Numerische Mathematik manuscript No.
(will be inserted by the editor)

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Convergence and Stability of a Numerical Method for Micromagnetics

Received: date / Revised: date

Abstract The convergence and stability of a numerical method, which applies a nonconforming finite element method and an artificial boundary method to a multi-atomic Young measure relaxation model, for micromagnetics are analyzed. By revealing some key properties of the solution sets of both the continuous and discrete problems, we show that our numerical method is stable, and the solution set of the continuous problem is well approximated by those of the discrete problems. The performance of our method is also illustrated by some numerical examples.

1 Introduction

In modeling micromagnetics, it is well known that [4], if the exchange energy is ignored, the stable magnetization field $\mathbf{m} : \Omega \to S^{n-1}$ of a rigid ferromagnetic body at a temperature below the Curie temperature, minimizes the Gibbs free energy functional

$$E(\mathbf{m}) = \int_\Omega \varphi(\mathbf{m})\,dx - \int_\Omega \mathbf{H}\cdot\mathbf{m}\,dx + \frac{1}{2}\int_{R^n} |\nabla u_{\mathbf{m}}|^2\,dx, \qquad n = 2,3, \tag{1.1}$$

The research was supported in part by the Major State Basic Research Projects (2005CB321701), NSFC projects (10431050, 10571006, 10528102 and 10871011) and RFDP of China.

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where $\varphi(\mathbf{m})$ is the magnetocrystalline-anisotropic energy density, $\mathbf{H}(x)$ is the applied static magnetic field, and $u_{\mathbf{m}}$ is the potential of the stray field energy which is related to $\mathbf{m}$ by the following Maxwell's equation

$$\operatorname{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_{\Omega}) = 0 \text{ in } H^{-1}(R^n); \quad \text{and } u_{\mathbf{m}} \to 0 \text{ as } |x| \to \infty \,. \tag{1.2}$$

However, since both the anisotropic energy density function φ and the set of admissible functions $\mathcal{A} := \{\mathbf{m} : \mathbf{m} \in (L^2(\Omega))^n \text{ and } |\mathbf{m}| = 1, \text{ a.e. } x \in \Omega\}$ are nonconvex, the variational problem

$$(\mathbf{P}) \begin{cases} \min\limits_{\mathbf{m}\in\mathcal{A}} E(\mathbf{m}) \\ s.t. \\ \operatorname{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_{\Omega}) = 0, & \text{in } H^{-1}(R^n), \\ u_{\mathbf{m}} \to 0, & \text{as } |x| \to \infty \,, \end{cases}$$

generally has no solution, and in such a case, the minimizing sequences generate finer and finer oscillations, which do not have cluster points in weak topology in the admissible set [14]. So, some relaxation problems are considered instead. As a mesoscopic model, the Young measure relaxation [8,15,19,21–23] can be used to capture some information on the underlying microstructure as well as the macroscopic information on the energy minimizing magnetization fields of the original problem $(\mathbf{P})$.

Denote $\nu = \{\nu_x\}_{x\in\Omega}$ a family of weakly measurable probability measures supported on S^{n-1}, *i.e.*, $\int_{S^{n-1}} \nu_x(dA) = 1$, $\forall x \in \Omega$ and, for any given $\varphi \in C(S^{n-1})$, the mapping $x \to \int_{S^{n-1}} \varphi(A)\nu_x(dA)$ is measurable. Let

$$\mathcal{A}^{\mu} = \left\{\nu = \{\nu_x\}_{x\in\Omega} : \operatorname{supp}\nu_x \subset S^{n-1}, a.e.\ x \in \Omega\right\}, \tag{1.3}$$

and define the relaxed energy functional by

$$E^{\mu}(\nu) = \int_{\Omega}\int_{S^{n-1}} \varphi(A)\nu_x(dA)dx - \int_{\Omega} \mathbf{H}\cdot\mathbf{m}\,dx + \frac{1}{2}\int_{R^n} |\nabla u_{\mathbf{m}}|^2 dx. \tag{1.4}$$

The Young measure relaxation problem $(\mathbf{RP})$ of problem $(\mathbf{P})$ is given by

$$(\mathbf{RP}) \begin{cases} \min\limits_{\nu\in\mathcal{A}^{\mu}} E^{\mu}(\nu) \\ s.t. \\ \mathbf{m}(x) = \int_{S^{n-1}} A\nu_x(dA), & \text{a.e. } x \in \Omega, \\ \operatorname{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_{\Omega}) = 0, & \text{in } H^{-1}(R^n), \\ u_{\mathbf{m}} \to 0, & \text{as } |x| \to \infty. \end{cases}$$

Numerical methods have been developed recently to approximate the Young measure in problem $(\mathbf{RP})$ (see for example [15,19]), and finite element methods have been applied to solve the corresponding Maxwell's equations (see for example [19,24]). Studies on the finite element methods for the Maxwell's equations in the Young measure relaxation problem $(\mathbf{RP})$ and the corresponding convex-hull relaxation problem $(\mathbf{RP1})$ (see section 2) revealed that an improper choice of the finite element space may generate artificial oscillations in the numerical solutions of the macroscopic magnetization field (see [19] for (**RP**) and [5] for (**RP1**) problems). To avoid the instability, a nonconforming finite element method was employed in [5], and a stabilized model was considered in [10]. However, the techniques used in [5, 10] in establishing the stability results are valid only for the convex-hull model (and even restricted to the uniaxial case), which is very restrictive in applications, since the convex-hull is in general not available explicitly. The stability of a nonconforming finite element method for (**RP**) in the uniaxial case, where the solution is unique, is proved in [24] by showing the uniqueness of the numerical solutions. For the general case, the situation is more challenging, since problem (**RP**) may admit infinitely many solutions which are embedded in a complicated manifold. To say a numerical method is stable, we mean that there is no artificial oscillations produced by the finite element discretization and the numerical solutions are approximately embedded in the manifold.

The main purpose of the present paper is to study the convergence and stability of a numerical method developed in [24] for the Young measure relaxation problem (**RP**). The method applies the multi-atomic Young measure to approximate the continuous Young measure and applies the Crouzeix-Raviart nonconforming finite element method coupled with an artificial boundary method to solve the Maxwell's equation. For simplicity, our numerical method and analysis are developed for problems in two dimensions. However, the method and the corresponding convergence and stability results can be extended to three dimensional problems without any difficulty.

In section 2, we establish some non-uniqueness results for the convex-hull relaxation of problem (**P**). In section 3, we analyze the Young measure relaxation and discuss its relationship with the convex-hull relaxation. In section 4, we present the semi-discrete multi-atomic Young measure relaxation method and the full discrete numerical method, which applies a nonconforming finite element method and an artificial boundary method to solve the Maxwell's equation in the semi-discrete multi-atomic Young measure relaxation problem. We also discuss the approximation properties of the finite element method. In section 5, we study the convergence and stability of the solutions of the fully discrete problem. In section 6, some numerical examples are given to illustrate the convergence and stability of the numerical method, which also serve to verify our analytical results.

2 Preliminaries: Non-uniqueness Results for the Convex-hull Relaxation

The convex-hull relaxation of problem (**P**) is defined as [5, 9, 21]

$$(\mathbf{RP1})\begin{cases} \min\limits_{\mathbf{m}\in\mathcal{A}^{**}} E^{**}(\mathbf{m}) \\ s.t. \\ \operatorname{div}(-\nabla u_{\mathbf{m}} + \mathbf{m}\chi_{\Omega}) = 0, & \text{in } H^{-1}(R^n), \\ u_{\mathbf{m}} \to 0, & \text{as } |x| \to \infty, \end{cases}$$

where $\mathcal{A}^{**} = \{\mathbf{m} \in (L^2(\Omega))^n : \ |\mathbf{m}| \le 1, \ \text{a.e. } x \in \Omega\}$ and

$$E^{**}(\mathbf{m}) = \int_\Omega \hat{\varphi}^{**}(\mathbf{m})dx - \int_\Omega \mathbf{H}\cdot\mathbf{m}dx + \frac{1}{2}\int_{R^n} |\nabla u_{\mathbf{m}}|^2 dx, \qquad n = 2. \tag{2.1}$$

Here $\hat{\varphi}^{**} = \min\{f : f \le \hat{\varphi} \ \text{ and } \ f \text{ is convex}\}$ is the convex hull of $\hat{\varphi} : R^n \to R^1 \cup \{\infty\}$, which is defined by

$$\hat{\varphi}(\mathbf{m}) = \begin{cases} \varphi(\mathbf{m}), & |\mathbf{m}| = 1; \\ +\infty, & \text{otherwise}. \end{cases}$$

In [9], it is shown that the minimizers of problem (**RP1**) are characterized by the following Euler-Lagrange equation:

$$\nabla u + D\hat{\varphi}^{**}(\mathbf{m}) + \lambda\mathbf{m} = \mathbf{H}, \qquad a.e. \text{ in } \Omega, \tag{2.2}$$

$$\lambda \ge 0, \ |\mathbf{m}| \le 1, \ \text{and} \ \ \lambda(1 - |\mathbf{m}|) = 0, \qquad a.e. \text{ in } \Omega, \tag{2.3}$$

where λ is a Lagrange multiplier, and u is the solution of the Maxwell's equation (1.2) with respect to $\mathbf{m}$, in other words, $u \in H^1(R^n)$, $u \to 0$ as $|x| \to \infty$ and

$$\int_{R^n} \nabla u \cdot \nabla v dx = \int_\Omega \mathbf{m} \cdot \nabla v dx, \qquad \forall v \in H^1(R^n). \tag{2.4}$$

For the sake of convenience of the readers, we introduce the following lemma.

