An Explicit Formulation for Two Dimensional Vector Partition Functions

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Abstract. In this paper, an explicit formulation for two dimensional multivariate truncated power functions is presented, and a simplified explicit formulation for two dimensional vector partition functions is given. Moreover, Popoviciu's formulation for restricted integer partition functions is generalized and the generalized Frobenius problem is also discussed.

1. Introduction

The *vector partition function* that we are interested in is in the form of

$$t(\mathbf{b}|M) = \#\{\mathbf{x} \in \mathbf{Z}_+^n | \mathbf{M}\mathbf{x} = \mathbf{b}\},$$

where, $\mathbf{Z}_+$ denotes the nonnegative integers, $\mathbf{M}$ is a fixed $s \times n$ integer matrix with columns $m_1, \cdots, m_n \in \mathbf{Z}^s$ and $\mathbf{b}$ is a variable vector in $\mathbf{Z}^s$. To guarantee that $t(\mathbf{b}|M)$ is finite, we require $[\{m_1, \cdots, m_n\}]$ does not contain the origin, where $[A]$ denotes the convex hull of a given set A. The vector partition function $t(\mathbf{b}|M)$, which is also called a *discrete truncated power*, has many applications in various mathematical areas including Algebraic Geometry [**26**], Representation Theory[**29**], Number Theory [**23**] , Statistics[**16**] and Randomized Algorithm [**32**] among others.

When $s = 1$, an explicit formulation for $t(\mathbf{b}|M)$, which counts the integer solutions for the linear Diophantine equation, is presented in [**1**]. Especially, Popoviciu gave a beautiful and surprising formulation for $t(n|M)$ ([**27**]), when $M = (a, b)$ where a and b are relatively prime.

For the general matrix M, the nature of $t(\mathbf{b}|M)$ is investigated and the piecewise structure of $t(\mathbf{b}|M)$ is given in [**15**] and [**31**]. Moreover, one is also interested in the explicit formulation of $t(\mathbf{b}|M)$. For the general matrix M, a powerful method for obtaining $t(\mathbf{b}|M)$ is described in [**8, 30**]. Another interesting algorithm for computing $t(\mathbf{b}|M)$ as a function of $\mathbf{b}$ is also introduced in [**3**]. When M is unimodular, where every nonsingular square submatrix has determinant ± 1, two algebraic algorithms for generating the explicit formulation for $t(\mathbf{b}|M)$ is presented in [**17**]. However, all of these methods depend on complex computation. In [**34**], based on multivariate truncated power functions $T(\mathbf{x}|M)$, an explicit formulation for $t(\mathbf{b}|M)$ is presented. However, the formulation involves multivariate truncated power functions $T(\mathbf{x}|M)$,

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which are not in explicit form, and high-dimensional Fourier-Dedekind sums, so we have to give an explicit form for $T(\mathbf{x}|M)$ and simplify high-dimensional Fourier-Dedekind sums, in order to predigest the explicit formulation for $t(\mathbf{b}|M)$. The goal of this paper is to generalize Popoviciu's formulation, and give a simplified explicit formulation for two dimensional vector partition functions. We believe that the results in this paper are not only helpful for understanding high dimensional vector partition functions but also useful for solving two linear Diophantine equations [**18, 35**].

The rest of the paper is organized as follows. To help make this paper self-contained we shall first introduce notations and definitions in Section 2. In Section 3, we recall previous results regarding vector partition functions $t(\mathbf{b}|M)$. Section 4 generalizes the Popoviciu's formulation to two dimension. In Section 5, the generalized Frobenius problem is investigated. Finally, Section 6 gives an explicit formulation for multivariate truncated powers in the case where $s = 2$ and show that a high-dimensional Fourier-Dedekind sum can be converted to a one-dimensional Fourier-Dedekind sum, which is convenient for computing. A simplified explicit formulation for two-dimension vector partition functions is then given.

2. Preliminaries

To describe the nature of $t(\mathbf{b}|\mathbf{M})$, we introduce several notations and definitions in which the common terminology of multiset theory is adopted. Intuitively, a *multiset* is a set with possible repeated elements; for instance $\{2, 2, 2, 3, 4, 4\}$. Let Y be an $s \times n$ matrix and Y can be considered as a multiset of elements of $\mathbf{R}^s$. The cone spanned by Y, denoted by $cone(Y)$, is the set

$$\{\sum_{y\in Y} a_y y : a_y \geq 0 \text{ for all } y\}.$$

Denote by $cone^\circ(Y)$ the relative interior of $cone(Y)$. Let $\mathcal{Y}(M)$ denote the set consisting of those submultisets Y of M for which $M\backslash Y$ does not span $\mathbf{R}^s$. Let the set $c(M)$ be the union of all sets $span(M \setminus Y)$ as Y runs over $\mathcal{Y}(M)$. A connected component of $cone^\circ(M)\backslash c(M)$, is called a *fundamental M-cone*. For a fundamental M-cone Ω, we set $v(\Omega|M) := \Omega - [[M))$. Here, $[[M)) := \{\sum_{j=1}^n a_j m_j : 0 \leq a_j < 1, \forall j\}, \Omega - [[M))$ is the set of all elements of the form $a - b$, where $a \in \Omega$ and $b \in [[M))$.

We shall use the standard multiindex notation. Specifically, an element $\alpha \in \mathbf{N}^m$ is called an $m-index$, and $|\alpha|$ is called the length of α. Define $z^\alpha := z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ for $z = (z_1, \cdots, z_m) \in \mathbf{C}^m$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m$. For $y = (y_1, \cdots, y_s) \in \mathbf{R}^s$ and a function f defined on $\mathbf{R}^s$, we denote by $D_y f$ the directional derivative of f in the direction y,i.e. $D_y = \sum_{j=1}^s y_j D_j$, where, D_j denote the partial derivative with respect to the jth coordinate. For $v := (v_1, \cdots, v_m) \in \mathbf{N}^m$, we let $D^v = D_1^{v_1} \cdots D_m^{v_m}$ and $v! = \prod_i v_i!$ Moreover, we let $e := (1, 1, \cdots, 1) \in \mathbf{Z}^s$.

Let $\mathcal{S}_k(M) = \{Y \subseteq M : \#Y = s + k, span(Y) = \mathbf{R}^s\}$ and $\mathcal{B}(Y) = \{X \subseteq Y : \#X = s, span(X) = \mathbf{R}^s\}$. If for any $Y \in \mathcal{S}_k(M)$, $gcd\{|det(X)|, X \in \mathcal{B}(Y)\} = 1$, then M is called a *$k-$ prime matrix.* In particular, when M is an $1-$prime matrix, M is also called a *pairwise relative prime matrix.* When $s = 1$, $k-$prime matrix means that no k of the integers $m_1, m_2, \cdots, m_n$ have a common factor, where $m_i, i = 1, \cdots, n$ are the elements in M. Moreover, we denote $e^{\frac{2\pi i}{k}}$ by W_k.

The multivariate truncated power $T(\cdot|M)$ associated with M, which was introduced by W.Dahmen [**10**] firstly, is the distribution given by the rule

$$T(\cdot|M) : \phi \mapsto \int_{\mathbf{R}^n_+} \phi(Mu)du, \phi \in \mathcal{D}(\mathbf{R}^s), \tag{2.1}$$

where $\mathcal{D}(\mathbf{R}^s)$ is the space of test functions on $\mathbf{R}^s$,i.e. the space of all compactly supported and infinitely differentiable functions on $\mathbf{R}^s$. In fact, $T(\cdot|M)$ agrees with some homogeneous polynomial of degree $n-s$ on each fundamental M-cone. When M is an $s \times s$ invertible matrix, $T(\cdot|M)$ agrees with the function on $\mathbf{R}^s$ which takes value $\frac{1}{|det(M)|}$ on $cone(M)$ and 0 elsewhere.

