CONSTRUCTION OF GEOMETRIC PARTIAL DIFFERENTIAL EQUATIONS IN COMPUTATIONAL GEOMETRY

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Abstract

It is well-known that mean curvature flow, surface diffusion flow and Willmore flow have played important roles in the field of geometry analysis. They are also widely used in the fields of computer aided geometric design, computer graphics and image processing. However, in the real applications one often needs to construct various flows according to specific requirements of the problems solved. In this paper, we propose a generic framework for constructing geometric partial differential equations, including $L^2$, $H^{-1}$ and $H^{-2}$ gradient flows. These flows are general, which contain mean curvature flow, surface diffusion flow and Willmore flow as their special cases.

Key words: computational geometry, energy functional, gradient descent flow, Euler–Lagrange operator


1 Introduction

Geometric flows, as a class of important geometric partial differential equations, have been highlighted in many fields of theoretical research and practical applications. Stimulated by the research of the mean curvature flow, Willmore flow and Ricci curvature flow, etc., the knowledge of geometric analysis, manifold theory, topology, complex analysis, partial differential equations, variational calculus, geometric measure theory, critical point theory, has been significantly enriched and developed. In the areas of physics, chemistry and biology, geometric flows have attracted extensive attention and are deeply studied. In practical applications, geometric flows have also greatly attracted engineers’ attention. In the recent two decades especially, geometric flows have been widely employed and show obvious superiority in the areas of Computer Aided Design (CAD) and Geometric Aided Geometric Design (CAGD), such as geometric modeling, surface processing and image processing. The theoretical study of the geometric flows is nowadays the focus of many subjects. But the construction of geometric flows...
flows is not a trivial task. Many geometric flows are manually manufactured, that is, they are constructed by combining several geometric entities and differential operators. Obviously, the geometric flows generated by such an approach are lack of physical or geometric meaning. The research topic of this paper is the construction of gradient descent flow—a another approach, which can generate a great variety of important geometric flows with obvious geometric meaning. Lacking of physical and geometric meaning is not the only drawback of the manufactured geometric flows, the choice of flow direction is also a puzzle in general. In practice, one often prescribes intuitively or tests to decide the flow direction. But the method we considered in this paper can overcome this shortcoming naturally. Gradient descent flow method can transform an optimization problem into an initial value (initial-boundary value) problem of an ordinary differential equation and thus is widely used in variational calculus. In analog, this method can be used to solve geometric optimization problems and construct many famous geometric flows.

The construction of gradient descent flow relates mainly to two issues, the definition of gradient and suitable choice of inner products. For a generic nonlinear energy functional, we define gradient by Gâteaux derivative. For the same energy functional, different inner products will generate different geometric flows. This paper is composed of the following contents. Firstly, we introduce some necessary preliminaries. Secondly, the construction of gradient descent flows for parametric surfaces, including $L^2$, $H^{-1}$, $H^{-2}$ gradient flows, is described in detail and a large number of examples are provided. In particular, $H^{-2}$ gradient flow and many examples are given for the first time. Lastly, a uniform framework for constructing the gradient descent flows of level set surfaces is presented.

2 Notations and Preliminary Materials

In this section, we summarize the notations and basic materials used in the following context. Contents on differential geometry and nonlinear functional analysis are included.

2.1 Differential Geometry for Parametric Surfaces

Let $\mathcal{M} := \{x(u^1, u^2) \in \mathbb{R}^3 : (u^1, u^2) \in \mathcal{D} \subset \mathbb{R}^2\}$ be a sufficiently smooth, regular and orientable parametric surface. Let $g_{\alpha\beta} = \langle x_{\alpha}, x_{\beta}\rangle$ and $b_{\alpha\beta} = \langle n, x_{\alpha}\rangle$ be the coefficients of the first and second fundamental forms of surface $\mathcal{M}$ with

$$x_{\alpha} = \frac{\partial x}{\partial u^\alpha}, \quad x_{\alpha\beta} = \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta}, \quad \alpha, \beta = 1, 2,$$

$$n = (x_u \times x_v)/||x_u \times x_v||, \quad (u, v) := (u^1, u^2),$$
where $(\cdot, \cdot)$, $\| \cdot \|$, $\cdot \times \cdot$ denote the usual inner product, norm and outer product in Euclidean space $\mathbb{R}^3$, respectively. Let

$[g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1}$, $[b^{\alpha\beta}] = [b_{\alpha\beta}]^{-1}$, $g = \det[g_{\alpha\beta}]$, $b = \det[b_{\alpha\beta}]$.

then using these notations, we can define the mean curvature $H$ and Gaussian curvature $K$ of surface $M$ as:

$$H = \frac{1}{2}[g^{\alpha\beta}][b_{\alpha\beta}] \quad \text{and} \quad K = \frac{g}{b},$$

here $A:B$ stands for the trace of $A^TB$. Let

$$H = Hn \quad \text{and} \quad K = Kn,$$

which are named as mean curvature normal and Gaussian curvature normal, respectively. Now let us introduce several used differential operators on surfaces.

**Tangential gradient operator.** Let $f \in C^1(M)$. Then the tangential gradient operator $\nabla_M$ acting on $f$ is given by

$$\nabla_M f = [x_u^1, x_u^2][g^{\alpha\beta}][f_u^1, f_u^2]^T \in \mathbb{R}^3. \quad (1)$$

For a vector-valued function $v = [v_1, v_2, v_3]^T \in C^1(M)^3$, we define

$$\nabla_M (v) = [\nabla_M (v_1), \nabla_M (v_2), \nabla_M (v_3)] \in \mathbb{R}^{3 \times 3}.$$

**Second tangent operator.** Let $f \in C^1(M)$. Then we introduce the second tangent operator $\diamond$ acting on $f$, which is defined by

$$\diamond f = [x_u^1, x_u^2][K b^{\alpha\beta}][f_u^1, f_u^2]^T \in \mathbb{R}^3. \quad (2)$$

**Tangential divergence operator.** Let $v$ be a $C^1$ smooth vector field on $M$. Then the tangential divergence operator $\text{div}_M$ acting on $v$ is defined by

$$\text{div}_M(v) = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2} \right]^T \left[ \sqrt{g} [g^{\alpha\beta}][x_u^\alpha, x_u^\beta]^T v \right]. \quad (3)$$

For a matrix-valued function $Q = [q_1, q_2, q_3] \in C^1(M)^{3 \times 3}$, we define

$$\text{div}_M(Q) = \left[ \text{div}_M(q_1), \text{div}_M(q_2), \text{div}_M(q_3) \right]^T \in \mathbb{R}^3.$$

**Laplace-Beltrami operator (LBO).** Let $f \in C^2(M)$. The Laplace-Beltrami operator (LBO) $\Delta_M$ applying to $f$ is defined by

$$\Delta_M f = \text{div}_M(\nabla_M f).$$
From (1) and (3), we can easily derive that

\[ \Delta Mf = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2} \right] \left[ \sqrt{g} \left[ g^{\alpha\beta} \left[ f_{u^\alpha}, f_{u^\beta} \right]^T \right] = [ g^{\alpha\beta} ] : [ f_{\alpha\beta} ], \right. \]

where

\[ f_{\alpha\beta} = f_{u^\alpha u^\beta} - (\nabla Mf)^T x_{u^\alpha u^\beta}, \quad \alpha, \beta = 1, 2, \]

are the second covariant derivatives of \( f \). It is easy to see that \( \Delta M \) is a second-order differential operator. LBO relates to the mean curvature normal by the relation

\[ \Delta M \mathbf{n} = 2 \mathbf{H}. \]

