Consistent approximations of several geometric differential operators and their convergence

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The numerical integration of geometric partial differential equations is used in many applications such as image processing, surface processing, computer graphics and computer-aided geometric design. Discrete approximations of several first- and second-order geometric differential operators, such as the tangential gradient operator, the second tangential operator, the Laplace–Beltrami operator and the Giaquinta–Hildebrandt operator, are utilized in the numerical integrations. In this paper, we consider consistent discretized approximations of these operators based on a quadratic fitting scheme. An asymptotic error analysis is conducted which shows that under very mild conditions the discrete approximations of the first- and second-order geometric differential operators have quadratic and linear convergence rates, respectively.

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1. Introduction

In many applications such as image processing, surface processing, computer graphics and computer-aided geometric design, discrete approximations of curvatures (mean and Gaussian curvatures) and various geometric differential operators are often required. For instance, to solve geometric partial differential equations (PDEs) using a divided-difference-like method, discrete approximations of surface normal, gradient, mean curvature, Gaussian curvature, Laplace–Beltrami and Giaquinta–Hildebrandt operators are a prerequisite (see [52,56]). The mean curvature and the Gaussian curvature relate to the Laplace–Beltrami operator and the Giaquinta–Hildebrandt operator, respectively, in a precise way (see Section 2). Discretizations of the mean and Gaussian curvatures yield discretizations of the Laplace–Beltrami and Giaquinta–Hildebrandt operators, and vice versa.

1.1. Previous work

In this subsection we briefly review the discretization schemes of the curvatures and the differential operators. Discretization schemes have been proposed for each of the differential operators from different points of views (see [20,21,24,27,33,35,38,50] for references).

Discretization of Laplace–Beltrami operator. The Laplace–Beltrami (LB) operator has many applications in computational geometry, computer graphics and image processing, including surface parametrization, modeling, editing, fairing, shape analysis, interpolation, segmentation, remeshing, compression, and matching [5,6,11,28,39]. Hence, discretization of the LB...
operator on a surface mesh is mostly often studied. Many discretization schemes of the LB operator on surface meshes have been proposed (a nice discussion on explaining the variety of discrete LB operators can be found in [47]), such as Taubin’s discretization (see [42,43]), Fujiwara’s discretization (see [17]), Desbrun et al.’s discretization (see [10]), Mayer’s discretization (see [32]) and Meyer et al.’s discretization (see [35]). All of these discretizations are in the linear form:

\[ \Delta f(x_i) = \sum_{j \in N(i)} w_{ij} \left[ f(x_j) - f(x_i) \right], \]  

(1.1)

where \( x_i \) and \( x_j \) are the vertices of a surface triangulation \( M \), \( N(i) \) is the index set of the one-ring neighbors of vertex \( x_i \), \( w_{ij} \) are constants depending on the mesh vertices, and are independent of \( f \). On the approximation of the mean curvature, there also exist several approaches, such as the ones proposed by Chen [8], Hamann [22] and Taubin [41], however these approaches do not yield the necessary linear form (1.1).

Among the various discretizations of LB operator, the most well-known one is the cotangent scheme for surfaces in \( \mathbb{R}^3 \), which was first proposed in [30] and was independently discovered by Duffin in [15], re-invented in [37], and modified in [10,32,35,50]. When computing discrete minimal surfaces, Pinkall and Polthier [37] employed the cotangent scheme to represent the discrete mean curvature vector. Desbrun et al. [10,35] used the cotangent formula to represent the gradient of the normal curvature and its square, Wollmann’s approach (see [49]) based on Euler’s theorem (see [12], p. 145) and the consistency order improves previous results reported for the mesh Laplacian. They further prove the consistency of the discrete Willmore energies that correspond to the discrete Laplace–Beltrami operators. Among these classes, there are some other approaches, e.g., Taubin’s approach [41] based on eigen-analyses, Watanabe and Belyaev’s approach (see [48]) based on integral formulas of the normal curvature and its square, Wollmann’s approach (see [49]) based on Euler’s theorem (see [12], p. 145) and Meusnier’s theorem (see [12], p. 142). Using the theory of normal cycles, Cohen–Steiner and Morvan derived an efficient and reliable curvature-estimation algorithm (see [9]). Error bounds of the estimated curvature are given in the case of restricted Delaunay triangulations.

**Convergence of cotangent scheme.** The cotangent scheme has several nice properties, such as being easy to implement and has weak convergence [24,45]. However, it has been shown [24,45,50,51,57] that the cotangent scheme does not provide pointwise convergence, while many applications rely on pointwise estimations, though there are some convergence results for certain special meshes and manifolds [50]. Existence of convergent schemes has been an open question [24]. Wardetzky showed a convergence result in his Ph.D. dissertation for spectra based on the cotangent scheme when the triangular surface satisfies some mild conditions [44]. Ref. [46] gives an overview of the approximation and convergence properties of the cotangent formula for discrete Laplacians and mean curvature vectors for polyhedral surfaces located in the vicinity of a smooth surface in \( \mathbb{R}^3 \). They show that mean curvature vectors converge in the sense of distributions, but fail to converge in \( L^2 \). In [2], the authors propose an algorithm for approximating the Laplacian operator of a surface from a mesh with pointwise convergence, which guarantees its applicability to arbitrarily meshed surfaces. In [23], the authors present a principle for constructing strongly consistent discrete Laplace–Beltrami operators based on cotangent weights. The consistency order improves previous results reported for the mesh Laplacian. They further prove the consistency of the discrete Willmore energies that correspond to the discrete Laplace–Beltrami operators.