In [**22**], an efficient method for computing the multivariate truncated power is presented.

THEOREM 2.1. *([**22**]) Let M be an $s \times n$ matrix with columns $m_1, \cdots, m_n \in \mathbf{Z}^s \setminus \{0\}$ such that the origin does not contain in $conv(M)$. For any $\lambda_1, \cdots, \lambda_n \in \mathbf{R}$, and $\mathbf{x} = \sum_{j=1}^n \lambda_j m_j$,*

$$T(\mathbf{x}|M) = \frac{1}{n-s} \sum_{j=1}^{n} \lambda_j T(x|M \setminus m_j). \tag{2.2}$$

For more detailed information about the function, the reader is referred to [**6**],[**10**].

A multivariate Box spline $B(\cdot|M)$ associated with M was introduced in [**5**] and [**6**], which is the distribution given by the rule

$$B(\cdot|M) : \phi \mapsto \int_{[0,1)^n} \phi(Mu)du, \phi \in \mathcal{D}(\mathbf{R}^s). \tag{2.3}$$

By taking $\phi = exp(-iy\cdot)$ in (2.3), we obtain the Fourier transform of $B(\cdot|M)$ as

$$\widehat{B}(\zeta|M) = \prod_{j=1}^{n} \frac{1 - exp(-i\zeta^T m_j)}{i\zeta^T m_j}, \zeta \in \mathbf{C}^s.$$

For more detail information about Box splines,the reader is referred to [**7**] .

REMARK 2.2. Our definition of a fundamental M-cone is slightly different from that presented in [**15**]. In [**15**], a fundamental M-cone is defined as a connected component of $cone^\circ(M) \setminus c(M)$, where $c(M)$ is the union of all sets $span(M \setminus Y)$ as Y runs over $\mathcal{Y}(M)$. In fact, the fundamental M-cone defined in this paper may be larger than the one defined in [**15**]. All the conclusions in [**15**], however, also hold for the larger fundamental M-cone. Prof. M. Vergne introduced this new definition of a fundamental M-cone in a private communication.

3. Vector partition functions

To describe the nature of $t(\mathbf{b}|M)$, we let $M_\theta := \{y \in M : \theta^y = 1\}$ and let $A(M) := \{\theta \in (\mathbf{C} \setminus \{0\})^s : span(M_\theta) = \mathbf{R}^s\}$. Recall $e = (1, 1, \cdots, 1) \in \mathbf{Z}^s$, as for any $y \in M, e^y = 1$, $e \in A(M)$.

The following qualitative result about $t(\cdot|M)$ is presented in [**15**].

THEOREM 3.1. *([**15**]) Let $M = \{m_1, \cdots, m_n\}$ be a multiset of integer vectors in $\mathbf{R}^s$ such that M spans $\mathbf{R}^s$ and the convex hull of M does not contain the origin. Then for any fundamental M-cone Ω, there exists a unique element $f_\Omega(\alpha|M) =$*

$\sum_{\theta\in A(M)} \theta^\alpha p_{\theta,\Omega}(\alpha)$ *such that* $f_\Omega(\alpha|M)$ *agrees with* $t(\alpha|M)$ *on* $v(\Omega|M)$, *where* $p_{\theta,\Omega}(\cdot)$ *is a polynomial with degree less than* $\#M_\theta - s$.

An explicit formulation for $p_{e,\Omega}(\alpha)$, which is the polynomial part for $t(\alpha|M)$, is presented in the following theorem.

THEOREM 3.2. *(*[**34**]*) Under the condition for Theorem 3.1,* $p_{e,\Omega}(x) = \sum_{k=0}^{n-s} p_{k,\Omega}(x)$, *where* $p_{k,\Omega}(x)$ *is homogeneous polynomial of degree* $n-s-k$, *defined inductively by*

$$p_{0,\Omega}(x) = T(x|M), p_{k,\Omega}(x) = -\sum_{j=0}^{k-1}(\sum_{|v|=k-j} D^v p_{j,\Omega}(x)(-i)^{|v|}D^v\widehat{B}(0|M)/v!), k\geq 1,$$

where, $x\in\Omega$.

More generally, an explicit formulation for $p_{\theta,\Omega}$ is also given as follows.

THEOREM 3.3. *(*[**34**]*) Given* $\theta_0\in A(M)\setminus e$, *under the condition for Theorem 3.1,* $p_{\theta_0,\Omega}(x) = \sum_{\mu=0}^{n-s-\kappa} p_{\mu,\Omega}^{\theta_0}(x)$, *where* $\kappa = \#(M\setminus M_{\theta_0}), p_{\mu,\Omega}^{\theta_0}(x)$ *is homogeneous polynomial of degree* $n-s-\kappa-\mu$, *defined inductively by*

$$\begin{aligned} p_{0,\Omega}^{\theta_0}(x) &= q_{0,r}^{\theta_0}(x), \\ p_{\mu,\Omega}^{\theta_0}(x) &= q_{\mu,r}^{\theta_0}(x) - \sum_{j=0}^{\mu-1}(\sum_{|v|=\mu-j} D^v p_{\mu,\Omega}^{\theta_0}(x)(-i)^{|v|}D^v\widehat{B}(0|\widehat{M_r})/v!), \mu\geq 1. \end{aligned}$$

Here, $q_{\mu,r}^{\theta_0}(x)$ *is a polynomial which is determined by the following conditions: when* $x\in\Omega$, $q_{\mu,r}^{\theta_0}(x) = \sum_{j_1+\cdots+j_\kappa=\mu}\prod_{i=1}^{\kappa}\frac{s_{1+j_i}(\theta_0^{-m_i})}{(j_i+1)!}\frac{1}{r^\kappa}D_{m_1}^{j_1}\cdots D_{m_\kappa}^{j_\kappa}T(x|M_{\theta_0})$, *where* $s_0(x) = \frac{x-x^r}{x-1}$, $s_j(x) = xs'_{j-1}(x), j\in\mathbf{Z}_+$.

In particular, when M is a 1-prime matrix, a simple formulation for $t(\cdot|M)$ is shown in the following theorem.

THEOREM 3.4. [**34**] *Under the condition for Theorem 3.1,when* M *is a* $1-$*prime matrix,*

$$f_\Omega(\alpha|M) = p_{e,\Omega}(\alpha|M) + \sum_{\theta\in A(M)\setminus e}\theta^\alpha\frac{1}{|det(M_\theta)|}\prod_{w\in M\setminus M_\theta}\frac{1}{1-\theta^{-w}}1_{cone(M_\theta)}(\Omega),$$

where $p_{e,\Omega}(\alpha|M)$ *is given in Theorem 3.2.*

For the convenience of description, throughout the rest of the paper, we suppose M is a 1-prime matrix without further declaration. According to Theorem 3.4, to give a simple explicit formulation for $t(\mathbf{b}|M)$, we have to present an explicit formulation for $T(\mathbf{x}|M)$. Moreover, to calculate the elements in $A(M)$ is a non-trivial problem, and hence, we have to predigest the non-polynomial part for $t(\mathbf{b}|M)$.

4. Generalized Popoviciu's formulation

In this section, we are interested in $t(\mathbf{n}|M)$, where $M = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \in \mathbf{Z}^{2\times 3}$, $\mathbf{n} = (n_1,n_2)^T \in \mathbf{Z}_+^2$. Without loss of generality, we suppose $\frac{y_1}{x_1} < \frac{y_2}{x_2} < \frac{y_3}{x_3}$. Obviously, for the matrix M, there exist two fundamental M-cones, i.e. $\Omega_1 = \{(x,y)^T|(x,y)^T\in cone(M), \frac{y_1}{x_1} < \frac{y}{x} < \frac{y_2}{x_2}\}$ and $\Omega_2 = \{(x,y)^T|(x,y)^T\in cone(M), \frac{y_2}{x_2} < \frac{y}{x} < \frac{y_3}{x_3}\}$ (See Fig. 1).