**Giaquinta-Hildebrandt operator.** Let \( f \in C^2(M) \). Then the Giaquinta-Hildebrandt operator (GHO) \( \Box \) acting on \( f \) is given by

\[ \Box f = \text{div}_M (\Box f). \]

From (2) and (3), we can easily derive that

\[ \Box f = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2} \right] \left[ \sqrt{g} \left[ K b^{\alpha\beta} \left[ f_{u^\alpha}, f_{u^\beta} \right]^T \right] = K \left[ b^{\alpha\beta} \right] : \left[ f_{\alpha\beta} \right]. \right. \]

This operator is a second-order differential operator, which relates to the Gaussian curvature normal by the equation

\[ \Box \mathbf{x} = 2 \mathbf{K}. \]

**Remark 2.1.** All the differential operators introduced above are geometric essential. That is, although they are defined using the local parameterization of surface, they do not depend on the concrete choice of the parameterization. We call this kind of property geometric essential.

**Green’s formula** (see [25]): Let \( \mathbf{v} \) be a three-dimensional smooth vector field on \( M \) and \( f \in C^1(M) \) with compact support. Then

\[ \int_M \langle \mathbf{v}, \nabla_M f \rangle dA = - \int_M f \text{div}_M (\mathbf{v}) dA. \quad (4) \]

### 2.2 Differential Geometry for Implicit Surfaces

Suppose \( M \) is defined by \( M = \{ \mathbf{x} \in \mathbb{R}^3 : \phi(\mathbf{x}) = 0 \} \), where \( \phi \) is a smooth level set function defined on \( \mathbb{R}^3 \). For simplicity, we assume \( ||\nabla \phi|| \neq 0 \) on \( M \). Thus, from the implicit function theorem, \( M \) is a smooth surface and the normal of the surface is well-defined at each of the surface points:

\[ \mathbf{n} = \frac{\nabla \phi}{||\nabla \phi||}. \]
The mean curvature and the Gaussian curvature of $M$ can be represented by level set function as

$$H = -\frac{1}{2} \text{div} \left( \frac{\nabla \phi}{\|\nabla \phi\|} \right) \quad \text{and} \quad K = -\|\nabla \phi\|^{-4} \text{det} \begin{pmatrix} \nabla^2 \phi & \nabla \phi \\ \nabla \phi^T & 0 \end{pmatrix}.$$  

These formulas can be found in [21], where $\nabla$ is the usual gradient operator and $\nabla^2 \phi$ is the gradient of $\nabla \phi$. For a vector-valued function $v = [v_1, v_2, v_3]^T \in \mathbb{R}^3$, we define

$$\nabla(v) = [\nabla(v_1), \nabla(v_2), \nabla(v_3)] \in \mathbb{R}^{3 \times 3}.$$  

**Tangential gradient operator.** Suppose $\|\nabla \phi\| \neq 0$ on some open neighborhood $\Omega$ of the level set surface $M = \{x : \phi(x) = 0\}$, $f$ is a differentiable function on $\Omega$, then the *tangential gradient operator* $\nabla_\phi$ acting on $f$ is defined by

$$\nabla_\phi f = P \nabla f,$$

where $P = I - nn^T$ is the projection operator to the tangential plane of the surface $M$ and $I$ is the identity mapping. Then, $P = P^T = P^2$ is valid. Note that, $\nabla_\phi f$ is well-defined on $\Omega$ everywhere.

**Divergence operator.** Suppose $\|\nabla \phi\| \neq 0$ on some open neighborhood $\Omega$ of the level set surface $M = \{x : \phi(x) = 0\}$, $v$ is a smooth vector field defined on $\Omega$, the *divergence operator* $\text{div}_\phi$ acting on $v$ is defined by

$$\text{div}_\phi(v) = \text{div}(v) - n^T (\nabla v) n,$$

where div is the classical divergence operator. Similar to the definition of the tangential gradient, $\text{div}_\phi(v)$ is well-defined in $\Omega$ everywhere.

**Laplace-Beltrami operator.** Suppose $\|\nabla \phi\| \neq 0$ on some open neighborhood $\Omega$ of the level set surface $M = \{x : \phi(x) = 0\}$ and $f$ is twice differentiable on $\Omega$, then the Laplace-Beltrami operator $\Delta_\phi$ acting on $f$ is defined by

$$\Delta_\phi f = \text{div}_\phi(\nabla_\phi f).$$

**Remark 2.2.** It can be proved that, the tangential gradient operator and Laplace-Beltrami operator defined on level set surfaces are consistent with their counterparts on parametric surfaces and the divergence operator on level set surfaces plays the same role on tangential vector field as the tangential divergence operator on parametric surfaces.

**Theorem 2.1 [Green's formula]** Given $\epsilon > 0$, for any $c \in [-\epsilon, \epsilon]$, let $M_c = \{x : \phi(x) = c\}$ be a closed surface and $\|\nabla \phi(x)\| > 0$, $\forall x \in \Omega := \bigcup_{|c| \leq \epsilon} \{x \in \mathbb{R}^3 : \phi(x) = c\}$. Assume that $v$ is a three-dimensional smooth vector field on $\Omega$ satisfying $\langle v, \nabla \phi \rangle = 0$, $f \in C^1(\Omega)$, then

$$\int_{\Omega} \langle v, \nabla_\phi f \rangle \|\nabla \phi\| \, dx = -\int_{\Omega} f \text{div}_\phi(v) \|\nabla \phi\| \, dx.$$

(5)
Proof. From the Green’s formula (4) for parametric surfaces, we have
\[ \int_{M_c} \langle v, \nabla_{M_c} f \rangle \, dA = - \int_{M_c} f \, \text{div}_{M_c}(v) \, dA. \]
Integrating both sides with respect to the level value \( c \), and noticing \( dA \, dc = \| \nabla \phi \| \, dx \) and Remark 2.2, we obtain (5).

2.3 Nonlinear Functional Analysis

Suppose \( X, Y \) are real normed linear spaces and \( U \) is an open set of \( X \).

Definition 2.1 Mapping \( f : U \to Y \) is called Gâteaux differentiable along the direction \( h \in X \) at \( x_0 \in U \), provided that the limit of
\[ Df(x_0; h) = \lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t} \]
exists. \( Df(x_0; h) \) is called the Gâteaux differential of \( f \) along the direction \( h \) at \( x_0 \). If \( f \) is Gâteaux differentiable along any direction at \( x_0 \), then \( f \) is called Gâteaux differentiable at \( x_0 \).

Definition 2.2 If \( f : U \to Y \) is Gâteaux differentiable at \( x_0 \in U \) and \( Df(x_0; h) \) is linear bounded with respect to \( h \), then \( Df(x_0; h) \) can be represented as
\[ Df(x_0; h) = Df(x_0)h, \quad Df(x_0) \in L(X, Y), \]
where \( L(X, Y) \) denotes the normed linear space consisting of all the bounded linear operators from \( X \) to \( Y \). We call \( Df(x_0) \) the Gâteaux derivative of \( f \) at \( x_0 \).

