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Multi-dimensional versions of a formula of Popoviciu

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Abstract In this paper, an explicit formulation for multivariate truncated power functions of degree one is given firstly. Based on multivariate truncated power functions of degree one, an formulation is presented which counts the number of non-negative integer solutions of $s \times (s+1)$linear Diophantine equations and it can be considered as a multi-dimensional versions of the formula counting the number of non-negative integer solutions of $ax + by = n$ which is given by Popoviciu in 1953.

Keywords: Multivariate Splines, Discrete Truncated Power , Linear Diophantine Equations

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1 Introduction

Let M be an $s\times n$ integer matrix with columns $m_1, \cdots, m_n \in \mathbf{Z}^s$ such that $[\{m_1, \cdots, m_n\}]$ does not contain the origin, where $[A]$ denotes the convex hull of a given set A. The vector partition functions $t(\mathbf{b}|M)$ is defined as follows

$$t(\mathbf{b}|M) = \#\{\mathbf{x} \in \mathbf{Z}_+^n | \mathbf{M}\mathbf{x} = \mathbf{b}\},$$

where $\#\{\cdot\}$ denotes the cardinality of the set $\{\cdot\}$, $\mathbf{Z}_+$ denotes the non-negative integer set and $\mathbf{b}$ is the vector in $\mathbf{Z}^s$. The vector partition function,which is also called *discrete truncated powers*, has many applications in various mathematical areas including Algebraic Geometry[1], Representation Theory[2], Number Theory[3], Statistics [4] and Randomized Algorithm [5] among others.

Suppose $a \in \mathbf{Z}_+$ and $b \in \mathbf{Z}_+$ are relatively prime and then we conside $t(n|(a,b)), n \in \mathbf{Z}$ which is a special case of the vector partition function. According to the definition of the vector partition function, $t(n|(a,b))$ counts the non-negative integer solutions of $ax + by = n$. In 1953, Popoviciu gave a very beautiful formulation for $t(n|(a,b))$,i.e.

$$t(n|(a,b)) = \frac{n}{ab} - \{\frac{b^{-1}n}{a}\} - \{\frac{a^{-1}n}{b}\} + 1,$$

where,$\{\cdot\}$ denotes the fraction part, i.e. $\{x\} := x - [x], b^{-1}, a^{-1}$ denote two integers satisfying $b^{-1}b \equiv 1 \ mod \ a, a^{-1}a \equiv 1 \ mod \ b$([6]). The formulation is surprising and has many important applications([7]). Furthermore, several different methods for proving the formulation are also given ([8,9]). But it seems to be difficult for extending the formulation to multi-dimension by using these methods.

The aim of the paper is to give a simple formulation for $t(\mathbf{b}|M)$ when M is an $s\times(s+1)$ integer matrix, which can be regarded as a generalization of Popoviciu's formulation.

We firstly review some results about $t(\mathbf{b}|M)$, where M is an $s\times n$ integer matrix. For the general matrix M, the nature of $t(\mathbf{b}|M)$ is investigated and the piecewise structure of $t(\mathbf{b}|M)$ is given in [10]. In [13], based on multivariate truncated power functions $T(\mathbf{x}|M)$, an explicit formulation for $t(\mathbf{b}|M)$ is presented. However, the formulation involves multivariate truncated power functions $T(\mathbf{x}|M)$, which are not in explicit form. Furthermore, the non-polynomial part in the formulation is also very complex.

In this paper, we firstly give an explicit formulation for $T(x|M)$ when M is an $s\times(s+1)$ integer matrix. Based on the formulation and the discrete Fourier transform, we generalize the Popoviciu's formulation to high dimension.