**Discretization of Gaussian curvature.** For Gaussian curvatures, there are also many discretized approaches. One class of approaches is based on local-fitting techniques, such as paraboloid fitting (see [22,26,40]), quadratic fitting (see [33,50]), higher-order fitting (see [7]), circular fitting (see [8,31]) and implicit fitting by a 3D function (see [14]). A second class of methods is based on a theorem for Gaussian maps (see [33,36]) or based on the Gauss–Bonnet theorem (see [1,16,25,33,40]). For a given neighborhood surrounding a surface point, the Gauss–Bonnet theorem states that the Gaussian curvature at the surface point can be approximated by the ratio of the area of the spherical image of the neighborhood and the area of the neighborhood itself. The approaches based on the Gauss–Bonnet theorem (see [1,16,25,33,40]) for triangulated surfaces are usually called angle deficit schemes or Gauss–Bonnet schemes. Apart from these classes, there are some other approaches, e.g., Taubin’s approach [41] based on eigen-analyses, Watanabe and Belyaev’s approach (see [48]) based on integral formulas of the normal curvature and its square, Wollmann’s approach (see [49]) based on Euler’s theorem (see [12], p. 145) and Meusnier’s theorem (see [12], p. 142). Using the theory of normal cycles, Cohen–Steiner and Morvan derived an efficient and reliable curvature-estimation algorithm (see [9]). Error bounds of the estimated curvature are given in the case of restricted Delaunay triangulations.

**Discrete differential operators from discrete differential geometry.** The methodology in [24] uses a variational approach, where energy functionals on polyhedral surfaces give rise to discrete curvatures and differential operators. Thin-shells are discretized using a discrete bending energy by Grinspun et al. [20,21]. From the discrete theory of smooth differential geometry, a key question is how to describe discrete geometric objects. A good discretization theory should preserve essential properties of the continuous theory. Relating to this, combinatorial approaches were introduced by Mercat [34] and Bobenko et al. [3] for discretizing the underlying conformal structure of Riemann surfaces.

**1.2. Motivations**

We present several facts that motivate the research of this paper.

1. All of the differential operators and the curvatures mentioned above are individually discretized based on various theorems in differential geometry; they are not generally consistent, meaning that they are not simultaneously computed from one and the same auxiliary surface.

2. Except for the schemes based on the interpolation or fitting, none of these schemes converge without any restriction on the regularity of the meshes considered.
3. When using a divided-difference-like method to solve geometric PDEs, the semi-implicit discretization of the equations requires that the discrete differential operators have a linear form, while the widely used discrete schemes, for instance the discrete approximation of the Gaussian curvature based on the Gauss–Bonnet theorem (see [1]), are not in this form.

Therefore, it is desirable to use a fitting scheme so that resultant discretized differential operators are consistent, convergent, and have the required forms. A quadratic fitting scheme is proposed in [50] that is based on a local parametrization technique, and it is frequently used to solve geometric partial differential equations and yields very desirable results (see [55, 56, 58]). Basing on an extensive numerical experiment, the author of [50] claims that the approximate mean curvature computed from the parametric quadratic fitting surface converges to the exact mean curvature. However, this fact has never been formally proved.

In this paper, several commonly used differential operators and curvatures are consistently approximated based on the parametric quadratic fitting algorithm. An asymptotic error analysis on the quadratic fitting is conducted, which firmly supports the claim made in [50]. We prove that the obtained discrete approximations of the first- and the second-order geometric differential operators from the quadratic fitting algorithm have quadratic and linear convergence rates, respectively, under a very mild condition on the distribution of the fitting knots. The convergence of the discrete second-order operators also implies the convergence of the corresponding discrete mean and Gaussian curvatures.

The rest of the paper is organized as follows: Section 2 introduces some notation and a set of geometric differential operators. Discretization schemes for these differential operators are considered in Section 3. A convergence analysis of these discrete differential operators is conducted in Section 4. Section 5 concludes the paper.

2. Geometric differential operators

In this section we introduce the notation and a set of geometric differential operators, including the gradient, the mean curvature, the Gaussian curvature, the Laplace–Beltrami operator and the Giaquinta–Hildebrandt operator.

Let \( S = (u, v), (u, v) \in \Omega \subset \mathbb{R}^2 \) be a regular smooth parametric surface in \( \mathbb{R}^3 \). For simplicity, we sometimes write \( (u, v) \) as \((u^1, u^2)^t\). Let \( g_{\alpha\beta} = (x_{\alpha\beta}, x_{\alpha\beta}) \) be the coefficients of the first fundamental form of \( S \) with \( x_{\alpha\beta} = \frac{\partial x}{\partial u^\alpha} \). Set \( g = \det[g_{\alpha\beta}], [g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1} \). Let \( b_{\alpha\beta} = (n, x_{\alpha\beta}) \) be the coefficients of the second fundamental form of \( S \) with \( x_{\alpha\beta} = \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} \) and \( n = (x_u \times x_v)/\|x_u \times x_v\| \).

**Curvatures.** The mean curvature \( H \) and the Gaussian curvature \( K \) are given by (see [12])

\[
H = \frac{b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11}}{2g}, \quad K = \frac{b_{11}b_{22} - b_{12}^2}{g}.
\]

The mean curvature vector and Gaussian curvature vector are \( H = Hn \) and \( K = Kn \), respectively.

**Tangential gradient operator.** Let \( f \in C^1(S) \). Then the *tangential gradient operator* \( \nabla \) acting on \( f \) is given by (see [12], p. 102)

\[
\nabla f = [x_\alpha, x_\beta] [g^{\alpha\beta}] [f_{\alpha}, f_{\beta}]^t \in \mathbb{R}^3
= g_\alpha^\beta f_\alpha + g_\beta^\gamma f_\gamma,
\]

where \( g_\alpha^\beta = \frac{1}{g}(g_{22}x_\alpha - g_{12}x_\beta) \) and \( g_\beta^\gamma = \frac{1}{g}(g_{11}x_\beta - g_{12}x_\alpha) \). Obviously, \( \nabla \) is a first-order differential operator.

**Second tangential operator.** Let \( f \in C^1(S) \). Then the *second tangential operator* \( \diamond \) acting on \( f \) is defined as (see [56])

\[
\diamond f = [x_\alpha, x_\beta] [Kb^{\alpha\beta}] [f_{\alpha}, f_{\beta}]^t \in \mathbb{R}^3
= g_\alpha^\beta f_\alpha + g_\beta^\gamma f_\gamma,
\]

where \( g_\alpha^\beta = \frac{1}{g}(b_{22}x_\alpha - b_{12}x_\beta) \) and \( g_\beta^\gamma = \frac{1}{g}(b_{11}x_\beta - b_{12}x_\alpha) \). It is easy to see that \( \diamond f \) is of the first order with respect to the function \( f \), but of the second order with respect to the surface. Hence, we classify it as a second-order operator.

