The Instability Degree In The Dimension of Spaces of Bivariate Spline *

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Abstract

In this paper, the dimension of the spaces of bivariate spline with degree less than $2r$ and smoothness order r on the Morgan-Scott triangulation is considered. The concept of the instability degree in the dimension of spaces of bivariate spline is presented. The results in the paper make us conjecture the instability degree in the dimension of spaces of bivariate spline is infinity.

Key Words: Bivariate splines, Morgan-Scott triangulation, Dimension

1. Introduction

Let Δ be a regular triangulation of a region $\Omega \subset \Re^2$. Suppose the triangles of Δ are denoted by $T_1, \cdots, T_N$. The space $S_d^r(\Delta)$ of bivariate splines with smoothness order r and degree d on Δ is defined as

$$S_d^r(\Delta) = \{s \in C^r(\Omega) : s|_{T_i} \in P_d, i = 1, \cdots, N\},$$

where P_d is the $((d+2)(d+1))/2$ dimensional space of bivariate polynomials of total degree d. In recent years there has been considerable interest in identifying the dimensions of such spaces. The dimensions of these spaces are known if d is large relative to r. Alfeld and Schumaker [2] gave dimension formulate for the case $d \geq 4r+1$. This was extended to $d \geq 3r+2$ by Hong [8]. But if d is small in relation to r, then the dimension of $S_d^r(\Delta)$ may depend on complicated geometric conditions. Morgan and Scott gave an example of a triangulation Δ_{ms} on which the dimension of $S_2^1(\Delta_{ms})$ depends on such geometric conditions [10]. The triangulation

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Δ_{ms} is shown in Fig.1. This problem has caused much interest of many experts and scholars in bivariate splines[5,6,7,11,14]. In [6], Diener showed that the dimension of $S^r_{2r}(\Delta_{ms})$ depended on the same type of geometric condition for all values of $r \geq 1$ and the dimension of $S^r_d(\Delta_{ms}), d > 2r$ was stable. But there is not any result about the dimension of $S^r_d(\Delta_{ms}), d \leq 2r-1$. The purpose of this paper is to discuss the dimension of $S^r_d(\Delta_{ms}), d \leq 2r-1$. Both upper and lower bounds on the dimension of spline spaces on general triangulation were given in Schumaker [12]. But the upper bound for the dimension of the space $S^r_d(\Delta_{ms}), d \leq 2r-1$ exceeds the real value greatly. In this paper we give an upper bound of $dimS^r_d(\Delta_{ms})$, for $d \leq 2r-1$. Many numerical results show the upper bound agrees with $dimS^r_d(\Delta^0_{ms})$, where Δ^0_{ms} is a symmetric Morgan-Scott triangulation. Specially, for $\frac{5}{3}r < d \leq \frac{5r+2}{3}$ we give a sufficient and necessary condition for the dimension of $S^r_d(\Delta_{ms})$ being equal to the upper bound. Moreover, we prove $dimS^r_d(\Delta_{ms}) = (d+1)(d+2)/2, d \leq \frac{5}{3}r$.

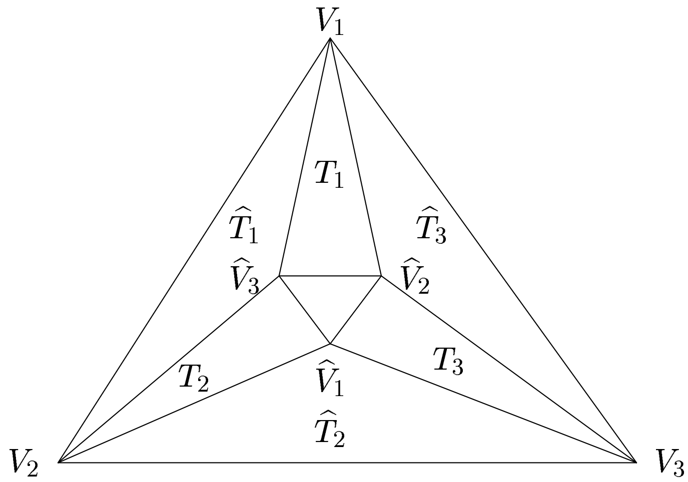

Fig.1. The Morgan-Scott Triangulation Δ_{ms}

The paper is organized as follows. In Section 2 we present two theorems that are the main results of the paper. Section 3 contains a brief statement of the relevant definitions. Section 4 contains the proofs of Theorem 1 and 2. In Section 5, we present the definition of instability degree and make a conjecture about the sharpness of our lower-bound and upper-bound formulas on the dimension for spline spaces over Morgan-Scott triangulation based on some numerical results. We conclude with some final remarks.

