Power-law scaling for solid-state dewetting of thin films: an Onsager variational approach

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We examine the kinetics of surface diffusion-controlled, solid-state dewetting by consideration of the retraction of the contact in a semi-infinite solid thin film on a flat rigid substrate. The analysis is performed within the framework of the Onsager variational principle applied to surface diffusion-controlled morphology evolution. Based on this approach, we derive a simple, reduced-order model to quantitatively analyse the power-law scaling of the dewetting process. Using asymptotic analysis and numerical simulations for the reduced-order model, we find that the retraction distance grows as the $2/5$ power of time and the height of the ridge.

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adjacent to the contact, grows as the $1/5$ power of time for late time. While the asymptotic analysis focuses on late time and a relative simple geometric model, the Onsager approach is applicable to all times and descriptions of the morphology of arbitrary complexity.

1 Introduction

Solid-state dewetting of thin films on substrates has been observed in a wide range of systems by many research groups over many decades\textsuperscript{1–8}. While solid-state dewetting is deleterious, in the sense that it leads to the destabilization/agglomeration of continuous deposited films, it can also be exploited to produce a controlled distribution of particles on a substrate. Recent examples include the formation of ordered arrays of nanoparticles and quantum dots, which have been exploited to produce sensors\textsuperscript{9,10}, optical and magnetic devices\textsuperscript{9,11} and for catalysts for the growth of carbon and semiconductor nanotubes and nanowires\textsuperscript{12,13}. Interest in such applications has driven research into the underlying mechanisms of solid-state dewetting\textsuperscript{14–25}.

Dewetting of thin solid films is similar in many aspects to the wetting/dewetting of liquid thin films\textsuperscript{26}. A major difference, however, is that in most applications, mass transport is dominated by surface diffusion rather than fluid dynamics\textsuperscript{19,21,27}. Typical features of surface diffusion-controlled solid-state thin film dewetting is the formation of a ridge and a valley area of the retracting film edge; the amplitudes of which increase with time/retraction distance\textsuperscript{4,15,16,28–30}. Experimental observations show that the edge retraction distance scales as the $2/5$ power of time (at long time)\textsuperscript{29,30}. While this power-law has been observed in numerical simulations of thin film dewetting\textsuperscript{15,19,21}, rigorous theoretical analysis has remained elusive (despite reasonable approximate solutions\textsuperscript{22,31}).
Brandon and Bradshaw (referred to as the BB model here) presented a simple geometric model for analyzing this problem more than half a century ago. Based on the observation that edge retraction and ridge growth are the main hallmarks of the experimental observation of surface diffusion-controlled solid state dewetting, BB described the discontinuous film cross-sectional profile as a semicircle that hits the substrate at right angles (i.e., a contact angle of 90°) and joins the finite thickness film. With this simple profile, they obtained two important scaling laws: (1) the radius of a growing hole increases with the $2/5$ power of time and the ridge height grows with the $1/5$ power of time. The correspondence of these results with experimental observations demonstrate that this simple profile is sufficient to capture the essential features of surface diffusion-controlled solid-state dewetting.

Zucker et al. re-examined the BB model with the generalization that the cross-section of the profile need not be semi-circular and the contact/Young angle $\theta_0$ need not be 90° (see Fig. 1). Their analytical solution, for volume-conserving (the BB solution does not conserve volume), surface diffusion-controlled dewetting reproduces the $2/5$ power-law. Their solution is both complicated and approximate.

In this paper, we present a new approach for analyzing the power-law scaling of surface diffusion-controlled, solid-state dewetting that is both rigorous (assuming the same film cross-section as in the Zucker et al.) and based upon the irreversible thermodynamics variational approach of Onsager for surface diffusion-controlled morphology evolution problems. In the next section, we briefly review the application of the Onsager variational principle to construct a
reduced-order model for an evolving, dissipative system. In Section 3, we apply this approach to
the power law retraction of the edge of a semi-infinite thin film on a substrate that results in an
ordinary differential equation (ODE) for surface diffusion-controlled solid-state dewetting. Then,
we provide both asymptotic analysis and numerical simulations for the resulted ODE, and recover
the $2/5$ experimentally and simulation-observed power-law.

