Theoretical Analysis for Meniscus Rise of a Liquid Contained between a Flexible Film and a Solid Wall

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Abstract – We study the dynamics of meniscus rise of a liquid contained in a narrow gap between a flexible film and a solid wall. We show that the meniscus rises indefinitely expelling liquid from the gap region, and that the height of the rising front $h(t)$ increases with time as $h(t) \propto t^{2/7}$, while the gap distance $e(t)$ decreases as $e(t) \propto t^{-3/7}$. These results are consistent with the experiments of Cambau et al. [1].

Introduction. – Capillary actions in flexible structures are relevant to a number of engineering and biological processes [2–4], such as printing, coating, lithography, blood flow in vessels etc. The phenomenon in which the capillary force is coupled with the elastic deformation is called elastocapillary effect, and many intriguing phenomena are discussed in a recent review paper of Roman and Bico [5].

A simple elastocapillary problem is the wetting of liquid in flexible fibers or sheets. In earlier works, the phenomenon was studied for flexible fibers [6] or sheets [7] which are fixed to a horizontal bar at the top. When such objects are inserted into a liquid bath, the liquid rises deforming the objects, and eventually takes an equilibrium state where capillary force, elastic force and gravitational force are balanced. The equilibrium state was studied both experimentally and theoretically. The dynamics of such system was also studied by Duprat, Aristoff and Stone [8] for the case that the liquid rises between flexible sheets.

Cambau, Bico and Reyssat [1] have shown that the behaviour of the meniscus rise changes dramatically if the flexible sheets are supported by vertical spacers as shown in fig. 1(a). Here a plastic sheet is placed on top of a slightly inclined glass plate supported by two vertical spacers (having thickness e_m), and the liquid rises between the sheet and the plate. They have shown that if the sheet is rigid, the meniscus stops rising at an equilibrium height, while the meniscus keeps rising if the sheet is flexible. This is because the gap distance e between the sheet and the plate can keep decreasing with the rise of the meniscus, while in the geometry of the ref. [6, 7] the gap distance cannot keep decreasing due to the constraint at the top. They reported that the height $h(t)$ increases with time t as a power law $h(t) \propto t^{\alpha}$, where α is close to (but less than) 1/3, but no theoretical explanation has been given.

In this paper, we study the dynamics of the capillary rise in the situation studied by Cambau et al. We shall show that the dynamics on a large time scale is governed by the surface tension and becomes independent of sheet elasticity and that the height $h(t)$ and gap distance $e(t)$ change with time as $h(t) \propto t^{2/7}$ and $e(t) \propto t^{-3/7}$.

Model. – We take the coordinate (x, z) as shown in fig. 1(a), where the origin is taken at the middle of the bottom edge. The flexible sheet has width $2w_m$ and length h_m, and the initial gap distance created by the spacer is e_m.

To simplify the analysis, we use the experimental observation by Cambau et al. [1]. They reported that when the sheet is inserted into the liquid bath, the central part of the sheet is bent inward due to the negative Laplace pressure created by the capillary force and that the meniscus takes a parabolic profile (as shown by the top part in fig. 1(b)). As time goes on, the wetting front moves both in x and z directions, reducing the gap distance further. Since the gap distance is constrained to e_m by the two spacers fixed at $x = \pm w_m$, the motion in x direction is

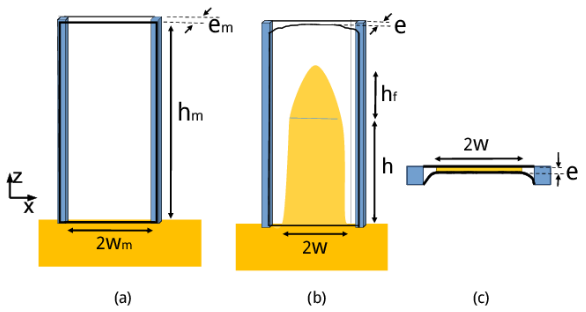

Fig. 1: Schematic illustration of the experiment in [1]: (a) A flexible polymer sheet (white) is separated from a rigid plate of width $2w_m$ by two spacers of thickness e_m. The flexible wall is simply supported on the spacers and the cell is very slightly tilted with respect to the vertical direction to prevent the flexible wall from falling. (b) When the cell is inserted into a liquid bath, the liquid rises by capillary force taking a form shown by yellow region. (c) The cross section of the cell in the rectangular part.e