2. Main Results

Let Δ_{ms} be the triangulation in Fig.1 with boundary vertices of V_1, V_2 and V_3 and interior vertices of $\widehat{V}_1, \widehat{V}_2$ and $\widehat{V}_3$, and with index arithmetic modulo 3. Assume that no pair of edges that meet at an interior vertex of Δ_{ms} are collinear. Let $T_1, T_2, T_3, \widehat{T}_1, \widehat{T}_2$ and $\widehat{T}_3$ be the triangles of Δ_{ms} with index arithmetic modulo 3. Let a_{ij}, b_{ij} and c_{ij} be the barycentric coordinates of V_j with respect to the triangle T_i,where

$$V_j = a_{i,j}V_i + b_{i,j}\widehat{V}_{i+1} + c_{i,j}\widehat{V}_{i-1}.$$

Using the dimension formulation in a cell [11,13], we have $dimS_d^r(\Delta_{ms}) = (d+2)(d+1)/2, d \leq r + [(r+1)/3]$. Throughout the paper, $[x]$ denotes the integer part of x. Hence, we only consider the case $d > r + [(r+1)/3]$. According to Schumaker's formulation [11], the lower bound for the dimension of the space $S_d^r(\Delta_{ms})$ is $\alpha + max(9\beta - 3\gamma + 3\sigma, 0)$, where $\alpha = (d+2)(d+1)/2, \beta = (d-r)(d-r+1)/2, \gamma = ((d+1)(d+2) - (r+1)(r+2))/2, \sigma = \sum_{j=1}^{d-r}(r+1-3j)_+$, where $(\cdot)_+ = max(\cdot, 0)$. We give a sharp upper bound for the dimension of the space $S_d^r(\Delta_{ms}), \frac{5}{3}r < d \leq 2r-1$ in the following theorems.

Theorem 1 *For all* $r \geq 1, \frac{5}{3}r < d \leq 2r-1, \alpha + max(9\beta - 3\gamma + 3\sigma, 0) \leq dimS_d^r(\Delta_{ms}) \leq \alpha + \sum_{1 \leq n \leq (d-r)/2}(3n-r-1)_+ + \sum_{(d-r)/2 < n \leq 2d-3r-1}(2d-3r-n)_+ + 1$. *Moreover,* $dimS_d^r(\Delta_{ms}) \leq \alpha + \sum_{1 \leq n \leq (d-r)/2}(3n-r-1)_+ + \sum_{(d-r)/2 < n \leq 2d-3r-1}(2d-3r-n)_+$ *unless the* a_{ij}, b_{ij} *and* c_{ij} *satisfy*

$$(a_{21}a_{32}a_{13})^{2d-3r}(c_{21}c_{32}c_{13})^{2r-d} = (a_{12}a_{23}a_{31})^{2d-3r}(b_{12}b_{23}b_{31})^{2r-d}. \qquad (1)$$

Theorem 2 *For all* $r \geq 1, d \leq \frac{5}{3}r, dimS_d^r(\Delta_{ms}) = (d+2)(d+1)/2$. *For* $\frac{5}{3}r < d \leq \frac{5r+2}{3}, (d+2)(d+1)/2 \leq dimS_d^r(\Delta_{ms}) \leq (d+2)(d+1)/2+1$. *Moreover,* $dimS_d^r(\Delta_{ms}) = (d+2)(d+1)/2+1$, *if and only if*

$$(a_{21}a_{32}a_{13})^{2d-3r}(c_{21}c_{32}c_{13})^{2r-d} = (a_{12}a_{23}a_{31})^{2d-3r}(b_{12}b_{23}b_{31})^{2r-d},$$

where $\frac{5}{3}r < d \le \frac{5r+2}{3}$.

For $d \le 2r-1$, the geometric characterization of (1) is equivalent to the condition that the three lines $V_1\widehat{V}_1, V_2\widehat{V}_2$ and $V_3\widehat{V}_3$ have a common point of intersection.

3. Preliminaries and Tools

To establish an upper bound on $dim S_d^r(\Delta_{ms})$, we shall use the Bernstein-Bézier representation of bivariate spline. Recall some notations(see [2]). Suppose that Δ is an arbitrary triangulation and that the triangles of Δ are denoted by $T_1, T_2, \cdots, T_N$. Each spline $s \in S_d^0(\Delta)$ can be written in the form

$$s(x,y) = s_l(x,y) \quad for(x,y) \in T_l, l = 1, \cdots, N,$$

where s_l is a polynomial of degree d, $l = 1, \cdots, N$. The Bernstein-Bézier representation of the polynomial s_l is

$$s_l(a,b,c) = \sum_{i+j+k=d} \beta_{i,j,k}^{[l]} \frac{d!}{i!j!k!} a^i b^j c^k,$$

where a, b and c are barycentric coordinates of a point (x,y) in the triangle T_l. The coefficient $\beta_{i,j,k}^{[l]}$ are called *Bézier ordinates.* Each Bézier ordinates $\beta_{i,j,k}^{[l]}$ is associated with a *domain point*

$$P_{i,j,k}^{[l]} = (iV_1 + jV_2 + kV_3)/d,$$

where V_1, V_2, and V_3 are the vertices of the triangle T_l. The set of all domain points is denoted by $B_d(\Delta)$. For $t = P_{i,j,k}^{[l]} \in B_d(\Delta)$, let λ_t denote the linear functional on $S_d^r(\Delta)$ defined by $\lambda_t s = \beta_{i,j,k}^{[l]}$. A set of domain points $P \subset B_d(\Delta)$ is called a *determining set* for $S_d^r(\Delta)$ if

$$\forall s \in S_d^r(\Delta) : \lambda_t s = 0, \forall t \in P \Longrightarrow s = 0.$$

If P is a determining set for $S_d^r(\Delta)$, then $dim S_d^r(\Delta) \le \#P$, where $\#P$ denotes the cardinality of the set P.