2 The Onsager variational principle

The Onsager variational principle, formulated in 1931$^{33,34}$, is based on the reciprocal sym-
metry in linear irreversible thermodynamics. This variational principle has found wide application
in deriving evolution equations in fluid dynamics$^{35–39}$ and soft matter physics$^{40–42}$. We apply this
variational approach to surface-diffusion controlled, solid-state dewetting$^{25}$.

Consider an isothermal system that may include interfaces (e.g., the solid-vapor interface,
solid-substrate interface, and vapor-substrate interface). If the system deviates from its equilibrium
state, then there will be spontaneous processes that tend to bring the system back to equilibrium.
In the linear response regime (i.e., not far from equilibrium), the time evolution of the system is
governed by a variational principle. Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$ be a set of state variables.
The time evolution of the system, may be described as the time derivatives of these state variables $\dot{\alpha}(t) = (\dot{\alpha}_1(t), \dot{\alpha}_2(t), \ldots, \dot{\alpha}_n(t))$ (a raised dot “$\cdot$” denotes a time derivative); it is determined by
minimizing the “Rayleighian” $R$ with respect to the rates $\{\dot{\alpha}_i\}$

$$R(\alpha, \dot{\alpha}) = \dot{W}(\alpha, \dot{\alpha}) + \Phi(\alpha, \dot{\alpha}).$$ (1)
Here, $W(\alpha) := W(\alpha_1, \alpha_2, \ldots, \alpha_n)$ represents the total free energy of the system (a state function) and $\dot{W}$ is the rate of change of $W$,

$$
\dot{W}(\alpha, \dot{\alpha}) = \sum_i \frac{\partial W}{\partial \alpha_i} \dot{\alpha}_i.
$$

(2)

$\Phi(\alpha, \dot{\alpha})$, in Eq. (1), is the free energy dissipation function; it is half the free energy dissipation rate. In the linear response regime, the dissipation function can be written as a quadratic function of the rates $\{\dot{\alpha}_i\}$; i.e.,

$$
\Phi(\alpha, \dot{\alpha}) = \frac{1}{2} \sum_{i,j} \zeta_{ij}(\alpha) \dot{\alpha}_i \dot{\alpha}_j,
$$

(3)

where the damping/friction coefficients $\zeta_{ij}$ form a symmetric, positive definite matrix. Minimizing the Rayleighian with respect to the rates $\{\dot{\alpha}_i\}$ yields a set of kinetic equations

$$
\sum_j \zeta_{ij} \dot{\alpha}_j = -\frac{\partial W}{\partial \alpha_i}, \quad i = 1, 2, \ldots, n.
$$

(4)

This describes the force balance between the potential force $-\frac{\partial W}{\partial \alpha_i}$ and the dissipative force $\frac{\partial \Phi}{\partial \alpha_i}$ (which is linear in the rates $\{\dot{\alpha}_i\}$). A simple calculation shows that the variational principle leads to $\dot{W} = -2\Phi$, which means $\Phi$ is half the rate of free energy dissipation, as asserted above. This variational principle for isothermal systems, outlined above, can be generalized to non-isothermal systems via the maximization of the Onsager-Machlup action$^{33,34}$.

The evolution of a dissipative system described by a set of field variables can be approximated by a finite set of state variables. The total free energy $W$ and dissipation function $\Phi$ can be obtained as functions of these state variables and their time derivatives. Application of the Onsager variational principle then gives a system of ordinary differential equations (ODEs) that describes
the time evolution of the state variables, i.e., the time evolution of the system \textsuperscript{36,37,42}. In our Onsager variational approach to the surface diffusion-controlled, solid-state dewetting problem, we can consider the field variables as a finite set of variables that represent a reduced-order description of the film profile. The purpose of the present work is to apply the Onsager’s variational principle to the dynamics of solid-state dewetting. While it is possible to use this approach to accurately describe the evolution of the entire film profile, we focus here on deriving the power law of the retraction of the thin film dewetting front for which a relative simple description of the film profile suffices. We derive a reduced model for the dynamics of the film by using the Onsager variational principle, then perform asymptotic analysis and numerical simulations to derive the power law scaling of the retraction front in the long-time limit.