Lemma 21 *[9] Let* $(\mathbf{m}, u, \lambda)$ *be a solution of the equations* (2.2), (2.3) *and* (2.4). *Then,* $\mathbf{m}$ *must be a minimizer of the convex-hull relaxation problem* (**RP1**).

With the help of the above lemma, we are able to show the following non-uniqueness result for problem (**RP1**). Denote $\mathcal{P}(\mathbf{m}) = \nabla\Delta^{-1}\text{div}\mathbf{m}$, where $\Delta^{-1}\text{div}\mathbf{m}$ is regarded as the solution of the Maxwell's equation (1.2) corresponding to $\mathbf{m}$. For a set $A \subset (L^\infty(\Omega))^n$, denote

$$\text{Int}_{\infty,\Omega'}(A) := \{\mathbf{a} \in A : \exists \delta > 0, B_{\infty,\Omega'}(\mathbf{a}, \delta) \subset A\},$$

where $B_{\infty,\Omega'}(\mathbf{a}, \delta) = \{\mathbf{b} \in (L^\infty(\Omega))^n : \mathbf{b} = \mathbf{a}, \text{ on } \Omega \setminus \Omega', \ \|\mathbf{b} - \mathbf{a}\|_{\infty,\Omega'} < \delta\}$.

Theorem 21 *If* $\mathcal{K} = \text{Int}_{\infty,\Omega}(\{\mathbf{m} : D\hat{\varphi}^{**}(\mathbf{m}) = \mathbf{0}, \ |\mathbf{m}| < 1, \ a.e.\, x \in \Omega\}) \ne \emptyset$*, and suppose that the applied magnetic field* $\mathbf{H}$ *is such that* $\mathbf{H}|_\Omega = \mathcal{P}(\mathbf{m})|_\Omega$ *for some* $\mathbf{m} \in \mathcal{K}$*, then, the corresponding problem* **(RP1)** *has infinitely many minimizers.*

Proof For $\mathbf{m} \in \mathcal{K}$, let $\mathbf{H} \in (L^2(\Omega))^n$ be such that $\mathbf{H}|_\Omega = \mathcal{P}(\mathbf{m})|_\Omega$, and let $u_{\mathbf{m}}$ be the solution of the Maxwell's equation (1.2) corresponding to $\mathbf{m}$. Then, notice that $D\hat{\varphi}^{**}(\mathbf{m}) = 0$ and $|\mathbf{m}(x)| < 1, a.e.\, x \in \Omega$, it is easily verified that the Euler-Lagrange equations (2.2), (2.3) and (2.4) hold for $\mathbf{H}$, $\mathbf{m}$, $\nabla u_{\mathbf{m}}$ and $\lambda = 0$, and thus $\mathbf{m}$ is a minimizer of the convex-hull relaxation problem (**RP1**) according to Lemma 21.

On the other hand, since $\mathbf{m} \in \mathcal{K}$, there exists an $\varepsilon > 0$, such that $D\hat{\varphi}^{**}(\mathbf{m}+\tilde{\mathbf{m}}) = 0$ and $|\mathbf{m}+\tilde{\mathbf{m}}| < 1$, for all $\tilde{\mathbf{m}}$ satisfying $\|\tilde{\mathbf{m}}\|_{\infty,\Omega} < \varepsilon$. Thus, if $\eta \in H_0^1(\Omega)$ is such that $\|\nabla\eta\|_{\infty,\Omega} < \varepsilon$, which implies also $\|\mathbf{curl}\,\eta\|_{\infty,\Omega} < \varepsilon$, then $\mathbf{m} + \mathbf{curl}\,\eta$, which corresponds to the same stray field $u_{\mathbf{m}}$ since $\mathrm{div}(\mathbf{curl}\,\eta\chi_\Omega) = 0$ in $H^{-1}(R^n)$, is also a minimizer of (**RP1**) with the applied magnetic field $\mathbf{H}$, because for such an η, it is easily seen that $\mathbf{m} + \mathbf{curl}\,\eta$, $u_{\mathbf{m}}$, $\mathbf{H}$ and $\lambda = 0$ still satisfy the Euler-Lagrange equations (2.2), (2.3), and (2.4).

It is worth noticing that the applied magnetic fields constructed in the above theorem are all curl-free in $H^{-1}(\Omega)$. In fact, if the applied magnetic field $\mathbf{H} \in H(\mathbf{curl}) = \{\mathbf{H} \in (L^2(\Omega))^n : \mathbf{curl}\,\mathbf{H} \in (L^2(\Omega))^n\}$ is such that $\mathbf{curl}\,\mathbf{H} \neq 0$ on a compact subset of $\Omega' \subset \Omega$, then, the corresponding minimizers $\mathbf{m}$ of problem (**RP1**) must not be in the set $\mathrm{Int}_{\infty,\Omega'}(\{\mathbf{m} \in \mathcal{A}^{**} : D\hat{\varphi}^{**}(\mathbf{m}) = \mathbf{0}, |\mathbf{m}| < 1,\ a.e.\ x \in \Omega'\})$. Since otherwise, for some sufficiently small $\varepsilon_0 > 0$, $\mathbf{m}+\mathbf{curl}\,\eta$ with $\eta = \varepsilon_0\mathbf{curl}\,\mathbf{H}$ would be in the set $\{\mathbf{m} \in \mathcal{A}^{**} : D\hat{\varphi}^{**}(\mathbf{m}) = \mathbf{0}, |\mathbf{m}| < 1,\ a.e.\ x \in \Omega'\}$, and thus, the same argument as in the proof of Theorem 21 would lead to $u_{\mathbf{m}} = u_{\mathbf{m}+\mathbf{curl}\,\eta}$ and we would have

$$E^{**}(\mathbf{m} + \mathbf{curl}\eta) = E^{**}(\mathbf{m}) - \int_{\Omega'} \varepsilon_0(\mathbf{curl}\mathbf{H})^2 dx < E^{**}(\mathbf{m}). \tag{2.5}$$

This contradicts the assumption that $\mathbf{m}$ is a minimizer.

As an application of Lemma 21 and Theorem 21, we have the following result for two important cases in physics.

Corollary 21 *a) For the uniaxial case, where* $\varphi(\boldsymbol{m}) = c_1 m_1^2 + c_2(1 - m_2^2)^2$ *with* $c_1 > 0$, $c_2 \geq 0$, *the convex-hull relaxation problem* (**RP1**) *has a unique solution* $\mathbf{m} \in \mathcal{A}^{**}$.

b) For the cubic case, where $\varphi(\boldsymbol{m}) = c_3 m_1^2 m_2^2$ *with* $c_3 > 0$, *suppose that the applied magnetic field* $\mathbf{H}$ *is such that* $\mathbf{H}|_\Omega = \mathcal{P}(\mathbf{m})|_\Omega$ *for some* $\mathbf{m} \in \mathcal{K} := \mathrm{Int}_{\infty,\Omega}(\{\mathbf{m} \in \mathcal{A}^{**} : D\hat{\varphi}^{**}(\mathbf{m}) = \mathbf{0},\ |\mathbf{m}| < 1,\ a.e.\ x \in \Omega\})$, *which is known to be non-empty, then, the corresponding problem* (**RP1**) *has infinitely many minimizers.*

Proof For the uniaxial case, it is proved in [6] that the solution to the Euler-Lagrange equations (2.2), (2.3), and (2.4) is unique. Thus, by Lemma 2.1, we have *a*).

For the cubic case, the convex-hull $\hat{\varphi}^{**}$ has no explicit form. However, it is known that $D\hat{\varphi}^{**}(\mathbf{m}) = 0$ in the square with vertexes $(\pm 1, 0), (0, \pm 1)$, which implies $\{\mathbf{m} : \mathbf{m}(x) \equiv (m_1, m_2) \in R^2, \text{ and } |m_1 \pm m_2| < 1\} \subset \mathcal{K}$. Thus the conclusion *b*) follows from Theorem 21.

As a consequence of Corollary 21, we conclude that, in the cubic case, if an applied magnetic field $\mathbf{H}$ has compact support in Ω with $|\mathbf{H}|$ being sufficiently small and $\mathrm{curl}\,\mathbf{H} = 0$, then, there are infinitely many minimizers for the corresponding convex-hull relaxation problem (**RP1**). In fact, for such an $\mathbf{H}$, there exists a function u such that $\mathbf{H} = \nabla u$, which has a compact support in Ω and satisfies the Maxwell's equation (1.2) with respect to $\mathbf{m} = \nabla u = \mathbf{H} \in \mathcal{K}$. This is an extension to the result [5] that, in the cubic case, the convex-hull relaxation problem (**RP1**) admits infinitely many minimizers for $\mathbf{H} \equiv 0$.