2 Multivariate Truncated Power Functions

To describe multivariate truncated power functions, we introduce some definitions and notations firstly. Let M be an $s\times n$ real matrix and $rank(M)=s$, whose column vectors are denoted as $m_1,\cdots,m_n\in\mathbf{Z}^s\setminus\{0\}$. M is also viewed as the multiset of its column vectors. We always assume that the convex hull of M does not contain the origin. After discarding the ith column vector in M we obtain a new matrix, which is denoted as M_i, i.e. $M_i=(m_1,\cdots,m_{i-1},m_{i+1},\cdots,m_n)$. For the vector m_i we use $m_{i,j}$ to denote the jth element in m_i. We use $cone(M)$ to denote the cone produced by M, i.e. $cone(M):=\{\sum_{m\in M}a_m m: a_m\geqslant 0 \text{ for all } m\}$ and use $cone^\circ(M)$ to denote the interior points of $cone(M)$. Moreover,we let $\mathcal{Y}(M)$ denote the set consisting of those submultisets Y of M for which $M\backslash Y$ does not span $\mathbf{R}^s$. Let A and B be two subsets of $\mathbf{R}^m$ and $c\in\mathbf{R}$. Then $A-B$ is the set of all elements of the form $a-b$, where $a\in A$ and $b\in B$. The sets $A+B$ and cA are defined analogously.

Let $c(M)$ be the union of all the sets cone$(M\setminus Y)$, as Y runs over $\mathcal{Y}(M)$. A connected component of $cone^\circ(M)\setminus c(M)$, according to [10], is called a *fundamental M-cone*. Given two sets D_1, D_2, let $1_{D_1}(D_2)=0$ if $D_1\cap D_2=\emptyset$, otherwise let $1_{D_1}(D_2)=1$.

The multivariate truncated power $T(\cdot|M)$ associated with M, first introduced by W. Dahmen [11], is the distribution given by the rule

$$\int_{\mathbf{R}^s}\phi(x)T(x|M)dx=\int_{\mathbf{R}^n_+}\phi(Mu)du,\forall\phi\in\mathcal{D}(\mathbf{R}^s) \tag{1}$$

where $\mathcal{D}(\mathbf{R}^s)$ is the space of test functions on $\mathbf{R}^s$, i.e. the space of all compactly supported and infinitely differentiable functions on $\mathbf{R}^s$.

According to the definition of $T(x|M)$, $T(x|M)=\frac{1}{|det(M)|}1_{cone(M)}(\{x\})$ when M is an $s\times s$ matrix. If $n>s$, there exists an element $\mathbf{m}$ such that $\{M\setminus\mathbf{m}\}$ still spans $\mathbf{R}^s$. Then we have the following recurrence formulation

$$T(\mathbf{x}|M)=\int_0^\infty T(\mathbf{x}-t\mathbf{m}|M\setminus\mathbf{m})dt,\mathbf{x}\in\mathbf{R}^s.$$

The following theorem describes the piecewise structure of the multivariate truncated power (see [12]).

THEOREM 1 *([12]) Let M be an $s\times n$ real matrix with rank$M=s\leqslant n$. Suppose the convex hull of M does not contain 0. Then $T(\cdot|M)$ agrees with some homogeneous polynomial of degree $n-s$ on each fundamental M-cone. $T(x|M)$ is a continuous function when $n>s$.*

In the rest of the paper, we always suppose $n = s+1$, i.e. M is an $s \times (s+1)$ matrix. We shall give an explicit formulation for $T(x|M)$. To describe conveniently, we suppose the column vectors in M are in the general position, i.e. for each submatrix $Y \subset M$ of cardinality s it follows that $det(Y) \neq 0$.

Before discussing the explicit formulation for $T(x|M)$ we deal with the fundamental M-cone. We can select a hyper-plane $\mathcal{H} \subset \mathbf{R}^s$ such that $\mathcal{H} \cap \{tm_i : t > 0\} \neq \emptyset$ since the convex hull of M does not contain the origin point, and we denote the intersection point as $\widetilde{m_i}$. Hence,$\widetilde{M} := \{\widetilde{m_1}, \widetilde{m_2}, \cdots, \widetilde{m}_{s+1}\}$ is the set of $s+1$ distinct points in $\mathcal{H}$, and M can be considered as the point set in $\mathbf{R}^{s-1}$ also. Consider the convex hull of the $s+1$ points on $\mathcal{H}$, i.e. $[\widetilde{M}]$. Then we have two cases: the first case is that s points locate the boundary of $[\widetilde{M}]$ and another point locates the interior of $[\widetilde{M}]$; the second case is that all the $s+1$ points locate the boundary of $[\widetilde{M}]$. Figure 1 shows the two cases when $M = (e_1, e_2, e_3, e_4)$, where, $e_i \in \mathbf{R}^3$.

