.

**Theorem 6** *Given a canal surface with polynomial spine curve* $m(t) = (m_1(t), m_2(t), m_3(t))$ *and polynomial radii function* $r(t)$*. Suppose* $GCD(m_1'(t), m_2'(t), m_3'(t), r(t)r'(t)) = 1$*. There exits parametrization form* $\Phi(t,u)$ *whose degree reach the low boundary presented in Corollary 3.*

*proof:* In [1], section 3.1 present a parametrization form $\Psi(t,u)$ with $n_u = 2$. By Corollary 2.3 in [1] $n_t = \max(m,n) - 1$ which reaches the low boundary in Corollary 3. □

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