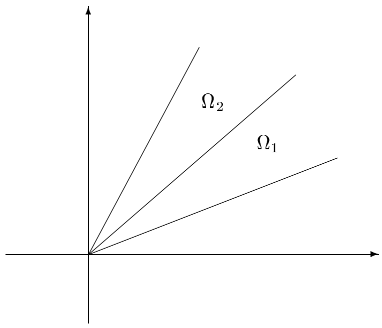

Fig. 1.The fundamental M-cones.

To describe conveniently, we let $M_{ij} = \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$, and let $Y_{ij} = det(M_{ij})$, where $i < j$. To describe the explicit formulation for $t(\mathbf{n}|M)$, we need to define the fractional part function $\{x\}$ which denotes the fractional part of x, i.e. $\{x\} = x - \lfloor x \rfloor$.

In this section, our goal is to generalize the following beautiful formula due to Popoviciu:

Theorem 4.1. [**27**] *If a and b are relatively prime,*

$$t(n|(a,b)) = \frac{n}{ab} - \{\frac{b^{-1}n}{a}\} - \{\frac{a^{-1}n}{b}\} + 1,$$

where $b^{-1}b \equiv 1 \mod a$, and $a^{-1}a \equiv 1 \mod b$, $n \in \mathbf{Z}_+$.

In order to generalize Theorem 4.1, we firstly consider the explicit formulation for $T(\mathbf{x}|M)$.

Lemma 4.2. *Suppose the matrix $M = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \in \mathbf{Z}^{2\times 3}$. When $\mathbf{x} = (x,y)^T \in \overline{\Omega}_1, T(\mathbf{x}|M) = \frac{yx_1 - xy_1}{(x_1y_2 - y_1x_2)(x_1y_3 - y_1x_3)}$; when $\mathbf{x} = (x,y)^T \in \overline{\Omega}_2, T(\mathbf{x}|M) = \frac{xy_3 - yx_3}{(x_2y_3 - y_2x_3)(x_1y_3 - y_1x_3)}$.*

Proof. Based on Theorem 2.1 and $T(\mathbf{x}|M_{ij}) = \frac{1}{det(M_{ij})}, \mathbf{x} \in cone(M_{ij}), i < j$, the Lemma can be proved easily after a brief calculation. □

The main theorem in this section is:

Theorem 4.3. *Suppose the 1-prime matrix $M = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$. When $\mathbf{n} = (n_1,n_2)^T \in \overline{\Omega}_1 \cap \mathbf{Z}^2$,*

$$\begin{aligned} t(\mathbf{n}|M) &= \frac{n_2x_1 - n_1y_1}{Y_{12}Y_{13}} - \{\frac{(f_{12}Y_{13} + g_{12}Y_{23})^{-1}(n_2(f_{12}x_1 + g_{12}x_2) - n_1(f_{12}y_1 + g_{12}y_2))}{Y_{12}}\} \\ &- \{\frac{(f_{13}Y_{12} + g_{13}Y_{23})^{-1}(n_2(f_{13}x_1 + g_{13}x_3) - n_1(f_{13}y_1 + g_{13}y_3))}{Y_{13}}\} + 1; \end{aligned}$$

when $\mathbf{n} = (n_1,n_2)^T \in \overline{\Omega}_2 \cap \mathbf{Z}^2$,

$$\begin{aligned} t(\mathbf{n}|M) &= \frac{n_1y_3 - n_2y_3}{Y_{23}Y_{13}} - \{\frac{(f_{23}Y_{13} + g_{23}Y_{12})^{-1}(n_1(f_{23}x_3 + g_{23}x_2) - n_2(f_{23}y_3 + g_{23}y_2))}{Y_{23}}\} \\ &- \{\frac{(f_{13}Y_{12} + g_{13}Y_{23})^{-1}(n_1(f_{13}x_1 + g_{13}x_3) - n_2(f_{13}y_1 + g_{13}y_3))}{Y_{13}}\} + 1, \end{aligned}$$

where, f_{12}, g_{12}, f_{13}*,* g_{13}*,* f_{23} *and* $g_{23} \in \mathbf{Z}$ *satisfy* $gcd(f_{12}Y_{13}+g_{12}Y_{23}, Y_{12}) = 1$ $gcd(f_{13}Y_{12}+g_{13}Y_{23}, Y_{13}) = 1$ *and* $gcd(f_{23}Y_{13}+g_{23}Y_{12}, Y_{23}) = 1$*, moreover,* $(f_{12}Y_{13}+g_{12}Y_{23})^{-1}(f_{12}Y_{13}+g_{12}Y_{23}) \equiv 1 \ mod \ Y_{12}, (f_{13}Y_{12}+g_{13}Y_{23})^{-1}(f_{13}Y_{12}+g_{13}Y_{23}) \equiv 1 \ mod \ Y_{13}, (f_{23}Y_{13}+g_{23}Y_{12})^{-1}(f_{23}Y_{13}+g_{23}Y_{12}) \equiv 1 \ mod \ Y_{23}$.

Proof. We only prove the case where $(n_1, n_2)^T \in \overline{\Omega}_1 \cap \mathbf{Z}^2$. Based on Theorem 3.2, $p_{e,\Omega_1}(\mathbf{x})$, which is the polynomial part for $t(\cdot|M)$ on Ω_1, is in the form of $p_{0,\Omega_1}(\mathbf{x}) + p_{1,\Omega_1}(\mathbf{x})$. Here, when $\mathbf{x} \in \Omega_1$, $p_{0,\Omega_1}(\mathbf{x}) = T(\mathbf{x}|M)$, $p_{1,\Omega_1}(\mathbf{x}) = -(\sum_{|v|=1} D^v p_{0,\Omega_1}(\mathbf{x})(-i)D^v \widehat{B}(0|M))$. By the explicit formulation for $T(\mathbf{x}|M)$, we have $p_{0,\Omega_1}(\mathbf{x}) = \frac{y_1 x - x_1 y}{(x_2y_1-y_2x_1)(x_1y_3-y_1x_3)}$. After a brief calculation, we have $p_{1,\Omega}(\mathbf{x}) = \frac{1}{2}(\frac{1}{Y_{13}} + \frac{1}{Y_{12}})$. Hence, the polynomial part for $t(\mathbf{n}|M)$ is $\frac{n_2x_1-n_1y_1}{Y_{12}Y_{13}} + \frac{1}{2}(\frac{1}{Y_{13}} + \frac{1}{Y_{12}})$. According to Theorem 3.4, we only need to consider the sum

$$\begin{aligned} &\sum_{\theta \in A(M)\backslash e} \frac{1}{|det(M_\theta)|} \prod_{w \in M \backslash M_\theta} \frac{\theta^n}{1-\theta^{-w}} 1_{cone(M_\theta)}(\Omega_1) \\ = &\frac{1}{Y_{12}} \sum_{\substack{\theta \in A(M)\backslash e \\ M_\theta = M_{12}}} \frac{\theta^n}{1-\theta^{-(x_3,y_3)}} + \frac{1}{Y_{13}} \sum_{\substack{\theta \in A(M)\backslash e \\ M_\theta = M_{13}}} \frac{\theta^n}{1-\theta^{-(x_2,y_2)}}. \end{aligned}$$

Recall $e^{\frac{2\pi i}{k}}$ is denoted by W_k. As pointed out in [**13**], the elements in the set $\{\theta | \theta \in A(M), M_\theta = M_{12}\}$ have the form $(W_{Y_{12}}^{\alpha_1^j}, W_{Y_{12}}^{\alpha_2^j})$, where $(\alpha_1^j, \alpha_2^j) \in \mathbf{Z}^2, 1 \leq j \leq Y_{12} - 1$.