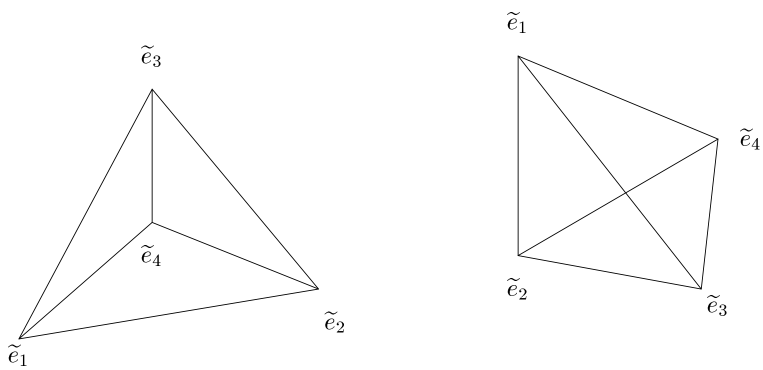

Fig.1. Two cases of the fundamental $M-$cones.

For fundamental M-cone Ω_i, suppose $\widetilde{\Omega_i} = \mathcal{H} \cap \Omega_i$. Notice that $\widetilde{\Omega_i}$ is a $s-1$ simplex,which has an edge which overlaps with one of edges of $[\widetilde{M}]$, for each case. Hence, Ω_i must have a face which overlaps with one of faces of $cone(M)$. Without loss of generality, we suppose the face is spanned by $\{m_1, \cdots, m_{s-1}\}$, i.e. $cone(m_1, \cdots, m_{s-1})$. So, the fundamental $M-$ cone $\Omega_i = cone(m_1, \cdots, m_{s-1}, m_s) \cap cone(m_1, \cdots, m_{s-1}, m_{s+1})$, i.e. $\Omega_i = cone(M_{s+1}) \cap cone(M_s)$, by the definition of fundamental $M-$cone.

According to the analysis above, we have

LEMMA 1 *Suppose M is an $s \times (s+1)$ integer matrix. For any fundamental $M-$cone Ω, there exist integers $j, i, 1 \leqslant j < i$ such that*

$$\Omega = cone(M_i) \cap cone(M_j).$$

In the rest of the paper, without loss of generality, we always suppose the fundamental $M-$cone is $cone(M_{s+1}) \cap cone(M_s)$. We can swap the column elements in M_s, M_{s+1} such that both $det(M_s)$ and $det(M_{s+1})$ are non-negative. To describe conveniently, we denote the matrixes after swaping column vectors as M_s, M_{s+1} still. If the column vector m_{s+1} in M_s is replaced by $x \in \mathbf{R}^s$, we obtain a new matrix which is denoted as $M_s(x)$, such as if $M_s = (m_{s+1}, m_1, \cdots, m_{s-1})$, then $M_s(x) = (x, m_1, \cdots, m_{s-1})$. Similarly, we use $M_{s+1}(x)$ to denote the matrix which is produced by replacing the element m_{s+1} by x in M_{s+1}.

According to the analysis of fundamental $M-$cone, we have

THEOREM 2 *Suppose the fundamental* $M-$*cone* $\Omega = cone(M_{s+1}) \cap cone(M_s)$. *Then,when* $x \in \overline{\Omega}$

$$T(x|M) = \frac{det(M_s(x))}{det(M_s)det(M_{s+1})}.$$

proof: Since $T(x|M)$is continuous, we only need to prove

$$T(x|M) = \frac{det(M_s(x))}{det(M_s)det(M_{s+1})},$$

when $x \in \Omega$. Notice that $M \setminus m_{s+1}$ is a square matrix. Hence

$$T(x|M\backslash m_{s+1}) = \frac{1}{|det(M \setminus m_{s+1})|}1_{cone(M\backslash m_{s+1})}(\{x\}) = \frac{1}{det(M_{s+1})}1_{cone(M\backslash m_{s+1})}(\{x\}).$$

Consider the iterative formulation for $T(x|M)$

$$T(x|M) = \int_0^\infty T(x - tm_{s+1}|M \setminus m_{s+1})dt.$$

Notice that $x \in cone(M_s)$. Hence, there exist $a_1, \cdots, a_{s-1}, t_0 \geqslant 0$, such that

$$x = a_1m_1 + \cdots + a_{s-1}m_{s-1} + t_0m_{s+1}. \qquad (2)$$

So,$\{x - tm_{s+1} : t \geqslant 0\}$ and $cone(m_1, \cdots, m_{s-1})$ intersect at $x - t_0m_{s+1}$.