3 Gradient Descent Flows for Parametric Surfaces

First of all, let us consider a classical minimization problem: searching \( x \), such that
\[ \min f(x), \]
where \( x \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R} \) is a twice continuous differentiable function. As is well-known, a necessary condition for \( x^* \) being the solution of the above problem is that
\[ \nabla f(x^*) = 0. \quad (6) \]
This is a nonlinear system in general with \( n \) equations and \( n \) unknowns. A commonly used method to solve this system is the so-called gradient descent flow approach, i.e., solving the following initial value problem of an ordinary differential equation
\[ \frac{dx(t)}{dt} = - \nabla f(x), \quad x(0) = x_0. \quad (7) \]
When the steady state solution is achieved, i. e.,
\[
\frac{dx(t)}{dt} = 0,
\]
we therefore obtain
\[
\nabla f(x^*) = 0.
\]

The effectiveness of this method can be made clearer by the following theorem (see [1]).

**Theorem 3.1** Consider that \( x^* \) is a point satisfying (6). Suppose that \( \nabla^2 f(x^*) \) is positive definite. If \( x_0 \) is close enough to \( x^* \), then \( x(t) \), solution of (7), tends to \( x^* \) as \( t \) goes to infinity.

The following theorem further shows that \( f(x(t)) \) is strictly decreasing.

**Theorem 3.2** Let \( x(t) \) be the solution of (7). For a fixed \( t_0 \) if \( \nabla f(x(t)) \neq 0 \) for all \( t > t_0 \), then \( f(x(t)) \) is strictly decreasing with respect to \( t \), for all \( t > t_0 \).

The idea described above for searching a minimizer can be used to construct geometric flows. Now we exposit this in detail.

### 3.1 \( L^2 \) Gradient Flows for Parametric Surfaces

Let \( M \) be a smooth parametric surface defined on \( \mathcal{D} \). Assume that the functions considered here are smooth on \( M \). Let \( L^2(M) \) be the space of measurable functions \( f \) on \( M \) for which
\[
\int_M |f|^2 dA < +\infty.
\]

In the space \( L^2(M) \), we introduce the usual inner product, and induced norm, given by
\[
(f, h) = \int_M fhdA, \quad \|f\|^2 = (f, f)
\]
for \( f, h \in L^2(M) \). With this inner product, \( L^2(M) \) is a Hilbert space. Similarly, for vector-valued functions \( f, g \in \mathbb{R}^3 \) defined on \( M \), we define inner product
\[
(f, g) = \int_M \langle f, g \rangle dA.
\]

The complete space induced by this inner product is a Hilbert space, denoted by \( \mathcal{L}^2(M) \). Let
\[
\mathcal{E}(M) = \int_M f(x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, \ldots) dA
\]
be a given energy functional, where \( x_u = \frac{\partial x}{\partial u} \), \( x_v = \frac{\partial x}{\partial v} \), and \( \cdots \) denotes the second or higher order derivatives of \( x \) possibly appeared. Let
\[
\mathcal{M}_e = \{ x_e = x + e\Theta : x \in \mathcal{M}_f \}, \quad \Theta \in C_0^\infty(M)^3, \quad |e| \ll 1,
\]
thus,

\[
\delta(\mathcal{E}(\mathcal{M}), \Theta) = \frac{d\left[\int_{\mathcal{M}} f(x + \epsilon \Theta, x_u + \epsilon \Theta_u, x_v + \epsilon \Theta_v, x_{uv} + \epsilon \Theta_{uv}, \cdot \cdot \cdot) dA\right]}{d\epsilon}\bigg|_{\epsilon = 0}
\]

\[
= \int_{\mathcal{M}} [\nabla_x f]^T \Theta + (\nabla_{x_u} f)^T \Theta_u + (\nabla_{x_v} f)^T \Theta_v] dA
\]

\[
+ \int_{\mathcal{M}} [\nabla_{x_{uv}} f]^T \Theta_{uv} + (\nabla_{x_v} f)^T \Theta_{uv} + \cdot \cdot \cdot] dA
\]

\[
+ \int_{\mathcal{M}} [f \ \text{tr}(\nabla_{\mathcal{M}} \Theta)] dA,
\]

where \(\nabla_y f\) is the gradient of \(f\) with respect to \(y\) (\(y = x, x_u, x_v, \cdot \cdot \cdot\)). Therefore, \(\delta(\mathcal{E}(\mathcal{M}), \Theta)\) is a linear functional defined on \(L^2(\mathcal{M})\). From Riesz representation theorem, there exists an element in \(L^2(\mathcal{M})\), denoted as \(E'_c(\mathcal{M})\), such that

\[
\delta(\mathcal{E}(\mathcal{M}), \Theta) = (E'_c(\mathcal{M}), \Theta).
\]

In the above equation, \(\delta(\mathcal{E}(\mathcal{M}), \Theta)\) is defined as the \textit{first-order complete variation} of \(\mathcal{E}(\mathcal{M})\) along the direction \(\Theta\), \(E'_c(\mathcal{M})\) is called the \(L^2\) \textit{gradient} or \textit{Euler–Lagrange operator} of \(\mathcal{E}(\mathcal{M})\).

\[
E'_c(\mathcal{M}) = 0
\]

is the Euler–Lagrange equation of functional (8). Then the necessary condition of \(\mathcal{M}^*\) being the minimizer of \(\mathcal{E}(\mathcal{M})\) is that \(\mathcal{E}'_c(\mathcal{M}^*) = 0\). To solve this Euler–Lagrange equation, we construct the following weak form gradient flow

\[
\int_{\mathcal{M}} \langle \frac{\partial x}{\partial t}, \Theta \rangle dA = -\int_{\mathcal{M}} \langle E'_c(\mathcal{M}), \Theta \rangle dA. \quad (9)
\]

This weak form is the bases of the finite element method. Subsequently, we obtain the geometric flow in the sense of \(L^2(\mathcal{M})\)

\[
\frac{\partial x}{\partial t} = -E'_c(\mathcal{M}) \in \mathbb{R}^3. \quad (10)
\]

The reason that minus symbol is selected before equation (9) is that the first-order variation is the increasing velocity of the energy and therefore minus symbol selection decreases the energy.

