A MULTISCALE FINITE ELEMENT METHOD FOR OSCILLATING NEUMANN PROBLEM ON ROUGH DOMAIN

PINGBING MING * AND XIANMIN XU †

Abstract. We develop a new multiscale finite element method for Laplace equation with oscillating Neumann boundary conditions on rough boundaries. The key point is the introduction of a new boundary condition that incorporates both the microscopically geometrical and physical information of the rough boundary. We prove the method has optimal convergence rate in the energy norm with a weak resonance term for periodic roughness. Numerical results are reported for both periodic and nonperiodic roughness.

Key words. Multiscale finite element method; Rough boundary; Homogenization

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1. Introduction. Many problems in nature and industry applications are described by partial differential equations in domain with multiscale boundary. Typical examples include fluid in rough domains [26, 11], wetting phenomena on plant leaves with tiny roughness [39, 9], wave scattering on objects with oscillating surfaces [41], and among many others. An example from industry [20] is the process of adoption of components of a chemically reacting flow on a surface with a given microstructure. In these examples, many problems have oscillatory Neumann or Robin boundary conditions on the rough boundary [20, 42].

Theoretical study for problems with rough boundary and oscillating boundary data mainly concerns the effective boundary conditions, which may be traced back to [28]. There are extensive work thereafter devoted to various topics in this field, including Poisson problem, eigenvalue problems, Stokes and Navier-Stokes equations with different types of boundary conditions; see [4, 10, 18, 26, 29, 32, 34, 35, 37] and the references therein. Recently, homogenization for problems with oscillating boundary data have arisen many interests [19, 27, 3]. Estimate for the boundary layers caused by the oscillating boundary data are the key to establish the homogenization results for such problems [19, 3].

Compared to the extensive work of theoretical study, numerical methods for the rough boundary problem have been received less attention, while there are many nu-
Numerical methods devoted to elliptic problems with rough coefficients [6, 7, 8, 5, 25, 23, 24, 15, 40, 16, 38, 12]. We refer to [14, Chapter 8] for a review. Only recently, a multiscale finite element method (MsFEM) was introduced to solve Laplace equation with homogeneous Dirichlet boundary value on rough domain [30, 33]. The multiscale basis functions are constructed for the elements near the rough boundary by solving a cell problem with the homogeneous Dirichlet condition on the rough edge and with linear nodal basis function as boundary condition on other edges, just as the standard MsFEM [23]. However, this approach can be applied neither to problems with non-Dirichlet boundary conditions over rough boundary, nor problems with inhomogeneous Dirichlet boundary value over the rough boundary.

We introduce a new multiscale finite element method for Laplace equation with oscillating Neumann boundary conditions on rough boundary. A special Neumann boundary condition that depends on the magnitude of the flux oscillation has to be imposed on the local cell problem posed on elements with rough edge. When the flux oscillation is of the same order of the roughness parameter, only the microscopical geometry of the rough boundary is needed in the boundary condition. Otherwise, one has to incorporate both the microscopical geometry of the rough boundary and the flux oscillation into the boundary condition. Such multiscale basis function coincides with the linear nodal basis functions for elements without a rough edge. This method is $H^1$-conforming, with degrees of freedom at the nodes of the mesh and the basis functions are solved over the elements near the rough boundary and can be computed off line. For periodic roughness, we prove that our method has optimal convergence rate in the energy norm besides a weak resonance term. The method also applies to problems with non-periodic roughness as demonstrated by the numerical experiments. The proof is based on certain homogenization results for Neumann rough boundary value problems, which refine the corresponding results in [18] by clarifying the dependence of the error bounds on the domain size.

The method is quite general in the sense that we do not impose any assumptions on the oscillating of the boundary. The method can be naturally generalized to the inhomogeneous Dirichlet boundary value problem over rough domain. It is also possible to combine the present method with the standard MsFEM method to deal with the problems with both oscillating coefficients and oscillating boundary data. These topics will be discussed in our future work.