**Divergence operator.** Let \( v \) be a \( C^1 \) smooth vector field on \( S \). Then the *divergence* of \( v \) is defined by

\[
\text{div}(v) = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^\beta} \right] [\sqrt{g} g^{\alpha\beta}] [x_\alpha, x_\beta]^t v.
\]

**Laplace–Beltrami operator.** Let \( f \in C^2(S) \). Then \( \nabla f \) is a smooth vector field on \( S \). The Laplace–Beltrami operator (LBO) \( \Delta \) applied to \( f \) is defined by (see [13])

\[
\Delta f = \text{div}(\nabla f).
\]

From the definition of \( \nabla \) and \( \text{div} \), it is easy to derive that
\[ \Delta f = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u} \sqrt{g} \frac{\partial}{\partial v} \right] \left[ \sqrt{g} \frac{\partial}{\partial v} \left[ f_u, f_v \right]^T \right] \]
\[ = g^{\alpha \beta} f_u + g^{\alpha \beta} f_v + g^{\alpha \beta} f_{uu} + g^{\alpha \beta} f_{uv} + g^{\alpha \beta} f_{vv}, \quad (2.3) \]

where
\[ g^{\alpha \beta} = -\left[ g_{11}(g_{22}g_{22}g_{12} - g_{12}g_{22}) + 2g_{12}(g_{12}g_{22}g_{12} - g_{22}g_{12}) + g_{22}(g_{12}g_{12} - g_{12}g_{22}) \right]/g^2, \]
\[ g^{\alpha \beta} = -\left[ g_{11}(g_{11}g_{22}g_{12} - g_{12}g_{22}) + 2g_{12}(g_{12}g_{22}g_{12} - g_{22}g_{12}) + g_{22}(g_{12}g_{12} - g_{12}g_{22}) \right]/g^2, \]
\[ g^{uu} = b_{22}/g, \quad g^{uv} = -2b_{12}/g, \quad g^{vv} = b_{11}/g. \]

The differential operator \( \Box \) was introduced by Giaquinta and Hildebrandt (see [18], pp. 82–85). Since \( b_{ij} \) involves the second-order derivatives of the surface considered, Eq. (2.4) might imply that \( \Box f \) is a third-order differential operator. However, (2.5) shows that it is actually second order, since the terms involving the third-order derivatives are canceled. Similar to the relation \( \Delta x = 2HN \), we have \( \Box x = 2K\mathbf{n} \) (see [54]).

The differential operators introduced above are frequently used to define various geometric partial differential equations. Using them, many geometric differential partial equations have been constructed (see [53]). For instance, to minimize a third-order energy functional \( f_h \| \nabla f (H, K) \|^2 dA \), we obtain the following sixth-order Euler–Lagrange equation (see [29])
\[ \Delta (f_h \Delta f) + 2\Box (f_h \Delta f) + 4KHf_h \Delta f + 4H^2 f_H \Delta f - 2Hf_h \Delta f - 2H \| \nabla f \|^2 + 2(\nabla f, \Box f) = 0, \]
where \( f \in C^2(S) \) is a given Lagrangian function. It is easy to see that almost all of the differential operators introduced above are included in this equation. However, it is still unclear if these operators are sufficient to describe all of the geometric partial equations up to the sixth order.

3. Discretizations of geometric differential operators

**Discrete surface.** Let \( M \) be a triangulation of surface \( S \). Let \( \{ x_i \}_{i=1}^N \) be the vertex set of \( S \). For vertex \( x_i \) with valence \( n \), denote by \( N(i) = \{ i_1, i_2, \ldots, i_n \} \) the set of the vertex indices of the one-ring neighbors of \( x_i \), and \( N_i(i) = \{ i_0, i_1, i_2, \ldots, i_n \} \) with \( i_0 = i \). We assume in the following that these \( i_1, \ldots, i_n \) are arranged such that the triangles \( [x_ix_{i_1}x_{i_0}] \) and \( [x_ix_{i_0}x_{i_1}] \) are in \( M \), and \( x_{i_0}, x_{i_1} \) opposite to the edge \( [x_ix_{i_0}] \).

We devote our attention to the discretization of geometric differential operators because of the need to solve various geometric partial differential equations. To solve geometric PDEs using a divided-difference-like method, discrete approximations of the gradient, the mean curvature, Gaussian curvature, Laplace–Beltrami operator and Giaquinta–Hildebrandt operator are required (see [56]). In order to use a semi-implicit scheme, the approximations of the above mentioned differential operators are required to have linear forms (see (3.3)–(3.7)). There are several discretization schemes of the Laplace–Beltrami operator and Gaussian curvature, as mentioned in Section 1. However, some of them are not in the required linear form and may not be consistent in the following sense.

**Definition 3.1.** A set of approximate geometric differential operators is said to be consistent if there exists a \( C^2 \) smooth surface \( S \) such that the approximate operators coincide with the exact counterparts of \( S \).

If a set of approximate differential operators are not consistent, some unexpected results may be obtained in the applications. For instance, if we compute the principal curvatures \( k_{1,2} = H \pm \sqrt{H^2 - K} \) from the approximated \( H \) and \( K \), there is the danger that \( H^2 - K < 0 \). One may forcibly take \( H^2 - K = 0 \) if \( H^2 - K < 0 \), but this will yield discontinuous principal curvatures.
Remark 3.1. It should be pointed out that the definition of consistency in Definition 3.1 is different from the concept of consistency in numerical solutions of partial differential equations, in which the consistency means the discrete operators converge to the corresponding continuous operators as the grid size goes to zero. In this paper, this convergence is called convergence.

Here we use a quadratic fitting of the surface data and function data to calculate the approximate differential operators. The algorithm we adopted is from [50]. Let \( \mathbf{x}_i \) be a vertex of \( M \) with valence \( n \), and let \( \mathbf{x}_j \) be the neighboring vertices for \( j \in \mathcal{N}(i) \).