Suppose that V is a vertex of Δ. We say the domain point $P_{i,j,k} = (iV + jV_2 + kV_3)/d$ is of *distance* $(d-i)$ from vertex V. The *ring of order* m around the vertex

V is

$$R_m(V) = \{\text{points which are distance } m \text{ from } v\},$$

and the *m-disk* around V is

$$D_m(V) = \cup_{j=0}^{m} R_j(V).$$

A set $P \subset D_m(V)$ is said to determine the disk $D_m(V)$ if

$$\lambda_t s = 0 \forall t \in P \Longrightarrow \lambda_t s = 0, \forall t \in D_m(V).$$

Now we deal specifically with the Morgan-Scott triangulation Δ_{ms}. Recall some notation (cf. [6]). Let $T_1, T_2, T_3, \widehat{T}_1, \widehat{T}_2$, and $\widehat{T}_3$ be the triangles of Δ_{ms}, with index arithmetic modulo 3. Let

$$P_{i,j,k}^{[l]} = (iV_l + j\widehat{V}_{l+1} + k\widehat{V}_{l-1})/d, i+j+k = d$$

be a domain point in the triangle T_l and let $\beta_{i,j,k}^{[l]}$ be the corresponding Bézier ordinates. At last, let $a_{i,j}, b_{i,j}$ and $c_{i,j}$ be the barycentric coordinates of V_j with respect to the triangle T_i, where

$$V_j = a_{i,j} V_i + b_{i,j} \widehat{V}_{i+1} + c_{i,j} \widehat{V}_{i-1}.$$

If the ordinates corresponding to domain points in the triangle T are zero, then the smoothness conditions across the edges of T are simply $\beta_{i,j,k}^{[l]} = 0, for \quad i \leq r$. According to the formulation in [4,3], the remaining smoothness conditions can be written as

$$\begin{aligned} \widehat{\beta}_{i,j,k}^{[l]} &= \sum_{\lambda+\mu+v=j, \lambda > r-k} \frac{j!}{\lambda!\mu!v!} \beta_{k+\lambda,\mu,i+v}^{[l]} a_{l,l+1}^{\lambda} b_{l,l+1}^{\mu} c_{l,l+1}^{v}, \text{for} \quad j = 1, \cdots, r, \\ \widehat{\beta}_{i,j,k}^{[l]} &= \sum_{\lambda+\mu+v=k, \lambda > r-j} \frac{k!}{\lambda!\mu!v!} \beta_{j+\lambda,\mu,i+v}^{[l+1]} a_{l+1,l}^{\lambda} b_{l+1,l}^{\mu} c_{l+1,l}^{v}, \text{for} \quad k = 1, \cdots, r. \end{aligned} \quad (2)$$

When $1 \leq j, k \leq r$, $\widehat{\beta}_{i,j,k}^{[l]}$ is determined by both equations. Subtracting the two expressions for $\widehat{\beta}_{i,j,k}^{[l]}$ results in the homogeneous system of equations

$$\sum_{\lambda+\mu+v=k, \lambda > r-j} \frac{k!}{\lambda!\mu!v!} \beta_{j+\lambda,i+\mu,v}^{[l+1]} a_{l+1,l}^{\lambda} b_{l+1,l}^{\mu} c_{l+1,l}^{v}$$

$$-\sum_{\lambda+\mu+v=j,\lambda>r-k}\frac{j!}{\lambda!\mu!v!}\beta^{[l]}_{k+\lambda,\mu,i+v}a^{\lambda}_{l,l+1}b^{\mu}_{l,l+1}c^{v}_{l,l+1}=0, \tag{3}$$
$$1\leq j,k\leq r, 0\leq i<r, i+j+k=d, l=1,2,3$$

Since $1 \leq j,k \leq r$, each ordinate $\widehat{\beta}^{[l]}_{i,j,k}$ appears in (2). The ordinates are determined consistently precisely when $\{\beta^{[l]}_{\lambda,\mu,v} : \lambda > r\}$ is a solution to (3). Hence, $dimS^r_d(\Delta_{ms}) = \alpha + \tau$, where τ is the dimension of the space of solutions to (3).

4. Proof of Main Results

To state the proof we need some more notations. Let

$$Lb(S^r_d(\Delta_{ms})) = \begin{cases} \alpha+max(9\beta-3\gamma+3\sigma,0), & d\geq\frac{5}{3}r \\ \alpha, & else \end{cases},$$

$$Ub(S^r_d(\Delta_{ms})) = \begin{cases} \alpha+\sum_{1\leq n\leq(d-r)/2}(3n-r-1)_+ +\sum_{(d-r)/2<n\leq 2d-3r-1}(2d-3r-n)_+ +1, & d\geq\frac{5}{3}r \\ \alpha, & else \end{cases}.$$

When (1) is not satisfied, we will build a determining set with the cardinality of $Ub(S^r_d(\Delta_{ms}))-1$ for $S^r_d(\Delta_{ms})$. Otherwise, a determining set with the cardinality of $Ub(S^r_d(\Delta_{ms}))$ is presented. Hence, Theorem 1,2 will be obtained.