It is worthwhile to mention that composite finite elements has been successfully applied to solve boundary value problems over complicated domain [21, 22]. However, the extension of composite finite elements to oscillatory boundary value problems posed on oscillating domain does not seem straightforward. In addition, we note that the heterogeneous multiscale method and MsFEM have been employed to solve partial differential equations on a rough surface in [1, 17], respectively, however, they did not treat the problems studied in this paper.

The structure of the paper is as follows. In Section 2, we describe the model
problem and the multiscale finite element method. In Section 3, we revisit the homogenization results for a Possion equation with an oscillatory Neumann boundary condition over rough domain. In Section 4, we estimate the convergence rate in energy norm of the proposed method. Numerical examples are illustrated in the last section.

2. The Model Problem and Multiscale Finite Element Method. Let \( \Omega_\varepsilon \subset \mathbb{R}^2 \) be a bounded domain with boundary \( \partial \Omega_\varepsilon \), a part of which is rough and denoted as \( \Gamma_\varepsilon \), where \( \varepsilon \) is a small parameter that characterizes the roughness of \( \Gamma_\varepsilon \).

We consider a model problem with Neumann boundary conditions on \( \Gamma_\varepsilon \): Given the source term \( f \) and the flux \( g_\varepsilon \) that is oscillatory, we find \( u \) satisfying

\[
\begin{align*}
-\Delta u &= f(x), & \text{in } \Omega_\varepsilon, \\
u &= 0, & \text{on } \Gamma_D, \\
\frac{\partial u}{\partial n} &= g_\varepsilon(x), & \text{on } \Gamma_\varepsilon.
\end{align*}
\]

For any measurable subset \( D \) of \( \Omega_\varepsilon \), we define

\[
V(D) = \{ v \in H^1(D) \mid v|_{\partial D \setminus \Gamma_\varepsilon} = 0 \}.
\]

Here \( H^1(D) \) is the standard Sobolev space, and the notations and definitions for Sobolev spaces can be found in [2]. To clarify the dependence of the roughness parameter \( \varepsilon \), we denote \( u^\varepsilon \) the solution of Problem (2.1), whose weak form is: Find \( u^\varepsilon \in V(\Omega_\varepsilon) \) such that

\[
a(u^\varepsilon, v) = (f, v) + (g_\varepsilon, v)_{\Gamma_\varepsilon} \quad \text{for all } v \in V(\Omega_\varepsilon),
\]

where

\[
a(u^\varepsilon, v) := \int_{\Omega_\varepsilon} \nabla u^\varepsilon \cdot \nabla v \, dx, \quad (f, v) := \int_{\Omega_\varepsilon} fv \, dx, \quad (g_\varepsilon, v)_{\Gamma_\varepsilon} := \int_{\Gamma_\varepsilon} g_\varepsilon v \, ds.
\]

We triangulate \( \Omega_\varepsilon \) by a shape regular mesh \( T_h \) in the sense of [13], with element \( \tau \in T_h \) be either a triangle or a quadrilateral, where \( h = \max_{\tau \in T_h} \text{diam } \tau \) with \( \text{diam } \tau \) the diameter of \( \tau \), and \( \mathcal{S}(\tau) \) is a suitable index set for nodes in \( \tau \). We assume that an element near \( \Gamma_\varepsilon \) has at most one rough edge on the rough boundary and denote such elements by \( \tau_\varepsilon \); see Fig. 2.1. For triangular mesh, there are elements that may have only one node on the rough boundary.