3 Non-uniqueness and Characterization of Minimizers for the Young Measure Relaxation

We first introduce a well-known result which establishes the equivalent relationship between the two relaxation problems (**RP**) and (**RP1**).

Lemma 31 *[21] If φ is a continuous, nonnegative function defined on S^{n-1}, then, the problems* **(P)**, **(RP)** *and* **(RP1)** *have the same infimum,*

$$\inf_{\boldsymbol{m}\in\mathcal{A}} \mathrm{E}(\boldsymbol{m}) = \inf_{\boldsymbol{m}\in\mathcal{A}^{**}} \mathrm{E}^{**}(\boldsymbol{m}) = \inf_{\nu\in\mathcal{A}^{\mu}} \mathrm{E}^{\mu}(\nu).$$

Furthermore, we have : a) if $\nu = \{\nu_x\}_{x\in\Omega}$ is a minimizer of E^{μ} in $\mathcal{A}^{\mu}$, then, its first moment $\boldsymbol{m}$, which is defined by (3.2)*, is a minimizer of E^{**} in $\mathcal{A}^{**}$, and*

$$\hat{\varphi}^{**}(\boldsymbol{m}(x)) = \int_{S^{n-1}} \varphi(A)\nu_x(dA); \tag{3.1}$$

b) if $\boldsymbol{m}$ is a minimizer of E^{**} in $\mathcal{A}^{**}$, then, there exists a $\nu_x \in \mathcal{A}^{\mu}$ such that the relations (3.1) *and*

$$\mathbf{m}(x) = \int_{S^{n-1}} A\nu_x(dA), \quad a.e\ x \in \Omega \tag{3.2}$$

hold, and $\nu = \{\nu_x\}_{x\in\Omega}$ is a minimizer of E^{μ} in $\mathcal{A}^{\mu}$.

Since the macroscopic magnetization fields of the two problems **(RP)** and **(RP1)** are essentially identical, they naturally share the same uniqueness or non-uniqueness property. In particular, as a consequence of Lemma 31, Theorem 21, and Corollary 21, we have the following non-uniqueness results for the Young Measure relaxation problem, including the uniqueness and non-uniqueness results for the two widely used cases in physics.

Theorem 31 *If φ is a continuous, nonnegative function defined on S^{n-1} and the set $\mathcal{K} = \mathrm{Int}_{\infty,\Omega}(\{\mathbf{m} : D\hat{\varphi}^{**}(\mathbf{m}) = \mathbf{0},\ |\mathbf{m}| < 1,\ a.e.\ x \in \Omega\}) \neq \emptyset$, then, the Young measure relaxation problem* **(RP)**, *with respect to an applied magnetic field $\mathbf{H}$ satisfying $\mathbf{H}|_{\Omega} = \mathcal{P}(\mathbf{m})|_{\Omega}$ for some $\mathbf{m} \in \mathcal{K}$, has infinitely many minimizers.*

Corollary 31 *a) For the uniaxial case, where $\varphi(\boldsymbol{m}) = c_1 m_1^2 + c_2(1 - m_2^2)^2$, with $c_1 > 0$ and $c_2 \geq 0$, the Young measure relaxation problem* **(RP)** *admits a unique macroscopic magnetization field $\mathbf{m}$.*

b) For the cubic case, where $\varphi(\boldsymbol{m}) = c_3 m_1^2 m_2^2$ with $c_3 > 0$, suppose that the applied magnetic field $\mathbf{H}$ is such that $\mathbf{H}|_{\Omega} = \mathcal{P}(\mathbf{m})|_{\Omega}$ for some $\mathbf{m} \in \mathcal{K} := \mathrm{Int}_{\infty,\Omega}(\{\mathbf{m} \in \mathcal{A}^{**} : D\hat{\varphi}^{**}(\mathbf{m}) = \mathbf{0},\ |\mathbf{m}| < 1,\ a.e.\, x \in \Omega\})$, in particular, for some $\mathbf{m} \in \mathcal{A}^{**}$ satisfying $|m_1(x) \pm m_2(x)| \leq \xi < 1$, $a.e.\, x \in \Omega$, then, the corresponding Young measure relaxation problem **(RP)** *has infinitely many minimizers.*

The fact that the Young measure relaxation **(RP)** may even have infinitely many minimizers makes the numerical computation and analysis for the problem more challenging. In the rest part of the section, a general description of the solution set of the Young measure relaxation problem (**RP**) is given, some key properties, which may be traced back to [9], of the Young measure relaxation solutions are revealed, followed by a brief direct proof.

Proposition 31 *The potential of the stray field energy* $u_{\mathbf{m}}$ *is uniquely determined by the minimizers of the Young measure relaxation problem* (**RP**).

Proof Let $\nu^{(1)}$, $\nu^{(2)}$ be two minimizers of problem (**RP**), let $\mathbf{m}^{(1)}(x) = \int_{S^{n-1}} A\nu_x^{(1)}(dA)$, $\mathbf{m}^{(2)}(x) = \int_{S^{n-1}} A\nu_x^{(2)}(dA)$, and let u_1 and u_2 be the potentials satisfying the Maxwell's equation (1.2) corresponding to $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$ respectively. We want to show that $u_1 = u_2$.

Suppose otherwise, that is $u_1 \neq u_2$, by defining $\nu^{(\lambda)} = \lambda\nu^{(1)} + (1-\lambda)\nu^{(2)}$, for $0 < \lambda < 1$, denoting $\mathbf{m}^{(\lambda)} := \int_{S^{n-1}} A\nu_x^{(\lambda)}(dA) = \lambda\mathbf{m}^{(1)} + (1-\lambda)\mathbf{m}^{(2)}$ and $u_\lambda = \lambda u_1 + (1-\lambda)u_2$, which is easily shown to be the potential corresponding to $\mathbf{m}^{(\lambda)}$, then, as a consequence of the convexity of $\int_\Omega \int_{S^{n-1}} \varphi(A)\nu_x(dA)dx - \int_\Omega \mathbf{H}\cdot\mathbf{m}\,dx$ with respect to the Young measures ν and the strict convexity of $\int_{R^n} |\nabla u|^2 dx$ with respect to u, we would have $E^\mu(\nu^{(\lambda)}) < \lambda E^\mu(\nu^{(1)}) + (1-\lambda)E^\mu(\nu^{(2)})$, which contradicts the assumption that $\nu^{(1)}$ and $\nu^{(2)}$ are the minimizers.

Proposition 32 *Suppose that* $\mathbf{m}'$ *and* $\mathbf{m}''$ *are two macroscopic magnetization fields corresponding to the minimizers of problem* (**RP**), *then, we have*

$$\mathbf{m}' - \mathbf{m}'' \in \mathcal{M} := \{\mathbf{m} \in (L^2(\Omega))^n : \int_{R^n} \mathbf{m}\chi_\Omega \cdot \nabla v dx = 0,\ \forall v \in H_0^1(R^n)\}. \tag{3.3}$$

Proof The conclusion follows directly from the Maxwell's equation (1.2) and Proposition 31.

As we will show in section 5, the relation (3.3) is crucial to the stability of the numerical methods for problem (**RP**), especially when the energy minimizing macroscopic magnetization field is not unique.

4 The discretization of the Young Measure relaxation

In this section, we describe briefly the discretization of problem **(RP)** and recall some known convergence properties.

First, let Γ_h be a regular triangulation of Ω [7] and denote by δ_A the Dirac measure supported on $A \in S^{n-1}$. We approximate the Young measure on each element $K \in \Gamma_h$ by a multi-atomic Young measure supported on k atoms:

$$\nu_K = \sum_{i=1}^{k} \lambda_{K,i}\delta_{A_{K,i}}. \tag{4.1}$$

Thus, the admissible Young measure set $\mathcal{A}^\mu$ is discretized to a set of piecewise constant multi-atomic Young measures:

$$\mathcal{A}^{\mu}_{h,k} = \Bigg\{ \nu^{h,k} = \{\nu^{h,k}|_K\}_{K\in\Gamma_h} : \nu^{h,k}|_K = \sum_{i=1}^{k} \lambda_{K,i}\delta_{A_{K,i}};\ A_{K,i} \in S^{n-1},$$

$$\lambda_{K,i} \geq 0,\ \forall 1 \leq i \leq k,\ \text{ and } \sum_{i=1}^{k} \lambda_{K,i} = 1,\ \forall K \in \Gamma_h \Bigg\}. \quad (4.2)$$

In general, it is known that problem (**RP**) admits a Young measure solution supported on no more than $n+1$ atoms [15], and the physical solutions are also known supported on very limited number of atoms, usually no more than the number of the energy wells of the anisotropic energy density φ. This fact makes the multi-atomic Young measure approximation a natural and highly efficient discretization method.