3 A reduced-order variational model

We assume the film geometry with a retracting contact and ridge proposed by Zucker et al.\textsuperscript{22}, as shown in Fig. 1. We assume that the contact satisfies the isotropic Young’s angle $\theta_0$, i.e.,

$$\cos \theta_0 = (\gamma_{VS} - \gamma_{FS})/\gamma_0,$$

where $\gamma_0$, $\gamma_{VS}$, $\gamma_{FS}$ represent the film-vapor, vapor-substrate and film-substrate interface energy per unit length. For simplicity in notations, we also introduce an angle $\theta_1(t)$ to characterize the position of the other end of the curve. We further assume that no energy is dissipated by the motion of the contact point, i.e., no contact drag (this is consistent with the assumption that the contact always satisfies the equilibrium Young’s angle condition).

In our application of the Onsager variational principle, we focus on the reduced order model
We assume that the profile of the film-vapor interface consists of a circular arc of radius $R := R(t)$ (center located at $(x_0(t), y_0(t))$), and a straight line representing the semi-infinite film of thickness $h_0$, measured from the substrate (i.e., the $x$-axis). The circular arc meets the substrate at the isotropic Young’s angle $\theta_0$. For convenience, all lengths are scaled by the film thickness.

where the film profile is characterized by the parameters $R(t), x_0(t), y_0(t)$ and $\theta_1(t)$ that evolve during the dewetting process. These parameters are not independent; they satisfy several geometric constraints such that there is only one independent variable. Examination of the geometry of Fig. 1 shows that

$$\begin{align*}
R \cos(\pi - \theta_0) &= y_0, \\
y_0 + R \cos \theta_1 &= h_0.
\end{align*}$$

This implies that both $y_0$ and $\theta_1$ can be written as functions of $R$. These equations imply

$$R(\cos \theta_1 - \cos \theta_0) = h_0. \tag{5}$$

Taking the time derivative of both sides of this equation, yields

$$(\cos \theta_1 - \cos \theta_0) \dot{R} = (R \sin \theta_1) \dot{\theta}_1. \tag{6}$$
which implies

\[ \dot{\theta}_1 = g_1(R, \theta_0, \theta_1) \dot{R}, \quad \text{where} \quad g_1(R, \theta_0, \theta_1) = \frac{\cos \theta_1 - \cos \theta_0}{R \sin \theta_1}. \quad (7) \]

Next, we consider conservation of the film mass. The area \( A := A(t) \) of the thin film enclosed by the circular arc and the rigid substrate is

\[ A = \frac{1}{2} R^2 (\theta_0 + \theta_1) - \frac{1}{2} R^2 \sin \theta_0 \cos \theta_0 + \frac{1}{2} R^2 \sin \theta_1 \cos \theta_1 - R^2 \cos \theta_0 \sin \theta_1. \quad (8) \]

The conservation of thin film mass implies

\[ \frac{dA}{dt} = \frac{d}{dt} \left[ h_0 (x_0 + R \sin \theta_1) \right]. \quad (9) \]

Inserting Eq. (8) into the above equation, we obtain a second important geometric relation

\[ h_0 \dot{x}_0 = \left[ R (\theta_0 + \theta_1 - \frac{1}{2} \sin 2\theta_0 + \frac{1}{2} \sin 2\theta_1 - 2 \cos \theta_0 \sin \theta_1) - h_0 \sin \theta_1 \right] \dot{R} \]

\[ + \left[ \frac{1}{2} R^2 (1 - 2 \cos \theta_0 \cos \theta_1 + \cos 2\theta_1) - h_0 R \cos \theta_1 \right] \dot{\theta}_1. \quad (10) \]

