Weighted Minimal Surfaces and Discrete Weighted Minimal Surfaces*

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Abstract

We introduce the concept of weighted minimal surface and define weighted mean curvature with explicit formulas for functional, parametric and implicit surfaces. We prove that the weighted minimal functional surface is stationary. The concept of the discretized weighted minimal surface is proposed. Examples of weighted minimal surfaces are presented.

Key words: Weighted minimal surface; Weighted mean curvature; Discretization.

1 Introduction

As is well known, a regular surface $\mathcal{M} \subset \mathbb{R}^3$ is minimal if the mean curvature of $\mathcal{M}$ vanishes everywhere. Equivalently, a minimal surface $\mathcal{M}$ is a critical point of the area functional, i. e.,

$$\mathcal{M} = \arg\left\{ \min_{\Sigma} \int_{\Sigma} dA \right\},$$

where $dA$ is the area element of surface $\Sigma$ in $\mathbb{R}^3$. In this paper, we study the weighted minimal surface $\mathcal{M}$ defined by zero weighted mean curvature everywhere, which is also a critical point of the weighted area functional, i. e.,

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\[
\mathcal{M} = \arg\left\{ \min_{\Sigma} \int_{\Sigma} \phi(p, q) dA \right\},
\]

where \( \phi(p, q) \) is a weight function and is also called anisotropic function. This weighted area functional appears in image analysis fields such as shape segmentation (see [5]), boundary detection (see [2]), tracking (see [4]).

In this paper, definitions of weighted mean curvature for various surface representations and stationary weighted minimal surface are discussed. We introduce the concept of discrete weighted minimal surface and give a discretization method. Brakke’s software Evolver [1] is used to conduct experiments. Comparisons of the weighted minimal surfaces with classical minimal surface are carried out.

2 Weighted mean curvature

2.1 Anisotropic function

The definition of the anisotropic function and its important properties are as follows.

**Definition 1** Let \( \phi : \mathbb{R}^3 \times \mathbb{R}^3 \backslash \{0\} \to (0, +\infty) \) be a \( C^2(\mathbb{R}^3 \times \mathbb{R}^3 \backslash \{0\}) \) function. If \( \phi \) is positively homogeneous of degree 1 for the second variable,

\[
\phi(p, \lambda q) = \lambda \phi(p, q), \quad p \in \mathbb{R}^3, q \in \mathbb{R}^3 \backslash \{0\}, \quad \lambda > 0
\]

and convex in the sense that there exists a constant \( c_0 \) such that

\[
\nabla^2_{qq} \phi(p, q) z \cdot z \geq c_0 |z|^2
\]

for all \( p, q, z \in \mathbb{R}^3 \) with \( q \cdot z = 0, \ |q| = 1 \),

then \( \phi \) is named as an anisotropic function. If \( \phi = \phi(q) \), then we say \( \phi \) is independent of the position.

**Proposition 1** Let \( \phi(p, q) \) be an anisotropic function. Then for any \( p \in \mathbb{R}^3, q \in \mathbb{R}^3 \backslash \{0\} \), we have

\[
\begin{align*}
\nabla_q \phi(p, q) \cdot q &= \phi(p, q), \\
\nabla^2_{qq} \phi(p, q) q &= 0, \\
\nabla^2_{qq} \phi(p, q) q &= \nabla_p \phi(p, q), \\
\n\nabla_q \phi(p, \lambda q) &= \nabla_q \phi(p, q), \\
\n\nabla^2_{qq} \phi(p, \lambda q) &= \frac{1}{\lambda} \nabla^2_{qq} \phi(p, q), \quad \forall \lambda > 0
\end{align*}
\]

\[
\nabla^2_{qq} \phi(p, q) z \cdot z \geq c_0 \left( \frac{|z|^2 - \left( \frac{q \cdot z}{|q|} \right)^2}{|q|} \right), \quad \forall z \in \mathbb{R}^3.
\]