stopped at a certain position $x = \pm w$ which is close to the spacers. Cambau et al. [1] gave a theoretical estimation for the value of $w - w_m$

$$w_m - w = \sqrt{1.5 L_{ec} e_m}\,, \tag{1}$$

where $L_{ec} = \sqrt{B/\gamma}$ is the elastocapillary length of the system, B is the bending modulus of the sheet and γ is the wetting energy of the fluid. They showed that eq. (1) is in agreement with their experiments.

At a later time, the wetted region looks as shown in fig. 1(b). It consists of a parabolic part (of vertical length h_f) and a rectangular part (of length $h(t)$ and width $2w$). The parabolic part is in a pseudo steady state (its shape does not change in time), and the meniscus rise of the liquid is due to the change of $h(t)$, the height of the rectangular part.

In this paper, we shall study the late stage of the meniscus rise and assume that the front part is in a steady state. Let $e(x, z, t)$ be the gap distance at position (x, z) and time t. Theoretical prediction for $e(x, z, t)$ is difficult. If the sheet is not stretchable in plane, $e(x, z, t)$ must be a linear function of z (since the Gaussian curvature must remain zero). However Cambau et al. [1] showed that $e(x, z, t)$ is not a linear function of z in the parabolic region $z > h(t)$. Therefore to analyse the front part, we need to consider both the stretching and the bending of the sheet. On the other hand, they have shown that $e(x, z, t)$ is independent of z in the rectangular region, and is almost independent of x except in the region close to the edge as shown in fig. 1(c).

We thus construct a model for the late stage of dynamics assuming that

(A) The parabolic region is in a steady state.

(B) The rectangular region has a constant width $w = w_m - \sqrt{1.5(L_{ec}e_m)}$, and a thickness $e(t)$ which depends on time but independent of x.

With these simplifications, we can derive the time evolution of $e(t)$ and $h(t)$ by considering the force balance for fluid and elastic sheet. Here we use the Onsager principle [9, 10] since it is simpler and gives the same result. The Onsager principle can be stated as follows. Let $a(t) = \{a_1(t), a_2(t), ...a_N(t)\}$ be the set of the parameters which specify the current state of the system. In the present problem, they are given by $(h(t), e(t))$. The time evolution of the system, i.e., the time derivative $\dot{a}(t) = \{\dot{a}_1(t), \dot{a}_2(t), ...\dot{a}_N(t)\}$ is determined by the minimum condition for the following function of $\dot{a}$

$$R(\dot{a}, a) = \Phi(\dot{a}, a) + \sum_i \frac{\partial A}{\partial a_i}\dot{a}_i\,, \tag{2}$$

where $A(a)$ is the potential energy of the system, and $\Phi(\dot{a}, a)$ is the energy dissipation function which is defined as the half of the energy dissipated per unit time in the fluid when $a(t)$ is changing at rate $\dot{a}$.

The energy of the system is given by the sum of the wetting energy, the gravitational energy of the fluid, and the elastic energy [11] of the sheet

$$A = -2\gamma h w + \frac{1}{2}\rho g e h^2 w + \frac{3}{2}\frac{B(e_m - e)^2}{(w_m - w)^3}h_m\,, \tag{3}$$

where γ and ρ stand for the surface tension and the density of the liquid, and g is the gravitational acceleration.