Next, we aim to solve the Maxwell's equation

$$\int_{R^n} \nabla u \cdot \nabla v dx = \int_{R^n} \mathbf{m}\chi_\Omega \cdot \nabla v dx, \quad \forall v \in H^1(R^n). \quad (4.3)$$

numerically by a method developed in [24], which is a combination of a nonconforming finite element method and an artificial boundary method.

Let $\Omega_i = B(0,R)$, where $R > 0$ is sufficiently large so that $\bar{\Omega} \subset \Omega_i$, and let $V = \{v \in H^1(\Omega_i) : \int_0^{2\pi} v(R,\theta)d\theta = 0\}$. Then, the solution of equation (4.3) can be obtained by solving the following variational problem defined on the bounded domain Ω_i [12,24]:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) + b(u,v) = f(v), \quad \forall v \in V, \end{cases} \quad (4.4)$$

where $a(u,v) = \int_{\Omega_i} \nabla u \cdot \nabla v dx$, $f(v) = \int_\Omega \mathbf{m} \cdot \nabla v dx$, and

$$b(u,v) = \sum_{j=1}^{\infty} \frac{j}{\pi} \int_0^{2\pi} \int_0^{2\pi} u(R,\theta)v(R,\phi)\cos j(\theta-\phi)\, d\phi\, d\theta. \quad (4.5)$$

In numerical computations, the infinite series $b(\cdot,\cdot)$ has to be replaced by its finite truncation

$$b_N(u,v) = \sum_{j=1}^{N} \frac{j}{\pi} \int_0^{2\pi} \int_0^{2\pi} u(R,\theta)v(R,\phi)\cos j(\theta-\phi)\, d\phi\, d\theta. \quad (4.6)$$

This leads to the approximation problem

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) + b_N(u,v) = f(v), \quad \forall v \in V. \end{cases} \quad (4.7)$$

Remark 41 *The approximation error between u_N and u, which are the solutions of the problems (4.7) and (4.4) respectively, has the following bound [12, 24]*

$$|u-u_N|_{1,\Omega_i} \le \frac{C}{(N+1)^{\alpha-1}}|u|_{\alpha-\frac{1}{2},\partial\Omega_i}.$$

In applications, u is usually very smooth near $\partial\Omega_i$, that is α is very large, thus a small N should be enough to obtain a sufficiently accurate approximation.

Let $\Gamma_h^{(i)}$ be a regular triangulation of Ω_i, which coincides with Γ_h on Ω, and let V_h be the Crouzeix-Raviart finite element function space, *i.e.*

$$\begin{aligned} V_h = \{v \in L^2(\Omega_i) : \int_0^{2\pi} v(R,\phi)\, d\phi = 0,\ v|_K \in P_1(K), \forall K \in \Gamma_h^{(i)},\ \text{and} \\ v\ \text{ is continuous at the midpoints of all interior element edges}\}. \end{aligned} \tag{4.8}$$

We define a non-conforming finite element problem of (4.7) as

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a_h(u_h, v_h) + b_N(u_h, v_h) = f_h(v_h), \quad \forall v_h \in V_h, \end{cases} \tag{4.9}$$

where $a_h(u_h, v_h) = \sum_{K\in\Gamma_h^{(i)}} \int_K \nabla u_h \cdot \nabla v_h\, dx$, $f_h(v_h) = \sum_{K\in\Gamma_h} \int_K \mathbf{m} \cdot \nabla v_h\, dx$. It is not difficult to show that the finite element problem (4.9) has a unique solution for any given $\mathbf{m} \in (L^2(\Omega))^n$ [12,24]. Furthermore, let $u_{\mathbf{m}}^N \in H^1(\Omega_i)$ and $u_{\mathbf{m}}^{h,N} \in V_h$ be the solutions to the problems (4.7) and (4.9) respectively for a given $\mathbf{m} \in (L^2(\Omega))^n$, then, we have [24]

$$\lim_{h\to 0} \|u_{\mathbf{m}}^N - u_{\mathbf{m}}^{h,N}\|_h = 0, \quad \text{uniformly for } \mathbf{m} \in (L^2(\Omega))^2, \tag{4.10}$$

where $\|v\|_h = (a_h(v,v) + b_N(v,v))^{\frac{1}{2}}$, for all $v \in V + V_h$.

Finally, the fully-discrete problem of (**RP**) can be defined as

$$(\mathbf{FDRP}){:}\ \min_{\nu^{h,k}\in\mathcal{A}_{h,k}^{\mu}} E_{h,N}^{\mu}(\nu^{h,k}). \tag{4.11}$$

where $E_{h,N}^{\mu}(\nu^{h,k})$ is given by

$$\begin{aligned} E_{h,N}^{\mu}(\nu^{h,k}) = \sum_{K\in\Gamma_h} \int_K \int_{S^{n-1}} \varphi(A)\nu^{h,k}|_K(dA)dx - \sum_{K\in\Gamma_h} \mathbf{H}_k \cdot \mathbf{m}_{h,k}|_K|K| \\ + \frac{1}{2}\sum_{K\in\Gamma_h} \int_K \mathbf{m}_{h,k}\chi_\Omega \cdot \nabla u_{\mathbf{m}_{h,k}}^{h,N}\, dx, \end{aligned} \tag{4.12}$$

and $u^{h,N}_{\mathbf{m}_{h,k}} \in V_h$ is the solution of the finite element problem (4.9) with

$$\mathbf{m}_{h,k}|_K = \int_{S^{n-1}} A\nu_{h,k}|_K(dA) = \sum_{i=1}^{k} \lambda_{K,i} A_{K,i}, \text{ and } \mathbf{H}_K = \frac{1}{|K|}\int_K H(x)dx.$$

It is known that the minimal energy of the discrete problem **(FDRP)** converges to that of the continuous problem **(RP)**, more precisely, we have [24]

$$\lim_{N\to\infty} \lim_{h\to 0, k\to\infty} \inf_{\nu^{h,k}\in\mathcal{A}^{\mu}_{h,k}} E^{\mu}_{h,N}(\nu^{h,k}) = \inf_{\nu\in\mathcal{A}^{\mu}} E^{\mu}(\nu). \tag{4.13}$$

5 The Convergence and Stability of the Numerical Method

In this section, we always assume that $\varphi \in C(S^{n-1})$ and $\varphi \geq 0$. For simplicity of the notation, we define the set of the energy minimizing macroscopic magnetization fields of the fully discrete relaxation problem **(FDRP)** by

$$\mathfrak{M}^N_{h,k} := \{\mathbf{m} \in (L^2(\Omega))^n : \text{ there exists a minimizer } \nu_{h,k} \text{ of } E^{\mu}_{h,N} \text{ in } \mathcal{A}^{\mu}_{h,k}, \quad \text{such that } \mathbf{m} = \int_{S^{n-1}} A\nu_{h,k}(dA)\}.$$

We first establish a convergence theorem for the discrete stray fields corresponding to the minimizers of the fully discrete problem **(FDRP)**. The theorem shows that the discrete stray fields converge to the unique continuous stray field corresponding to the minimizers of the continuous problem **(RP)**.

Theorem 51 *Let $u^{h,N}_{\mathbf{m}_{h,k,N}}$ be the finite element solution of (4.9) with respect to $\mathbf{m}_{h,k,N} \in \mathfrak{M}^N_{h,k}$, let u be the potential of the stray field corresponding to the minimizers of problem* **(RP)**. *Then, we have*

$$\lim_{N\to\infty} \lim_{h\to 0,\, k\to\infty} \|u^{h,N}_{\mathbf{m}_{h,k,N}} - u\|_h = 0.$$

Proof For fixed N, it follows from (4.10) that $\|u^{N}_{\mathbf{m}_{h,k,N}} - u^{h,N}_{\mathbf{m}_{h,k,N}}\|_h \to 0$ uniformly for k as $h \to 0$, where $u^{N}_{\mathbf{m}_{h,k,N}}$ is the solution of (4.7) with respect to $\mathbf{m}_{h,k,N}$. On the other hand, we have $\|u^{N}_{\mathbf{m}_{h,k,N}} - u_{\mathbf{m}_{h,k,N}}\|_h \to 0$ uniformly for h and k as $N \to \infty$, where $u_{\mathbf{m}_{h,k,N}}$ is the solution of (4.4) with respect to $\mathbf{m}_{h,k,N}$ [12]. Notice also that, by the definition of $b_N(\cdot,\cdot)$, we have $\|u_{\mathbf{m}_{h,k,N}} - u\|_h \leq |u_{\mathbf{m}_{h,k,N}} - u|_{1,R^n}$. So, we need only to prove that

$$\lim_{N\to\infty} \lim_{h\to 0,\, k\to\infty} |u_{\mathbf{m}_{h,k,N}} - u|_{1,R^n} = 0. \tag{5.1}$$