The proof of Proposition 1 is quite straightforward and notations are only explained. Suppose \( p = (p_1, p_2, p_3)^T \) and \( q = (q_1, q_2, q_3)^T \) are two vectors in
\[ \mathbb{R}^3 \]. We use \( p \cdot q \) and \( p \times q \) to denote the usual inner product and outer product, respectively. \( \nabla_p \phi(p, q) \) represents the gradient of \( \phi(p, q) \) about variable \( p \). \( \nabla^2_{qq} \phi(p, q) \) stands for the gradient of \( \nabla_q \phi(p, q) \) about \( p \) variable whose \( ij \) entry is \( \phi_{q_ipj}(p, q) \). \( \nabla^2_{qq} \phi(p, q) \) is the hessian of \( \phi(p, q) \) about variable \( q \) with \( \phi_{q_iq_j} \) as its \( ij \) entry.

2.2 Weighted mean curvature

The Euler equation of the weighted area functional

\[ E_\phi(M) = \int_M \phi(p, \vec{n}) dA, \]

will be given and the definition of weighted mean curvature is formulated for three types of surface \( M \), where \( p \) and \( \vec{n} \) are surface point and the unit normal vector, respectively.

**Functional Surface.** Suppose surface \( M = \{(x_1, x_2, u(x_1, x_2)); (x_1, x_2) \in \Omega \subset \mathbb{R}^2\} \). The weighted area of \( M \) is calculated by

\[ E_\phi(u) = \int_{\Omega} \phi((x_1, x_2, u), (\nabla u, -1)) dudv. \]

The Euler equation of this weighted area functional is

\[ \phi_{p_3} - \sum_{j=1}^{2} (\phi_{q_ip_j} + \phi_{q_jp_i} u_{x_j}) - \sum_{i,j=1}^{2} \phi_{q_iq_j} u_{x_i x_j} = 0. \] (9)

For the first two terms of (9) depend on the position of the surface, we use the last term of (9) to define the weighted mean curvature for a graph as

\[ H_\phi = \sum_{i,j=1}^{2} \phi_{q_iq_j} ((x_1, x_2, u(x_1, x_2)), (\nabla u, -1)) u_{x_i x_j}. \]

**Parametric Surface.** For a regular parametric surface \( M \) parameterized as \( r(u, v) \in \mathbb{R}^3 \) on a domain \( \Omega \subset \mathbb{R}^2 \), the weighted area is expressed as

\[ E_\phi(M) = \iint_{\Omega} \phi(r, \vec{n}) \cdot |r_u \times r_v| \ du dv = \iint_{\Omega} \phi(r, r_u \times r_v) \ du dv. \]

The Euler equation of area functional is

\[
\begin{align*}
\nabla_p \phi(r, \vec{n}) \cdot \vec{n} - \left( \vec{n} \times r_v \right) \cdot (\nabla^2_{qpr} \phi(r, \vec{n}) r_u) + \left( r_u \times \vec{n} \right) \cdot (\nabla^2_{qp} \phi(r, \vec{n}) r_v) \\
- \dfrac{\left( \vec{n} \times r_v \right) \cdot \nabla^2_{pq} \phi(r, \vec{n}) (r_u \times r_v) + \left( r_u \times \vec{n} \right) \cdot (\nabla^2_{qp} \phi(r, \vec{n}) (r_u \times r_v) v)}{|r_u \times r_v|^2} = 0,
\end{align*}
\]

\[ L \phi = \sum_{i,j=1}^{2} \phi_{q_iq_j} (\nabla u, -1) u_{x_i x_j}. \]
and the weighted mean curvature is defined as
\[
H_\phi = \left[ (\vec{n} \times r_v) \cdot \nabla_{qq}^2 \phi(r, \vec{n}) (r_u \times r_v)_u + (r_u \times \vec{n}) \cdot (\nabla_{qq}^2 \phi(r, \vec{n}) (r_u \times r_v)_v) \right] / |r_u \times r_v|^2.
\]