For any measurable subset \( D \) of \( \Omega_\varepsilon \), we define a localized version of \( a \) as

\[
a_D(v, w) := \int_D \nabla v \cdot \nabla w \, dx \quad v, w \in H^1(D).
\]

For each \( p \in \mathcal{S}(\tau) \) we construct nodal basis functions \( \Phi_p^{MS} \), whose restriction to each element \( \tau \) is denoted by \( \Phi_{p,\tau}^{MS} \), which satisfies

\[
a_\tau(\Phi_{p,\tau}^{MS}, v) = (\theta_p, v)_{\partial \tau \cap \Gamma_\varepsilon} \quad \text{for all } v \in V(\tau),
\]

\[
(2.3)
\]
supplemented with a suitable boundary condition
\[ \Phi_{p,\tau}^{MS} = \phi_{p,\tau} \text{ on } \partial\tau \setminus \Gamma_\varepsilon, \quad \Phi_{p,\tau}^{MS}(x_q) = \delta_{p,q} \text{ for all } p, q \in \mathcal{I}(\tau), \quad (2.4) \]
where \( \phi_{p,\tau} \) is the restriction of the standard linear nodal basis function \( \phi_p \) on \( \tau \).

The flux \( \theta_{p,\tau} \) is defined as follows. If \( \tau \) has no edge on \( \Gamma_\varepsilon \), then we let \( \theta_{p,\tau} = 0 \). Problem (2.3) changes to a Dirichlet boundary value problem with a unique solution \( \Phi_{p,\tau}^{MS} = \phi_{p,\tau} \). If \( \tau_\varepsilon \) has one edge on \( \Gamma_\varepsilon \), then
\[ \theta_{p,\tau}(x) := \begin{cases} \frac{\partial_n \phi_{p,\tau_0}}{r(x)} g_\varepsilon(x) & \text{if } \|g_\varepsilon - \langle g \rangle\|_{L^\infty(\partial\tau_\varepsilon)} \leq C_\varepsilon, \\ \frac{\partial_n \phi_{p,\tau_0}}{r(x)} \langle g \rangle(x) & \text{otherwise.} \end{cases} \]
In this case, the local problem has mixed boundary conditions. Here the parameter \( r = |s_\varepsilon|/|s_0| \) with \( s_\varepsilon \) being the rough edge of \( \tau_\varepsilon \), while \( s_0 \) being the homogenized rough edge, and \( \partial_n \) is the unit outer normal of \( s_0 \), \( \tau_0 \) is the homogenized element of \( \tau_\varepsilon \) and \( \langle g_\varepsilon \rangle = f_{s_\varepsilon} \), \( g_\varepsilon \) is the mean of \( g_\varepsilon \) over \( s_\varepsilon \).

The flux \( \theta_{p,\tau} \) in (2.5) also includes both geometrical and physical information of the boundary. If the flux has no oscillations, i.e., \( g_\varepsilon = \langle g \rangle \), then (2.5) changes to the first one, which does not contain the physical information any more. Furthermore, if the boundary is also smooth, i.e., \( r = 1 \), then the unique solution of the cell problem is the standard nodal basis functions, and the method automatically changes to the standard finite element method.

The bound \( C_\varepsilon \) in (2.5) is a threshold for whether the physical information is incorporated into the cell problem. When \( \|g_\varepsilon - \langle g \rangle\|_{L^\infty(\partial\tau_\varepsilon)} \) is as small as \( O(\varepsilon) \), we need not any physical information but the geometrical information of the rough boundary. Otherwise, we need include the physical information in the cell problem. This is consistent with our intuition as seen from the example below.

**Example 2.1.** If \( g_\varepsilon(x) = \varepsilon \sin(x/\varepsilon) \), it is clear that \( \|g_\varepsilon - \langle g \rangle\|_{L^\infty} = \varepsilon \), then we may use (2.5)1. On the other hand, if \( g_\varepsilon(x) = 1 + \sin(x/\varepsilon) \), then \( \|g_\varepsilon - \langle g \rangle\|_{L^\infty} = 1 \).
and we have to use (2.5). There are some special cases beyond our definition (2.5), e.g., \( g_\varepsilon(x) = \sin(x/\varepsilon) \) so that \( ||g_\varepsilon||_{\infty} = 1 \) but \( \langle g \rangle = 0 \). In this case, the method works as well if we decompose \( g_\varepsilon \) as \( g_\varepsilon = g_1 + g_2 \) with \( g_1 := 1 \) and \( g_2 := \sin(x/\varepsilon) - 1 \) and split Problem (2.1) into two problems with boundary conditions \( g_1 \) and \( g_2 \), respectively.