Let P be the set consisting of the following domain points.

(1) All domain points in the triangle T.

(2)The domain points

$$P^{[l]}_{r+m,n-m,d-n-r}, \qquad 1\leq n\leq\frac{d-r}{2}, 1\leq m\leq(3n-r-1)_+, l=1,2,3.$$

(3)The domain points

$$P^{[l]}_{r+n-j,j,d-r-n}, \frac{d-r}{2}<n\leq 2d-3r-1, 2r+1-d\leq j\leq d-r-n, l=1,2,3.$$

Lemma 1 *For $\frac{5}{3}r<d<2r$, $\frac{d-r}{2}<n\leq 2d-3r-1$,suppose that the ordinates with domain points in the triangle T and in the $(r+n-1)$-disks about the interior vertices of Δ_{ms} are zero. If for $l=1,2,3$ the ordinates $\beta^{[l]}_{r+n-j,j,d-r-n}$, $2r+1-d\leq j\leq d-r-n$ vanish, then all ordinates with domain points in the $(r+n)-$disk about the interior vertices of Δ_{ms} are zero.*

proof. We remove the terms with ordinates that are assumed to vanish. The set of equations in (3) restricted to the $(r+n)-$disk about the vertex $\widehat{V}_{l-1}$ is reduced to

$$\sum_{v=0}^{d-r-n-1}\binom{k}{v}\beta^{[l+1]}_{r+n-v,d-r-n,v}a^{k-v}_{l+1,l}c^{v}_{l+1,l}-\sum_{\mu=0}^{2r-d}\binom{r+n-k}{\mu}\beta^{[l]}_{r+n-\mu,\mu,d-r-n}a^{r+n-k-\mu}_{l,l+1}b^{\mu}_{l,l+1}=0 \tag{4}$$

$$n\le k\le r.$$

This is a system of $r-n+1$ equations in $r-n+1$ variables. We can rewrite the system in the form $A\beta=0$, where $A=(A_1,-B_1)$

$$A_1=\begin{pmatrix}\binom{n}{0}a^{n}_{l+1,l} & \binom{n}{1}a^{n-1}_{l+1,l}c_{l+1,l} & \cdots & \binom{n}{d-r-n-1}a^{n-(d-r-n-1)}_{l+1,l}c^{d-r-n-1}_{l+1,l}\\ \binom{n+1}{0}a^{n+1}_{l+1,l} & \binom{n+1}{1}a^{n}_{l+1,l}c_{l+1,l} & \cdots & \binom{n+1}{d-r-n-1}a^{n+1-(d-r-n-1)}_{l+1,l}c^{d-r-n-1}_{l+1,l}\\ \vdots & \vdots & \cdots & \vdots\\ \binom{r}{0}a^{r}_{l+1,l} & \binom{r}{1}a^{r-1}_{l+1,l}c_{l+1,l} & \cdots & \binom{r}{d-r-n-1}a^{r-(d-r-n-1)}_{l+1,l}c^{d-r-n-1}_{l+1,l}\end{pmatrix},$$

$$B_1=\begin{pmatrix}\binom{r}{0}a^{r}_{l,l+1} & \binom{r}{1}a^{r-1}_{l,l+1}b_{l,l+1} & \cdots & \binom{r}{2r-d}a^{d-r}_{l,l+1}b^{2r-d}_{l,l+1}\\ \binom{r-1}{0}a^{r-1}_{l,l+1} & \binom{r-1}{1}a^{r-2}_{l,l+1}b_{l,l+1} & \cdots & \binom{r-1}{2r-d}a^{d-r-1}_{l,l+1}b^{2r-d}_{l,l+1}\\ \vdots & \vdots & \cdots & \vdots\\ \binom{n}{0}a^{n}_{l,l+1} & \binom{n}{1}a^{n-1}_{l,l+1}b_{l,l+1} & \cdots & \binom{n}{2r-d}a^{n-2r+d}_{l,l+1}b^{2r-d}_{l,l+1}\end{pmatrix},$$

$$\beta=(\beta^{[l+1]}_{r+n,d-r-n,0},\beta^{[l+1]}_{r+n-1,d-r-n,1},\cdots,\beta^{[l+1]}_{2(r+n)-d+1,d-r-n,d-r-n-1},$$
$$\beta^{[l]}_{r+n,0,d-r-n},\beta^{[l]}_{r+n-1,1,d-r-n},\cdots,\beta^{[l]}_{d+n,2r-d,d-r-n})^T.$$

To prove Lemma 1 it is sufficient to show that the matrix A is nonsingular.