Suppose otherwise, there would exist an $\varepsilon_0 > 0$ and a sequence $\mathbf{m}_i \in \mathfrak{M}_{h_i,k_i}^{N_i}$ with $h_i \to 0$ and $k_i, N_i \to \infty$, such that

$$\lim_{i\to\infty} \inf_{\nu^{h_i,k_i}\in\mathcal{A}^{\mu}_{h_i,k_i}} E^{\mu}_{h_i,N_i}(\nu^{h_i,k_i}) = \inf_{\nu\in\mathcal{A}^{\mu}} E^{\mu}(\nu) \tag{5.2}$$

holds for the sequence and

$$|u_{\mathbf{m}_i} - u|_{1,R^n} > \varepsilon_0. \tag{5.3}$$

Since $\mathcal{A}^{**}$ is sequentially weakly* compact in $(L^{\infty}(\Omega))^n$ (*c.f.* Alaoglu's theorem [26]), without loss of generality, we may assume that $\mathbf{m}_i \rightharpoonup \tilde{\mathbf{m}}$ in $(L^2(\Omega))^n$ for some $\tilde{\mathbf{m}} \in \mathcal{A}^{**}$, and consequently

$$\nabla u_{\mathbf{m}_i} \rightharpoonup \nabla u_{\tilde{\mathbf{m}}}, \quad \text{in } (L^2(\Omega))^n, \tag{5.4}$$

where $u_{\tilde{\mathbf{m}}}$ is the solution of the Maxwell's equation (1.2) with respect to $\tilde{\mathbf{m}}$. By the convexity, and consequently the sequentially weakly lower semi-continuity, of $\hat{\varphi}^{**}$ and $\int_{R^n} |\nabla v|^2 dx$, we would have

$$\inf_{\mathbf{m}\in\mathcal{A}^{**}} E^{**}(\mathbf{m}) = \inf_{\nu\in\mathcal{A}^{\mu}} E^{\mu}(\nu) = \lim_{i\to\infty} E^{\mu}(\nu^{h_i,k_i}) \geq \lim_{i\to\infty} E^{**}(\mathbf{m}_i) \geq E^{**}(\tilde{\mathbf{m}}),$$

where ν^{h_i,k_i} are the Young measure solutions of the fully discrete problem (**FDRP**) corresponding to $\mathbf{m}_i$, and hence

$$\inf_{\mathbf{m}\in\mathcal{A}^{**}} E^{**}(\mathbf{m}) = \lim_{i\to\infty} E^{\mu}(\nu^{h_i,k_i}) = \lim_{i\to\infty} E^{**}(\mathbf{m}_i) = E^{**}(\tilde{\mathbf{m}}). \tag{5.5}$$

Thus, by the uniqueness of the potential of the stray field energy of problem (**RP**) (see Theorem 31), (5.5) implies that

$$u_{\tilde{\mathbf{m}}} \equiv u. \tag{5.6}$$

Notice that, by the convexity of $\hat{\varphi}^{**}$ and $\int_{R^n} |\nabla v|^2 dx$, (5.4), (5.5) and (5.6) imply also $|u_{\mathbf{m}_i}|_{1,R^n} \to |u|_{1,R^n}$ and thus $|u_{\mathbf{m}_i} - u|_{1,R^n} \to 0$, which contradicts the inequality (5.3).

As a direct consequence of Theorem 51, we have

Corollary 51 *For the discrete potentials $u^{h,N}_{\mathbf{m}_{h,k,N}}$ of the stray field energy with respect to the minimizers $\mathbf{m}_{h,k,N} \in \mathfrak{M}^N_{h,k}$ of the fully discrete multi-atomic Young measure relaxation problem* (**FDRP**)*, we have*

$$\lim_{N\to\infty} \lim_{h\to 0,\, k\to\infty} \sup_{\mathbf{m}'_{h,k,N},\, \mathbf{m}''_{h,k,N}\in\mathfrak{M}^N_{h,k}} \|u^{h,N}_{\mathbf{m}'_{h,k,N}} - u^{h,N}_{\mathbf{m}''_{h,k,N}}\|_h \to 0. \tag{5.7}$$

With similar arguments as in the proof of Theorem 51, we have the following convergence result for the macroscopic magnetization field.

Theorem 52 *Let* $\mathbf{m}_i \in \mathfrak{M}^{N_i}_{h_i,k_i}$ *be a minimizing sequence with* $h_i \to 0$, $k_i \to \infty$, $N_i \to \infty$ *such that* (5.2) *holds. Then, the sequence is weakly precompact in* $\mathcal{A}^{**}$, *and any cluster point of the sequence is a minimizer of* E^{**} *in* $\mathcal{A}^{**}$, *and hence, by Lemma 31, is also a energy minimizing macroscopic magnetization field of the Young measure relaxation problem* **(RP)**.

Next, we consider the stability property of the discrete macroscopic magnetization field. Recall that, for the Young measure relaxation problem (**RP**), any two energy minimizing macroscopic magnetization fields $\mathbf{m}'$ and $\mathbf{m}''$ satisfy $\mathbf{m}' - \mathbf{m}'' \in \mathcal{M}$, where $\mathcal{M}$ is the set of divergence free functions defined by (3.3). To measure how well the discrete solutions preserve this important property, we introduce an $(h,\,\varepsilon)$-divergence function set for the discrete problem:

$$\mathcal{M}^{\varepsilon}_h := \{\mathbf{m} \in (L^2(\Omega))^n : |\int_{\Omega_i} \mathbf{m}\chi_\Omega \cdot \nabla v_h\, dx| \le \varepsilon,\ \forall v_h \in V_h, \|v_h\|_h = 1\}. \tag{5.8}$$

Notice that by the discrete Helmholtz's decomposition [1], any function $\mathbf{m}_h \in \mathcal{M}^{\varepsilon}_h \cap \mathbf{V}^0_h$, where $\mathbf{V}^0_h$ is the space of piecewise constant vector functions, can be decomposed as $\mathbf{m}_h = \nabla\tilde{v}_h + \mathbf{curl}\,\tilde{w}_h$, for some $\tilde{v}_h \in V_h$ and $\tilde{w}_h \in V^c_h$, where V^c_h is the corresponding piecewise linear conforming finite element space. Thus, by the definition of $\mathcal{M}^{\varepsilon}_h$, the L^2-norm of $\nabla\tilde{v}_h$, which determines the divergence of $\mathbf{m}_h$, is less than ϵ. That is why we could consider $\mathcal{M}^{\varepsilon}_h$ as a set of nearly divergence free functions.

Theorem 53 *For any given* $h > 0$ *and integers* $k \ge 1$, $N \ge 1$, *there exists an* $\varepsilon(h,k,N) \ge 0$ *such that* $\lim_{N\to\infty} \lim_{h\to 0,\, k\to\infty} \varepsilon(h,k,N) = 0$, *and*

$$\mathbf{m}'_{h,k,N} - \mathbf{m}''_{h,k,N} \in \mathcal{M}^{\varepsilon(h,k,N)}_h \tag{5.9}$$

holds for any pair $\mathbf{m}'_{h,k,N}$, $\mathbf{m}''_{h,k,N} \in \mathfrak{M}^N_{h,k}$.

Proof Let $u^{h,N}_{\mathbf{m}'_{h,k,N}}$ and $u^{h,N}_{\mathbf{m}''_{h,k,N}}$ be the solutions of equation (4.9) corresponding to $\mathbf{m}'_{h,k,N}$ and $\mathbf{m}''_{h,k,N}$. Then, we have

$$\begin{aligned} |\int_{\Omega_i} (\mathbf{m}'_{h,k,N} - \mathbf{m}''_{h,k,N})\chi_\Omega \cdot \nabla v_h dx| = |a_h(\nabla u^{h,N}_{\mathbf{m}'_{h,k,N}} - \nabla u^{h,N}_{\mathbf{m}''_{h,k,N}}, \nabla v_h) + \\ b_N(u^{h,N}_{\mathbf{m}'_{h,k,N}} - u^{h,N}_{\mathbf{m}''_{h,k,N}}, v_h)| \le \|u^{h,N}_{\mathbf{m}'_{h,k,N}} - u^{h,N}_{\mathbf{m}''_{h,k,N}}\|_h \|v_h\|_h,\ \ \forall v_h \in V_h. \end{aligned} \tag{5.10}$$

The conclusion follows now from (5.7).

Theorem 53 states that the discrete energy minimizing magnetization fields $\mathbf{m}_{h,k,N}$ of the fully discrete problem **(FDRP)** all lie on a manifold with a pair-wise asymptotically divergence free property. This suggests that the discrete problem **(FDRP)** asymptotically keeps the key property of the continuous problem **(RP)**, where the magnetization fields lie on a manifold with a pair-wise divergence free property (see Proposition 32). More precisely, we have the following result.

Corollary 52 *For the energy minimizing macroscopic magnetization fields of the fully discrete relaxation problem* **(FDRP)**, *we have*

$$\lim_{N\to\infty} \lim_{h\to 0,\, k\to\infty} \sup_{\mathbf{m}',\, \mathbf{m}''\in\mathfrak{M}_{h,k}^N} |\int_\Omega (\mathbf{m}' - \mathbf{m}'')\cdot\nabla v dx| = 0, \quad \forall v\in H^1(R^n).$$

Proof Notice that

$$\int_\Omega (\mathbf{m}' - \mathbf{m}'')\cdot\nabla v dx = \int_\Omega (\mathbf{m}' - \mathbf{m}'')\cdot\nabla v_h dx + \int_\Omega (\mathbf{m}' - \mathbf{m}'')\cdot\nabla (v - v_h) dx.$$

The conclusion of the corollary follows immediately from (5.10) and the approximation property of the non-conforming finite element function space V_h.