Let \(\mathbf{n}, \mathbf{e}_1, \mathbf{e}_2\) be an orthonormal frame of surface \(\mathcal{M}\), then \(\Theta\) can be decomposed as

\[
\Theta = \vartheta \mathbf{n} + \alpha \mathbf{e}_1 + \beta \mathbf{e}_2, \quad \vartheta, \alpha, \beta \in C^\infty(\mathcal{M}).
\]

Thus under the assumption that \(\mathcal{E}(\mathcal{M})\) is geometric essential, we can prove

\[
\mathbf{e}_1^T E'_c(\mathcal{M}) = \mathbf{e}_2^T E'_c(\mathcal{M}) = 0.
\]
Therefore (9) turns out to be
\[
\int_M \left( \frac{\partial x}{\partial t} \cdot \partial n + \alpha e_1 + \beta e_2 \right) dA = -\int_M \mathcal{E}'_n(M) \partial n dA,
\]
where
\[\mathcal{E}'_n(M) = n^T \mathcal{E}'_c(M).\]
Considering the parameters \(\vartheta, \alpha, \beta\) can be freely chosen, we therefore have
\[e_1^T \frac{\partial x}{\partial t} = e_2^T \frac{\partial x}{\partial t} = 0.\]
Thus (9) becomes
\[
\int_M \left( \frac{\partial x}{\partial t} \cdot \partial n \right) dA = -\int_M \mathcal{E}'_n(M) \partial n dA, \quad \vartheta \in C^\infty_0(M).
\]
The corresponding geometric flow in the sense of \(L^2(M)\) inner product is therefore
\[
\frac{\partial x}{\partial t} = -\mathcal{E}'_n(M)n \in \mathbb{R}^3. \tag{11}
\]
This kind of geometric flows is frequently used.

**Remark 3.1.** \(\mathcal{E}'_c(M)\) in (10) is independent of surface orientation, while \(\mathcal{E}'_n(M)\) depends on it. However, \(\mathcal{E}'_n(M)n\) no longer relates to the surface orientation.

The choice of function \(f\) in (8) is almost free. But the following special choices can yield a large number of commonly used geometric flows.

**Case 1:** Choose \(f = f(x, n)\). The geometric flow (10) in the \(L^2(M)\) sense as we have derived in [25] is
\[
\frac{\partial x}{\partial t} = -(\nabla_x f + \text{div}_M(\nabla_n f)n + \nabla_M(n \nabla_n f - \text{div}_M(f \nabla_M x))). \tag{12}
\]
Geometric flow (11) in the \(L^2(M)\) sense is
\[
\frac{\partial x}{\partial t} = -(n^T \nabla_x f + \text{div}_M(\nabla_n f) - 2fH)n. \tag{13}
\]

**Example 1. Mean curvature flow.** Let \(f(x, n) = 1\). Then (12) turns out to be
\[
\frac{\partial x}{\partial t} = \Delta_M x,
\]
and (13) becomes
\[
\frac{\partial x}{\partial t} = 2H.
\]
This is the classical mean curvature flow. Although the research of minimal surface, the steady solution of the mean curvature flow, can be dated back to 250 years ago, the mean curvature
flow is originally introduced by Mullins in 1956 (see [18]). This flow has been deeply studied in theory and plenty of results have been obtained. Furthermore, it has also been used in many application areas.

**Example 2. Weighted mean curvature flow.** Let \( f(x, n) \) be a positively homogeneous function of degree \( t \) with respect to the second variable, i.e.,

\[
 f(x, \lambda n) = \lambda^t f(x, n), \quad \forall x \in \mathbb{R}^3, n \in \mathbb{R}^3 \setminus \{0\}, \quad \lambda > 0.
\]

Then (13) becomes

\[
 \frac{\partial x}{\partial t} = -(n^T \nabla_x f + 2fH(t-1) + \nabla_M(x)(\nabla_{xn}^2 f) + \nabla_M n(\nabla_{nn}^2 f)) n,
\]

where

\[
 \nabla_{xn}^2 f = \nabla_x \nabla_n f \in \mathbb{R}^{3 \times 3}, \quad \nabla_{nn}^2 f = \nabla_n \nabla_n f \in \mathbb{R}^{3 \times 3}.
\]

The detail derivation of equation (14) can be found in [25]. In particular, when \( t = 1 \), Taylor in [22] named the scalar part of the right hand side of (14) as weighted mean curvature. But considering the first part relates to the position vector, the authors in [7, 27] took the last term, i.e., \( \nabla_M n(\nabla_{nn}^2 f) \) as the definition of the weighted mean curvature. This flow has been used in anisotropic surface processing (see [8]), computer vision (see [16]), and weighted minimal surface research (see [27]).

**Case 2:** Choose \( f = f(H, K) \). As we have derived in [25], geometric flow (10) is

\[
 \frac{\partial x}{\partial t} = - (\Delta_M f + 1) \nabla_M f - \text{div}_M (f \nabla_M n) - \text{div}_M [(f - 2KfK) \nabla_M x],
\]

Based on [15], (11) is

\[
 \frac{\partial x}{\partial t} = -(\Box fK + \frac{1}{2} \Delta_M fH + \text{div}_M (fH \nabla_M n) - \text{div}_M [(f - 2KfK) \nabla_M x]).
\]

**Example 3. Willmore flow.** Let \( f = H^2 \), then (16) turns out to be

\[
 \frac{\partial x}{\partial t} = -(\Delta_M H + 2H(H^2 - K)) n.
\]

This is the well-known Willmore flow. There exists a lot of studies on this flow and it has been widely used in the field of computational geometry.

**Example 4. Gauss curvature flow.** Let \( f = H \), then (15) becomes

\[
 \frac{\partial x}{\partial t} = \frac{1}{2} \Box x.
\]

(16) turns out to be

\[
 \frac{\partial x}{\partial t} = K.
\]
This is the Gauss curvature flow, which was introduced by Firey [14] in 1974 as a model of the wearing process undergone by a pebble on a beach. Notice that when the surface is non-convex, the Lagrange function is not always positive. Thus the minimizing process is meaningless and ill-posed. Therefore the initial surface is always required to be strictly convex (see [6, 23]) or convex (see [2]). To the best of the authors’ knowledge, this is the first time to understand the Gauss curvature flow from gradient flow viewpoint.

Case 3: Choose \( f = \sum_{\ell=1}^{m} ||\nabla_{M} f^{\ell}(H, K)||^2 \). Geometric flow (11), as derived in [17], becomes

\[
\frac{\partial x}{\partial t} = -\left[ \sum_{\ell=1}^{m} (2H||\nabla_{M} f^{\ell}||^2 - 2\langle \nabla_{M} f^{\ell}, \nabla f^{\ell} \rangle - 4HK f^{\ell}_{K} \Delta_{M} f^{\ell} - 2(2H^2 - K) f^{\ell}_{H} \Delta_{M} f^{\ell} - 2\Box(f^{\ell}_{H} \Delta_{M} f^{\ell} - \Delta_{M}(f^{\ell}_{H} \Delta_{M} f^{\ell})) \right] n.
\]

which has been used in surface processing.

Example 5. Minimal mean-curvature-variation flow. Let \( f = ||\nabla_{M} H||^2 \). In [26] we derive

\[
\frac{\partial x}{\partial t} = (\Delta_{M}^2 H + 2(2H^2 - K)\Delta_{M} H + 2\langle \nabla_{M} H, \nabla H \rangle - 2H ||\nabla_{M} H||^2 ) n.
\]

and use this flow in surface modeling.

The \( L^2 \) gradient flow discussed above originates from the classical inner product. If we use other types of inner product, we can derive different geometric flows for the same first-order variation of an identical energy functional. In the following, we will take two cases into account.