**Implicit Surface.** If surface \( \mathcal{M} \) is expressed by a level set of a function \( \varphi \), i.e., \( \mathcal{M} = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3; \varphi(x_1, x_2, x_3) = c \} \), where \( c \) is a constant, then the weighted area of the surface \( \mathcal{M} \) can be written as ([3], page 15)
\[
E_\phi(\mathcal{M}) = \int_{\mathbb{R}^3} \phi(x, \vec{n}) \delta(\varphi(x)) |\nabla \varphi| \, dx = \int_{\mathbb{R}^3} \phi(x, \nabla \varphi) \delta(\varphi(x)) \, dx,
\]
where \( \delta \) is one-dimensional Dirac delta function. After calculating, the Euler equation of weighted area functional is
\[
\sum_{i=1}^{3} \phi_{qq_i}(x, \nabla \varphi) \delta(\varphi) + \text{tr}(\nabla_{qq}^2 \phi(x, \nabla \varphi) \nabla^2 \varphi(x)) \delta(\varphi) = 0,
\]
where \( \text{tr}(\cdot) \) stands for the trace of a square matrix. The weighted mean curvature of \( \mathcal{M} \) is defined as
\[
H_\phi = \text{tr}(\nabla_{qq}^2 \phi(x, \nabla \varphi) \nabla^2 \varphi(x)), \tag{10}
\]
because \( \delta(\varphi) \neq 0 \) only on \( \mathcal{M} \).

### 3 Stationary weighted minimal surface

From now on, we restrict the anisotropic function \( \phi(p, q) \) to be independent of the first variable \( p \).

**Definition 2 (weighted minimal surface)** A surface \( \mathcal{M} \) is called a weighted minimal surface if its weighted mean curvature \( H_\phi \) is zero everywhere.

**Definition 3 (stationary weighted minimal surface)** For any variation \( \mathcal{M}_t \) of a weighted minimal surface \( \mathcal{M} \) with fixed boundary, if the second variation of the weighted area of the surface is positive, i.e.,
\[
\left. \frac{d^2 E_\phi(\mathcal{M}_t)}{dt^2} \right|_{t=0} > 0,
\]
then \( \mathcal{M} \) is named as a stationary weighted minimal surface.

Now suppose the weighted minimal surface \( \mathcal{M} \) is described by a graph \( u(x) \) as in section 2. We can compute the second variation of the weighted area for
\( \eta \in C^\infty_0(\Omega) \) as follows:

\[
\frac{d^2}{dt^2} E_\phi(u + t\eta) \bigg|_{t=0} = \int_{\Omega} \sum_{i,j=1}^{2} \frac{1}{\sqrt{1 + |\nabla u|^2}} \phi_{ij} \left( \frac{(\nabla u, -1)}{\sqrt{1 + |\nabla u|^2}} \right) \eta_x \eta_{x_j}
\]

\[
\geq \int_{\Omega} \frac{c_0}{\sqrt{1 + |\nabla u|^2}} \left( |\nabla \eta|^2 - \left( \nabla \eta \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)^2 \right) > 0,
\]

where \( \geq \) is valid because of the property (8) of anisotropic function \( \phi \). Then we obtain the following:

**Theorem 1** The weighted minimal graph is a stationary weighted minimal surface.

This stationary property of the weighed minimal graph is quite similar to the classical minimal graph.

### 4 Discrete weighted minimal surface

For a discrete triangular surface mesh \( M \),

\[
\vec{H}_\phi(v_i) \approx -\frac{\nabla A_\phi(v_i)}{A(v_i)},
\]

is a discrete approximation to the weighted mean curvature, where \( A(v_i) \) and \( A_\phi(v_i) \) are classical area and weighted area around a vertex \( v_i \in M \), respectively. Let \([v_i v_j v_{j+1}]\) be a neighbor triangle of vertex \( v_i \). The usual area of the triangle \([v_i v_j v_{j+1}]\) is \( A^{(j)}(v_i) = \frac{1}{2}|(v_j - v_i) \times (v_{j+1} - v_i)| \) and the weighted area can be computed by \( A^{(j)}_\phi(v_i) = \frac{1}{2}\phi((v_j - v_i) \times (v_{j+1} - v_i)) \). So the area and weighted area of the one ring neighborhood of vertex \( v_i \) are