To show that Theorem 53 is related to the stability of the numerical method, we give the following corollary,

Corollary 53 *For* $\mathbf{m}'_{h,k,N}$, $\mathbf{m}''_{h,k,N} \in \mathfrak{M}_{h,k}^N$, *we have*

$$\mathbf{m}'_{h,k,N} = \mathbf{m}''_{h,k,N}, \qquad \text{if } u^{h,N}_{\mathbf{m}'_{h,k,N}} = u^{h,N}_{\mathbf{m}''_{h,k,N}}. \tag{5.11}$$

Proof If $u^{h,N}_{\mathbf{m}'_{h,k,N}} = u^{h,N}_{\mathbf{m}''_{h,k,N}}$, it follows from (5.10) that

$$\int_{\Omega_i} (\mathbf{m}'_{h,k,N} - \mathbf{m}''_{h,k,N})\chi_\Omega \cdot \nabla v_h dx = 0, \quad \forall v_h \in V_h.$$

This implies $\mathbf{m}'_{h,k,N} = \mathbf{m}''_{h,k,N}$.

Corollary 53 shows that there are no artificial numerical oscillations on the element by element scale as is known to appear in the conforming finite element approach [5,19]. In particular, in the uniaxial case, where the discrete stray field of the fully discrete problem **(FDRP)** is unique [24], Corollary 53 implies that the energy minimizing magnetization field is also unique in such a case.

6 Numerical Implementation and Examples

First, we rewrite the set $\mathcal{A}^{\mu}_{h,k}$ and deduce the discrete problem to an unconstrained nonconvex optimization problem. Let $k = 2^j$ where $j \geq 1$ is an integer. For $K \in \Gamma_h(\Omega)$ and $i = 1, 2, \cdots, 2^j$, let $\theta_{K,i} \in [-\pi, \pi]/\{-\pi, \pi\}$, i.e. $-\pi$ and π are considered to be the same point in the set, define $A(\theta_{K,i}) \in S^1$ by

$$A(\theta_{K,i}) = \begin{pmatrix} \cos(\theta_{K,i}) \\ \sin(\theta_{K,i}) \end{pmatrix}. \tag{6.1}$$

For $K \in \Gamma_h(\Omega)$, let $\alpha_K = \{\alpha_{K,l}\}_{\ell=1}^{j}$ with $\alpha_{K,\ell} \in [-\pi/2, \pi/2]/\{-\pi/2, \pi/2\}$, and let $i = 1 + i_1 2^0 + i_2 2^2 + \cdots + i_j 2^j$ with $i_\ell \in \{0, 1\}$ for $\ell = 1, 2, \cdots, j$, define

$$\lambda(\alpha_K, i) = \prod_{\ell=1}^{j} cs(i_\ell, \alpha_{K,\ell}) \tag{6.2}$$

where

$$cs(\xi, \beta) = \begin{cases} \cos^2(\beta), & \text{if } \xi = 0 \ ; \\ \sin^2(\beta), & \text{if } \xi = 1 \ . \end{cases} \tag{6.3}$$

It is not difficult to see that the $\lambda(\alpha_K, I)$ satisfy

$$0 \leq \lambda(\alpha_K, i) \leq 1, \quad \text{and} \quad \sum_{i=1}^{k} \lambda(\alpha_K, i) = 1.$$

Denote $\boldsymbol{\theta} = \{\theta_{K,i} \mid K \in \Gamma_h(\Omega)\ ,\ i = 1, \cdots, k\}$ and $\boldsymbol{\alpha} = \{\alpha_{K,\ell} \mid K \in \Gamma_h(\Omega), \ell = 1, \cdots, j\}$. It is easily verified that

$$\mathcal{A}^{\mu}_{h,k} = \mathcal{A}^{\mu}_{h,k}(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \Big\{ \nu^{h,k} = \big\{ \nu^{h,k}(\boldsymbol{\theta}, \boldsymbol{\alpha})|_K \big\}_{K \in \Gamma_h} : \nu^{h,k}(\boldsymbol{\theta}, \boldsymbol{\alpha})|_K = \sum_{i=1}^{k} \lambda(\alpha_K, i) \delta_{A(\theta_{K,i})} \Big\}. \tag{6.4}$$

Now, the full discrete problem (**FDPR**) can be rewritten as

$$(\mathbf{FDRP}'): \text{To minimize } E^{\nu}_h(\boldsymbol{\theta}, \boldsymbol{\alpha}) := E^{\nu}_h(\nu^{h,k}(\boldsymbol{\theta}, \boldsymbol{\alpha})) \text{ in } \mathcal{A}^{\mu}_{h,k}(\boldsymbol{\theta}, \boldsymbol{\alpha}). \tag{6.5}$$

The following algorithm can be applied to this unconstrained nonconvex optimization problem [19]:

1. set $j = j_0 \geq 1$, set $k = 2^j$, give the initial mesh;
2. set $(\boldsymbol{\theta}, \boldsymbol{\alpha}) = (\boldsymbol{\theta}_0, \boldsymbol{\alpha}_0)$;
3. compute $E^{\nu}_h(\boldsymbol{\theta}, \boldsymbol{\alpha})$ by (6.1)-(6.3) and by solving (4.9) ;
4. compute $\mathbf{d}(\boldsymbol{\theta}, \boldsymbol{\alpha}) = \frac{\partial E^{\nu}_h(\boldsymbol{\theta}, \boldsymbol{\alpha})}{\partial(\boldsymbol{\theta}, \boldsymbol{\alpha})}$;

5. if $\|\mathbf{d}(\boldsymbol{\theta}, \boldsymbol{\alpha})\| < TOL$, go to step 7;
6. search for a minimizer $(\boldsymbol{\theta}_1, \boldsymbol{\alpha}_1)$ of E_h^ν along the conjugate gradient direction. Let $(\boldsymbol{\theta}, \boldsymbol{\alpha}) = (\boldsymbol{\theta}_1, \boldsymbol{\alpha}_1)$, go to step 3;
7. if j is not sufficiently large, then set $j = j + 1$ and $k = 2^j$, distribute the new atoms accordingly, then go to step 3;

Notice that the equation (4.9) is in fact a system of linear equations of the form

$$Tu_{\mathbf{m}} = G\mathbf{m}, \tag{6.6}$$

where T is symmetric and positive definite, and thus we have

$$\int_\Omega \mathbf{m} \cdot \nabla u_{\mathbf{m}} dx = u_{\mathbf{m}}^T T u_{\mathbf{m}} = \mathbf{m}^T G^T T^{-1} G\mathbf{m}, \tag{6.7}$$

and

$$\frac{\partial \int_\Omega \mathbf{m} \cdot \nabla u_{\mathbf{m}} dx}{\partial(\boldsymbol{\theta}, \boldsymbol{\alpha})} = 2G^T T^{-1} G\mathbf{m}. \tag{6.8}$$

In step 3 and 4, Equation (6.7) and (6.8) are used to compute the corresponding items.

The criteria for increasing j in step 7 may depend on the problem we solve. In general, j may be increased, if the number of the actual active atoms is greater than 2^j.The necessary number of active atoms can be estimated analytically in some special cases, for example two-atomic Young measures are sufficient in the uniaxial case [24], or by numerical experiments in general cases. Our numerical experiments show that, if more atoms than actually necessary are used in the computation, then, either the supports of some of the atoms will merge, or the volume fractions λ of some of the atoms will go to 0, i.e. the corresponding atoms turn to inactive.

In the following, we present some numerical examples, which show that our new method is efficient and avoids the artificial oscillations.

Example 1 *Let* $\Omega = \{(x, y) : \ x^2 + y^2 < 0.6^2\}$ *and* $\Omega_i = \{(x, y) : \ x^2 + y^2 < 1\}$, $\varphi(\mathbf{m}) = 0.5(m_2^2 + (m_1^2 - 1)^2)$, *which corresponds to the uniaxial case in physics. In this case, we have* $\hat{\varphi}^{**}(\mathbf{m}) = 0.5(m_2^2 + m_2^4)$, *for all* $|\mathbf{m}| \le 1$. *Let*

$$\mathbf{m}_0(x, y) = \begin{cases} \mathbf{0}, & \text{if } r > 0.5; \\ -\dfrac{1}{2}\nabla \exp(-\dfrac{1}{4r^2 - 1}), & \text{if } r \le 0.5, \end{cases}$$

where $r = (x^2 + y^2)^{1/2}$. *Let* $u_{\mathbf{m}_0}$ *be the corresponding stray field. Let* $\mathbf{H}_0$ *be given by the equation* (2.2), *noticing that* $\nabla u_{\mathbf{m}_0} = \mathbf{m}_0$ *in this case. Then, it is easily verified that* $\mathbf{m}_0$ *is the solution to problem (**RP**) with the applied field* $\mathbf{H}_0$. *In our numerical experiments, we set* $j = 2$ *so that* $k = 4$, *that is 4-atomic Young measures are used. In addition, we set* $TOL = 10^{-10}$ *and* $N = 9$ *in Equation* (4.9).