Solving linear equations (2), we have

$$t_0 = \frac{det(M_s(x))}{det(M_s)}.$$

Hence,when $x \in \Omega$,

$$T(x|M) = \int_0^{t_0} T(x - tm_{s+1}|M \setminus m_{s+1})dt = \frac{det(M_s(x))}{det(M_s)det(M_{s+1})}.$$

Remark Notice that the simplex in $\mathbf{R}^{s-1}$ has s faces. So, the numbers of fundamental $M-$ cones of the first case and the second case are s and $2s - 2$ respectively.

3 VECTOR PARTITION FUNCTIONS

In this section, we shall describe some results about vector partition functions. We suppose M is an $s \times n$ integer matrix. For $\theta := (\theta_1, \cdots, \theta_s) \in \mathbf{C}^s$, let $M_\theta := \{y \in M : \theta^y = 1\}$, where $\theta^y := \theta_1^{y_1} \cdots \theta_s^{y_s}$. Furthermore, we let $A(M) := \{\theta \in (\mathbf{C} \setminus \{0\})^s : span(M_\theta) = \mathbf{R}^s\}, e := (1, 1, \cdots, 1) \in \mathbf{Z}^s$. Since $e^y = 1$ for any $y \in M$, we have $e \in A(M)$. Let $[[M]] := \{\sum_{j=1}^n a_jm_j : 0 \leqslant a_j \leqslant 1, \forall j\}$ and $v(\Omega|M) := \Omega - [[M]]$,

The piecewise structure is given in the following theorem.

THEOREM 3 *([10]) Let* $M = \{m_1, \cdots, m_n\} \in \mathbf{Z}^{s\times n}$. *Suppose the convex hull of* M *does not contain the origin. For any fundamental* $M-$ *cone* Ω, *there exists a unique element* $f_\Omega(\alpha|M) = \sum\limits_{\theta\in A(M)} \theta^\alpha p_{\theta,\Omega}(\alpha)$, *such that such that* $f_\Omega(\alpha|M)$ *agrees with* $t(\alpha|M)$ *on* $v(\Omega|M)$, *where* $p_{\theta,\Omega}(\cdot)$ *is polynomial with degree not more than* $\#M_\theta - s$.

Furthermore, an explicit formulation for $f_\Omega(\alpha|M)$ is presented in [13,14]. The formulation can be simplified when M satisfy some special properties. To describe the results, we firstly introduce the definition of $1-$ prime matrix. We call the M as the $1-$ prime matrix provided for any $s\times(s+1)$ submatrix Q in M it follows that $gcd\{|det(Y)| : Y \in \mathcal{B}(Q)\} = 1$, where $\mathcal{B}(Q)$ denotes the set consisting of the $s \times s$ matrix in Q. Especially, the $1-$prime matrix M means the integers in M are relative prime when $s = 1$.

When M is a $1-$ prime matrix, an explicit formulation for $f_\Omega(\alpha|M)$ is given in the following theorem

THEOREM 4 *([13]) Under the condition of Theorem 3, when M is a 1-prime matrix,*

$$f_\Omega(\alpha|M) = P_e f_\Omega(\alpha|M) + \sum_{\theta\in A(M)\setminus e} \theta^\alpha \frac{1}{|det(M_\theta)|} \prod_{w\in M\setminus M_\theta} \frac{1}{1-\theta^{-w}} 1_{cone(M_\theta)}(\Omega),$$

where,$P_e f_\Omega(\alpha|M)$ is defined inductively as follow $P_e f_\Omega(\alpha|M) = \sum_{k=0}^{n-s} p_{k,\Omega}(x)$, where $p_{k,\Omega}(x)$ is homogeneous polynomial of degree $n-s-k$ defined inductively as follows

$$p_{k,\Omega}(x) = \begin{cases} T(x|M), k = 0, \\ -\sum_{j=0}^{k-1}(\sum_{|v|=k-j} D^v p_{j,\Omega}(x)(-i)^{|v|} D^v \widehat{B}(0|M)/v!), 1 \leqslant k \leqslant n-s, \end{cases}$$

where, $x \in \Omega$.