From the above relation, and making use of Eq. (7), we obtain

\[ \dot{x}_0 = g_2 \dot{R}, \quad (11) \]

where \( g_2 \) is given as

\[ g_2 = \left[ \frac{R}{h_0} \left( \theta_0 + \theta_1 - \frac{1}{2} \sin 2\theta_0 + \frac{1}{2} \sin 2\theta_1 - 2 \cos \theta_0 \sin \theta_1 \right) - \sin \theta_1 \right] \]

\[ + \left[ \frac{R}{2h_0} (1 - 2 \cos \theta_0 \cos \theta_1 + \cos 2\theta_1) - \cos \theta_1 \right] \frac{\cos \theta_1 - \cos \theta_0}{\sin \theta_1}. \]

Substituting Eq. (5) into this expression for \( g_2 \), we obtain \( g_2 \) as a function of \( \theta_0 \) and \( \theta_1 \),

\[ g_2 := g_2(\theta_0, \theta_1) = \frac{1}{\cos \theta_1 - \cos \theta_0} (\theta_0 + \theta_1 - \sin \theta_0 \cos \theta_0 - \sin \theta_1 \cos \theta_0). \quad (12) \]
With the above geometrical relations in hand, we derive a reduced model for describing the
dynamics of the dewetting process by application of the Onsager variational principle. We choose
\( R(t) \) as the only free variable. The total interfacial free energy of this system \( W := W(t) \) can be
written as
\[
W = \gamma_0 \left[ R(\theta_0 + \theta_1) - (x_0 + R \sin \theta_1) \right] + \gamma_0 \cos \theta_0 (x_0 - R \sin \theta_0).
\] (13)
Taking its time derivative and making use of Eqs. (7) and (11), we have
\[
\dot{W} = \frac{\partial W}{\partial R} \dot{R},
\] (14)
where
\[
\frac{\partial W}{\partial R} = \gamma_0 \left[ (\theta_0 + \theta_1 - \sin \theta_1 - \cos \theta_0 \sin \theta_0) + R(1 - \cos \theta_1) g_1 - (1 - \cos \theta_0) g_2 \right].
\] (15)
Next, we compute the energy dissipation function for the evolving profile. For this, we
parameterize the circular section of the film/vapor profile as
\[
\begin{cases}
x(\theta, t) = x_0(t) + R(t) \sin \theta, \\
y(\theta, t) = R(t)(\cos \theta - \cos \theta_0),
\end{cases}
\] (16)
where \( \theta \in [-\theta_0, \theta_1] \). The procedure for obtaining the dissipation function \( \Phi \) is similar to that
presented previously \(^{25}\). We write the normal velocity of the interface curve \( v_n(\theta) \) as
\[
v_n(\theta) = \dot{x}_0 \sin \theta + \dot{R}(1 - \cos \theta_0 \cos \theta), \quad \theta \in [-\theta_0, \theta_1].
\] (17)
The corresponding (mass) flux \( j := j(\theta), \theta \in [-\theta_0, \theta_1] \) along the circular arc is

\[
j(\theta) = \int_{-\theta_0}^{\theta} v_n(\theta) Rd\theta
\]

\[
= -R(\cos \theta - \cos \theta_0)\dot{x}_0 + R(\theta + \theta_0 - \sin \theta \cos \theta_0 - \sin \theta_0 \cos \theta_0)\dot{R}
\]

\[
= g_3(\theta, \theta_0, \theta_1) R \dot{R},
\tag{18}
\]

where

\[
g_3(\theta, \theta_0, \theta_1) = \left[-(\cos \theta - \cos \theta_0)\left(\theta_0 + \theta_1 - (\sin \theta_0 + \sin \theta_1) \cos \theta_0\right)
\right.
\]

\[
+ (\cos \theta_1 - \cos \theta_0)\left(\theta + \theta_0 - (\sin \theta + \sin \theta_0) \cos \theta_0\right)] \frac{1}{\cos \theta_1 - \cos \theta_0}.
\tag{19}
\]

Here, we employed Eqs. (11) and (12) and imposed the zero-mass flux boundary condition at the contact point \( j(-\theta_0) = 0 \) (this implies that the total area/mass is conserved during the evolution)\(^{21,25}\).