In numerical implementation, k and N can be properly chosen so that there are sufficiently many atoms involved in the computation and the approximation error of u_N to u reduces to the machine precision (see Remark 41). In such a case, the convergence behavior of our scheme essentially depends only on the parameter of $\Gamma_h^{(i)}$. For $k = 4$ and $N = 9$, the convergence behavior is shown in Figure 6.1, and it is clearly seen that the convergence rates for the errors $err_m = \|\mathbf{m} - \mathbf{m}_h\|_{0,\Omega}$ and $err_u = \|u - u_h\|_h$ are both about $N_e^{1/2}$, where N_e is the total number of elements in the triangulation $\Gamma_h^{(i)}(\Omega_i)$. In addition, numerical results shows that the Young measures are always supported on two atoms. This is consistent with the theoretical results in [24]. In our numerical experiments, the atoms A_1 and A_3 merge into one atom, so do the atoms A_2 and A_4, while the the corresponding volume fractions converge and lead to the relation $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$. Some typical Young measures produced by our algorithm are shown in Table 6.1. It is worth noticing here that the strong convergence of $\mathbf{m}$ indicates that there is no element-wise artificial oscillation in the discrete magnetization field.

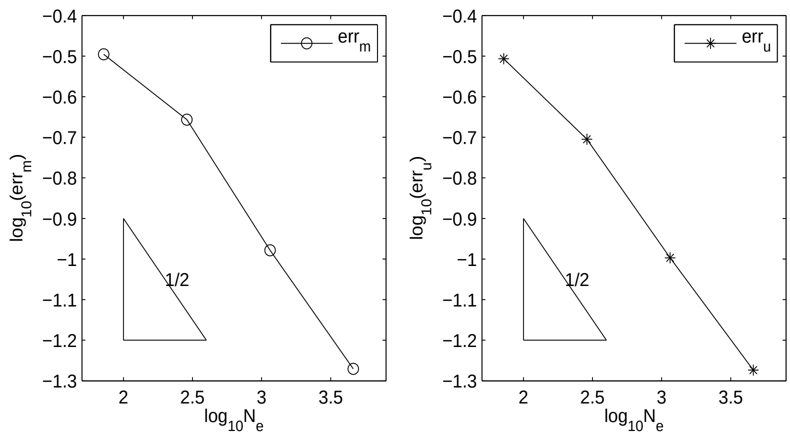

Fig. 6.1 Convergence Behavior for Example 1.

Table 6.1 Some typical Young measure results for example 1.

$A_1(A_3)$	$\lambda_1(\lambda_3)$	$A_2(A_4)$	$\lambda_2(\lambda_4)$
(0.937357, 0.34837)	0.350676	(-0.937357, 0.34837)	0.149324
(0.975485, 0.220068)	0.331261	(-0.975485, 0.220068)	0.168739
(0.934356, 0.35634)	0.339631	(-0.934356, 0.35634)	0.160369
(0.867438, 0.497545)	0.387645	(-0.867438, 0.497545)	0.112355
(0.929412, 0.369043)	0.310461	(-0.929412, 0.369043)	0.189539
(0.861992, 0.506922)	0.313476	(-0.861992, 0.506922)	0.186524
(0.773746, 0.633496)	0.355084	(-0.773746, 0.633496)	0.144916
(0.788827, 0.614615)	0.392117	(-0.788827, 0.614615)	0.107883

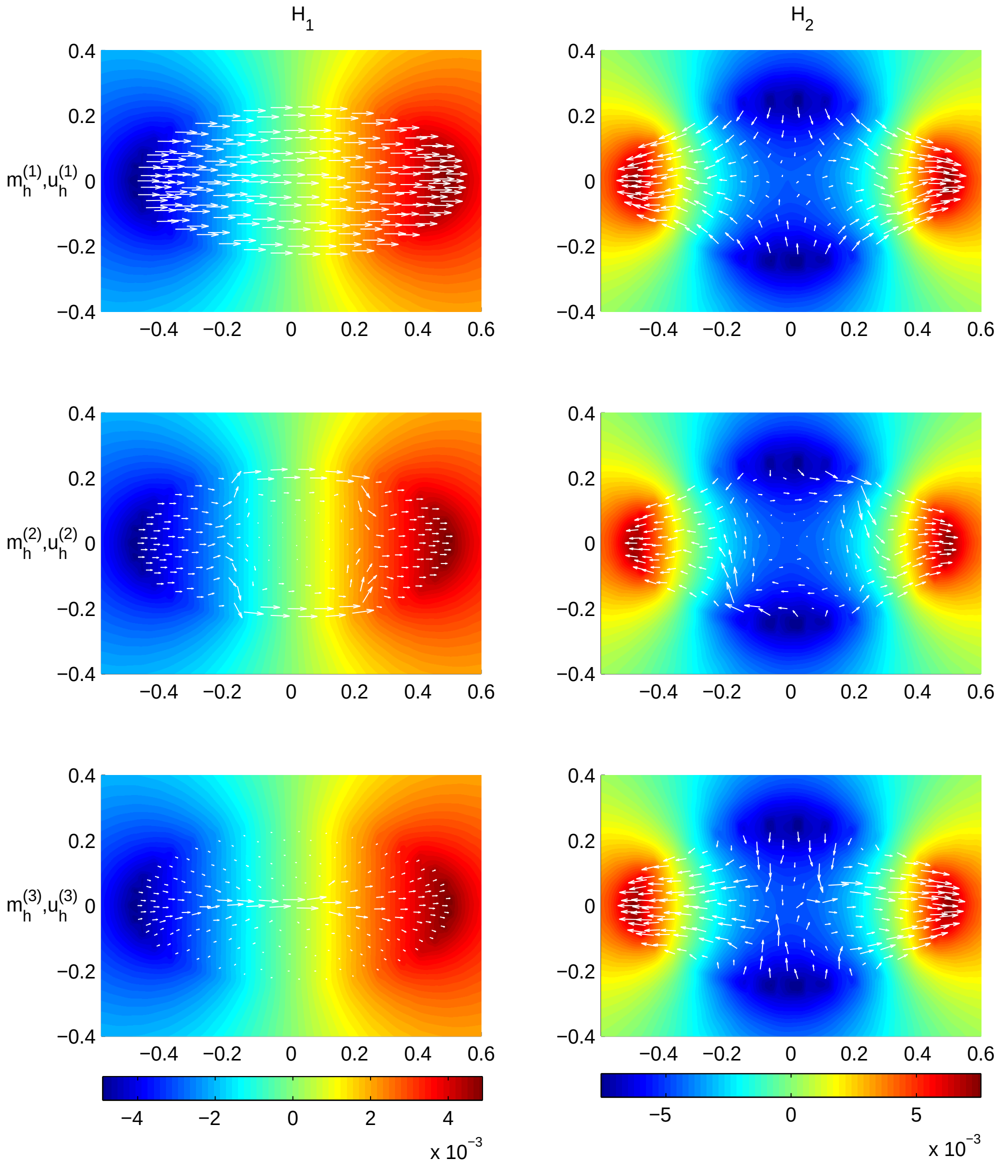

Fig. 6.2 $\mathbf{m}_h^{(i)}$ and $u_h^{(i)}$ obtained with various initial distributions $\mathbf{m}_0^{(i)}$ and applied fields $\mathbf{H}_1$ and $\mathbf{H}_2$ for Example 2.

Example 2 *Let* $\Omega = \{(x,y) : \frac{x^2}{0.5^2} + \frac{y^2}{0.2^2} < 1\}$. *Consider the cubic case with* $\varphi(\mathbf{m}) = 10^{-2} m_1^2 m_2^2$, *and consider the applied fields* $\mathbf{H}_1 = (10^{-2}, 0)$ *and* $\mathbf{H}_2 = 0.1 \times (x, -y)$, *which are both sufficiently small and curl-free.*

We set $j = 2$ (so that $k = 2^j = 4$), $TOL = 10^{-10}$ and $N = 9$, which turn out to be sufficient, and run the minimizer searching algorithm with three different initial macroscopic magnetization fields $\mathbf{m}_0^{(1)} = (0,0)$, $\mathbf{m}_0^{(2)} = 0.02 \times (x, -y)$ and $\mathbf{m}_0^{(3)} = 0.02 \times (y, -x)$. The numerical results of the macroscopic magnetization fields $\mathbf{m}_h^{(i)}$ and the corresponding stray field potentials $u_h^{(i)}$ are shown in Figure 6.2, where $\mathbf{m}_h^{(i)}$ are visualized by the arrays of arrows and $u_h^{(i)}$ are plotted in pseudocolor contour maps.