Under the conditions (2.3), (2.4) and (2.5), we have

\[
\sum_{p \in \mathcal{P}^h(\tau)} \Phi_{p,\tau}^{MS} = 1, \quad \forall \tau \in \mathcal{T}_h.
\]

The basis function \( \phi_{p,\tau}^{MS} \) is continuous across the element boundary so that

\[
V_h := \text{span} \{ \phi_{p,\tau}^{MS} | p \in \mathcal{P}(\Omega_\varepsilon) \} \subset V(\Omega_\varepsilon).
\]

The MsFEM approximation of Problem (2.1) is to find \( u_h \in V_h \) such that

\[
a(u_h, v) = (f, v) + (g_\varepsilon, v)_{\Gamma_\varepsilon} \quad \text{for all } v \in V_h. \tag{2.6}
\]

This is a conforming method, and the existence and uniqueness of the solution follow from Lax-Milgram theorem. Moreover, we have

\[
||\nabla(u^\varepsilon - u_h)||_{L^2(\Omega_\varepsilon)} = \inf_{v \in V_h} ||\nabla(u^\varepsilon - v)||_{L^2(\Omega_\varepsilon)}. \tag{2.7}
\]

The error estimate now boils down to the interpolate error estimate, which will be the focus of the later sections.

Before stepping further, we remark that the proposed method can be easily generalized to the problem with oscillatory inhomogeneous Dirichlet boundary conditions on the rough surface. In this case, we assume \( u^\varepsilon = g_\varepsilon \) on \( \Gamma_\varepsilon \). The MsFEM basis functions are defined as follows.

\[
a_\varepsilon(\Phi_{p,\tau}^{MS}, v) = 0 \quad \text{for all } v \in H^1_0(\tau), \tag{2.8}
\]

supplemented with \( \Phi_{p,\tau}^{MS} = \theta_{p,\tau}(x) \) on the rough edge \( \partial\tau \cap \Gamma_\varepsilon \) with

\[
\theta_{p,\tau}(x) := \begin{cases} 
\frac{\phi_{p,\tau_0}}{r(x)} & \text{if } ||g_\varepsilon - \langle g \rangle||_{L^\infty(\partial\tau)} \leq C, \\
\frac{\phi_{p,\tau_0} g_\varepsilon(x)}{r(x)} & \text{otherwise},
\end{cases} \tag{2.9}
\]

and \( \Phi_{p,\tau}^{MS} = \phi_{p,\tau} \) on \( \partial\tau \setminus \Gamma_\varepsilon \). The detailed analysis of the MsFEM for inhomogeneous Dirichlet boundary value problem will be addressed in a separate paper.

3. Error Estimates for the Homogenization Problem. In this section, we revisit some homogenization results for Problem (2.1), which have been established in [18], while we clarify the dependence of the estimates on the domain size, which is crucial for studying the accuracy of the proposed method. Our approach is different from that in [18] for estimate of the first order approximation.
In what follows, we assume that $\Omega_\varepsilon \subset \mathbb{R}^2$ is a bounded domain with
$$\Omega_\varepsilon := \{ x \in \mathbb{R}^2 \mid 0 < x_1 < 1, \varepsilon \gamma(x_1/\varepsilon) < x_2 < 1 \} ,$$
and the oscillating bottom boundary $\Gamma_\varepsilon$ is given by
$$\Gamma_\varepsilon = \{ x \in \overline{\Omega}_\varepsilon \mid 0 < x_1 < 1, x_2 = \varepsilon \gamma(x_1/\varepsilon) \}$$
with $\gamma$ a positive smooth 1-periodic function. We assume that $g_\varepsilon(x_1) = g(x_1/\varepsilon)$ with $g$ a smooth 1-periodic function and satisfying
$$\| g - \langle g \rangle \|_{L^\infty(\Sigma)} \leq C(\| \langle g \rangle \| + \varepsilon), \tag{3.1}$$
where $\Sigma = \{ \xi \in \mathbb{R}^2 \mid 0 < \xi_1 < 1, \xi_2 = \gamma(\xi_1) \}$ with $\langle g \rangle$ the mean of $g$ over $\Sigma$:
$$\langle g \rangle = \frac{1}{r} \int_0^1 g(t)[1 + (\gamma'(t))^2]^{1/2} \, dt \quad \text{and} \quad r = \int_0^1 [1 + (\gamma'(t))^2]^{1/2} \, dt.$$