**Algorithm 3.1 (Quadratic fit).**

1. **Local parametrization.** Compute angles
   \[
   \alpha_k = \cos^{-1} \left( \frac{\mathbf{x}_{ik} - \mathbf{x}_i}{\|\mathbf{x}_{ik} - \mathbf{x}_i\|} \right), \quad k = 1, \ldots, n,
   \]
   and then compute the angles
   \[
   \beta_k = 2\pi \alpha_k / \sum_{j=1}^n \alpha_j, \quad k = 1, \ldots, n.
   \]
   Set \( \mathbf{q}_0 = [0, 0]^T \), \( \theta_1 = 0 \) and \( \mathbf{q}_k = \|\mathbf{x}_{ik} - \mathbf{x}_i\| [\cos \theta_k, \sin \theta_k]^T \), \( \theta_k = \beta_1 + \cdots + \beta_{k-1}, k = 1, \ldots, n \).

2. **Local fitting.** Take the basis functions
   \[
   \left\{ \mathbf{B}_l(u, v) \right\}_{l=0}^5 = \left\{ 1, u, v, \frac{1}{2} u^2, uv, \frac{1}{2} v^2 \right\},
   \]
   and determine the coefficient \( \mathbf{c}_l \in \mathbb{R}^3 \) of \( \mathbf{x}(u, v) = \sum_{l=0}^5 \mathbf{c}_l \mathbf{B}_l(u, v) \) such that
   \[
   \sum_{l=0}^5 \mathbf{c}_l \mathbf{B}_l(\mathbf{q}_k) = \mathbf{x}_i, \quad k = 0, \ldots, n
   \]
in the least-squares sense. This system is solved by solving the normal equation. Let \( \mathbf{A} = (\mathbf{B}_l(\mathbf{q}_k))_{k=0,l=0}^{n,5} \in \mathbb{R}^{(n+1) \times 6} \), and let
   \[
   \mathbf{C} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \in \mathbb{R}^{6 \times (n+1)},
   \]
   then \( [\mathbf{c}_0, \ldots, \mathbf{c}_5] = [\mathbf{x}_0, \ldots, \mathbf{x}_n] \mathbf{C}^T \).

Remark 3.2. The construction algorithm above may fail if the coefficient matrix of the normal equation is singular or nearly singular. In this case, we can look for a least-squares solution with a minimal normal. Let \( \mathbf{A}^T \mathbf{A} = \mathbf{b} \) be the linear system in matrix form. We find a least-squares solution \( \mathbf{x} \) such that \( \| \mathbf{x} \|_2 = \min \). That is, we replace \( (\mathbf{A}^T \mathbf{A})^{-1} \) in (3.1) with \( (\mathbf{A}^T \mathbf{A})^+ \), which is the Moore–Penrose inverse. It is well known that \( (\mathbf{A}^T \mathbf{A})^+ \) can be computed by the SVD of \( \mathbf{A} \) (see [19], Chapter 5). Let \( \mathbf{V} = \text{diag}(\sigma_1, \ldots, \sigma_6) \), where \( \sigma_1 \geq \cdots \geq \sigma_6 \geq 0 \) are the singular values of \( \mathbf{A} \). If the computed singular value \( \sigma_i < 10^{-8} \), we regard this singular value as zero (we use double-precision arithmetic operations). In practice, \( \|\mathbf{x}_i - \mathbf{x}_k\| \) may be very small and the singular values are also small. Then, the truncation of the singular values mentioned above may be misleading. To overcome this difficulty, matrix \( \mathbf{A} \) is normalized by being multiplied by a diagonal matrix \( \text{diag}(1, h^{-1}, h^{-1}, h^{-2}, h^{-2}, h^{-2}) \) on the right, where \( h = \max_{i,k} \|\mathbf{x}_i - \mathbf{x}_k\| \).

Remark 3.3. It should be noted that, unless the valence of the vertex \( \mathbf{x}_i \) is less than 5, the matrix \( \mathbf{A}^T \mathbf{A} \) has a very small chance of being singular for the randomly distributed grid points. When the valence of the vertex \( \mathbf{x}_i \) is 3 or 4, the coefficient matrix of the fitting problem is singular. An effective way to solve the singularity problem of the matrix is to increase the number of neighboring vertices of \( \mathbf{x}_i \). We know that the triangulation of a given set of discrete points on the surface is not unique. One-ring neighbors of a vertex can be subjectively specified. Hence, we can select more points in the range of a geodesic distance than one-ring neighbors in order to obtain a nonsingular coefficient matrix of the fitting problem.

**Partial derivatives.** Given discrete function values \( f(\mathbf{x}_i) \), the fitting function is \( f(u, v) = \sum_{l=0}^5 d_l \mathbf{B}_l(u, v) \) with \( [d_0, \ldots, d_5]^T = \mathbf{C}^T [f(\mathbf{x}_0), \ldots, f(\mathbf{x}_n)] \). We compute partial derivatives of \( \mathbf{x}(u, v) \) and \( f(u, v) \) at the origin up to the second order. Denote the second, third, fourth, fifth and sixth rows of \( \mathbf{C} \) as \( \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_{11}, \mathbf{C}_{12} \) and \( \mathbf{C}_{22} \), respectively, then we can see that...
\(\mathbf{x}_w = [\mathbf{x}_{i0}, \ldots, \mathbf{x}_{in}]^T, \quad \alpha = 1, 2,\)
\[
\frac{\partial f}{\partial u^\alpha} = [f(\mathbf{x}_{i0}), \ldots, f(\mathbf{x}_{in})]^T, \quad \alpha = 1, 2,
\]
\[
x_{w\alpha} = [\mathbf{x}_{i0}, \ldots, \mathbf{x}_{in}]^T, \quad 1 \leq \alpha \leq \beta \leq 2,
\]
\[
\frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} = [f(\mathbf{x}_{i0}), \ldots, f(\mathbf{x}_{in})]^T, \quad 1 \leq \alpha \leq \beta \leq 2. \tag{3.2}
\]

**Tangential gradient operator.** Substituting (3.2) into (2.1), we get an approximation of tangential gradient operator as follows:
\[
\nabla f(\mathbf{x}_i) \approx \sum_{j \in N_1(i)} w_{ij}^T f(\mathbf{x}_j), \quad w_{ij}^T = g_{u}^\beta c_1^{(j)} + g_{v}^\beta c_2^{(j)} \in \mathbb{R}^3. \tag{3.3}
\]

Here \(c_\alpha^{(j)}\) are the \(j\)-th component of \(C_u\).