### 3.2 \( H^{-1} \) Gradient Flows for Parametric Surfaces

For simplicity, we consider a compact and closed surface \( M \) without boundary. Define an inner product as

\[
(f, h)_{H^{-1}} = \int_{M} \langle \nabla_{M}(\Delta_{M}^{-1} f), \nabla_{M}(\Delta_{M}^{-1} h) \rangle dA, \quad \forall f, h \in \hat{H}(M),
\]

where

\[
\hat{H}(M) := \left\{ f : \int_{M} f dA = 0 \right\}
\]

\( \Delta_{M}^{-1} f : \varphi_{f} \) denotes a function on \( M \), satisfying the Poisson differential equation on \( M \)

\[
\Delta_{M} \varphi_{f} = f.
\]

For this equation, there exists a unique solution \( \varphi_{f} \) up to a constant (see [3], page 104). To assure the inner product \( (f, h)_{H^{-1}} \) is well-defined, we need to verify that \( (f, f)_{H^{-1}} = 0 \) iff \( f = 0 \). On the one hand, if

\[
(f, f)_{H^{-1}} = \int_{M} \langle \nabla_{M}(\Delta_{M}^{-1} f), \nabla_{M}(\Delta_{M}^{-1} f) \rangle dA = 0,
\]
then we have

$$\nabla_M(\Delta^{-1}_M f) = 0.$$ 

Therefore

$$0 = \text{div}_M[\nabla_M(\Delta^{-1}_M f)] = \Delta_M(\Delta^{-1}_M f) = f.$$ 

On the other hand, if \( f = 0 \),

\[
(f, f)_{H^{-1}} = \int_M \langle \nabla_M(\Delta^{-1}_M f), \nabla_M(\Delta^{-1}_M f) \rangle dA \\
= - \int_M \text{div}_M[\nabla_M(\Delta^{-1}_M f)]\Delta^{-1}_M f dA \\
= - \int_M f\Delta^{-1}_M f dA \\
= 0.
\]

We denote the space \( \hat{H}(M) \) equipped with inner product \((\cdot, \cdot)_{H^{-1}} \) by \( H^{-1}(M) \) after completion, which is a Hilbert space. In the same way, in space

$$\hat{H}(M) := \left\{ f \in \mathbb{R}^3 : \int_M f dA = 0 \right\},$$

we can define inner product

$$\langle f, h \rangle_{H^{-1}} = \int_M \nabla_M(\Delta^{-1}_M f):\nabla_M(\Delta^{-1}_M h) dA,$$

where \( \Delta^{-1}_M f := \Phi_t \) denotes a vector-valued function on \( M \), satisfying the following Poisson differential equation on \( M \)

$$\Delta_M \Phi_t = f,$$

This kind of function \( \Phi_t \) exists uniquely up to a constant vector and the inner product is well-defined. We denote space \( \hat{H}(M) \) with inner product \((\cdot, \cdot)_{H^{-1}} \) by \( H^{-1}(M) \) after completion, which is a Hilbert space.

In analogy with \( L^2 \) gradient flow before, we take complete variation of surface into account first. Let \( \Theta \in \hat{H}(M) \cap C_0^\infty(M)^3 \). We can obtain weak \( H^{-1} \) gradient flow from (9)

$$\left\langle \frac{\partial x}{\partial t}, \Theta \right\rangle_{H^{-1}} = \langle \Delta_M \phi'_c(M), \Theta \rangle_{H^{-1}},$$

and \( H^{-1} \) gradient flow

$$\frac{\partial x}{\partial t} = \Delta_M \phi'_c(M) \in \mathbb{R}^3.$$ (17)
Let $\delta_n(\mathcal{E}(M), \theta)$ be the first-order normal direction variation of $\mathcal{E}(M)$ with $\theta \in \dot{H}(M) \cap C_0^\infty(M)$. Thus

$$
\delta_n(\mathcal{E}(M), \theta) = \int_M \mathcal{E}_n'(M) \theta \, dA
= \int_M \Delta_M^{-1} [\Delta_M(\mathcal{E}_n'(M))] \Delta_M(\Delta_M^{-1}\theta) \, dA
= \int_M -\langle \nabla_M \Delta_M^{-1} [\Delta_M(\mathcal{E}_n'(M))], \nabla_M(\Delta_M^{-1}\theta) \rangle \, dA
= -\langle \Delta_M(\mathcal{E}_n'(M)), \theta \rangle_{H^{-1}}.
$$

Subsequently we construct $H^{-1}$ gradient flow in a weak sense and $H^{-1}$ gradient flow as follows

$$
\left( n^T \frac{\partial x}{\partial t}, \theta \right)_{H^{-1}} = \langle \Delta_M(\mathcal{E}_n'(M)), \theta \rangle_{H^{-1}},
$$

$$
\frac{\partial x}{\partial t} = \Delta_M(\mathcal{E}_n'(M)) \, n.
$$

(18)

**Example 6. Quasi surface diffusion flow.** Let $f(x, n) = 1$. Then $\mathcal{E}_c'(M) = -\Delta_M x$. Thus from (17), we obtain quasi surface diffusion flow

$$
\frac{\partial x}{\partial t} = -\Delta_2 M x.
$$

The flow was introduced in [24] for surface processing and yields desirable results.

**Example 7. Surface diffusion flow.** Let $f(x, n) = 1$. Then $\mathcal{E}_c'(M) = -2H$. From (18), we obtain the well-known surface diffusion flow

$$
\frac{\partial x}{\partial t} = -2\Delta_M H n.
$$

This is a fourth-order partial differential equation, which was introduced by Mullins ([19]) in 1957 to study the kinetic laws of the interface during grain growth in physics. It should be pointed out that it was Fife who first regarded the surface diffusion flow as the gradient descent flow of area functional in [12, 13].

### 3.3 $H^{-2}$ Gradient Flows for Parametric Surfaces

The $H^{-2}$ gradient flow introduced in this subsection can be used to explain the rationality of several existing sixth-order geometric flows. In analogy with the subsection above, we define inner product

$$
(f, h)_{H^{-2}} = \int_M \Lambda^{-1}_M f \Lambda^{-1}_M h \, dA, \quad \forall f, h \in \dot{H}(M),
$$

where $\Lambda^{-1}_M$ is the inverse of the Laplace-Beltrami operator $\Lambda_M$.
and denote $\check{H}(M)$ with inner product $(\cdot, \cdot)_{H^2}$ by $H^{-2}(M)$ after completion. In the same way, we can define the following inner product

$$(f, h)_{H^{-2}} = \int_M \langle \Delta_M^{-1} f, \Delta_M^{-1} h \rangle dA,$$

in $\check{H}(M)$ and denote it $H^{-2}(M)$. Spaces $H^{-2}(M)$ and $H^{-2}(M)$ are all Hilbert spaces.