In Theorem 4,$\widehat{B}(\zeta|M)$ denotes the Fourier transform of Box spline,i.e.

$$\widehat{B}(\zeta|M) = \prod_{j=1}^{n} \frac{1-exp(-i\zeta^T m_j)}{i\zeta^T m_j}, \zeta \in \mathbf{C}^s.$$

4 Multi-dimensional versions of a formula of Popoviciu

In this section, we shall generalize Popoviciu's formulation to multi-dimension. To describe conveniently, we suppose the determinants of any two $s \times s$ submatrix in M are relatively prime. We have

THEOREM 5 *Let $M = \{m_1, \cdots, m_{s+1}\} \in \mathbf{Z}^{s\times(s+1)}$. Suppose the convex hull of M does not contain the origin point and the determinants of any two $s \times s$ submatrix in M are relatively prime. Moreover,suppose the fundamental $M-$cone $\Omega = cone(M_s) \cap cone(M_{s+1})$. Then, when $\mathbf{n} \in v(\Omega|M) \cap \mathbf{Z}^s$,*

$$\begin{aligned} t(\mathbf{n}|M) &= \frac{det(M_s(\mathbf{n}))}{det(M_s)det(M_{s+1})} - \{\frac{det(M_{s+1})^{-1}det(M_s(\mathbf{n}))}{det(M_s)}\} \\ &- \{\frac{det(M_s)^{-1}det(M_{s+1}(\mathbf{n}))}{det(M_{s+1})}\} + 1, \end{aligned}$$

where $det(M_s)^{-1}det(M_s) \equiv 1 \ \ mod \ \ det(M_{s+1}), det(M_{s+1})^{-1}det(M_{s+1}) \equiv 1 \ \ mod \ \ det(M_s)$.

proof: Notice that $\Omega \subset v(\Omega|M)$ and the function in the form of $\sum_{\theta\in A(M)} \theta^\alpha p_{\theta,\Omega}(\alpha)$ can be determined by the values on $\Omega \cap \mathbf{Z}^s$. Hence, we only need to prove the formulation holds on $\Omega \cap \mathbf{Z}^s$.

According to Theorem 4, the polynomial part of $t(\mathbf{n}|M)$is $p_{0,\Omega}(\mathbf{n})+p_{1,\Omega}(\mathbf{n})$ when $\mathbf{n} \in \Omega\cap\mathbf{Z}^s$, where, $p_{0,\Omega}(\mathbf{n}) = T(\mathbf{n}|M), p_{1,\Omega}(\mathbf{n}) = -(\sum_{|v|=1} D^v p_{0,\Omega}(\mathbf{n})\,(-i)^{|v|}D^v\widehat{B}(0|M)/v!)$. Since $T(\cdot|M)$ is continuous, when $\mathbf{n} \in \overline{\Omega} \cap \mathbf{Z}^s$

$$T(\mathbf{n}|M) = \frac{det(M_s(\mathbf{n}))}{det(M_s)det(M_{s+1})},$$

by Theorem 2.

Notice that

$$\widehat{B}(\zeta|M) = \prod_{j=1}^{n} \frac{1 - exp(-i\zeta^T m_j)}{i\zeta^T m_j}, \zeta \in \mathbf{C}^s.$$

After a brief calculation, we have $p_{1,\Omega}(\mathbf{n}) = \frac{1}{2}(\frac{1}{det(M_s)} + \frac{1}{det(M_{s+1})})$. Hence,when $\mathbf{n} \in \Omega$, the polynomial part of $t(\mathbf{n}|M)$ is $\frac{det(M_s(\mathbf{n}))}{det(M_s)det(M_{s+1})} + \frac{1}{2}(\frac{1}{det(M_s)} + \frac{1}{det(M_{s+1})})$.