Consider firstly

$$\frac{1}{Y_{12}} \sum_{\substack{\theta \in A(M)\backslash e \\ M_\theta = M_{12}}} \frac{\theta^{\mathbf{n}}}{1-\theta^{-(x_3,y_3)}} = \frac{1}{Y_{12}} \sum_{j=1}^{Y_{12}-1} \frac{W_{Y_{12}}^{n_1\alpha_1^j + n_2\alpha_2^j}}{1 - W_{Y_{12}}^{-(x_3\alpha_1^j + y_3\alpha_2^j)}}. \tag{4.1}$$

We set $x_3\alpha_1^j + y_3\alpha_2^j \equiv k \ mod \ Y_{12}$. Since M is a 1-prime matrix, $x_3\alpha_1^j + y_3\alpha_2^j \not\equiv x_3\alpha_1^m + y_3\alpha_2^m \ mod \ Y_{12}$ when $j \neq m$. Hence, k runs over $[1, Y_{12}-1] \cap \mathbf{Z}$.

For $\theta \in \{\theta | \theta \in A(M), M_\theta = M_{12}\}$, we have $\theta^{(x_1,y_1)} = \theta^{(x_2,y_2)} = 1$. Hence,

$$x_1\alpha_1^j + y_1\alpha_2^j \equiv 0 \ mod \ Y_{12}, \tag{4.2}$$

$$x_2\alpha_1^j + y_2\alpha_2^j \equiv 0 \ mod \ Y_{12}, \tag{4.3}$$

$$x_3\alpha_1^j + y_3\alpha_2^j \equiv k \ mod \ Y_{12}. \tag{4.4}$$

By x_1 on both sides of (4.4), we have

$$x_1x_3\alpha_1^j + x_1y_3\alpha_2^j \equiv x_1 k \ mod \ Y_{12}. \tag{4.5}$$

According to (4.2), we obtain

$$x_1\alpha_1^j \equiv -y_1\alpha_2^j \ mod \ Y_{12}. \tag{4.6}$$

Substituting (4.6) into (4.5), we have

$$\alpha_2^j Y_{13} \equiv x_1 k \ mod \ Y_{12}. \tag{4.7}$$

By using similar method,

$$\alpha_2^j Y_{23} \equiv x_2 k \ mod \ Y_{12}. \tag{4.8}$$

Since M is a 1-prime matrix, there exist $f_{12}, g_{12} \in \mathbf{Z}$ such that $gcd(f_{12}Y_{13} + g_{12}Y_{23}, Y_{12}) = 1$. Combining (4.7) and (4.8), we have

$$\alpha_2^j(f_{12}Y_{13} + g_{12}Y_{23}) \equiv (f_{12}x_1 + g_{12}x_2)k \ \ mod \ \ Y_{12}.$$

Hence, $\alpha_2^j \equiv (f_{12}Y_{13} + g_{12}Y_{23})^{-1}(f_{12}x_1 + g_{12}x_2)k \ \ mod \ \ Y_{12}$.

Similarly, $\alpha_1^j \equiv -(f_{12}Y_{13} + g_{12}Y_{23})^{-1}(f_{12}y_1 + g_{12}y_2)k \ \ mod \ \ Y_{12}$. Hence,(4.1) is reduced to

$$\frac{1}{Y_{12}} \sum_{k=1}^{Y_{12}-1} \frac{W_{Y_{12}}^{(n_2(f_{12}x_1+g_{12}x_2)-n_1(f_{12}y_1+g_{12}y_2))(f_{12}Y_{13}+g_{12}Y_{23})^{-1}k}}{1 - W_{Y_{12}}^{-k}}. \tag{4.9}$$

According to discrete Fourier transforms,

$$-\{\frac{t}{a}\} = \frac{1-a}{2a} + \frac{1}{a}\sum_{k=1}^{a-1} \frac{W_a^{tk}}{1 - W_a^{-k}}, \tag{4.10}$$

(4.9) can be reduced to

$$\{\frac{(n_2(f_{12}x_1 + g_{12}x_2) - n_1(f_{12}y_1 + g_{12}y_2))(f_{12}Y_{13} + g_{12}Y_{23})^{-1}}{Y_{12}}\} + \frac{1}{2} - \frac{1}{2Y_{12}}.$$

Hence

$$\begin{aligned} & \frac{1}{Y_{12}} \sum_{\substack{\theta \in A(M) \\ M_\theta = Y_{12}}} \frac{\theta^n}{1 - \theta^{-(x_3, y_3)}} \\ = \quad & -\{\frac{(n_2(f_{12}x_1 + g_{12}x_2) - n_1(f_{12}y_1 + g_{12}y_2))(f_{12}Y_{13} + g_{12}Y_{23})^{-1}}{Y_{12}}\} + \frac{1}{2} - \frac{1}{2Y_{12}}. \end{aligned}$$

By using similar method, we have

$$\begin{aligned} & \frac{1}{Y_{13}} \sum_{\substack{\theta \in A(M) \setminus e \\ M_\theta = Y_{13}}} \frac{\theta^{\mathbf{n}}}{1 - \theta^{-(x_2, y_2)}} \\ = \quad & -\{\frac{(n_2(f_{13}x_1 + g_{13}x_3) - n_1(f_{13}y_1 + g_{13}y_3))(f_{13}Y_{12} + g_{13}Y_{23})^{-1}}{Y_{13}}\} + \frac{1}{2} - \frac{1}{2Y_{13}}. \end{aligned}$$

Hence, when $(n_1, n_2)^T \in v(\Omega_1|M) \cap \mathbf{Z}^2$,

$$\begin{aligned} & t(\mathbf{n}|M) \\ = \quad & \frac{n_2x_1 - n_1y_1}{Y_{12}Y_{13}} - \{\frac{(f_{12}Y_{13} + g_{12}Y_{23})^{-1}(n_2(f_{12}x_1 + g_{12}x_2) - n_1(f_{12}y_1 + g_{12}y_2))}{Y_{12}}\} \\ - \quad & \{\frac{(f_{13}Y_{12} + g_{13}Y_{23})^{-1}(n_2(f_{13}x_1 + g_{13}x_3) - n_1(f_{13}y_1 + g_{13}y_3))}{Y_{13}}\} + 1. \end{aligned}$$

Note that $\overline{\Omega}_1 \subset v(\Omega_1|M)$. Hence, when $\mathbf{n} \in \overline{\Omega}_1 \cap \mathbf{Z}^2$, the theorem holds. □

Remark 4.4. If $f_{12}, g_{12}, f_{13}, g_{13}, f_{23}$ and g_{23} satisfy $f_{12}Y_{23}+g_{12}Y_{13} = gcd(Y_{23}, Y_{13})$, $f_{13}Y_{12} + g_{13}Y_{23} = gcd(Y_{12}, Y_{23})$, $f_{23}Y_{13} + g_{23}Y_{12} = gcd(Y_{13}, Y_{12})$, then $gcd(f_{12}Y_{13} + g_{12}Y_{23}, Y_{12}) = 1$, $gcd(f_{13}Y_{12} + g_{13}Y_{23}, Y_{13}) = 1$ and $gcd(f_{23}Y_{13} + g_{23}Y_{12}, Y_{23}) = 1$. Hence,one can determine $f_{12}, g_{12}, f_{13}, g_{13}, f_{23}$ and g_{23} by Euclidean algorithm. But in some special cases, such as Y_{12}, Y_{13} and Y_{23} are pairwise relative prime, there exists a simpler method for obtaining them.