The dissipation function \( \Phi \) can be written as\(^{25}\)

\[
\Phi = \frac{1}{2} \frac{k_B T}{D_s \nu \Omega_0^2} \int_{-\theta_0}^{\theta_1} j^2(\theta) R \, d\theta,
\tag{20}
\]

where \( D_s \) is the surface diffusivity, \( \nu \) is the number of diffusing atoms per unit area, \( \Omega_0 \) is the atomic volume, and \( k_B T \) is the thermal energy. By inserting (18) into this expression and making use of (11), it can be recast into the following quadratic form with respect to the rate function \( \dot{R} \)

\[
\Phi := \Phi(R, \dot{R}) = \frac{1}{2} \zeta(R, \theta_0, \theta_1) \dot{R}^2,
\tag{21}
\]

where the friction coefficient \( \zeta := \zeta(R, \theta_0, \theta_1) \) is

\[
\zeta(R, \theta_0, \theta_1) = \frac{k_B T}{D_s \nu \Omega_0^2} R^3 \int_{-\theta_0}^{\theta_1} g_3(\theta, \theta_0, \theta_1)^2 \, d\theta.
\tag{22}
\]
Applying the Onsager variational principle\textsuperscript{25,41}, we write the Rayleighian of our system in terms of the free energy \( W := W(R) \) and dissipation function \( \Phi := \Phi(R, \dot{R}) \):

\[
\mathcal{R}(R, \dot{R}) = \frac{\partial W}{\partial R} \dot{R} + \Phi(R, \dot{R}).
\] (23)

Minimization of the Rayleighian \( \mathcal{R} \) with respect to the rate variable \( \dot{R} \) yields the following evolution equation for the radius function \( R := R(t) \),

\[
\zeta(R, \theta_0, \theta_1) \dot{R} = -\frac{\partial W}{\partial R},
\] (24)

where the function \( \theta_1 := \theta_1(t) \) is updated according to Eq. (5). This ODE (24) governs the interface evolution of a retracting semi-infinite thin film depicted by Fig. 1; this is a reduced-order variational model for the dewetting of a solid film on a substrate via surface diffusion. An alternative approach is to solve the coupled ODEs (24) and (7) with respect to \( R \) and \( \theta_1 \) to obtain the interface evolution.

4 Asymptotic analysis

We first perform an asymptotic analysis of the ODE Eq. (24) to obtain a simple, power-law description of dewetting. For simplicity of presentation, we focus on the special case of \( \theta_0 = \pi/2 \). Numerical results are presented for other Young’s angles below. For \( \theta_0 = \pi/2 \), the expressions for \( \partial W/\partial R \) and \( \zeta(R, \theta_0, \theta_1) \) in (24) can be simplified. Inserting Eqs. (7) and (12) into Eq. (15) gives

\[
-\frac{\partial W}{\partial R} = -\gamma_0 \left\{ \frac{1}{\cos \theta_1} \left[ (\frac{\pi}{2} + \theta_1)(\cos \theta_1 - 1) - \sin \theta_1 \cos \theta_1 \right] + \frac{1}{\sin \theta_1} (1 - \cos \theta_1) \cos \theta_1 \right\}. \] (25)

\( \zeta(R, \theta_0, \theta_1) \) in Eq. (22) simplifies by writing \( g_3 = -(\pi/2 + \theta_1) \cos \theta / \cos \theta_1 + \theta + \pi/2 \) and

\[
\int_{-\theta_0}^{\theta_1} g_3(\theta, \frac{\pi}{2}, \theta_1) \sin \theta \, d\theta = \frac{1}{2} \left( \frac{\theta_1 + \pi/2}{\cos \theta_1} \right)^2 \left( \frac{\pi}{2} + \theta_1 + \sin \theta_1 \cos \theta_1 \right) + \frac{1}{3} \left( \frac{\theta_1 + \pi/2}{\cos \theta_1} \right)^3 - 2 \left( \frac{\theta_1 + \pi/2}{\cos \theta_1} \right) \left( \frac{\pi}{2} \sin \theta_1 + \theta_1 \sin \theta_1 + \cos \theta_1 \right). \] (26)
In the long-time limit, we can assume $R \gg h_0$ such that $\cos \theta_1 = h_0/R \ll 1$. This implies that $\theta_1 \approx \pi/2 + \mathcal{O}(h_0/R)$ and $\sin \theta_1 \approx 1 + \mathcal{O}((h_0/R)^2)$. With these approximations, Eq. (25) reduces to