Equation (3) gives the following expression for $\dot{A}$

$$\begin{aligned}\dot{A} &= -2\gamma\dot{h}w + \rho g e h\dot{h}w \\ &\quad + \frac{1}{2}\rho g\dot{e}h^2 w - 3B\frac{(e_m - e)}{(w_m - w)^3}h_m\dot{e}.\end{aligned} \tag{4}$$

To calculate the energy dissipation function, we use the lubrication approximation. Let $\boldsymbol{v}(x, z)$ be the depth averaged velocity of the fluid, then the energy dissipation function is given by

$$\Phi = \frac{1}{2}\int_0^h dz \int_0^w dx \frac{12\eta}{e}\boldsymbol{v}^2\,, \tag{5}$$

where η is the viscosity of the fluid. In the present problem $\boldsymbol{v}$ has z component v_z only. The incompressible condition of the fluid is written as

$$\dot{e} = -\frac{\partial}{\partial z}(v_z e)\,, \tag{6}$$

which gives

$$v_z = \dot{h} + \frac{\dot{e}}{e}(h - z)\,, \tag{7}$$

where we have used the condition that $v_z = \dot{h}$ at $z = h$.

Equations (5) and (7) give the following expression for the energy dissipation function

$$\Phi = \frac{2\eta h w}{e}\left(3\dot{h}^2 + 3\frac{h}{e}\dot{h}\dot{e} + \frac{h^2}{e^2}\dot{e}^2\right). \tag{8}$$

The minimum condition of $R = \Phi + \dot{A}$ with respect to $\dot{h}$ and $\dot{e}$ gives the following equations

$$\begin{aligned}
\frac{2\eta h w}{e}\left(6\dot{h} + \frac{3h}{e}\dot{e}\right) &= 2\gamma w - \rho g e h w\,, \\
\frac{2\eta h w}{e}\left(\frac{3h}{e}\dot{h} + \frac{2h^2}{e^2}\dot{e}\right) &= -\frac{1}{2}\rho g h^2 w + \frac{3B(e_m - e)h_m}{(w_m - w)^3}\,,
\end{aligned}$$

which are solved for $\dot{h}$ and $\dot{e}$ as

$$\frac{\dot{h}}{U} = \frac{2}{3}\frac{e}{h} - \frac{1}{12}\kappa_e^2 e^2 - \sqrt{\frac{2L_{ec}}{3e_m}}\frac{(e_m - e)}{e_m}\frac{h_m}{w}\frac{e^2}{h^2}\,, \tag{9}$$

$$\frac{\dot{e}}{U} = -\frac{e^2}{h^2} + \sqrt{\frac{8L_{ec}}{3e_m}}\frac{(e_m - e)}{e_m}\frac{h_m}{w}\frac{e^3}{h^3}\,, \tag{10}$$

where $U = \gamma/\eta$ represents the characteristic capillary velocity, and $\kappa_e^2 = \rho g/\gamma$ ($1/\kappa_e$ is the capillary length).

The first terms on the right hand side in eqs. (9) and (10) represent the capillary effect, while the last terms represent the effect of the bending elasticity of the sheet. It can be seen that the bending elastic term is smaller than the capillary term by factor e/h, and can be ignored at long times. If we ignore these terms, eqs. (9) and (10) indicate that the amount of liquid contained in the wetted region decreases in time since

$$\frac{d(eh)}{dt} = -Ue\left[\frac{1}{3}\frac{e}{h} + \frac{1}{12}\kappa_e^2 e^2\right] < 0. \tag{11}$$

Hence at long times, the gravitational force, the second term on the right hand side in eq. (9), can be ignored. Therefore the governing equations at long times become very simple

$$\dot{h} = \frac{2}{3}\frac{e}{h}U\,, \tag{12}$$

$$\dot{e} = -\frac{e^2}{h^2}U. \tag{13}$$

The solutions of these two equations with the initial conditions $h(0) = h_0$ and $e(0) = e_0$ are

$$h(t) = h_0\left(1 + \frac{7}{3}\frac{e_0 U t}{h_0^2}\right)^{2/7}, \tag{14}$$

$$e(t) = e_0\left(1 + \frac{7}{3}\frac{e_0 U t}{h_0^2}\right)^{-3/7}. \tag{15}$$

Equations (12) and (13) can be derived quite simply if we assume that capillary rise takes place between two parallel plates the gap distance of which is adjustable (i.e., the gap distance is determined automatically by the force balance acting on the plates).