**Remark 3.1.** The assumption $\gamma \geq 0$ ensures $\Omega_\varepsilon \subset \Omega_0$. This may make the presentation slightly simpler. All the results in this section remain valid for general $\gamma$, we refer to [18, §8] and [34] for related discussions.

**3.1. The zeroth order approximation.** Let
$$\Omega_0 = \{ x \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < 1 \} \quad \text{and} \quad \Gamma_0 = \partial \Omega_0 \cap \{ x \in \mathbb{R}^2 \mid x_2 = 0 \} .$$
Define $\Gamma_D := \partial \Omega_0 \setminus \Gamma_0$ and
$$V(\Omega_0) = \{ v \in H^1(\Omega_0) \mid v|_{\Gamma_D} = 0 \} .$$
The zeroth order approximations $u^0$ of $u^\varepsilon$ is such that $u^0 \in V(\Omega_0)$ and
$$a_{\Omega_0}(u^0, v) = (f, v)_{\Omega_0} + (r \langle g \rangle, v)_{\Gamma_0} \quad \text{for all} \ v \in V(\Omega_0) . \tag{3.2}$$
We start with an extension result that will be frequently used later on, which slightly refines that in [31, Appendix A].

**Lemma 3.2.** There exists an extension operator $E : H^1(\Omega_\varepsilon) \to H^1(\Omega_0)$ such that, for any $\phi \in H^1(\Omega_\varepsilon)$,

$$
\|E\phi\|_{H^1(\Omega_0)} \leq \sqrt{2 + 2A^2 + 2A\sqrt{1 + A^2} \|\phi\|_{H^1(\Omega_\varepsilon)}},
$$

where $A = \|\gamma\|_{L^\infty(0,1)}$.

**Proof.** For any $\phi \in H^1(\Omega_\varepsilon)$, we define $E\phi$ by reflection with respect to $\Gamma_\varepsilon$.

$$
E\phi(x) = \begin{cases}
\phi(x) & x \in \Omega_\varepsilon, \\
\phi(x_1, 2\varepsilon\gamma(x_1/\varepsilon) - x_2) & x \in \Omega_0 \backslash \Omega_\varepsilon.
\end{cases}
$$

A direct calculation gives

$$
\partial_{x_1} E\phi(x) = \begin{cases}
\partial_{x_1} \phi(x) & x \in \Omega_\varepsilon, \\
\partial_{x_1} \phi + 2\partial_{x_2} \phi \gamma'(x_1/\varepsilon) & x \in \Omega_0 \backslash \Omega_\varepsilon.
\end{cases}
$$

and

$$
\partial_{x_2} E\phi(x) = \begin{cases}
\partial_{x_2} \phi(x) & x \in \Omega_\varepsilon, \\
-\partial_{x_2} \phi(x_1, 2\varepsilon\gamma(x_1/\varepsilon) - x_2) & x \in \Omega_0 \backslash \Omega_\varepsilon.
\end{cases}
$$