Corollary 4.5. *Suppose Y_{12}, Y_{13} and Y_{23} are pairwise relative prime. When $\mathbf{n} = (n_1, n_2)^T \in \overline{\Omega}_1 \cap \mathbf{Z}^2$,*

$$t(\mathbf{n}|M) = \frac{n_2x_1 - n_1y_1}{Y_{12}Y_{13}} - \{\frac{Y_{13}^{-1}(n_2x_1 - n_1y_1)}{Y_{12}}\} - \{\frac{Y_{12}^{-1}(n_2x_1 - n_1y_1)}{Y_{13}}\} + 1,$$

where $Y_{13}^{-1}Y_{13} \equiv 1 \mod Y_{12}$ and $Y_{12}^{-1}Y_{12} \equiv 1 \mod Y_{13}$. When $\mathbf{n} = (n_1, n_2)^T \in \overline{\Omega}_2 \cap \mathbf{Z}^2$,

$$t(\mathbf{n}|M) = \frac{n_1y_3 - n_2x_3}{Y_{23}Y_{13}} - \{\frac{Y_{13}^{-1}(n_1x_3 - n_2y_3)}{Y_{23}}\} - \{\frac{Y_{23}^{-1}(n_1x_3 - n_2y_3)}{Y_{13}}\} + 1,$$

where $Y_{13}^{-1}Y_{13} \equiv 1 \mod Y_{23}$ and $Y_{23}^{-1}Y_{23} \equiv 1 \mod Y_{13}$.

Proof. We firstly consider the case where $\mathbf{n} \in \overline{\Omega}_1 \cap \mathbf{Z}^2$. Since $gcd(Y_{12}, Y_{13}) = 1$, M is a 1-prime matrix. In Theorem 4.3, we may set $f_{12} = 1, g_{12} = 0, f_{13} = 1$, and $g_{13} = 0$. Hence, When $\mathbf{n} = (n_1, n_2)^T \in \overline{\Omega}_1 \cap \mathbf{Z}^2$,

$$t(\mathbf{n}|M) = \frac{n_2x_1 - n_1y_1}{Y_{12}Y_{13}} - \{\frac{Y_{13}^{-1}(n_2x_1 - n_1y_1)}{Y_{12}}\} - \{\frac{Y_{12}^{-1}(n_2x_1 - n_1y_1)}{Y_{13}}\} + 1.$$

Using similar method, when $\mathbf{n} = (n_1, n_2)^T \in \overline{\Omega}_2 \cap \mathbf{Z}^2$,

$$t(\mathbf{n}|M) = \frac{n_1y_3 - n_2x_3}{Y_{23}Y_{13}} - \{\frac{Y_{13}^{-1}(n_1x_3 - n_2y_3)}{Y_{23}}\} - \{\frac{Y_{23}^{-1}(n_1x_3 - n_2y_3)}{Y_{13}}\} + 1.$$

□

Remark 4.6. An interesting observation is that the formulation presented in Corollary 6.2 is remarkably similar with Popoviciu's formulation.

We now turn to consider the special case where $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. Without loss of generality, we suppose $M = \begin{pmatrix} kx_1 & lx_1 & x_3 \\ ky_1 & ly_1 & y_3 \end{pmatrix}$, where $k, l \in \mathbf{Z}$. In this case, there exists only one fundamental M-cone, which is denoted as Ω. Moreover, since M is a 1-prime matrix, we have $gcd(k, l) = 1, x_1y_3 - y_1x_3 = 1$. Then we have

Theorem 4.7. *Suppose $\frac{y_1}{x_1} < \frac{y_3}{x_3}$. When $M = \begin{pmatrix} kx_1 & lx_1 & x_3 \\ ky_1 & ly_1 & y_3 \end{pmatrix}$, $t(\mathbf{n}|M) = \frac{x_3n_2 - y_3n_1}{kl} - \{\frac{l^{-1}}{k}(n_1y_3 - n_2x_3)\} - \{\frac{k^{-1}}{l}(n_1y_3 - n_2x_3)\} + 1$, where $\mathbf{n} = (n_1, n_2)^T \in \overline{\Omega} \cap \mathbf{Z}^2, k^{-1}k \equiv 1 \mod l, l^{-1}l \equiv 1 \mod k$.*

Proof. By using the recurrence formulation for $T(\mathbf{x}|M)$, we have $T(\mathbf{x}|M) = \frac{x_3y - y_3x}{kl}$. Hence, the polynomial part of $t(\cdot|M)$ is $\frac{x_3y - y_3x}{kl} + \frac{1}{2}(\frac{1}{k} + \frac{1}{l})$. We now only need to consider the sums

$$\frac{1}{k} \sum_{\theta: M_\theta = Y_k} \frac{\theta^n}{1 - \theta^{-(lx_1, ly_1)}}, \frac{1}{l} \sum_{\theta: M_\theta = Y_l} \frac{\theta^n}{1 - \theta^{-(kx_1, ky_1)}}.$$

By the similar method with the one presented in the proof of Theorem 4.3, we have $\frac{1}{k} \sum_{\substack{\theta \in A(M)\setminus e \\ M_\theta = Y_k}} \frac{\theta^n}{1 - \theta^{-(lx_1, ly_1)}} = -\{l^{-1}\frac{n_1y_3 - n_2x_3}{k}\} + \frac{1}{2} - \frac{1}{2k}$, $\frac{1}{l} \sum_{\substack{\theta \in A(M) \\ M_\theta = Y_l}} \frac{\theta^n}{1 - \theta^{-(kx_1, ky_1)}} = -\{k^{-1}\frac{n_1y_3 - n_2x_3}{k}\} + \frac{1}{2} - \frac{1}{2l}$. Note that $\overline{\Omega} \subset v(\Omega|M)$. According to Theorem 3.4, when $\mathbf{n} = (n_1, n_2)^T \in \overline{\Omega}$,

$$\begin{aligned} & t(\mathbf{n}|M) \\ = \; & \frac{x_3 y - y_3 x}{kl} + \frac{1}{2}\left(\frac{1}{k} + \frac{1}{l}\right) + \frac{1}{k}\sum_{\theta: M_\theta = Y_k} \frac{\theta^n}{1 - \theta^{-(lx_1, ly_1)}} + \frac{1}{l}\sum_{\theta: M_\theta = Y_l} \frac{\theta^n}{1 - \theta^{-(kx_1, ky_1)}} \\ = \; & \frac{x_3 n_2 - y_3 n_1}{kl} - \left\{\frac{l^{-1}}{k}(n_1 y_3 - n_2 x_3)\right\} - \left\{\frac{k^{-1}}{l}(n_1 y_3 - n_2 x_3)\right\} + 1. \end{aligned}$$

□

REMARK 4.8. When the matrix M is of the form $\begin{pmatrix} x_1 & kx_2 & lx_2 \\ y_1 & ky_2 & ly_2 \end{pmatrix}$, a similar result can be obtained using the same method with the one presented in Theorem 4.7.

5. Linear Diophantine problem of Frobenius

Consider the linear Diophantine equation

$$x_1 a_1 + \cdots x_n a_n = N, \tag{5.1}$$

where, $a_i \in \mathbf{Z}_+, gcd(a_1, \cdots, a_n) = 1$.

It is well known that for all sufficiently large N the equation has solutions. The Frobenius problems asks us to find the largest integer for which no solution exists. We call the largest integer the Frobenius number and denote it by $f(a_1, \cdots, a_n)$. For $n = 2$ the largest N for which no solution exists can be explicitly written as $a_1 a_2 - a_1 - a_2$, i.e. $f(a_1, a_2) = a_1 a_2 - a_1 - a_2$. But,to our knowledge, no such formula exists for $n \geq 3$.