According to Theorem 4, the non-polynomial part of $t(\mathbf{n}|M)$ is

$$\begin{aligned}
&\sum_{\theta\in A(M)\setminus e} \theta^{\mathbf{n}} \frac{1}{|det(M_\theta)|} \prod_{w\in M\setminus M_\theta} \frac{1}{1-\theta^{-w}} 1_{cone(M_\theta)}(\Omega) \\
=& \sum_{\substack{\theta\in A(M)\setminus e\\ M_\theta = M_{s+1}}} \theta^{\mathbf{n}} \frac{1}{det(M_{s+1})} \frac{1}{1-\theta^{-m_{s+1}}} + \sum_{\substack{\theta\in A(M)\setminus e\\ M_\theta = M_s}} \theta^{\mathbf{n}} \frac{1}{det(M_s)} \frac{1}{1-\theta^{-m_s}}.
\end{aligned}$$

To prove the theorem, we need to simplify the summation formulation. We firstly consider

$$\sum_{\substack{\theta\in A(M)\setminus e\\ M_\theta = M_{s+1}}} \theta^{\mathbf{n}} \frac{1}{det(M_{s+1})} \frac{1}{1-\theta^{-m_{s+1}}}. \tag{3}$$

According to the conclusions in [15], the elements in $\{\theta|\theta \in A(M) \setminus e, M_\theta = M_{s+1}\}$ are in the form of

$$\theta = (e^{2\pi i\alpha_1^j/det(M_{s+1})}, \cdots, e^{2\pi i\alpha_s^j/det(M_{s+1})}), \tag{4}$$

where, $\alpha^j := (\alpha_1^j, \cdots, \alpha_s^j) \in \mathbf{Z}^s, 1 \leqslant j \leqslant det(M_{s+1}) - 1$.

Consider the $\theta^{m_{s+1}} := e^{2\pi i\sum_{l=1}^s \alpha_l^j m_{s+1,l}/det(M_{s+1})}$ in (3). Let $k := \sum_{l=1}^s \alpha_l^j m_{s+1,l}$. When θ runs over the set $\{\theta|\theta \in A(M)\setminus e, M_\theta = M_{s+1}\}$, k runs over $\mathbf{Z}\cap[1, det(M_{s+1})-1]$, since the determinant of $s \times s$ submatrix in M are relatively prime.

Notice that $\theta^{m_l} = 1, 1 \leqslant l \leqslant s$ when $\theta \in \{\theta|\theta \in A(M), M_\theta = M_{s+1}\}$. Hence, the integer vector α^j satisfy the following equations

$$\begin{aligned}
&\alpha_1^j m_{1,1} + \alpha_2^j m_{1,2} + \cdots + \alpha_s^j m_{1,s} \equiv 0 \ \ mod\ \ det(M_{s+1}),\\
&\alpha_1^j m_{2,1} + \alpha_2^j m_{2,2} + \cdots + \alpha_s^j m_{2,s} \equiv 0 \ \ mod\ \ det(M_{s+1}),\\
&\vdots\\
&\alpha_1^j m_{s,1} + \alpha_2^j m_{s,2} + \cdots + \alpha_s^j m_{s,s} \equiv 0 \ \ mod\ \ det(M_{s+1}),\\
&\alpha_1^j m_{s+1,1} + \alpha_2^j m_{s+1,2} + \cdots + \alpha_s^j m_{s+1,s} \equiv k \ \ mod\ \ det(M_{s+1}).
\end{aligned} \tag{5}$$

Notice that the determinant of $s \times s$ submatrixes in M are relatively prime. We can discard any equation in the first s equations of (5) and the solutions of (5) are not changed.

Here, we discard the sth equation and hence obtain a new linear equations which have the same solution with (5). Using Cramer rule for solving the new linear equations and substituting the solutions into $\theta^{\mathbf{n}} := e^{2\pi i \sum_{l=1}^{s} \alpha_l^j n_l / det(M_{s+1})}$, we have

$$\sum_{l=1}^{s-1} n_l \alpha_l^j \equiv det(M_s)^{-1} det(M_{s+1}(\mathbf{n}))k \;\; mod \;\; det(M_{s+1}).$$

Hence, (3) can be written as

$$\frac{1}{det(M_{s+1})} \sum_{k=1}^{det(M_{s+1})-1} e^{2\pi i det(M_s)^{-1} det(M_{s+1}(\mathbf{n}))k/det(M_{s+1})} \frac{1}{1 - e^{-2\pi i k/det(M_{s+1})}}. \quad (6)$$

