Using saddlepoint approximations to estimate the maximum value of box splines

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Three research topics

- Mahler conjecture

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- Mahler conjecture (discrete geometry)

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- box splines

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- box splines (approximation theory)

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there being equality when K has $2n$ facets (parallelepiped).

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It is well known that any centrally symmetric polytope with $2n+2$ facets is a section of an $(n+1)$-dimensional cube.

A special case of Mahler's conjecture

K. Ball (1995):

$$\mathrm{vol}(H \cap [-1/2, 1/2]^n) \cdot \frac{\sum_{\epsilon \in \{1,-1\}^n} |\sum_i \epsilon_i a_i|}{2^n} \geq 1,$$

for all unit vectors a and $H = \langle a \rangle^\perp$.

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Lopez and S. Reisner proved the result holds when $n \leq 8$ (1998).

A special case of Mahler's conjecture

Using the language of box splines, we can say

$$\max_x B(x|(a_1,\ldots,a_n)) \geq \frac{2^n}{\sum_{\epsilon\in\{1,-1\}^n} |\sum_i \epsilon_i a_i|}.$$

Box splines

Box splines $B(\cdot|A)$ associated with A are the distribution given by the rule

$$\int_{\mathbb{R}^s} B(x|A)\phi(x)dx = \int_{[0,1)^n} \phi(Au)du, \ \phi \in \mathscr{D}(\mathbb{R}^s).$$

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C. de Boor, K. Höllig and S. Riemenschneider, Box Splines, Springer-Verlag, New York, 1993.

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Set $A := (a_1, \ldots, a_n)$ with positive entries and $\sum_j a_j^2 = 1$. We have

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$$\max_x B(x|A) = B((a_1 + \cdots + a_n)/2|A).$$

The maximum value of box splines

Set $C_A := \max_x B(x|A)$.

Lemma 1: For any non-negative and increasing functions $f(x)$ on $[0,\infty)$, one has

$$C_A \cdot \int_0^{\frac{1}{2C_A}} f(x)dx \;\le\; \int_0^{\infty} f(x)B(x|A)dx.$$

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$$\frac{1}{24 \cdot C_A} \ \leq \ \frac{1}{24} \Longrightarrow C_A \geq 1.$$

The proof of Lemma 1

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\geq \; & f(\frac{1}{2C_A}) \int_0^{\frac{1}{2C_A}} (B(x|A) - C_A)dx + f(\frac{1}{2C_A}) \int_{\frac{1}{2C_A}}^{\infty} B(x|A)dx
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= \; & f(\frac{1}{2C_A}) \left(\int_0^{\infty} B(x|A)dx - \int_0^{\frac{1}{2C_A}} C_A dx \right) = 0.
\end{aligned}
$$

The maximum value of box splines

$$C_A \geq \frac{2^n}{\sum_{\epsilon \in \{1,-1\}^n} \left| \sum_i \epsilon_i a_i \right|}.$$

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The saddlepoint approximation to $g(x)$ is given as

$$\frac{1}{\sqrt{2\pi K''(\hat{s})}} \exp(K(\hat{s}) - \hat{s}x),$$

where $K'(\hat{s}) = x$.

Saddlepoint approximations of box splines

Set

$$A := (a_1, \ldots, a_n), \quad \text{where } \sum_j a_j^2 = 1.$$

$$B(x|A) \simeq \sqrt{\frac{6}{\pi}} \exp(-6(x - \sum_j a_j/2)^2).$$

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U. Haagerup (1992): There is a positive integer N_0 so that

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Condition: $\max A / \min A$ is bounded.

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The limit of the maximum value

Suppose that A is an $s \times n$ matrix and $\det(AA^T) = 1$. We have

$$\max_x B(x|A) \simeq \left(\frac{6}{\pi}\right)^{s/2} .$$

Thank you!