In analogy with $L^2$ gradient flow, we consider complete variation of surface first. Let $\Theta \in \check{H}(M) \cap C^\infty_0(M)$. Similarly to (9), we can construct weak form $H^{-2}$ gradient flow as

$$\langle \frac{\partial x}{\partial t}, \Theta \rangle_{H^{-2}} = -\langle \Delta^2_M (E'_c(M)), \Theta \rangle_{H^{-2}},$$

and $H^{-2}$ gradient flow

$$\frac{\partial x}{\partial t} = -\Delta^2_M (E'_c(M)) \in \mathbb{R}^3.$$  \hspace{.5cm} (19)

Let $\delta_n(E(M), \theta)$ be the first-order normal direction variation of $E(M)$ with $\theta \in \check{H}(M) \cap C^\infty_0(M)$. Then

$$\delta_n(E(M), \theta) = \int_M E'_n(M) \theta dA$$

$$= \int_M \Delta^{-1}_M [\Delta_M(E'_n(M))] \Delta_M(\Delta^{-1}_M \theta) dA$$

$$= \int_M \Delta^{-1}_M [\Delta^2_M(E'_n(M))] (\Delta^{-1}_M \theta) dA$$

$$= \left( \Delta^2_M(E'_n(M)), \theta \right)_{H^{-2}}.$$ 

Therefore the weak form $H^{-2}$ gradient flow and $H^{-2}$ gradient flow are

$$\left( n^T \frac{\partial x}{\partial t}, \theta \right)_{H^{-2}} = -\left( \Delta^2_M (E'_n(M)), \theta \right)_{H^{-2}}$$

and

$$\frac{\partial x}{\partial t} = -\Delta^2_M (E'_n(M)) n,$$  \hspace{.5cm} (20)

respectively.

**Example 8. Quasi Xuguo flow.** Let $f(x, n) = 1$. Then $E'_n(M) = -\Delta_M x$. From (19), we obtain

$$\frac{\partial x}{\partial t} = \Delta^3_M x.$$

The flow was introduced in [24] and is used for $C^2$ surface modeling. For appellation convenience, we name this flow as quasi Xuguo flow. In [4], the steady status form of this equation is
used. It has been illustrated by experiments that this flow is more powerful in surface processing than second-order and fourth-order flows. And it was the first time to deal with surfaces with such a high order flow.

**Example 9. Xuguo flow.** Let $f(x, n) = 1$. Then $\mathcal{E}_n(M) = -2H$. From (20), we obtain

$$\frac{\partial x}{\partial t} = -2\Delta^2_M H n.$$ 

This is a sixth-order differential equation (see [24]). For convenience, we name it Xuguo flow. Experiments indicate that this flow can achieve $G^2$ continuity on boundary. It is such a higher continuity that can fulfill the requirements of practical engineering applications such as the shape design of streamlined surfaces of aircrafts, ships and cars. When the general normal deformation of surface is studied in [20], the short time existence problem of this flow is referred.

### 4 Gradient Descent Flows for Level Set Surfaces

Now let us consider gradient descent flows for level set surfaces. In equation (8), the energy functional is defined on a parametric surface $M$. Assume now $M := \{x : \phi(x) = 0\}$ and $\|\nabla \phi\| \neq 0$ on $M$. To compute the variation of the energy functional for level set surfaces, we should extend the domain of the energy functional to a three-dimensional region around $M$. Two approaches are available. One is introducing the Dirac’s $\delta$ function $\delta(x)$, which is defined as the derivative of Heaviside function

$$\mathcal{H}(x) = \begin{cases} 
1, & x \geq 0, \\
0, & x < 0
\end{cases}$$

in the distribution sense. Thus, energy functional can be written as

$$\mathcal{E}(\phi) = \int_M f(x)dA = \int_{\mathbb{R}^3} f(x)\delta(\phi(x))dx.$$ 

The other approach is considering a family of level set surfaces

$$M_c = \{x : \phi(x) = c\}, \ c \in [-\varepsilon, \varepsilon].$$

Therefore, using co-area formula (see [10]), the energy functional can be defined as

$$\mathcal{E}_\varepsilon(\phi) = \int_{-\varepsilon}^\varepsilon \left[ \int_{M_c} f(x)dA \right] dc = \int_{\Omega} f(x)\|\nabla \phi(x)\|dx,$$

where

$$\Omega = \bigcup_{c \in [-\varepsilon, \varepsilon]} \{x \in \mathbb{R}^3 : \phi(x) = c\}.$$
Since \(\|\nabla \phi(x)\| > 0, \forall x \in M\), we can assume \(\|\nabla \phi(x)\| > 0, \forall x \in \Omega\) without loss of generality. It is easy to see that, as \(\varepsilon \rightarrow 0\),
\[
\frac{1}{2\varepsilon} \mathcal{E}_\varepsilon(\phi) \rightarrow \mathcal{E}(\phi).
\]
Computational results indicate that these two kinds of functionals yield the identical Euler–Lagrange equations. But the computation of the second energy functional is simpler, we adapt it.

### 4.1 \(L^2\) Gradient Flows for Level Set Surfaces

To address the dependence of \(f\) on \(\phi\) and its partial derivatives of any order, we denote it by
\[
f(x) = F(\phi(x), \nabla \phi(x), \nabla^2 \phi(x), \cdots).
\]
Let
\[
\phi_\varepsilon(x) = \phi(x) + \varepsilon \psi(x), \quad |\varepsilon| \ll 1.
\]
Then,
\[
\delta_\phi(\mathcal{E}_\varepsilon(\psi), \psi) = \mathcal{E}_\varepsilon(\phi) = \int_{\Omega} \frac{d}{d\varepsilon} F(\phi_\varepsilon, \nabla \phi_\varepsilon, \nabla^2 \phi_\varepsilon, \cdots) \|\nabla \phi_\varepsilon\| \bigg|_{\varepsilon=0} \, dx,
\]
with \(\psi \in C^\infty_0(\Omega)\). Thus, \(\delta_\phi(\mathcal{E}_\varepsilon(\psi), \psi)\) is a linear functional defined on \(H(\Omega)\). Here \(H(\Omega)\) is the Hilbert space consists of all continuous functions on \(\Omega\) under the inner product
\[
(g, h) = \int_{\Omega} gh \, dx.
\]
Obviously, \(H(\Omega) \subset L^2(\Omega)\). From Riesz representation theorem, there exists an element in \(H(\Omega)\), denoted as \(\mathcal{E}_\varepsilon'(\phi)\), such that
\[
\delta_\phi(\mathcal{E}_\varepsilon(\phi), \psi) = \int_{\Omega} \mathcal{E}_\varepsilon'(\phi) \psi \, dx = (\mathcal{E}_\varepsilon'(\phi), \psi).
\]
\(\mathcal{E}_\varepsilon'(\phi)\) in the above equation is called the \(L^2\) gradient or Euler–Lagrange operator of \(\mathcal{E}_\varepsilon(\phi)\).

Thus a necessary condition for \(\phi^*\) being the minimizer of \(\mathcal{E}_\varepsilon(\phi)\) is that \(\mathcal{E}_\varepsilon'(\phi^*) = 0\). We call
\[
\mathcal{E}_\varepsilon'(\phi) = 0
\]
the Euler–Lagrange equation. To solve this equation, let
\[
x_\varepsilon = x + \varepsilon n(x) \theta(x) + O(\varepsilon^2)
\]
be the motion of $x$ in the normal direction of $M$ caused by the movement of $\phi$ to $\phi + \epsilon\psi$.