**Second tangential operator.** Substituting (3.2) into (2.2), we get an approximation of second tangential operator as follows:
\[
\diamond f(\mathbf{x}_i) \approx \sum_{j \in N_1(i)} w_{ij}^\Delta f(\mathbf{x}_j), \quad w_{ij}^\Delta = g_{u}^{\Delta,1} c_1^{(j)} + g_{v}^{\Delta,2} c_2^{(j)} \in \mathbb{R}^3. \tag{3.4}
\]

**Laplace–Beltrami operator.** Substituting (3.2) into (2.3), we get an approximation of LBO as follows:
\[
\Delta f(\mathbf{x}_i) \approx \sum_{j \in N_1(i)} w_{ij}^\nabla f(\mathbf{x}_j), \tag{3.5}
\]
where \(w_{ij}^\nabla = g_{u}^{\nabla,1} c_1^{(j)} + g_{v}^{\nabla,2} c_2^{(j)} + g_{uu}^{\nabla,11} c_1^{(j)} + g_{uv}^{\nabla,12} + g_{vv}^{\nabla,22} c_2^{(j)}\) and \(c_\alpha^{(j)}\) are the \(j\)-th component of \(C_{u\beta}\).

**Giaquinta–Hildebrandt operator.** Substituting (3.2) into (2.5), we get an approximation of GHO as follows:
\[
\Box f(\mathbf{x}_i) \approx \sum_{j \in N_0(i)} w_{ij}^\Box f(\mathbf{x}_j), \tag{3.6}
\]
where \(w_{ij}^\Box = g_{u}^{\Box,1} c_1^{(j)} + g_{v}^{\Box,2} c_2^{(j)} + g_{uu}^{\Box,11} c_1^{(j)} + g_{uv}^{\Box,12} + g_{vv}^{\Box,22} c_2^{(j)}\).

**Mean curvature vector and mean curvature.** Using the relation \(\Delta \mathbf{x} = 2 \mathbf{H}\), we have
\[
\mathbf{H}(\mathbf{x}_i) \approx \frac{1}{2} \sum_{j \in N_1(i)} w_{ij}^\nabla \mathbf{x}_j, \quad \mathbf{H}(\mathbf{x}_i) \approx \frac{1}{2} \sum_{j \in N_1(i)} w_{ij}^\Delta \mathbf{n}(\mathbf{x}_j)^T \mathbf{x}_j, \tag{3.7}
\]
where \(\mathbf{n}(\mathbf{x}_j)\) is the surface normal at \(\mathbf{x}_j\), which is computed as \((\mathbf{x}_j \times \mathbf{x}_i)/\|\mathbf{x}_j \times \mathbf{x}_i\|\).

**Gaussian curvature vector and Gaussian curvature.** Using the relation \(\Box \mathbf{x} = 2 K \mathbf{n}\), we have
\[
\mathbf{K}(\mathbf{x}_i) \mathbf{n}(\mathbf{x}_i) \approx \frac{1}{2} \sum_{j \in N_1(i)} w_{ij}^\Box \mathbf{x}_j, \quad \mathbf{K}(\mathbf{x}_i) \approx \frac{1}{2} \sum_{j \in N_1(i)} w_{ij}^\nabla \mathbf{n}(\mathbf{x}_j)^T \mathbf{x}_j.
\]


It is well known that for a given function \(f(x, y)\), the errors of the coefficients \(c_{ij}\) of the Lagrangian interpolation polynomial \(\sum_i j \leq n c_i x^i y^j\) of degree \(n\) versus the Taylor expansion of \(f(x, y)\) around the origin are bounded by (see [7])
\[
|c_{ij} - f_{ij}(0)/i! j!| \leq C h^{n+1-(i+j)},
\]
where \(f_{ij} = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}\), \(C\) is a constant and \(h\) is the maximal distance of the interpolation nodes to the origin. When approximating by least-squares fitting, a similar result holds (see [7]). However, these results do not explicitly imply the convergence for our quadratic fitting algorithm because of the following two reasons.

1. Our fitting surface is in the parametric form.
2. The nodes of the fitting are determined in a way that is different from the function case (see Step 1 of Algorithm 3.1).

In the following, we analyzed the asymptotic properties of the fitting surface generated by Algorithm 3.1. The analysis strategy is as follows: we first determine the asymptotic properties (see Lemma 4.2) for the data sampled from a functional surface. Then this result is extended to the graph data (see Lemma 4.6). Finally, the asymptotic result is extended to the data sampled from a parametric surface (see Theorem 4.1).
Lemma 4.1. Since the problem of the quadratic fitting function of \( f \) on the node set \( K \) is well-posed, then we say that the node set \( K \) is well-posed for the problem of the quadratic fitting: Determining a bivariate polynomial \( G(x, y) = \sum_{i+j \leq 2} a_{ij} x^i y^j / i! j! \) such that

\[
G(q_i) = f(q_i), \quad i = 0, 1, \ldots, n,
\]
in the least-squares sense, for a given function \( f(x, y) \).

If the node set \( K \) is well-posed, then the quadratic fitting problem has a unique solution. In the following, we assume that \( q_0, q_1, \ldots, q_n \) are mutual distinct and further assume that \( q_0 = [0, 0]^T \) for simplicity.

Lemma 4.1. Let \( K := \{q_i = [x_i, y_i]^T \in \mathbb{R}^2\}_{i=0}^n \) be a well-posed node set for the quadratic fitting problem. Then the node set \( K(h) := hK := \{hq_i\}_{i=0}^n \) is also well-posed for any \( h > 0 \). More generally, let \( L \in \mathbb{R}^{2 \times 2} \) be a nonsingular matrix. Then the node set \( K(L) := \{Lq_i\}_{i=0}^n \) is well-posed.