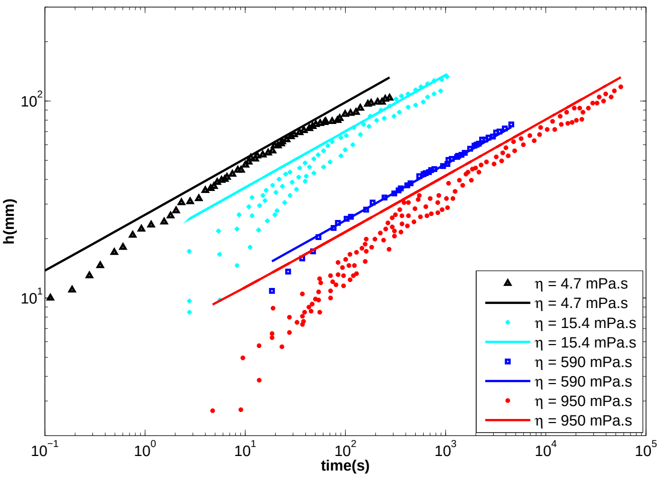

Fig. 2: The prediction (solid lines) of eq. (14) is compared with the experimental data (symbols) of [1] for silicon oils with different viscosities.

Let $p(x, z, t)$ be the pressure in the liquid at point (x, z). The liquid velocity $\boldsymbol{v}$ is related to p by $\boldsymbol{v} = (e^2/12\eta)\boldsymbol{\nabla}p$. The incompressible condition (6) gives

$$\dot{e} = -\frac{e^3}{12\eta}\frac{\partial^2 p}{\partial z^2}\,, \tag{16}$$

which can be solved with the boundary conditions $p = 0$ at $z = 0$, and $p = 2\gamma/e$ at $z = h$. Finally, the condition that the total force acting on the sheet is zero, i.e.,

$$\int_0^h dz p(z, t) = 0 \tag{17}$$

gives eq. (13).

Equation (14) indicates that $h(t)$ increases with time following the power law $h(t) \propto t^{2/7}$. Cambau et al. reported that the exponent in the power law is close to 1/3, the value observed for the capillary rise at a corner of intersecting walls [12–14], but less than 1/3. Fig. 2 shows the comparison of the theoretical curves with the experimental results for various silicone oils having different viscosities, $\eta = 4.7mPas, 15.4mPas, 590mPas, 950mPas$. The theoretical curves are drawn assuming that the surface tensions γ is equal to $21mN/m$, the initial thickness e_0 is equal to $e_m = 1mm$, and that the initial height h_0 is equal to the equilibrium height (known as Jurin's law), $h_0 = 2\gamma/\rho g e_0$. At long times, the prediction (solid lines) for the evolution of the height $h(t)$, equation (14), is in quantitative agreement with the experimental data in [1] (dotted lines) within a wide range of viscosities (different colors) as represented in fig. 2.

Discussion and conclusion. – We have studied the evolution of meniscus rise between flexible sheet and rigid wall. It is shown that the long time behaviour is described by a simple set of equations (eqs. (12) and (13)) and that they explain the power law $h(t) \propto t^{2/7}$ observed experimentally. While such equations can be derived straightforwardly once the assumptions for the model are made, the present analysis indicates several important points. (a) The long time behaviour is governed by the capillary force, and the elasticity of the sheet plays no role (it only affects the initial condition). (b) At a later stage, the meniscus rises expelling liquid from the gap region; therefore the total amount of liquid contained between the sheet and the plate will show a peak as a function of time. These predictions can be checked experimentally.

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