According to the discrete Fourier formulation

$$-\{\frac{t}{a}\} = \frac{1-a}{2a} + \frac{1}{a} \sum_{k=1}^{a-1} \frac{e^{\frac{2\pi i t k}{a}}}{1 - e^{\frac{-2\pi i k}{a}}}, t, a \in \mathbf{Z}, \quad (7)$$

(5) can be reduced to

$$-\{\frac{det(M_s)^{-1} det(M_{s+1}(\mathbf{n}))}{det(M_{s+1})}\} - \frac{1}{2det(M_{s+1})} + \frac{1}{2}.$$

By similar method, we can prove

$$\begin{aligned} &\sum_{\substack{\theta \in A(M) \setminus e \\ M_\theta = M_s}} \theta^{\mathbf{n}} \frac{1}{det(M_s)} \frac{1}{1 - \theta^{-m_s}} \\ = \;\; & -\{\frac{det(M_{s+1})^{-1} det(M_s(\mathbf{n}))}{det(M_s)}\} - \frac{1}{2det(M_s)} + \frac{1}{2}. \end{aligned}$$

After a brief calculation, we have $\mathbf{n} \in v(\Omega|M) \cap \mathbf{Z}^s$,

$$\begin{aligned} t(\mathbf{n}|M) \;\; &= \;\; \frac{det(M_s(\mathbf{n}))}{det(M_s) det(M_{s+1})} - \{\frac{det(M_{s+1})^{-1} det(M_s(\mathbf{n}))}{det(M_s)}\} \\ &- \;\; \{\frac{det(M_s)^{-1} det(M_{s+1}(\mathbf{n}))}{det(M_{s+1})}\} + 1. \end{aligned}$$

Remark:

1.We require the determinants of all the $s \times s$ square matrix in M are relatively prime. In fact, we can give a similar formulation for $t(\cdot|M)$ when M is the $1-$ prime matrix by using the similar method.

2. If an $s \times s$ matrix is singular, we can also present a simple formulation for $t(\cdot|M)$ by using the similar method in this paper.

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References

1. Pommersheim J., Toric varieties, lattice points, and Dedekind sums, Math. Ann. 1993, 295(1),1-24.
2. Schmidt J.R. , Bincer, A. The Kostant partition function for simple Lie algebras, J. Mathematical Physics 1984,25(8),2367-2373.

3. Nijehuis A. , Wilf H., Representation of integers by linear forms in nonnegative integers, J. Number Theory 1972,4(1),98-106.
4. Dinwoodie I.,Stochastic simulation on integer constraint sets, SIAM J. Optimization 1999,9(1),953-61.
5. Welsh D., Approximate counting , in R.A. Bailey eds: Surveys in Combinatorics, London Mathematical Society Lecture Notes, 1997,241.
6. Popoviciu T., Asupra unei probleme de patitie a numerelor, Acad. Republicii Populare Romane, Filiala Cluj, Studii si cercetari stiintifice 1953,4,7-58.
7. Beck M. , Robins S., Computing the continuous discretely integer point enumeration in polyhedron, to appear in Springer Undergraduate Texts in Mathematics.
8. Sertöz S. , On the number of solutions of the Diophantine equation of Frobenius , Diskret. Math. 1988,10(2),62-71.
9. Tripathi A., The number of solutions to ax+by=n , Fibonacci Quart. 2000,38(4),290-293.
10. Dahmen W., Micchelli C.A., The number of solutions to linear Diophantine equations and multivariate splines, Trans. Amer. Math. Soc.1988, 308(2),509-532.
11. Dahmen W., On multivariate B-splines, SIAM J. Numer. Anal. 1980,17(2),179-191.
12. Jia R. Q., Symmetric magic squares and multivariate splines,Linear Algebra Appl.,1997,250(1-3),69-103.
13. Xu Z., Discrete Truncated Powers, Volume of convex polytopes and Ehrhart polynomials, http://arxiv.org/abs/math.CO/0505129.
14. Wang R., Xu Z., Multivariate splines and lattice points in rational polytope, J. Comp. Appl. Math.,2003,159(1) 149-159.
15. Dahmen W., Micchelli C.A., On the solution of certain systems of partial difference equations and linear dependence of translates of box splines, Trans. Amer. Math. Soc. 1985,292(1),305-320.