Differentiating both sides of the equation

$$\phi(x_\epsilon) + \epsilon\psi(x_\epsilon) = c$$

with respect to $\epsilon$, then taking $\epsilon = 0$, we have

$$[\nabla \phi(x)]^T \mathbf{n}(x) \theta(x) + \psi(x) = 0.$$  

Thus

$$\theta(x) = -\frac{\psi(x)}{||\nabla \phi(x)||},$$

and

$$x_\epsilon = x - \epsilon \mathbf{n}(x) \frac{\psi(x)}{||\nabla \phi(x)||} + O(\epsilon^2).$$

This equation indicates that, the first-order variation $\delta_x \phi_x(\epsilon, \psi)$ of energy functional $\phi_x(\phi)$ with respect to $\phi$ along the direction $\psi$ coincides with the first-order variation

$$\delta_x \left( \phi_x(\phi), -\mathbf{n}(x) \frac{\psi(x)}{||\nabla \phi||} \right) = \int_{-\epsilon}^{\epsilon} \left( \int_{M_\epsilon} \frac{d[f(x - \epsilon \mathbf{n}(x) \frac{\psi(x)}{||\nabla \phi||})]}{d\epsilon} \bigg|_{\epsilon=0} \right) dA \big|_{\epsilon=0}$$

of the energy with respect to $x$. Therefore we construct a weak form geometric flow as follows

$$\int_{-\epsilon}^{\epsilon} \left( \int_{M_\epsilon} \frac{\partial x}{\partial t}, -\mathbf{n} \frac{\psi(x)}{||\nabla \phi||} \right) dA = -\delta_x \left( \phi_x(\phi), -\mathbf{n}(x) \frac{\psi(x)}{||\nabla \phi||} \right)$$

or

$$\left( -\mathbf{n} \frac{\partial x}{\partial t}, \psi \right) = -(\phi_x^\epsilon(\phi), \psi).$$

Since $\mathbf{n} = \frac{\nabla \phi}{||\nabla \phi||}$, the above equation can be written as

$$\left( -(\nabla \phi)^T \frac{\partial x}{\partial t} ||\nabla \phi||^{-1}, \psi \right) = -(\phi_x^\epsilon(\phi), \psi). \quad (21)$$

Now we write $\phi$ as $\phi(x(t), t)$ to address its dependence on time $t$. Let us consider the level set equation

$$\phi(x(t), t) = c.$$

Differentiating both sides with respect to $t$, we have

$$\frac{\partial \phi}{\partial t} + (\nabla \phi)^T \frac{\partial x}{\partial t} = 0. \quad (22)$$
Substituting (22) into (21), we obtain
\[
\left( \frac{\partial \phi}{\partial t} \|\nabla \phi\|^{-1}, \psi \right) = -\left( \varepsilon'_x(\phi), \psi \right).
\] (23)

This weak form is the basis of finite element method. Then we get the following geometric flow
\[
\frac{\partial \phi}{\partial t} = -\varepsilon'_x(\phi)\|\nabla \phi\|.
\] (24)

which is the motion equation of \( \phi \).

Similar to subsection 3.1, there are infinitely many possibilities for selecting the Lagrange function. Here we merely list several special choices.

**Case 1**: Choose \( f = f(x, n) \). Then the weak form geometric flow (23) is
\[
\left( \frac{\partial \phi}{\partial t} \|\nabla \phi\|^{-1}, \psi \right) = -\int_{\Omega} \left[ f \left( \frac{\langle \nabla \phi, \nabla \psi \rangle}{\|\nabla \phi\|} - \frac{\langle \nabla_x f, \nabla \phi \rangle}{\|\nabla \phi\|} \right) + \langle \nabla_n f, P \nabla \psi \rangle \right] \|\nabla \phi\| \, dx,
\]
and geometric flow (24) is
\[
\frac{\partial \phi}{\partial t} = \left[ \text{div} \left( \frac{f \nabla \phi}{\|\nabla \phi\|} \right) + \frac{\langle \nabla_x f, \nabla \phi \rangle}{\|\nabla \phi\|} \right] \|\nabla \phi\|.
\] (25)

**Example 10. Mean curvature flow.** Let \( f = 1 \). Then from (25) we obtain the classical mean curvature flow
\[
\frac{\partial \phi}{\partial t} = \text{div} \left( \frac{\nabla \phi}{\|\nabla \phi\|} \right) \|\nabla \phi\|.
\]
The flow has been deeply studied in the literatures (see [5], [11] for references).

**Case 2**: Choose \( f = f(H, K) \). The corresponding weak form geometric flow (23) is
\[
\left( \frac{\partial \phi}{\partial t} \|\nabla \phi\|^{-1}, \psi \right) = -\int_{\Omega} \left[ f \left( \frac{\langle \nabla \phi, \nabla \psi \rangle}{\|\nabla \phi\|} - \frac{\langle \nabla_x f, \nabla \phi \rangle}{\|\nabla \phi\|} \right) + \langle \nabla_n f, P \nabla \psi \rangle \right] \|\nabla \phi\| \, dx,
\]
and geometric flow (24) is
\[
\frac{\partial \phi}{\partial t} = \left[ \text{div} \left( \frac{f \nabla \phi}{\|\nabla \phi\|} \right) + \frac{\langle \nabla_x f, \nabla \phi \rangle}{\|\nabla \phi\|} - \text{div} \left( \frac{4fK \nabla \phi}{\|\nabla \phi\|} \right) \right] \|\nabla \phi\|.
\] (25)

where
\[
\Psi = \begin{pmatrix} \nabla^2 \psi & \nabla \psi \\ \nabla \psi^T & 0 \end{pmatrix}, \quad A = \begin{pmatrix} B & B_1 \\ B_2^T & B_2 \end{pmatrix}
\] is the adjoint of matrix of
\[
\begin{pmatrix} \nabla^2 \phi & \nabla \phi \\ \nabla \phi^T & 0 \end{pmatrix},
\]
\( B \in \mathbb{R}^{3 \times 3} \), \( B_1 \in \mathbb{R}^3 \) and \( B_2 \in \mathbb{R}^3 \).
Example 11. Willmore flow. Let $f = H^2$. From (26) we obtain

$$\frac{\partial \phi}{\partial t} = \text{div} \left( \frac{H^2 \nabla \phi}{\|\nabla \phi\|} \right) + \text{div} \left( \frac{P \nabla (H \|\nabla \phi\|)}{\|\nabla \phi\|} \right) \|\nabla \phi\|.$$  

This flow is introduced by Droske and Rumpf in [9] and named as the level set form of Willmore flow.