$$
-\frac{\partial W}{\partial R} = -\gamma_0 \left[ \left( \frac{R}{h_0} \right) \left( \pi + \mathcal{O} \left( \frac{h_0}{R} \right) \right) \left( \frac{h_0}{R} - 1 \right) - \left[ 1 + \mathcal{O} \left( \frac{h_0}{R} \right)^2 \right] \frac{h_0}{R} \right] + \frac{1 - h_0/R}{1 + \mathcal{O}((h_0/R)^2)} \left( \frac{h_0}{R} \right)
\approx \pi \gamma_0 \left( \frac{R}{h_0} \right),
$$

(27)

where in the last line we keep only the leading order term. In the same limit, Eq. (26) reduces to

$$
\int_{-\theta_0}^{\theta_1} g_3(\theta, \theta_0, \theta_1) \, d\theta = \frac{1}{2} \left[ \pi + \mathcal{O} \left( \frac{h_0}{R} \right)^2 \right] \left( \frac{R}{h_0} \right)^2 \left( \pi + \mathcal{O} \left( \frac{h_0}{R} \right) \right) + \left[ 1 + \mathcal{O} \left( \frac{h_0}{R} \right)^2 \right] \frac{h_0}{R}

-2 \left[ \pi + \mathcal{O} \left( \frac{h_0}{R} \right) \right] \left( \frac{R}{h_0} \right) \left[ \frac{\pi}{2} \left[ 1 + \mathcal{O} \left( \frac{h_0}{R} \right)^2 \right] \right]

+ \left[ \frac{\pi}{2} + \mathcal{O} \left( \frac{h_0}{R} \right) \right] \left( 1 + \mathcal{O} \left( \frac{h_0}{R} \right)^2 \right) + \frac{\hbar_0}{R} + \frac{1}{3} \left( \pi + \mathcal{O} \left( \frac{h_0}{R} \right)^3 \right)
\approx \frac{\pi^3}{2} \left( \frac{R}{h_0} \right)^2.
$$

Inserting this expression into Eq. (22), we obtain

$$
\zeta(R, \pi/2, \theta_1) \approx \frac{\pi^3 k_B T}{2 D_s \nu \Omega_0^2 h_0^2 R^5}.
$$

Finally, by inserting this expression and Eq. (27) into the ODE describing dewetting Eq. (24), we obtain a simplified (leading-order) ODE for dewetting:

$$
R^4 \dot{R} = a,
$$

(28)

where $a = 2B \gamma_0 h_0/\pi^2$ and $B = D_s \nu \Omega_0^2/k_B T$ is a material constant.

Eq. (28) demonstrates that

$$
R(t) = \left( 5a t + C_1 \right)^{1/5} \propto (5a)^{1/5} t^{1/5},
$$

12
where $C_1$ is a constant that depends on the initial value of $R$. This implies that the ridge adjacent to the moving contact grows with time in a power law fashion as $t^{1/5}$. This power-law exponent is consistent with previous analysis\textsuperscript{22,31} and experiment.