Note the module of the Jacobian of the substitution

$$(x_1, x_2) \mapsto (x_1, 2\varepsilon\gamma(x_1/\varepsilon) - x_2)$$

is equal to one. Observe that the inequality

$$
\|E\phi\|^2_{L^2(\Omega_0)} \leq 2 \|\phi\|^2_{L^2(\Omega_\varepsilon)}
$$

holds, where we have used

$$
\|E\phi\|^2_{L^2(\Omega_0)} = \int_0^1 \int_{\gamma(x_1/\varepsilon)} \int_{\gamma(x_1/\varepsilon)} |\phi(x_1, 2\varepsilon\gamma(x_1/\varepsilon) - x_2)|^2 \, dx
$$

$$
= \int_0^1 \int_{\gamma(x_1/\varepsilon)} |\phi(x_1, x_2)|^2 \, dx
$$

$$
\leq \|\phi\|^2_{L^2(\Omega_\varepsilon)}.
$$

Next, we estimate $\partial_{x_1} E\phi$ and $\partial_{x_2} E\phi$.

$$
\|\partial_{x_1} E\phi\|^2_{L^2(\Omega_0)} \leq \|\partial_{x_1} \phi\|^2_{L^2(\Omega_\varepsilon)} + \|\partial_{x_1} \phi + 2\gamma' \partial_{x_2} \phi\|^2_{L^2(\Omega_\varepsilon)}
$$

$$
\leq (2 + t) \|\partial_{x_1} \phi\|^2_{L^2(\Omega_\varepsilon)} + 4A^2(1 + 1/t) \|\partial_{x_2} \phi\|^2_{L^2(\Omega_\varepsilon)}
$$

with any $t > 0$, and

$$
\|\partial_{x_2} E\phi\|^2_{L^2(\Omega_0)} \leq 2 \|\partial_{x_2} \phi\|^2_{L^2(\Omega_\varepsilon)}.
$$
The above three estimates imply
\[ \|E\|_{H^1(O_\omega)}^2 \leq (2 + t) \|\partial_{x_1}\phi\|_{L^2(O_\omega)}^2 + (2 + 4A^2(1 + 1/t)) \|\partial_{x_2}\phi\|_{L^2(O_\omega)}^2 + 2 \|\phi\|_{L^2(O_\omega)}^2. \]

Hence,
\[ \|E\|_{H^1(O_\omega)}^2 \leq \max\{2 + t, 2 + 4A^2(1 + 1/t)\} \|\phi\|_{H^1(O_\omega)}^2. \]

Optimizing the maximum with respect to parameter \( t \), we obtain the maximum is minimal if \( t = 2A^2 + 2\sqrt{A^2 + 1} \). This completes the proof. \( \square \)

To estimate the error between \( u^0 \) and \( u^\epsilon \), we need an auxiliary result [37, Lemma 1.5, p.7]. The present form can be found in [36, inequality (17) in Lemma 10].

**Lemma 3.3.** Let \( \Omega \) be a Lipschitz domain, and \( S_\varepsilon = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq \varepsilon \} \), then for any \( 1/2 < \kappa \leq \mu \leq 1 \) and \( v \in H^\mu(\Omega) \), there holds
\[ \|v\|_{L^2(S_\varepsilon)} \leq C \left( \sqrt{\varepsilon} \|v\|_{H^\kappa(\Omega)} + \varepsilon^\mu \|v\|_{H^\mu(\Omega)} \right). \quad (3.4) \]

**Lemma 3.4.** Let \( u^\epsilon \) and \( u^0 \) be the solutions to Problems (2.1) and (3.2), respectively. Then
\[ \|\nabla(u^\epsilon - u^0)\|_{L^2(\Omega)} \leq C \sqrt{\varepsilon} \|f\|_{L^2(\Omega_\omega)} + \|\nabla u^0\|_{H^1(\Omega_\omega)} + \|g_\epsilon\|_{L^2(\Gamma_\epsilon)}. \quad (3.5) \]