Proof. Since \( h > 0 \) and

\[
A(K(h)) = A(K)A, \quad A = \text{diag}\{1, h, h^2, h^3, h^4\},
\]
the first conclusion of the lemma follows. To prove the second conclusion, let \( L = (l_{ij})_{i,j=1}^2 \). Then under the transform \( L \),

\[
A(K(L)) = A(K) \text{diag} \begin{bmatrix} 1, L, \begin{bmatrix} l_{11}^2 & l_{12}^2 \ l_{12} & l_{22}^2 \end{bmatrix} \end{bmatrix}.
\]

It is not difficult to calculate that the determinant of the last \( 3 \times 3 \) block matrix above is \( (l_{11}l_{22} - l_{12}l_{21})^3 \neq 0 \), since \( \det(L) = l_{11}l_{22} - l_{12}l_{21} \neq 0 \). Hence the lemma is proved. \( \square \)

Lemma 4.2. Suppose \( f(x, y) \) is a sufficiently smooth bivariate function in the neighborhood of the origin \( q_0 \). Let \( K := \{q_i\}_{i=0}^n \) be a well-posed node set, and

\[
G^{(h)}(x, y) = \sum_{i+j \leq 2} a_{ij}^{(h)} x^i y^j / i! j!
\]
the quadratic fitting function of \( f \) on the node set \( K^{(h)} := hK \), \( h > 0 \). Then

\[
|a_{ij}^{(h)} - f_{ij}(q_0)| \leq c_{ij} \left\| A(K)^T A(K) \right\|^{-1} A(K) \left\| h^{3-i-j}, \right. (4.3)
\]
where \( c_{ij} \) are constants depending on \( f \), independent of \( K^{(h)} \).

Proof. Let

\[
X = \begin{bmatrix} a_{00}^{(h)}, a_{10}^{(h)}, a_{01}^{(h)}, a_{20}^{(h)}, a_{11}^{(h)}, a_{02}^{(h)} \end{bmatrix}^T, \quad F = \begin{bmatrix} f(q_0), f(hq_1), \ldots, f(hq_n) \end{bmatrix}^T.
\]
Then from the fitting problem (4.1), we have

\[
A(K^{(h)})^T A(K^{(h)}) X = A(K^{(h)})^T F,
\]
and, by (4.2), we further have

\[
A(K)^T A(K) X = A(K)^T F. \quad (4.4)
\]
Since

\[
f(hq_i) = f(q_0) + hq_i^T \nabla f(q_0) + \frac{1}{2} h^2 q_i^T \nabla^2 f(q_0) q_i + O(h^3),
\]
we have

\[
F = A(K) AF_0 + O(h^3).
\]
Proof. Since Lemma 4.3. in the neighborhood of the origin. Then there exists a constant C, such that points in the neighborhood of the origin, and For this 

\[ \text{Lemma 4.4.} \]

and

\[ X = F_0 + A^{-1} [A(K)^T A(K)]^{-1} A(K)^T O(h^3). \]

Then (4.3) follows. \( \square \)

**Corollary 4.1.** Suppose \( f \) is a sufficiently smooth bivariate function in the neighborhood of the origin. Let \( K = \{q_i\}_{i=0}^n \) be a well-posed node set. Let \( G^{(h)} \) be the quadratic fitting function of \( f \) on the node set \( K^{(h)} \). Then we have

\[ \|n(G^{(h)}) - n(f)\| \leq C_0 h^2, \quad \|k_i(G^{(h)}) - k_i(f)\| \leq C_i h, \quad i = 1, 2, \]

where \( n(f) \) and \( k_i(f) \) denote the normal and the principal curvatures of \( f \) at \( q_0 \), respectively.

**Remark 4.1.** The corollary says that the normal \( n(G^{(h)}) \) has a quadratic convergence rate, while the curvatures have linear convergence rates. These conclusions match Meek and Walton’s results (see [33], Lemma 4.1).

**Lemma 4.3.** Let \( K = \{q_i\}_{i=0}^n \) be a well-posed node set. Then for any \( B > \|A(K)^T A(K)\|^{-1} \), there exists an \( \epsilon > 0 \), such that

\[ \text{(i) } q_0 \notin \Omega_i, \Omega_i \cap \Omega_j = \emptyset, i \neq j, i, j \geq 1, \text{ where } \Omega_i = \{q \in \mathbb{R}^3: \|q - q_i\| < \epsilon\}; \]

\[ \text{(ii) For any node set } R := \{q_i \in \Omega_i\}_{i=0}^n, \text{ we have } \|A(R)^T A(R)\|^{-1} \|A(R)\| < B; \]

\[ \text{(iii) For this } \epsilon, \text{ let } \Omega_i^{(h)} = \{q \in \mathbb{R}^3: \|q - q_i\| < \epsilon h\} (h > 0). \text{ Then the quadratic fitting problem on the node set } R^{(h)} := \{r_i \in \Omega_i^{(h)}\}_{i=0}^n \text{ has a unique solution} \]

\[ C^{(h)}(x, y) = \sum_{i+j \leq 2} a^{(h)}_{ij} x^i y^j / i! j! \]

and

\[ \|a^{(h)}_{ij} - f_{ij}(q_0)\| \leq c^{(h)}_{ij} h^{3-i-j}, \quad i + j \leq 2, \quad (4.5) \]

where \( c_{ij} \) are constants depending on \( f \), independent of \( R^{(h)} \).

**Proof.** Since \( K \) is a well-posed node set, \( q_i \) are distinct. It is obvious that there exists an \( \epsilon_1 > 0 \) such that (i) holds. Notice that the elements of the matrix \( A(R) \) are continuous functions of \( r_0, \ldots, r_n \). Hence the inverse of \( A(R)^T A(R) \) exists in the neighborhood of \( q_0, \ldots, q_n \). Then, there exists an \( \epsilon \leq \epsilon_1 \) such that (ii) holds. Similar to the proof of Lemma 4.2, (4.5) can be derived. \( \square \)

**Lemma 4.4.** Suppose \( f(x, y) \) is a smooth function around the origin \( q_0 \), satisfying \( f(q_0) = 0 \) and \( \nabla f(q_0) = 0 \). Let \( q \in \mathbb{R}^2 \) be a point in the neighborhood of the origin. Then there exists a constant \( C \), such that

\[ \sqrt{\|q - q_0\|^2 + f(q)^2} - \|q - q_0\| \leq C \|q - q_0\|^3. \]

**Proof.** Since

\[ f(q) = f(q_0) + (q - q_0)^T \nabla f(q_0) + O(\|q - q_0\|^2) = O(\|q - q_0\|^2), \]

we have

\[ \sqrt{\|q - q_0\|^2 + f(q)^2} = \sqrt{\|q - q_0\|^2 + O(\|q - q_0\|^2)} = \|q - q_0\| + O(\|q - q_0\|^3). \quad \square \]