Next, we examine how the contact point $x_c := x_c(t)$ moves at long time. The contact point evolution is related to $R(t)$ by

$$x_c(t) = x_0(t) - R(t) \sin \theta_0 = x_0(t) - R(t)$$  \hspace{1cm} (29)

and

$$\dot{x}_c = (g_2 - 1) \dot{R}.$$  \hspace{1cm} (30)

Eq. (12) implies

$$g_2 - 1 = \frac{1}{\cos \theta_1} \left( \frac{\pi}{2} + \theta_1 \right) - 1 = \left( \frac{h_0}{R} \right)^{-1} \left( \pi + O \left( \frac{h_0}{R} \right) \right) - 1 \approx \pi \left( \frac{h_0}{R} \right)^{-1} = \frac{\pi R}{h_0},$$  \hspace{1cm} (31)

such that, in the long-time limit (i.e., to leading-order),

$$\dot{x}_c = \frac{\pi R}{h_0} \dot{R}.$$  \hspace{1cm} (32)

Integration of this expression leads to

$$x_c(t) = \frac{\pi R^2(t)}{2h_0} + C_2 \propto b^{2/5},$$  \hspace{1cm} (33)

where $C_2$ is also a constant determined by the initial location of the contact point and $b^{5} = (5B\gamma_0)^2 \pi/(2h_0)^3$. This is consistent with earlier results for the power-law dependence of the retraction distance with time in surface diffusion-controlled, solid-state dewetting\textsuperscript{15,19,21,22,29}. 
5 Numerical results

The full ODE, Eq. (24), can be solved numerically for all times. We employ the classical fourth-order Runge-Kutta method for any initial condition and Young’s angle, $\theta_0$. Here, because our main focus is on the power-law scaling of surface diffusion-controlled, solid state dewetting kinetics, we examine only one initial values (e.g., $R(0) = 2h_0$) for different isotropic Young’s angles. Note that although the initial value of $R$ (e.g., $\theta_0 = 15^\circ, 30^\circ$) may be inconsistent with the chosen value of $\theta_1$, the evolution is consistent after a short time.

Figures 2 and 3 show numerical results for the evolution of the ridge radius and contact retraction velocity for six different isotropic Young’s angles $\theta_0 = 15^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ, \text{and } 150^\circ$. As shown in Fig. 2, a log-log plot of the ridge radius $R$ versus time clearly exhibits a 1/5 power-law at late times for all six different Young’s angles. The ridge height (i.e., $R + y_0 = R(1 - \cos \theta_0)$) also follows the same power-law during dewetting. Figure 3 shows that the contact retraction velocity $\dot{x}_c$ is consistent with the predicted power-law $\dot{x}_c \propto t^{-3/5}$ at late times, which indicates that the retraction distance $x_c(t)$ satisfies a 2/5 power-law with time.

6 Summary and conclusion

We examine the dependence of the surface diffusion-controlled, solid-state dewetting of a semi-infinite thin film on a flat substrate. Our approach is based upon the Onsager variational principle and motivated by earlier, simple geometric models that are consistent with the power-law retraction behavior observed in more complete numerical studies of the evolving film profile during
Figure 2: The ridge radius $R$ (i.e., proportional to ridge height) versus time $t$ for six different isotropic Young’s angles $\theta_0 = 15^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ,$ and $150^\circ$. The late time data are consistent with a power-law of the form $R \propto t^{1/5}$ for all Young’s angles.
Figure 3: The retraction velocity of the contact point $\dot{x}_c$ versus time $t$ for the same six isotropic Young’s angles as in Fig. 2, where a power law that $\dot{x}_c \propto t^{-3/5}$ is clearly shown. The late time data are consistent with a power-law of the form $x_c \propto t^{1/5}$ for all Young’s angles.
solid-state dewetting. The simplified nature of the film geometry, allows us to derive a reduced-order variational model, the evolution of which is governed by an ordinary differential equation, ODE. Asymptotic analysis and numerical simulations of the ODE reproduced the $2/5$ power-law of the retraction distance with time, and the $1/5$ time exponent for the height of the ridge adjacent to the moving contact line. While these power-laws have been predicted previously based on analysis and numerical simulation, the present results demonstrate the simplicity and applicability of the Onsager variational principle to describe surface diffusion-controlled morphology evolution problems in materials science. While the geometric model employed here is very simple, the Onsager variational principle approach is applicable to much more complex representations of the geometry; such generalizations yield a system of ODEs rather than a single ODE as examined here.

References


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**Competing Interests**  The authors declare that they have no competing financial interests.

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