**Proof.** Denote \( e = u^\epsilon - u^0 \in V(\Omega_\epsilon) \). For any \( \phi \in V(\Omega_\epsilon) \), we have
\[
\int_{\Omega_\epsilon} e \nabla \phi \, dx = - \int_{\Omega_\epsilon} (f - \nabla u^0) \cdot \nabla \phi \, dx + \int_{\Gamma_\epsilon} g_\epsilon \phi \, d\sigma - \int_{\Gamma_\epsilon} r(g) \phi \, dx_1
\]
\[
= - \int_{\Omega_\epsilon} (f - \nabla u^0) \cdot \nabla \phi \, dx
\]
\[
+ \int_0^1 \left( g(x_1/\epsilon) \phi(x_1, \epsilon) \gamma(x_1/\epsilon) \right) [1 + (\gamma'(x_1/\epsilon))^2]^{1/2} - r(g) \phi(x_1, 0) \, dx_1
\]
\[
= I_1 + I_2.
\]

Using (3.4), we bound \( I_1 \) as
\[
|I_1| \leq \|f\|_{L^2(\Omega_\omega \setminus \Omega_\epsilon)} \|\phi\|_{L^2(\Omega_\omega \setminus \Omega_\epsilon)} + \|\nabla u^0\|_{L^2(\Omega_\omega \setminus \Omega_\epsilon)} \|\nabla \phi\|_{L^2(\Omega_\omega \setminus \Omega_\epsilon)}
\]
\[
\leq C \sqrt{\varepsilon} \|f\|_{L^2(\Omega_\omega \setminus \Omega_\epsilon)} \|\phi\|_{H^1(\Omega_\omega \setminus \Omega_\epsilon)} + C \sqrt{\varepsilon} \|\nabla u^0\|_{H^1(\Omega_\omega \setminus \Omega_\epsilon)} \|\nabla \phi\|_{L^2(\Omega_\omega \setminus \Omega_\epsilon)}
\]
\[
\leq C \sqrt{\varepsilon} \left( \|f\|_{L^2(\Omega_\omega)} + \|\nabla u^0\|_{H^1(\Omega_\omega)} \right) \|\nabla \phi\|_{L^2(\Omega_\omega)}, \quad (3.6)
\]

where in the last step we have used Poincaré’s inequality for \( \phi \).

Denote by \( p^\epsilon(x_1) = p(x_1/\epsilon) \) with \( p(t) = g(t)[1 + \gamma'(t)^2]^{1/2} \), we have
\[
I_2 = \int_0^1 (p^\epsilon(x_1) \phi(x_1, \epsilon) \gamma(x_1/\epsilon)) - (p) \phi(x_1, 0) \, dx_1.
\]
where \( \langle p \rangle \) denotes the average of \( p \) and \( \langle p \rangle = r(g) \), i.e., \( \langle p \rangle = \int_0^1 g(t)[1 + \gamma'(t)^2]^\frac{1}{2} \, dt \).

It is clear to see

\[
I_2 = \int_0^1 p^\varepsilon(x_1) (\phi(x_1, \varepsilon \gamma(x_1/\varepsilon)) - \phi(x_1, 0)) \, dx_1 + \int_0^1 (p^\varepsilon - \langle p \rangle) \phi(x_1, 0) \, dx_1.
\]

A direct calculation gives that

\[
\left| \int_0^1 p^\varepsilon(x_1) (\phi(x_1, \varepsilon \gamma(x_1/\varepsilon)) - \phi(x_1, 0)) \, dx_1 \right| = \left| \int_0^1 p^\varepsilon(x_1) \int_0^{\gamma(x_1/\varepsilon)} \frac{\partial \phi}{\partial x_2} \, dx_2 \, dx_1 \right|
\leq \varepsilon \| \nabla \phi \|_{L^2(\Omega_0 \cup \Omega)} \left\| p^\varepsilon \gamma^{1/2} \right\|_{L^2(\Gamma_0)}
\leq C \varepsilon \| \nabla \phi \|_{L^2(\Omega_0)} \| g_{\varepsilon} \|_{L^2(\Gamma_0)},
\]

where we have used

\[
\left\| p^\varepsilon \gamma^{1/2} \right\|^2_{L^2(\Gamma_0)} \leq A \| \gamma \|_{L^\infty(0, 1)} \| g_{\varepsilon} \|^2_{L^2(\Gamma_0)}.
\]