As pointed out in [**34**], when $gcd\{|Y| : Y \in \mathcal{B}(M)\} = 1$, for all sufficiently large N the linear Diophantine equations $Mx = N\mathbf{n}$ has solution, where $\mathbf{n} \in cone(M)$. Naturally, we hope to find the largest integer N for which no solution exist, which is denoted as $f(M, \mathbf{n})$. In particular, we are interested in the linear Diophantine equations $M_0 x = N\mathbf{n}$, where $M_0 = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \in \mathbf{Z}^{2\times 3}, \mathbf{n} \in cone(M_0)$. In fact, the generalized Frobenius number $f(M_0, \mathbf{n})$ is a generalization of $f(a_1, a_2)$.

Recall $M_{ij} = \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$ and $Y_{ij} = det(M_{ij})$. In the following theorem, we shall present an upper boundary for $f(M_0, \mathbf{n})$.

THEOREM 5.1. *Suppose Y_{12}, Y_{13} and Y_{23} are pairwise relative prime. For $\mathbf{n} \in \overline{\Omega}_1 \cap \mathbf{Z}^2$, $f(M_0, \mathbf{n}) < \frac{Y_{12}Y_{13} - Y_{12} - Y_{13} + 1}{n_2 x_1 - n_1 y_1}$. For $\mathbf{n} \in \overline{\Omega}_2 \cap \mathbf{Z}^2$, $f(M_0, \mathbf{n}) < \frac{Y_{23}Y_{13} - Y_{23} - Y_{13} + 1}{n_1 y_3 - n_2 x_3}$.*

PROOF. We only prove the case where $\mathbf{n} \in \overline{\Omega}_1 \cap \mathbf{Z}^2$. Note $t(N\mathbf{n}|M) = \frac{N(n_2 x_1 - n_1 y_1)}{Y_{12}Y_{13}} - \{\frac{(Y_{13})^{-1}(N(n_2 x_1 - n_1 y_1))}{Y_{12}}\} - \{\frac{(Y_{12})^{-1}(N(n_2 x_1 - n_1 y_1))}{Y_{13}}\} + 1 = t(N(n_2 x_1 - n_1 y_1)|(Y_{12}, Y_{13}))$. When $N(n_2 x_1 - n_1 y_1) \geq Y_{12}Y_{13} - Y_{12} - Y_{13} + 1$, $t(N(n_2 x_1 - n_1 y_1)|Y_{12}, Y_{13}) = t(N\mathbf{n}|M) > 0$. Hence, when $N \geq \frac{Y_{12}Y_{13} - Y_{12} - Y_{13} + 1}{(n_2 x_1 - n_1 y_1)}$, $t(N\mathbf{n}|M) > 0$. So, $f(M, \mathbf{n}) < \frac{Y_{12}Y_{13} - Y_{12} - Y_{13} + 1}{n_2 x_1 - n_1 y_1}$. □

REMARK 5.2. Theorem 5.1 only gives an upper boundary for $f(M_0, \mathbf{n})$. According to the proof of Theorem 5.1, giving the exact value of $f(M_0, \mathbf{n})$ is equivalent for any given $b_0 \in \mathbf{Z}$ determining the largest integer N for which the Diophantine equation $x_1 a_1 + x_2 a_2 = N b_0$ no solution exist.

6. Two-dimension vector partition functions

We now turn to the general case. We let $M = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ be a $2 \times n$ integer matrix and $\frac{y_{i-1}}{x_{i-1}} < \frac{y_i}{x_i}, i = 2, \cdots, n$.

For the matrix M, there exist $n-1$ fundamental M-cones. Denote them as $\Omega_i := \{(x,y)^T | (x,y)^T \in cone(M), \frac{y_i}{x_i} < \frac{y}{x} < \frac{y_{i+1}}{x_{i+1}}\}, i = 1, \cdots, n-1$ respectively. In this section, we shall discuss the explicit formulation for $t(\mathbf{b}|M)$. First, we present an explicit formulation for $T(\mathbf{x}|M)$.

Theorem 6.1. *For* $\mathbf{x} = (x,y)^T \in \mathbf{R}^2$,

$$T(\mathbf{x}|M) = \frac{1}{(n-2)!} \sum_{i=1}^{n} \frac{(y_i x - x_i y)_+^{n-2}}{\prod_{j \neq i}(y_i x_j - y_j x_i)},$$

where, $(y_i x - x_i y)_+ = \begin{cases} y_i x - x_i y, & y_i x - x_i y \geq 0, \\ 0, & otherwise. \end{cases}$

Proof. According to the definition of $(y_i x - x_i y)_+$ we only need to prove that when $\mathbf{x} \in \Omega_k$, $T(\mathbf{x}|M) = \frac{1}{(n-2)!} \sum_{i=k+1}^{n} \frac{(y_i x - x_i y)^{n-2}}{\prod_{j \neq i}(y_i x_j - y_j x_i)}$.

We argue by induction on n. Initially, when $n = 2, 3$ the theorem certainly holds. In the inductive step, we assume that when $n = n_0$ the theorem holds and we consider the case when $n = n_0 + 1$.

According to the definition of $(y_i x - x_i y)_+$ we only need to prove that for $\mathbf{x} \in \Omega_k, T(\mathbf{x}|M) = \frac{1}{(n_0-1)!} \sum_{i=k+1}^{n_0+1} \frac{(y_i x - x_i y)^{n_0-1}}{\prod_{j \neq i}(y_i x_j - y_j x_i)}$, where M is a $2 \times (n_0+1)$ matrix.

After a brief calculation, it is easy for obtaining $\mathbf{x} = \frac{x y_{k+1} - x_{k+1} y}{y_{k+1} x_k - y_k x_{k+1}} (x_k, y_k)^T + \frac{x y_k - x_k y}{y_{k+1} x_k - y_k x_{k+1}} (x_{k+1}, y_{k+1})^T$. Based on the recurrence formulation of $T(\cdot|M)$, we have

$$\begin{aligned} T(\mathbf{x}|M) = \frac{1}{n_0 - 1} (& \frac{x y_{k+1} - x_{k+1} y}{y_{k+1} x_k - y_k x_{k+1}} T(\mathbf{x}|M \setminus (x_k, y_k)^T) + \\ & \frac{x y_k - x_k y}{y_{k+1} x_k - y_k x_{k+1}} T(\mathbf{x}|M \setminus (x_{k+1}, y_{k+1})^T)). \end{aligned}$$