After a series of primary transformation , the matrix A can become:

$$\widehat{A}_1=\begin{pmatrix}\binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{d-r-n-1} & \binom{r}{0}a^{r-n} & \binom{r}{1}a^{r-n} & \cdots & \binom{r}{2r-d}a^{r-n}\\ \binom{n+1}{0} & \binom{n+1}{1} & \cdots & \binom{n+1}{d-r-n-1} & \binom{r-1}{0}a^{r-n-1} & \binom{r-1}{1}a^{r-n-1} & \cdots & \binom{r-1}{2r-d}a^{r-n-1}\\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ \binom{r}{0} & \binom{r}{1} & \cdots & \binom{r}{d-r-n-1} & \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{2r-d}\end{pmatrix},$$

where $a=a_{l+1,l}a_{l,l+1}$. According to the formulation $\binom{n+1}{k}-\binom{n}{k}=\binom{n-1}{k-1}$, matrix $\widehat{A}_1$ can become:

$$\widehat{A}_2=\begin{pmatrix}D & B\\ 0 & C\end{pmatrix},$$

where D is a diagonal matrix with 1 on the diagonal, $C=(c_{gh}), c_{gh}=\frac{a^h((1-a)^{d-r-n}a^{2r-d+n-g})^{(h)}}{a^n h!}$, $0\le g\le 2r-d, 0\le h\le 2r-d$. To see that C is nonsingular, consider the matrix M associated with the problem of finding a univariate polynomial $p=$

$\sum_{i=0}^{2r-d+n} c_i(1-x)^{d-r-n}x^i + \sum_{j=0}^{d-r-n-1} c_{2r-d+n+j}(1-x)^j$ with prescribed values of

$D^v p(a) = 0, v = 0, \cdots, 2r-d, D^v p(0) = 0, v = 0, \cdots, n-1, D^v p(1) = 0, v = 0, \cdots, d-r-n-1.$

The matrix M has the form

$$M = \begin{pmatrix} G_1 & \widehat{C} & G_2 \\ H_1 & 0 & G_3 \\ 0 & 0 & H_2 \end{pmatrix},$$

where H_1, H_2 is a nonsingular matrix. Since the associated interpolation is unisolvent, M is nonsingular also. Hence, $\widehat{C}$ is nonsingular. The matrix C can be obtained from $\widehat{C}$ by multiplying nonzero constant and transposing the resulting matrix. Therefor, C is nonsingular. The Lemma is obtained.$\square$

Lemma 2 *For $\frac{5}{3}r < d < 2r, 2d-3r+1 \leq n \leq d-r$, if the ordinates with domain points in the triangle T and in the $(r+n-1)-$disks about the interior vertices of Δ_{ms} are zero, then all ordinates with domain points in the $(r+n)-$disk about the interior vertices of Δ_{ms} are zero.*

proof. After the terms with ordinates that are assumed to vanish are removed, the set of equations in (3) restricted to the $(r+n)-$disk about the vertex $\widehat{V}_{l-1}$ is reduced to

$$\sum_{v=0}^{d-r-n} \binom{k}{v} \beta^{[l+1]}_{r+n-v,d-r-n,v} a^{k-v}_{l+1,l} c^{v}_{l+1,l} - \sum_{\mu=0}^{d-r-n} \binom{r+n-k}{\mu} \beta_{r+n-\mu,\mu,d-r-n} a^{r+n-k-\mu}_{l,l+1} b^{\mu}_{l,l+1} = 0,$$
$$n \leq k \leq r.$$

This is a system of $r-n+1$ equations and $2(d-r-n+1)$ variables. Obviously, when $2d-3r+1 \leq n \leq d-r$, we have $2(d-r-n+1) \leq r-n+1$. For $2d-3r+1 \leq n \leq d-r$, we only consider the first $2(d-r-n+1)$ equations in the system,i.e. $n \leq k \leq 2d-2r-n+1$. The system can be written as $\widetilde{A}\widetilde{\beta} = 0$, where

$$\widetilde{\beta} = (\beta^{[l+1]}_{r+n,d-r-n,0}, \beta^{[l+1]}_{r+n-1,d-r-n,1}, \cdots, \beta^{[l+1]}_{2(r+n)-d,d-r-n,d-r-n},$$
$$\beta^{[l]}_{r+n,0,d-r-n}, \beta^{[l]}_{r+n-1,1,d-r-n}, \cdots, \beta^{[l]}_{2(r+n)-d,d-r-n,d-r-n}).$$

Using the same way with Lemma 1, we can prove $\widetilde{A}$ is nonsingular. So the system has only zero solution. The Lemma is obtained. $\square$

proof of Theorem 1. Suppose the domain points in the triangle T and item (2) are chosen, following Schumaker [13], they determine the $(r+[\frac{d-r}{2}])-$disks about the interior vertices of Δ_{ms}. By induction on n, using Lemma 1, it is easy to see that all ordinates with domain points in the $(2d-3r-1)-$ disks of the interior vertices of Δ_{ms} must vanish. When $n = 2d-3r$, the set of equations in (3) restricted to the $(r+n)-$disk about the vertex $\widehat{V}_{l-1}$ is reduced to

$$\sum_{v=0}^{2r-d}\binom{k}{v}\beta^{[l+1]}_{2d-2r-v,2r-d,v}a^{k-v}_{l+1,l}c^{v}_{l+1,l}-\sum_{\mu=0}^{2r-d}\binom{2d-2r-k}{\mu}\beta^{[l]}_{2d-2r-\mu,\mu,2r-d}a^{2d-2r-k-\mu}_{l,l+1}b^{\mu}_{l,l+1}=0,$$
$$2d-3r\le k\le r.$$