\[
A(v_i) = \sum_{j \in N(i)} A^{(j)}(v_i) \quad \text{and} \quad A_\phi(v_i) = \sum_{j \in N(i)} A^{(j)}_\phi(v_i),
\]

respectively, where \( N(i) \) is the index set of the one ring neighbor vertices of vertex \( v_i \). We can compute the gradient of \( A_\phi(v_i) \) to vertex \( v_i \) as

\[
\nabla_{v_i} A_\phi(v_i) = \frac{1}{2} \sum_{j \in N(i)} (v_j - v_{j+1}) \times \nabla \phi((v_j - v_i) \times (v_{j+1} - v_i)). \quad (11)
\]

Hence

\[
\vec{H}_\phi(v_i) \approx -\frac{1}{2A(v_i)} \sum_{j \in N(i)} (v_j - v_{j+1}) \times \nabla \phi((v_j - v_i) \times (v_{j+1} - v_i)). \quad (12)
\]
Definition 4  Let $\phi$ be a given anisotropic function, and $M$ be a triangular surface mesh with interior vertices $\{v_i\}$. If
\[
\sum_{j \in N(i)} (v_j - v_{j+1}) \times \nabla \phi((v_j - v_i) \times (v_{j+1} - v_i)) = 0,
\]
for all interior vertices $v_i$, then we say $M$ is a weighted discrete minimal surface.

5 Examples of weighted minimal surfaces

We present the classical Scherk’s minimal surface and weighted Scherk’s minimal surfaces generated by the following anisotropic functions
\[
\phi_1(\vec{n}) = (n_1^2 + 0.25n_2^2 + 16n_3^2)^{1/2} \quad \text{and} \quad \phi_2(\vec{n}) = (n_1^4 + 4n_2^4 + 4n_3^4)^{1/4}.
\]
by Brakke’s software Evolver. The results for these three minimal surfaces are shown in Table 1 and Fig. 1 to Fig. 3. Let us explain the various entries that appear in Table 1. The left column, refine, is the number of refining, facets, is the number of facet of the mesh. The third column, iters, is approximate number of iterations. The next three columns with each one including two columns, area, which is the classical discrete area of the whole mesh, and energy, the whole weighted discrete area. From this table, we can see that weighted minimal surface is not minimal surface, because when energy is decreasing, the area may be increasing. These three weighted or classical minimal surfaces are stationary because these three surfaces are graphs defined on a domain. For example, Scherk’s minimal surface can be written in function form as
\[
z = \ln \frac{\cos y}{\cos x}, \quad -\pi/2 < x, y < \pi/2.
\]
This verified that the weighted minimal graph is stationary which we have proved in section 3.

Table 1: Discrete minimal Scherk’s surface and weighted discrete minimal Scherk’s surfaces

<table>
<thead>
<tr>
<th>refine</th>
<th>facets</th>
<th>iters</th>
<th>DM Scherk’s area</th>
<th>DM Scherk’s energy</th>
<th>WDM Scherk’s I area</th>
<th>WDM Scherk’s I energy</th>
<th>WDM Scherk’s II area</th>
<th>WDM Scherk’s II energy</th>
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<td>44.674</td>
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</tr>
</tbody>
</table>

References

Fig. 1. Discrete classical Scherk’s minimal surface corresponding to the DM Scherk’s in Table 1: (a) is the original model. (b) is obtained after refining the initial mesh once and iterating $O(10)$ times. (c) is obtained after further refining and iterating. Another refinement and more iterations result in (d).

Fig. 2. Discrete minimal Scherk’s surface I weighted by anisotropic function $\phi_1(\bar{n})$ corresponding to WDM Scherk’s I in Table 1: (a) is initial model. (b) is obtained after refining once and iterating. (c) and (d) are showed after further refinement and iteration.

Fig. 3. Discrete minimal Scherk’s surface II weighted by function $\phi_2(\bar{n})$ corresponding to WDM Scherk’s II in Table 1. Other explanations are similar to Fig. 2.