By the inductive hypothesis, $T(\mathbf{x}|M\setminus(x_k,y_k)^T)=\frac{1}{(n_0-2)!}\sum\limits_{i=k+1}^{n_0+1}\frac{(y_ix-x_iy)^{n_0-2}}{\prod\limits_{j\neq i,j\neq k}(y_ix_j-y_jx_i)}$, $T(\mathbf{x}|M\setminus(x_{k+1},y_{k+1})^T)=\frac{1}{(n_0-2)!}\sum\limits_{i=k+2}^{n_0+1}\frac{(y_ix-x_iy)^{n_0-2}}{\prod\limits_{j\neq i}(y_ix_j-y_jx_i)}$. Then we obtain

$$\begin{aligned}
T(\mathbf{x}|M) &= \frac{1}{(n_0-1)!}\Big(\frac{xy_{k+1}-x_{k+1}y}{y_{k+1}x_k-y_kx_{k+1}}\sum_{i=k+1}^{n_0+1}\frac{(y_ix-x_iy)^{n_0-2}(x_ky_i-y_kx_i)}{\prod\limits_{j\neq i}(y_ix_j-y_jx_i)}\\
&+ \frac{xy_k-x_ky}{y_{k+1}x_k-y_kx_{k+1}}\sum_{i=k+2}^{n_0+1}\frac{(y_ix-x_iy)^{n_0-2}(x_{k+1}y_i-y_{k+1}x_i)}{\prod\limits_{j\neq i}(y_ix_j-y_jx_i)}\Big)\\
&= \frac{1}{(n_0-1)!}\Big(\frac{(y_{k+1}x-x_{k+1}y)^{n_0-1}}{\prod\limits_{j\neq k+1}(y_{k+1}x_j-y_jx_{k+1})}+\frac{1}{y_{k+1}x_k-y_kx_{k+1}}\sum_{i=k+2}^{n_0+1}(y_ix-x_iy)^{n_0-2}\\
&\quad\Big(\frac{(xy_{k+1}-x_{k+1}y)(x_ky_i-y_kx_i)-(xy_k-x_ky)(x_{k+1}y_i-y_{k+1}x_i)}{\prod\limits_{j\neq i}(y_ix_j-y_jx_i)}\Big)\Big)\\
&= \frac{1}{(n_0-1)!}\sum_{i=k+1}^{n_0+1}\frac{(y_ix-x_iy)^{n_0-1}}{\prod\limits_{j\neq i}(y_ix_j-y_jx_i)}.
\end{aligned}$$

Thus, when $n=n_0+1$ the theorem holds also, which completes the inductive step and the proof. □

The following statements follow from Theorem 6.1.

Corollary 6.2.

$$D^{v_1,v_2}T(\mathbf{x}|M)=\frac{1}{(n-2-v_1-v_2)!}\sum_{i=1}^{n}\frac{(y_ix-x_iy)_+^{n-2-v_1-v_2}}{\prod\limits_{j\neq i}(y_ix_j-y_jx_i)}y_i^{v_1}(-x_i)^{v_2}.$$

We now turn to the non-polynomial part for $t(\cdot|M)$. We firstly recall the definition of a Fourier-Dedekind sum(see [**1**]), which is defined as $\sigma_t(C;n)=\frac{1}{n}\sum\limits_{\lambda^n=1\neq\lambda}\frac{\lambda^t}{\prod_{c\in C}(\lambda^c-1)}$, where C is an integer multiset and n is an integer. To simplify the non-polynomial part for $t(\cdot|M)$, we naturally arrived at the sums

$$\frac{1}{Y_{ij}}\sum_{\theta^{M_{ij}}=1,\theta\neq e}\theta^n\prod_{\omega\in M\setminus M_{ij}}\frac{1}{1-\theta^{-\omega}}, \tag{6.1}$$

which is considered as a generalized Fourier-Dedekind sum. Here, $\theta^{M_{ij}}=1$ means $\theta^m=1$ for any $m\in M_{ij}$. In fact, it is a non-trivial problem for computing all complex vectors satisfying $\theta^{M_{ij}}=1$. In the following Lemma, we shows the generalized Fourier-Dedekind sums (6.1) can be converted into the 1-dimensional Fourier-Dedekind sums.

Lemma 6.3. *When M is a 1-prime matrix, for any given integer m, $1\le m\le n, m\neq i,j$,*

$$\frac{1}{Y_{ij}}\sum_{\substack{\theta^{M_{ij}}=1\\ \theta\neq e}}\theta^n\prod_{\omega\in M\setminus M_{ij}}\frac{1}{1-\theta^{-\omega}}=\sigma_{t_{ij}}(C_{ij};Y_{ij}).$$

Here, $C_{ij} = \cup_{1\le h\le n, h\ne i, h\ne j}\{(fY_{im}+gY_{jm})^{-1}(-(fy_i+gy_j)x_h+(fx_i+gx_j)y_h)\}, t_{ij} = (fY_{im}+gY_{jm})^{-1}(-(fy_i+gy_j)n_1+(fx_i+gx_j)n_2)+\sum_{c\in C_{ij}} c$, *where* $f, g \in \mathbf{Z}$ *satisfy* $gcd(fY_{im}+gY_{jm}, Y_{ij}) = 1$, $(fY_{im}+gY_{jm})^{-1}(fY_{im}+gY_{jm}) \equiv 1 \mod Y_{ij}$.

PROOF. As pointed out in [**13**], the elements in the set $\{\theta|\theta \in A(M), M_\theta = M_{ij}\}$ have the form $(W_{Y_{ij}}^{\alpha_1^l}, W_{Y_{12}}^{\alpha_2^l})$, where $(\alpha_1^l, \alpha_2^l) \in \mathbf{Z}^2, 1 \le l \le Y_{ij}$.

Hence,

$$\frac{1}{Y_{ij}} \sum_{\substack{\theta^{M_{ij}}=1\\ \theta\ne e}} \theta^n \prod_{\omega\in M\setminus M_{ij}} \frac{1}{1-\theta^{-\omega}} = \frac{1}{Y_{ij}} \sum_{l=1}^{Y_{ij}-1} \frac{W_{Y_{ij}}^{n_1\alpha_1^l+n_2\alpha_2^l}}{\prod_{h\ne i, h\ne j}(1-W_{Y_{ij}}^{-(x_h\alpha_1^l+y_h\alpha_2^l)})}. \tag{6.2}$$

Noting $m \ne i, m \ne j$, we set $x_m\alpha_1^l + y_m\alpha_2^l \equiv k \ mod\ Y_{ij}$. Since M is a 1-prime matrix, k runs over $[1, Y_{ij}-1] \cap \mathbf{Z}$. Using the similar method with the one in the proof of Theorem 4.3, we have

$$\begin{aligned} \alpha_1^l &\equiv -(f_{ij}Y_{im}+g_{ij}Y_{jm})^{-1}(f_{ij}y_i+g_{ij}y_j)k \ mod\ Y_{ij}, \\ \alpha_2^l &\equiv (f_{ij}Y_{im}+g_{ij}Y_{jm})^{-1}(f_{ij}x_i+g_{ij}x_j)k \ mod\ Y_{ij}. \end{aligned}$$

Hence, (6.2) is reduced to

$$\begin{aligned} &\frac{1}{Y_{ij}} \sum_{k=1}^{Y_{ij}-1} \frac{W_{Y_{ij}}^{(n_2(f_{ij}x_i+g_{ij}x_j)-n_1(f_{ij}y_i+g_{12}y_j))(f_{ij}Y_{im}+g_{ij}Y_{jm})^{-1}k}}{\prod_{h\ne i,h\ne j}(1-W_{Y_{ij}}^{-(f_{ij}Y_{im}+g_{ij}Y_{jm})^{-1}(-x_h(f_{ij}y_i+g_{ij}y_j)+y_h(f_{ij}x_i+g_{ij}x_j))k})} \\ = \;& \sigma_{t_{ij}}(C_{ij}; Y_{ij}). \end{aligned}$$

□

REMARK 6.4. When $|det(M_{ij})| = 1$, the terms in $\sigma_{t_{ij}}(C_{ij} : Y_{ij})$ disappear,since $\{\theta : \theta^{M_{ij}} = 1\} = \{e\}$.

Combining Theorem 3.2, Theorem 3.4, Theorem 6.1 and Lemma 6.3, we can present a simplified formulation for $t(\cdot|M)$.