As is predicted by the theory established in section 2 and section 3, our numerical experiments show that the magnetization fields $\mathbf{m}_h^{(i)}$ obtained with different initial macroscopic magnetization fields are dramatically different, while the corresponding stray field potentials $u_h^{(i)}$ are numerically asymptotically identical.

To exhibit the stability result stated in Theorem 53, we define a norm

$$|\mathbf{m}_h|_{h^*} = \sup_{v_h \in \mathcal{V}_h} \frac{|\int_\Omega \mathbf{m}_h \cdot \nabla v_h dx|}{\|v_h\|_h}, \quad \text{for piecewise constant } \mathbf{m}_h,$$

where $\mathcal{V}_h = \{$all of the base functions of the finite element space $V_h\}$. To compare with the theoretical results of Corollary 51 and Theorem 53, the convergence behavior of $\mathrm{diff}(\mathbf{m}_h) := |\mathbf{m}_h - \tilde{\mathbf{m}}_h|_{h^*}$ for two sequences of discrete solutions $\{\mathbf{m}_h\}$ and $\{\tilde{\mathbf{m}}_h\}$, which converge to different macroscopic magnetization fields, and that of $\mathrm{diff}(u_h) := \|u_h - \tilde{u}_h\|_h$ for the corresponding discrete potentials of the stray field energy are shown in Figure 6.3. In the computation, we use a sequence of meshes with numbers of elements $N_e = 246, 418, 882, 1264$, respectively. The numerical results show that $N = 9$ is sufficiently large to achieve machine accuracy, and that the problem admits 4-atomic Young measure solutions in all of the six cases, in fact, 4-atomic discrete Young measures were always obtained with $k \geq 4$ for the example.

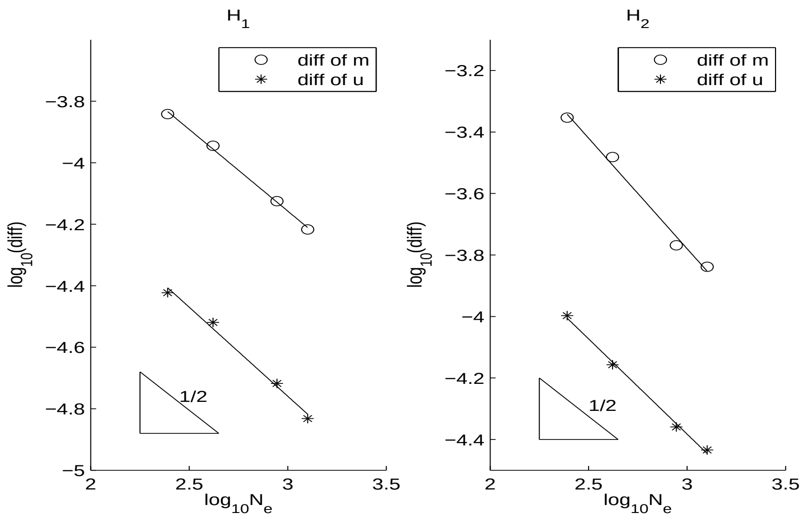

Fig. 6.3 The convergence behavior of $\|\mathrm{diff}(\mathbf{m}_h)\|_{h^*}$ and $\|\mathrm{diff}(u_h)\|_h$ for different applied fields $\mathbf{H}_1$ and $\mathbf{H}_2$.

Example 3 *Consider an applied field $\mathbf{H}_3 = 0.1 \times (y, -x)$, which is still sufficiently small but no longer curl-free, and let everything else be the same as in Example 1.*

Our numerical experiments show that both the macroscopic magnetization field $\mathbf{m}_h^{(i)}$ and the potential of the stray field energy $u_h^{(i)}$ are numerically identical, with the relative errors in maximum norm bounded by 0.05. This somehow indicates that, in such a case, the macroscopic magnetization field $\mathbf{m}$ is unique. Again, we have a 4-atomic Young measure minimizer. A typical numerical result on $\mathbf{m}_h^{(i)}$ and $u_h^{(i)}$ is shown in Figure 6.4.

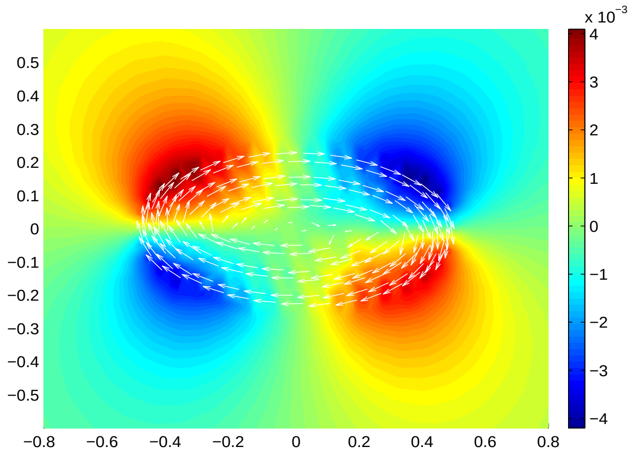

Fig. 6.4 A typical numerical result on $\mathbf{m}_h^{(i)}$ and $u_h^{(i)}$ obtained with various initial distributions $\mathbf{m}_0^{(i)}$ for Example 3.

Example 4 *Consider a uniaxial anisotropic energy density $\varphi(\mathbf{m}) = 10^{-2}(m_1^2 + (1 - m_2^2)^2)$ and the applied field $\mathbf{H}_1$. Let everything else be the same as in Example 1.*

As is predicted by the theory established in section 3, our numerical experiments show that both the macroscopic magnetization field $\mathbf{m}_h^{(i)}$ and the potential of the stray field energy $u_h^{(i)}$ are numerically identical, with the relative errors in maximum norm bounded by 0.01, and a 2-atomic Young measure minimizer is obtained in this case. A typical numerical result on $\mathbf{m}_h^{(i)}$ and $u_h^{(i)}$ is shown in Figure 6.5.

7 Conclusions

The convergence and stability theorems are rigourously established in this paper for a numerical method developed in [24] for the Young measure relaxation problem in micromagnetics,

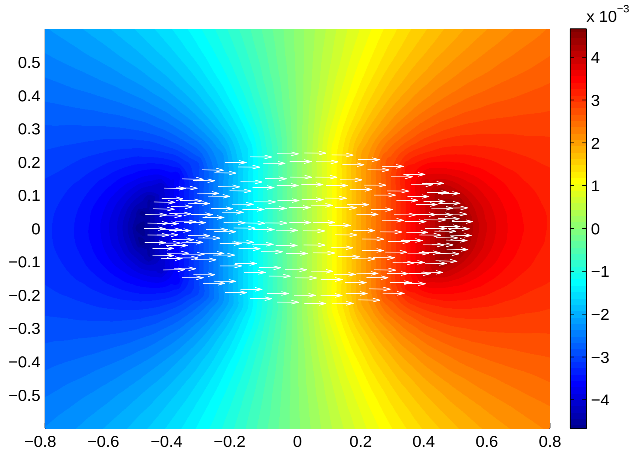

Fig. 6.5 A typical numerical result on $\mathbf{m}_h^{(i)}$ and $u_h^{(i)}$ obtained with various initial distribution $\mathbf{m}_0^{(i)}$ for Example 4.

which applies the multi-atomic Young measures to approximate the continuous Young measure and applies the Crouzeix-Raviart nonconforming finite element method coupled with an artificial boundary method to solve the Maxwell's equation.

By regarding the very limited number of atoms as unknowns in S^{n-1} and determining the positions and volume fractions of these atoms in the process of energy minimization, our method provides better approximation and is more efficient as compared with Kruzik-Prohl's method [15], since the atoms do not have to be represented on tremendous number of fixed nodal points and singled out by a relatively costing adaptive process.

As compared with Li-Wu's work [19], in which a conforming finite element method is applied to solve the Maxwell's equation and discrete energy minimizing magnetization field is found to have obvious element-wise artificial oscillations, our method is shown to have an asymptotically pair-wise divergence free property for the discrete energy minimizing magnetization fields, which turns out to be the key feature to diminishing the element-wise artificial oscillations and guaranteeing the stability and strong convergence of the method.

In addition, the method is easy to implement in practise. Traditional optimization methods for large nonlinear problems, such as the conjugate gradient method, can be used directly.

Some interesting new results are also obtained in this paper, for example, the non-uniqueness for the energy minimizing magnetization fields under a nontrivial applied field, the convergence of the discrete stray fields, etc.. Some numerical examples are given to verify our analytical results and to show the convergence and stability of the numerical scheme.

In the end, we would like to remark that there is no theoretical difficulties to extend our numerical method to three-dimensional case $n = 3$, although we confined ourself to $n = 2$ for simplicity in this paper. For example, we could choose three-dimensional artificial boundary conditions [13] in solving the Maxwell's equation (1.2) and construct a non-constraint optimization problem like **(FDRP')** in a similar way. In addition, mesh adaptivity based on a posteriori error estimates could be applied to solve the discrete Maxwell's equation (4.9) [25] to further increase the efficiency of the computation.

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