This is a system of $4r-2d+1$ equations in $4r-2d+2$ variables that can be written in the form $H_l\beta_l=0$, where $H_l=(\widetilde{A}_l,-\widetilde{B}_l)$,

$$\widetilde{A}_l=\begin{pmatrix}\binom{2d-3r}{0}a^{2d-3r}_{l+1,l} & \binom{2d-3r}{1}a^{2d-3r-1}_{l+1,l}c_{l+1,l} & \cdots & \binom{2d-3r}{2r-d}a^{2d-3r}_{l+1,l}c^{2r-d}_{l+1,l}\\ \binom{2d-3r+1}{0}a^{2d-3r+1}_{l+1,l} & \binom{2d-3r+1}{1}a^{2d-3r}_{l+1,l}c_{l+1,l} & \cdots & \binom{2d-3r+1}{2r-d}a^{2d-3r+1}_{l+1,l}c^{2r-d}_{l+1,l}\\ \vdots & \vdots & \cdots & \vdots\\ \binom{r}{0}a^{r}_{l+1,l} & \binom{r}{1}a^{r-1}_{l+1,l} & \cdots & \binom{r}{2r-d}a^{d-r}_{l+1,l}c^{2r-d}_{l+1,l}\end{pmatrix},$$

$$\widetilde{B}_l=\begin{pmatrix}\binom{r}{0}a^{r}_{l,l+1} & \binom{r}{1}a^{r-1}_{l,l+1}b_{l,l+1} & \cdots & \binom{r}{2r-d}a^{d-r}_{l,l+1}b^{2r-d}_{l,l+1}\\ \binom{r-1}{0}a^{r-1}_{l,l+1} & \binom{r-1}{1}a^{r-2}_{l,l+1}b_{l,l+1} & \cdots & \binom{r-1}{2r-d}a^{d-r-1}_{l,l+1}b^{2r-d}_{l,l+1}\\ \vdots & \vdots & \cdots & \vdots\\ \binom{2d-3r}{0}a^{2d-3r}_{l,l+1} & \binom{2d-3r}{1}a^{2d-3r-1}_{l,l+1}b_{l,l+1} & \cdots & \binom{2d-3r}{2r-d}a^{2d-3r}_{l,l+1}b^{2r-d}_{l,l+1},\end{pmatrix},$$

$$\beta_l=(\beta^{[l+1]}_{2d-2r,2r-d,0},\beta^{[l+1]}_{2d-2r-1,2r-d,1},\cdots,\beta^{[l+1]}_{3d-4r,2r-d,2r-d},$$
$$\beta^{[l]}_{2d-2r,0,2r-d},\beta^{[l]}_{2d-2r-1,1,2r-d},\cdots,\beta^{[l]}_{3d-4r,2r-d,2r-d})^T.$$

Since $l=1,2,3$, we have

$$H_1\beta_1=0\qquad H_2\beta_2=0\qquad H_3\beta_3=0.$$

We rewrite the three systems as a homogeneous system

$$\widehat{H}\widehat{\beta}=0, \tag{5}$$

where

$$\widehat{H}=\begin{pmatrix}0&0&0&0&0&\widetilde{A}^{(1)}_1&\widetilde{A}^{(0)}_1&-\widetilde{B}^{(1)}_1&-\widetilde{B}^{(0)}_1\\ 0&0&\widetilde{A}^{(1)}_2&\widetilde{A}^{(0)}_2&-\widetilde{B}^{(1)}_2&0&-\widetilde{B}^{(0)}_2&0&0\\ \widetilde{A}^{(1)}_3&-\widetilde{B}^{(1)}_3&0&-\widetilde{B}^{(0)}_3&0&0&0&0&\widetilde{A}^{(0)}_3\end{pmatrix},$$

$\widehat{\beta}=(\beta^{(1)}_{2d-2r,2r-d,0},\beta^{(1)}_{2d-2r-1,2r-d,1},\cdots,\beta^{(1)}_{3d-4r+1,2r-d,2r-d-1},\beta^{(3)}_{2d-2r,0,2r-d},\cdots,\beta^{(3)}_{3d-4r+1,2r-d-1,2r-d},$
$\beta^{(3)}_{2d-2r,2r-d,0},\beta^{(3)}_{2d-2r-1,2r-d,1},\cdots,\beta^{(3)}_{3d-4r,2r-d,2r-d},\beta^{(2)}_{2d-2r,0,2r-d},\cdots,\beta^{(2)}_{3d-4r+1,2r-d-1,2r-d},$
$\beta^{(2)}_{2d-2r,2r-d,0},\beta^{(2)}_{2d-2r-1,2r-d,1},\cdots,\beta^{(2)}_{3d-4r,2r-d,2r-d},\beta^{(1)}_{2d-2r,0,2r-d},\cdots,\beta^{(1)}_{3d-4r,2r-d,2r-d})^T,$