Case 3: Choose $f = \sum_{\ell=1}^m \|\nabla \phi f^\ell(H, K)\|^2$. Then the corresponding geometric flow (23) is

$$\left( \frac{\partial \phi}{\partial t}, \nabla \phi \right)^{-1}, \psi \right) = -\int_\Omega \sum_{\ell=1}^m \left( \frac{\langle \nabla \phi, \nabla \psi \rangle \|\nabla \phi f^\ell\|^2}{\|\nabla \phi\|} - \frac{2\langle \nabla f^\ell, \nabla \phi \rangle \langle \nabla \phi f^\ell, \nabla \psi \rangle}{\|\nabla \phi\|} \right) dx$$

and geometric flow (24) is

$$\frac{\partial \phi}{\partial t} = \sum_{\ell=1}^m \text{div} \left( \frac{\|P \nabla f^\ell\|^2 \nabla \phi}{\|\nabla \phi\|} \right) - 2 \text{div} \left( \frac{\langle \nabla f^\ell, \nabla \phi \rangle P \nabla f^\ell}{\|\nabla \phi\|} \right)$$

$$- \text{div} \left( \frac{P \frac{\partial f^\ell}{\partial x} \text{div} (\|\nabla \phi\| P \nabla f^\ell)}{\|\nabla \phi\|^2} \right) + 8 \text{div} \left( \frac{f^\ell K \text{div} (\|\nabla \phi\| P \nabla f^\ell) \nabla \phi}{\|\nabla \phi\|^2} \right)$$

$$- 2 \text{div} \left( \frac{f^\ell K \text{div} (\|\nabla \phi\| P \nabla f^\ell) B_1}{\|\nabla \phi\|^4} \right) + 4 \text{div} \left( \frac{f^\ell K \text{div} (\|\nabla \phi\| P \nabla f^\ell) B_1}{\|\nabla \phi\|^4} \right) \|\nabla \phi\|. \quad (27)$$

Example 12. Minimal mean-curvature-variation flow. Let $\ell = 1$ and $f^\ell = H$. From (27) we obtain

$$\frac{\partial \phi}{\partial t} = \left[ \text{div} \left( \frac{\|P \nabla H\|^2 \nabla \phi}{\|\nabla \phi\|} \right) - 2 \text{div} \left( \frac{\langle \nabla H, \nabla \phi \rangle P \nabla H}{\|\nabla \phi\|} \right) - \text{div} \left( \frac{P \nabla \text{div} (\|\nabla \phi\| P \nabla H)}{\|\nabla \phi\|} \right) \right] \|\nabla \phi\|.$$  

This flow is named as minimal mean-curvature-variation flow.

4.2 $H^{-1}$ and $H^{-2}$ Gradient Flows for Level Set Surfaces

The $L^2$ gradient flow discussed above originates from classical inner product. If we utilize other types of inner products, we can get different geometric flows from the same first-order variation of the given energy. In the following, we consider two cases. First, let us define $H^{-1}$ inner product

$$(f, h)_{H^{-1}} = \int_\Omega \langle \nabla \phi (\Delta_\phi^{-1} f), \nabla \phi (\Delta_\phi^{-1} h) \rangle \|\nabla \phi\| dx, \quad \forall f, h \in \hat{H}(\Omega),$$

where $\hat{H}(\Omega) := \{ f : \int_\Omega f dx = 0 \}$, $\Delta_\phi^{-1} f := \varphi_f$ denotes a function on $\Omega$ satisfying the Poisson differential equation

$$\Delta \varphi_f = f.$$
There exists a unique function $\varphi_f$ up to a constant. We denote the space $\hat{H}(\Omega)$ with inner product $(\cdot, \cdot)_{H^{-1}}$ by $H^{-1}(\Omega)$ after completion, which is a Hilbert space. Similarly, we can define $H^{-2}$ inner product

$$(f, h)_{H^{-2}} = \int_\Omega \Delta_{\phi}^{-1} f \Delta_{\phi}^{-1} h \|\nabla \phi\| \, dx, \quad \forall f, h \in \hat{H}(\Omega),$$

and denote the Hilbert space induced by this inner product as $H^{-2}(\Omega)$. Thus from

$$\delta \varphi(E(\varphi), \psi) = \int_\Omega E'_{\varepsilon}(\varphi) \psi \, dx$$

$$= \int_\Omega \Delta_{\phi}^{-1} \left[\Delta \varphi \left(E'_{\varepsilon}(\varphi)\right) \right] \Delta \varphi \left(\Delta_{\phi}^{-1}(\|\nabla \phi\|^{-1} \psi)\right) \|\nabla \phi\| \, dx$$

$$= \int_\Omega \left(-\Delta \varphi \Delta_{\phi}^{-1} \left[\Delta \varphi \left(E'_{\varepsilon}(\varphi)\right) \right], \nabla \varphi \left(\Delta_{\phi}^{-1}(\|\nabla \phi\|^{-1} \psi)\right)\right) \|\nabla \phi\| \, dx$$

$$= -\left(\Delta \varphi \left(E'_{\varepsilon}(\varphi)\right), \|\nabla \phi\|^{-1} \psi\right)_{H^{-1}},$$

we construct the weak form $H^{-1}$ gradient flow as

$$\left(\frac{\partial \varphi}{\partial t} \|\nabla \phi\|^{-1}, \|\nabla \phi\|^{-1} \psi\right)_{H^{-1}} = \left(\Delta \varphi \left(E'_{\varepsilon}(\varphi)\right), \|\nabla \phi\|^{-1} \psi\right)_{H^{-1}}$$

and $H^{-1}$ gradient flow

$$\frac{\partial \varphi}{\partial t} = \Delta \varphi \left(E'_{\varepsilon}(\varphi)\right)\|\nabla \phi\|. \quad (28)$$

Similarly, from

$$\delta \varphi(E(\varphi), \psi) = \int_\Omega E'_{\varepsilon}(\varphi) \psi \, dx$$

$$= \int_\Omega \Delta_{\phi}^{-1} \left[\Delta \varphi \left(E'_{\varepsilon}(\varphi)\right) \right] \Delta \varphi \left(\Delta_{\phi}^{-1}(\|\nabla \phi\|^{-1} \psi)\right) \|\nabla \phi\| \, dx$$

$$= \int_\Omega \Delta_{\phi}^{-1} \left[\Delta \varphi \left(E'_{\varepsilon}(\varphi)\right) \right] \Delta \varphi \left(\Delta_{\phi}^{-1}(\|\nabla \phi\|^{-1} \psi)\right) \|\nabla \phi\| \, dx$$

$$= \left(\Delta_{\phi} \left(E'_{\varepsilon}(\varphi)\right), \|\nabla \phi\|^{-1} \psi\right)_{H^{-2}},$$

we can construct the weak form $H^{-2}$ gradient flow as

$$\left(\frac{\partial \varphi}{\partial t} \|\nabla \phi\|^{-1}, \|\nabla \phi\|^{-1} \psi\right)_{H^{-2}} = -\left(\Delta_{\phi}^2 \left(E'_{\varepsilon}(\varphi)\right), \|\nabla \phi\|^{-1} \psi\right)_{H^{-2}}$$

and $H^{-2}$ gradient flow

$$\frac{\partial \varphi}{\partial t} = -\Delta_{\phi}^2 \left(E'_{\varepsilon}(\varphi)\right)\|\nabla \phi\|. \quad (29)$$

**Example 13. Surface diffusion flow.** Let $f = 1$. We obtain from (28) the classical surface diffusion flow

$$\frac{\partial \varphi}{\partial t} = -\Delta_{\phi} \left(\text{div} \left(\frac{\nabla \phi}{\|\nabla \phi\|}\right)\right) \|\nabla \phi\|. \quad (28)$$
Example 14. Xuguo flow. Let \( f = 1 \). We obtain from (29) Xuguo flow

\[
\frac{\partial \phi}{\partial t} = \Delta \phi \left[ \text{div} \left( \frac{\nabla \phi}{|| \nabla \phi ||} \right) \right] || \nabla \phi ||.
\]

Remark 4.1. The gradient descent flows obtained in this section for level set surfaces are completely consistent with the gradient descent flows for parametric surfaces.

References