**Lemma 4.5.** Let \( f(x, y) \) be a smooth function around the origin \( q_0 \), satisfying \( f(q_0) = 0 \) and \( \nabla f(q_0) = 0 \). Let \( q_1, q_2 \in \mathbb{R}^2 \) be two points in the neighborhood of the origin, and \( q_i \neq 0 \). Set

\[ \theta = \cos^{-1} \left( \frac{q_1 \cdot q_2}{\|q_1\| \|q_2\|} \right), \quad \theta_f^{(h)} = \cos^{-1} \left( \frac{x_1 \cdot x_2}{\|x_1\| \|x_2\|} \right), \]

where \( x_i = [h q_i^T, f(h q_i)]^T, i = 1, 2. \) Then there exists a constant \( C \), such that

\[ \|\theta_f^{(h)} - \theta\| \leq C h^2. \]
Proof. Since
\[
f(q_i) = f(q_0) + h q_i^T \nabla f(q_0) + \frac{1}{2} h^2 q_i^T \nabla^2 f(q_0) q_i + O(h^3)
\]
we have
\[
\theta_f(h) = \cos^{-1} \left( \frac{x_1, x_2}{\|x_1\| \|x_2\|} \right) = \cos^{-1} \left( \frac{q_1, q_2}{\|q_1\| \|q_2\|} + O(h^2) \right)
\]
\[
\omega = \theta + O(h^2).
\]

Lemma 4.6. Let $f$ be a smooth function around the origin $q_0$, satisfying $\nabla f(q_0) = 0$. Let $K = \{q_i\}_{i=0}^n$ be a well-posed node set, and
\[
G(h)(x, y) = \sum_{i+j \leq 2} a_{ij}^h x^i y^j / i! j!
\]
the quadratic fitting function generated by Algorithm 3.1 using the sampling data $[h q_i^T, f(h q_i)]^T \in \mathbb{R}^3$. Then for any given $B > \|[A(K)^T A(K)]^{-1} A(K)\|$, there exists an $h_0 > 0$, such that
\[
\|a_{ij}^h - F_{ij}(q_0)\| \leq C_{ij} h^{3-i-j}, \quad i + j \leq 2, \quad \text{as } 0 \leq h < h_0,
\]
where $C_{ij}$ are constants depending on $f, F(x, y) = [x, y, f(x, y)]^T$ and $F_{ij} = \frac{\partial^i f}{\partial x^i y^j}$.

Proof. Without loss of generality (WOLOG), we may assume $f(q_0) = 0$. Let $q_i^{(h)}$ be the fitting nodes defined by Algorithm 3.1, using the sampling data $[h q_i^T, f(h q_i)]^T$. Let
\[
q_i^{(h)} = q_i h = r_i^{(h)} [\cos \theta_i^{(h)}, \sin \theta_i^{(h)}]^T, \quad q_i^{(h)} = r_i^{(h)} [\cos \tilde{\theta}_i^{(h)}, \sin \tilde{\theta}_i^{(h)}]^T.
\]
Then by Lemma 4.4 and Lemma 4.5, we have
\[
|r_i^{(h)} - r_i^{(h)}| < Ch^3, \quad |\theta_i^{(h)} - \tilde{\theta}_i^{(h)}| < Ch^2.
\]
Note that, $r_i^{(h)} = O(h), r_i^{(h)} = O(h)$. Hence
\[
\|q_i^{(h)} - q_i^{(h)}\| \leq Ch^3.
\]
Therefore, when $h$ is small enough, $q_i^{(h)} \in \Omega_i^{(h)}$. Then by Lemma 4.3, (4.5) holds for the third component of the vector-valued function $F(x, y) = [x, y, f(x, y)]^T$. Similarly, (4.5) holds for the first and second components of $F(x, y)$. Therefore, (4.6) holds. □

Lemma 4.6 is for the sampling data of a function. Next we consider the sampling data of a parametric surface. Let $x(u, v)$ be a smooth parametric surface in the $xyz$-space $\mathbb{R}^3$. Let $q_0, q_1 \in \mathbb{R}^2$, and $q_0 = 0, q_1 \neq 0$. Suppose $x(q_0) = 0$. First we define a linear map $\sigma_{q_1}(u, v)$ from $uv$-plane to the tangent plane of $x(u, v)$ at $q_0$, which is defined as follows. In the $xyz$-space, take another Descartes coordinate system $XYZ$, such that the origin of the system is $x(q_0)$, the unit $Z$, $X$ and $Y$-direction are
\[
e_3 = \frac{x_u(q_0) \times x_v(q_0)}{\|x_u(q_0) \times x_v(q_0)\|}, \quad e_1 = \frac{[x_u(q_0), x_v(q_0)]q_1}{\|[x_u(q_0), x_v(q_0)]q_1\|} \quad \text{and} \quad e_2 = e_1 \times e_3.
\]
respectively. Obviously, the $XY$-plane is the tangent plane of $x(u, v)$ at $q_0$. Define
\[
\sigma_{q_1}(u, v) : (u, v) \rightarrow (X, Y) \quad \text{such that} \quad [e_1, e_2][X, Y]^T = [x_u(q_0), x_v(q_0)][u, v]^T.
\]
That is
\[
\sigma_{q_1}(u, v) = [e_1, e_2]^T [x_u(q_0), x_v(q_0)][u, v]^T.
\]
Now let us define another map $\sigma(u, v)$ from $uv$-plane to $XY$-plane as follows:
Theorem 4.1. Let \( x(u, v) \in \mathbb{R}^3 \) be a smooth parametric surface around the origin \( q_0, K = \{ q_i \}_{i=0}^n \) a well-posed node set on the uv-plane. Let

\[
G^{(h)}(X, Y) = \sum_{i+j \leq 2} s_{ij}^{(h)} X^i Y^j / i! j!
\]

be the quadratic fitting surface generated by Algorithm 3.1 using the sampling data \( x(hq_i) \). Then we have

\[
|s_{ij}^{(h)} - x_{ij}(q_0)| \leq C_{ij} B h^{3-i-j}, \quad i + j \leq 2, \tag{4.7}
\]

where \( C_{ij} \) are constants depending merely on \( x(u, v) \), \( B \) is a given constant as in Lemma 4.6.