$A_l=(\widetilde{A}^{(1)}_l,\widetilde{A}^{(0)}_l),B_l=(\widetilde{B}^{(1)}_l,\widetilde{B}^{(0)}_l)$. The order of $\widetilde{A}^{(1)}_l$ and $\widetilde{A}^{(0)}_l$ are $(4r-2d+1)\times(2r-d),(4r-2d+1)\times 1$ respectively. The order of $\widetilde{B}^{(1)}_l$ and $\widetilde{B}^{(0)}_l$ equal the order of $\widetilde{A}^{(1)}_l$ and $\widetilde{A}^{(0)}_l$ respectively. According to $H_l\beta_l=0$ and Cramer's rule, we have $\beta^{[l+1]}_{3d-4r,2r-d,2r-d}=\beta^{[l]}_{3d-4r,2r-d,2r-d}\frac{det(\widetilde{A}^{(1)}_l,\widetilde{B}^{(0)}_l,\widetilde{B}^{(1)}_l)}{det(\widetilde{A}^{(1)}_l,\widetilde{A}^{(0)}_l,\widetilde{B}^{(1)}_l)},l=1,2,3$. Hence, the system $\widehat{H}\widehat{\beta}=0$ has a nonzero solution if and only if $\frac{\prod_{l=1}^{3}det(\widetilde{A}^{(1)}_l,\widetilde{B}^{(0)}_l,\widetilde{B}^{(1)}_l)}{\prod_{l=1}^{3}det(\widetilde{A}^{(1)}_l,\widetilde{A}^{(0)}_l,\widetilde{B}^{(1)}_l)}=1$. Using the same transformation with in the proof of Lemma 1, we have $\frac{\prod_{l=1}^{3}det(\widetilde{A}^{(1)}_l,\widetilde{B}^{(0)}_l,\widetilde{B}^{(1)}_l)}{\prod_{l=1}^{3}det(\widetilde{A}^{(1)}_l,\widetilde{A}^{(0)}_l,\widetilde{B}^{(1)}_l)}=\frac{(a_{21}a_{32}a_{13})^{2d-3r}(c_{12}c_{23}c_{31})^{2r-d}}{(a_{12}a_{23}a_{31})^{2d-3r}(b_{12}b_{23}b_{31})^{2r-d}}$. Hence the system $\widehat{H}\widehat{\beta}=0$ forces the remaining ordinates to vanish unless

$$(a_{21}a_{32}a_{13})^{2d-3r}(c_{12}c_{23}c_{31})^{2r-d}=(a_{12}a_{23}a_{31})^{2d-3r}(b_{12}b_{23}b_{31})^{2r-d}.$$

According to Lemma 2, if (1) does not hold, then P is a determining set for $S^r_d(\Delta_{ms}),d\le 2r-1$. If any of the ordinates $\beta^{[l]}_{3d-4r,2r-d,2r-d}$ is zero, then (5) implies the other ordinates in (5) must also vanish. Therefore, when (1) is satisfied, one of the domain points $P^{[l]}_{3d-4r,2r-d,2r-d}$ can be added to the set P to form a determining set for $S^r_d(\Delta_{ms})$. □

Lemma 3 *For $d\le\frac{5}{3}r,\frac{d-r}{2}<n\le d-r$,suppose that the ordinates with domain points in the triangle T and in the $(r+n-1)$-disks about the interior vertices of Δ_{ms} are zero. Then all ordinates with domain points in the $(r+n)-$disk about the interior vertices of Δ_{ms} are zero.*

The proof is similar to the proof of Lemma 2. Hence, we omit it.

Proof of Theorem 2. Suppose the domain points in triangle T is chosen. According to Schumaker[13], they determine the $(r+[\frac{r+1}{3}])-$disks about the interior vertices of Δ_{ms}. For $d\le\frac{5}{3}r$, we have $\frac{d-r}{2}\le[\frac{r+1}{3}]$. By induction on n, using Lemma 3, it is easy to see that all ordinates with domain points in the triangulation vanish. Hence, for $d \leq \frac{5}{3}r$, when the domain points in the triangle T are chosen, all the domain points in the triangulation vanish, i.e. $dimS_d^r(\Delta_{ms}) = \frac{(d+2)(d+1)}{2}, d \leq \frac{5}{3}r$. When $\frac{5}{3}r < d \leq \frac{5r+2}{3}$, we have $2d-3r-1 \leq \frac{d-r}{2}$ and $\frac{d-r}{2} \leq [\frac{r+1}{3}]$. Using Lemma 2, we only need to consider the system (5). It follows that, for $\frac{5}{3}r < d \leq \frac{5r+2}{3}$, (1) give necessary and sufficient conditions for $dimS_d^r(\Delta_{ms}) = \frac{(d+2)(d+1)}{2} + 1$. $\square$

5. The Instability Degree In The Dimension of Bivariate Spline

For a given triangulation Δ_0, let $V_1, \cdots, V_I$ denote the vertex of Δ_0 and $N(\Delta_0, V_i)$ denote the number of edges with different slopes that meet at $V_i, i = 1, \cdots, I$. The set of triangulation $\widetilde{\Delta_0}$ is defined as

$\widetilde{\Delta_0} = \{\Delta | \Delta$ share the same topological structrue with $\Delta_0, N(\Delta, V_i) = N(\Delta_0, V_i), i = 1, \cdots, I\}$.