THEOREM 6.5. *Suppose* $M = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ *is a* $2\times n$ *integer 1-prime matrix and* $\frac{y_i}{x_i} < \frac{y_{i+1}}{x_{i+1}}$. *When* $\mathbf{n} = (n_1, n_2)^T \in \Omega_k \cap \mathbf{Z}^2$,

$$t(\mathbf{n}|M) = p_{e,\Omega_k}(\mathbf{n}) + \sum_{(i,j)\in\{(i,j):i\le k<j\}} \sigma_{t_{ij}}(C_{ij}; Y_{ij}),$$

where, $p_{e,\Omega_k}(\mathbf{x}) = \sum_{j=0}^{n-2} p_{j,\Omega_k}(\mathbf{x}), p_{0,\Omega_k}(\mathbf{x}) = \frac{1}{n-2}\sum_{l=k+1}^{n} \frac{(y_lx-x_ly)^{n-2}}{\prod_{j\ne l}(y_lx_j-y_jx_l)}, p_{j,\Omega_k}(\mathbf{x}) = -\sum_{l=0}^{j-1}(\sum_{|v|=j-l} D^v p_{l,\Omega_k}(\mathbf{x})(-i)^{|v|}\frac{D^v\widehat{B}(0|M)}{v!})$, t_{ij} *and* C_{ij} *are defined in Lemma 6.3.*

PROOF. Based on Theorem 3.4, when $\mathbf{n} \in \Omega_k \cap \mathbf{Z}^2$,

$$t(\mathbf{n}|M) = p_{e,\Omega_k}(\mathbf{n}) + \sum_{\theta\in A(M)\setminus e} \theta^{\mathbf{n}} \frac{1}{|det(M_\theta)|} \prod_{w\in M\setminus M_\theta} \frac{1}{1-\theta^{-w}} 1_{cone(M_\theta)}(\Omega_k),$$

where, the p_{e,Ω_k} can be determined easily. Since M is a 1-prime matrix,

$$\begin{aligned}&\sum_{\theta\in A(M)\setminus e}\theta^{\mathbf{n}}\frac{1}{|det(M_\theta)|}\prod_{w\in M\setminus M_\theta}\frac{1}{1-\theta^{-w}}1_{cone(M_\theta)}(\Omega_k)\\ =\ &\sum_{i<j}\frac{1}{Y_{ij}}\sum_{\substack{\theta^{M_{ij}}=1\\ \theta\neq e}}\theta^{\mathbf{n}}\prod_{\omega\in M\setminus M_{ij}}\frac{1}{1-\theta^{-\omega}}1_{cone(M_{ij})}(\Omega_k).\end{aligned}$$

Based on Lemma 6.3, the above sum becomes as follows:

$$\sum_{i<j}\sigma_{t_{ij}}(C_{ij}:Y_{ij})1_{cone(M_{ij})}(\Omega_k). \tag{6.3}$$

Since when $k\geq j$ or $k<i$, $cone(M_{ij})\cap\Omega_k=\emptyset$. Hence, (6.3) is converted into

$$\sum_{(i,j)\in\{(i,j):i\leq k<j\}}\sigma_{t_{ij}}(C_{ij}:Y_{ij}). \tag{6.4}$$

The theorem holds. □

Using Theorem 6.5, we shall present an explicit formulation for an actual example, which is the same with the one presented in [**3**]. By using Theorem 6.5, it is indeed easier for obtaining the explicit formulation for the actual vector partition function.

EXAMPLE 6.6. Let $A=\begin{pmatrix}1&2&1&0\\0&1&1&1\end{pmatrix}$. We denote by A_{ij} the square matrix containing the ith and the jth columns in A.

For the matrix A, there exist three fundamental cones, which are denoted as Ω_1,Ω_2 and Ω_3 respectively. We shall discuss the explicit formulation for $t(\mathbf{n}|A)$. After a brief calculation, we have

$$T(\mathbf{x}|A)=\begin{cases}\frac{y^2}{2}, & \mathbf{x}\in\Omega_1,\\ \frac{1}{4}(-x^2+4xy-2y^2), & \mathbf{x}\in\Omega_2,\\ \frac{x^2}{4}, & \mathbf{x}\in\Omega_3.\end{cases}$$

Hence, $p_{0,\Omega_1}=\frac{y^2}{2}$. According to Theorem 6.1, $p_{1,\Omega_1}=3/2y$ and $p_{2,\Omega_1}=1$ respectively. Since for any $1<j\leq 3, |det(A_{1j})|=1$, accord to Remark 6.4,the terms in the Fourier-Dedekind sum shall not appear when $\mathbf{n}\in\Omega_1\cap\mathbf{Z}^2$. Based on Theorem 6.1, we have when $\mathbf{n}\in\Omega_1\cap\mathbf{Z}^2, t(\mathbf{n}|\mathbf{A})=\frac{n_2^2}{2}+\frac{3n_2}{2}+1$.

Similarly, $p_{0,\Omega_2}=\frac{1}{4}(-x^2+4xy-2y^2), p_{1,\Omega_2}=\frac{x+y}{2}, p_{2,\Omega_2}=\frac{7}{8}$. Based on Lemma 6.3, the non-polynomial part is $\frac{1}{Y_{23}}\sum_{\theta^{A_{23}}=1,\theta\neq e}\theta^n\prod_{\omega\in A\setminus A_{ij}}\frac{1}{1-\theta^{-\omega}}=(-1)^{n_1}$. Hence, when $\mathbf{n}\in\Omega_2\cap\mathbf{Z}^2, t(\mathbf{n}|A)=n_1n_2-\frac{n_1^2}{4}-\frac{n_2^2}{2}+\frac{n_1+n_2}{2}+\frac{7}{8}+\frac{(-1)^{n_1}}{8}$.

Using the same method with the above, we obtain $p_{0,\Omega_3}=\frac{x^2}{4}, p_{1,\Omega_3}=x, p_{2,\Omega_3}=\frac{7}{8}$.

Hence, $t(\mathbf{n}|A)=\begin{cases}\frac{n_2^2}{2}+\frac{3n_2}{2}+1, & \mathbf{n}\in\Omega_1\cap\mathbf{Z}^2\\ n_1n_2-\frac{n_1^2}{4}-\frac{n_2^2}{2}+\frac{n_1+n_2}{2}+\frac{7}{8}+\frac{(-1)^{n_1}}{8}, & \mathbf{n}\in\Omega_2\cap\mathbf{Z}^2\\ \frac{n_1^2}{4}+n_1+\frac{7}{8}+\frac{(-1)^{n_1}}{8}, & \mathbf{n}\in\Omega_3\cap\mathbf{Z}^2.\end{cases}$

Remark 6.7. The explicit formulation presented in Theorem 6.5 contains $D^v\widehat{B}(0|M)$. Note

$$\widehat{B}(\zeta|M)=\prod_{j=1}^{n}\frac{1-exp(-i\zeta^T m_j)}{i\zeta^T m_j},\zeta\in \mathbf{C}^s.$$

The following assertion is obvious:

$$D^{v_1,v_2}\widehat{B}(0|M)=(-i)^{v_1+v_2}\sum_{k_1+\cdots+k_n=v_1}\sum_{l_1+\cdots+l_n=v_2}\frac{v_1!}{k_1!\cdots k_n!}\frac{v_2!}{l_1!\cdots l_n!}\prod_{j=1}^{n}\frac{x_j^{k_j}y_j^{l_j}}{k_j+l_j+1}.$$

Remark 6.8. In Theorem 6.5, when the case of $\frac{y_i}{x_i}=\frac{y_j}{x_j}$ happens, the explicit formulation for $T(\mathbf{x}|M)$ can be obtained by taking the limit. Using similar method with the one in the proof of Theorem 4.7, an explicit formulation for $t(\mathbf{n}|M)$ can be given also.

Remark 6.9. To simplify any-dimensional vector partition functions, we have to give an explicit formulation for multivariate truncated power functions $T(\mathbf{x}|M)$ and compute the chamber complex consisting of the fundamental M-cones, which are indeed challenging problems.

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