Proof. WOLOG, we may assume \( x(q_0) = 0 \). Using the linear map \( \sigma_{q_i} (u, v) \), the well-posed node set \( K = \{ q_i \}_{i=0}^n \) on the uv-plane is mapped to another well-posed node set \( K' = \{ q_i \}_{i=0}^n \) on the XY-plane. Using map \( \sigma(u, v) \), we can identify the parametric surface \( x(q) \) as the graph \( [X, Y, f(X, Y)]^T \) of a function \( f(X, Y) \) on the XY-plane, where \( f(X, Y) \) is defined as follows.

For any point \( x(q) \) on the parametric surface, its independent variable on the XY-plane is the projection of the point. Denoting the projection point as \( P(x(q)) \), we have

\[
P(x(q)) = [e_1, e_2][X, Y]^T, \quad [X, Y]^T = \sigma(u, v).
\]

The function value \( f(X, Y) \) at \( P(x(q)) \) is \( e_3^T x(q) - P(x(q)) = e_3^T x(q) \). Hence

\[
x(q) = [e_1, e_2, e_3][X, Y, f(X, Y)]^T. \tag{4.8}
\]

Since

\[
P(x(hq_i)) = [x_u(q_0), x_v(q_0)][g_{u\beta}^{-1}][x_u(q_0), x_v(q_0)]^T x(hq_i)
\]

\[
= [x_u(q_0), x_v(q_0)][g_{u\beta}^{-1}][x_u(q_0), x_v(q_0)]^T [(x_u(q_0), x_v(q_0)]hq_i + O(h^2)]
\]

\[
= [x_u(q_0), x_v(q_0)hap_i + O(h^2),
\]

and

\[
q'_i = [x_u(q_0), x_v(q_0)]q_i.
\]

we have

\[
\| \sigma(hq_i) - \sigma_{q_i}(hq_i) \| = \|hq_i - P(x(hq_i))\| = O(h^2).
\]

Therefore, \( x(hq_i) \) can be regarded as the sampling of \( f(X, Y) \) on the XY-plane. The difference of the sampling points \( \{ P(x(hq_i)) \}_{i=0}^n \) on the XY-plane are bounded by \( O(h^2) \). Hence, by Lemma 4.3 we know that, if \( h \) is small enough, \( \{ P(x(hq_i)) \}_{i=0}^n \) is also a well-posed node set on the XY-plane. In Lemma 4.6, take the function \( f(x, y) \) as \( f(X, Y) \), and let \( \sum_{i+j \leq 2} s_{ij}^{(h)} X^i Y^j / i! j! \) be the fitting surface, then (4.6) holds. Let

\[
s_{ij}^{(h)} = [e_1, e_2, e_3]a_{ij}^{(h)}.
\]

From (4.8), we have

\[
[e_1, e_2, e_3]F_{ij}(X, Y) = \frac{\partial^{i+j}[e_1, e_2, e_3]F(X, Y)}{\partial X^i Y^j} = \frac{\partial^{i+j}x(u, v)}{\partial X^i Y^j} = \frac{\partial^{i+j}x(\sigma^{-1}(X, Y))}{\partial X^i Y^j}.
\]

Therefore, (4.7) holds. \( \square \)

Theorem 4.1 states that the first-order partial derivatives of the fitting surface have quadratic convergence rates, while the second-order partial derivatives have linear convergence rates. Therefore, we have the following corollary:
Corollary 4.2. Under the conditions of Theorem 4.1, the discretized first- and second-order geometric differential operators have quadratic and linear convergence rates, respectively.

It should be pointed out that except for the tangential gradient operator, which is a first-order geometric differential operator, all of the other geometric differential operators introduced in Section 2 are second-order.

Remark 4.2. The numerical experiments conducted in [50] show that for certain domain triangulations, the discrete approximation of the mean curvature, as derived from the quadratic fitting algorithm, has a quadratic convergence rate. This convergence rate is one order higher (super-convergence) than what we obtained from the theoretical analysis. Therefore, an open problem we are left with is: “what is the necessary and sufficient condition for domain triangulation, under which the super-convergence occurs for any smooth parametric surface to be sampled?”. We believe that if $n = 2m$, $m > 2$, and the node set $K = \{q_i\}_i=1^m$ satisfies:

$$q_{i+m} + q_i = 0, \quad i = 1, \ldots, m,$$

then the quadratic fitting surface obtained from the sampling data of a smooth parametric surface on the node set $K^{(h)} = hK$, will have super-convergence properties.

Remark 4.3. In Algorithm 3.1, the local parametrization step is crucial to the quality of the fitting surface. Here an interesting problem merits further study is: “is there another local parametrization technique which yields an even better fitting surface?”

Remark 4.4. Though Algorithm 3.1 is described for one vertex, it can be applied to any other vertices. Consider a sufficiently smooth parametric surface $x(u,v)$, $(u,v) \in \Omega = [0,1]^2$. Suppose $\Omega$ is uniformly partitioned into triangles by the lines: $u = i/2^l$, $i = 0, \ldots, 2^l$, $v = j/2^l$, $j = 0, \ldots, 2^l$, and $u - v = k/2^l$, $k = -2^l + 1, \ldots, 2^l - 1$. Then a surface triangulation is obtained by mapping the triangulation of $\Omega$ on the surface. At each surface point $x(i/2^l, j/2^l)$, Algorithm 3.1 can be applied. The approximated differential operators computed from the fitting surface will converge to the exact ones as $l$ goes to infinity.

5. Conclusions

In the literature, many approximation schemes on geometric differential operators have been proposed. Except for schemes based on interpolation or fitting, none of these schemes converge without any restriction on the regularity of the meshes considered. It has been claimed in [50] that the mean curvature computed from a parametric quadratic fitted surface converges. However, this fact has never been formally proved. In this paper, several commonly used geometric differential operators have been approximated based on a parametric quadratic fitting. We have proved that under very mild conditions these approximated differential operators are convergent in linear (for second-order operators) or quadratic (for first-order operators) rates with respect to mesh size.

References