The instability degree of $S_d^r(\Delta_0)$ is defined as

$$Id(S_d^r(\Delta_0)) = \max_{\Delta \in \widetilde{\Delta_0}} dimS_d^r(\Delta) - \min_{\Delta \in \widetilde{\Delta_0}} dimS_d^r(\Delta).$$

We are interested in the *instability degree on the dimension of bivariate spline*, which is defined as

$$Id = \max_{d,r,\Delta} Id(S_d^r(\Delta)).$$

Obviously $Id \geq \max_{d,r} Id(S_d^r(\Delta_{ms}))$. To date, some results only show $Id \geq 1$ [10,14].

Using the applet in *http://www.math.utah.edu/∼alfeld/MDS* (cf.[1]), we compute $dimS_d^r(\Delta_{ms}^1), dimS_d^r(\Delta_{ms}^0)$, where Δ_{ms}^1 and Δ_{ms}^0 are the generic Morgan-Scott and the symmetric Morgan-Scott triangulation in the applet respectively. We compare these results with the upper bound and low boundary in the paper. See Table.1, the low and upper boundary agree with them. Therefore, we conjecture $Lb(S_d^r(\Delta_{ms})) = dimS_d^r(\Delta_{ms}^1), Ub(S_d^r(\Delta_{ms})) = dimS_d^r(\Delta_{ms}^0)$. Obviously,if the conjecture holds,$Id \geq Lb(S_d^r(\Delta_{ms})) - Ub(S_d^r(\Delta_{ms}))$. Name the function $F_r(d) = UbS_d^r(\Delta_{ms}) - LbS_d^r(\Delta_{ms})$ as the *instability function of degree r*. It is easy for proving $\lim_{r\to+\infty} F_r([\frac{11}{6}r]) = +\infty$, which implies $Id = +\infty$.

Table.1

r	d	$dimS_d^r(\Delta_{ms}^1)$	$dimS_d^r(\Delta_{ms}^0)$	$Lb(S_d^r(\Delta_{ms}))$	$Ub(S_d^r(\Delta_{ms}))$
4	7	36	37	36	37
5	9	55	56	55	56
6	11	78	82	78	82
7	13	108	112	108	112
7	12	91	92	91	92
8	15	142	146	142	146
9	16	153	157	153	157
9	17	183	187	183	187
10	18	190	197	190	197
11	20	231	241	231	241

In general, the value of instability function of degree r increases along with d increasing firstly, then it decreases along with d increasing. Near $\frac{11}{6}r$, the function reachs the maximum value (see fig.2).

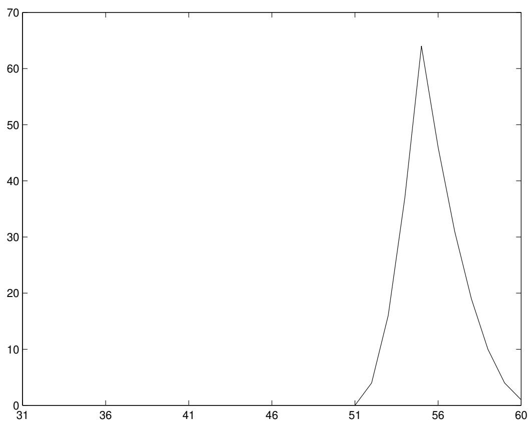

Fig.2. The graph of instability function of degree 30

Remark

(1) When r is not divisible by 3, there is a unique d satisfying $\frac{5}{3}r < d \leq \frac{5r+2}{3}$. Otherwise, we can not find d such that $\frac{5}{3}r < d \leq \frac{5r+2}{3}$.

(2) If $r = 1$ or $r = 2$, then d, satisfying $\frac{5}{3}r < d \leq \frac{5r+2}{3}$,is equal to $2r$. Hence, Theorem 2 gives necessary and sufficient conditions for $dimS^r_{2r} = Ub(S^r_{2r}(\Delta_{ms})), r \leq 2$, which agrees with the Diener's result [6].

(3) Theorem 1 does not give $dimS^r_d(\Delta_{ms}), d \leq 2r-1$. We conjecture that (1) gives necessary and sufficient conditions for $dimS^r_d(\Delta_{ms}) = Ub(S^r_d(\Delta_{ms})), d \leq 2r-1$. Moreover, we are interested in whether we can find a Morgan-Scott triangulation Δ^2_{ms} such that $Lb(S^r_d(\Delta_{ms})) < dimS^r_d(\Delta^2_{ms}) < Ub(S^r_d(\Delta_{ms}))$.

(4) According to Theorem 2,$dimS^r_d(\Delta_{ms}), d \leq \frac{5}{3}r$ is stable. Can we find such a triangulation Δ_0 that $dimS^r_d(\Delta_0), d \leq \frac{5}{3}r$ is instable?

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