Denote by \( s_0 = 0 < s_1 = \varepsilon < \cdots < s_N = N \varepsilon = 1 \), using the fact that

\[
\langle p \rangle = \langle p^\varepsilon \rangle = \int_{s_0}^{s_{N+1}} p^\varepsilon(x_1) \, dx_1,
\]

we decompose the second term into

\[
\int_{\Gamma_0} (p^\varepsilon(x_1) - \langle p \rangle) \phi(x_1, 0) \, dx_1 = \sum_{i=0}^{N-1} \int_{s_i}^{s_{i+1}} p^\varepsilon(x_1) (\phi(x_1, 0) - \langle \phi \rangle) \, dx_1,
\]

where \( \langle \phi \rangle = \int_{s_i}^{s_{i+1}} \phi(x_1, 0) \, dx_1 \). Using *Poincaré’s inequality*, we obtain

\[
\left| \int_{\Gamma_0} (p^\varepsilon(x_1) - \langle p \rangle) \phi(x_1, 0) \, dx_1 \right| \leq \sum_{i=0}^{N-1} \| p^\varepsilon \|^2_{L^2(s_i, s_{i+1})} \| \phi - \langle \phi \rangle \|_{L^2(s_i, s_{i+1})}
\leq C \varepsilon \sum_{i=0}^{N-1} \| p^\varepsilon \|^2_{L^2(s_i, s_{i+1})} \| \phi \|_{H^{1/2}(s_i, s_{i+1})}
\leq C \varepsilon \left( \sum_{i=0}^{N-1} \| p^\varepsilon \|^2_{L^2(s_i, s_{i+1})} \right)^{1/2} \left( \sum_{i=0}^{N-1} \| \phi \|^2_{H^{1/2}(s_i, s_{i+1})} \right)^{1/2}
\leq C \varepsilon \| g_{\varepsilon} \|_{L^2(\Gamma_0)} \| \phi \|_{H^{1/2}(\Gamma_0)}
\leq C \varepsilon \| g_{\varepsilon} \|_{L^2(\Gamma_0)} \| \phi \|_{H^1(\Omega_0)},
\]

where we have used the fact that \( \sum_{i=0}^{N-1} \| p^\varepsilon \|^2_{L^2(s_i, s_{i+1})} \leq (1 + A) \| g_{\varepsilon} \|^2_{L^2(\Gamma_0)} \). This implies

\[
|I_2| \leq C \varepsilon \| \phi \|_{H^1(\Omega_0)} \| g_{\varepsilon} \|_{L^2(\Gamma_0)}.
\]

(3.7)

Using the extension result (3.3), we have

\[
\| u^\varepsilon - u^0 \|_{H^1(\Omega_0)} \leq C \| u^\varepsilon - u^0 \|_{H^1(\Omega_0)},
\]

9
where $C$ only depends on $\|\gamma\|_{L^\infty(0,1)}$. This inequality together with (3.6) and (3.7) implies (3.5).

The above lemma shows that $u^0$ approximates to $u^\varepsilon$ in $H^1$ seminorm with rate $\mathcal{O}(\sqrt{\varepsilon})$. The convergence rate is inadequate in many applications. We step to the first order approximation in the next part.

### 3.2. Some auxiliary problems

To find the next order approximation of Problem (2.1), we define a semi-infinite tube as

$$Z_{bl}: = \{ \xi \in \mathbb{R}^2 \mid 0 < \xi_1 < 1, \xi_2 > \gamma(\xi_1) \}$$

with a curved boundary $\Sigma: = \{ \xi \in \mathbb{R}^2 \mid 0 < \xi_1 < 1, \xi_2 = \gamma(\xi_1) \}$.

Three auxiliary problems are defined as follows. Let $\beta_0$, $\beta_1$ and $\beta_2$ be three unknown functions posed on $Z_{bl}$, which are periodic in $\xi_1$ with period 1 and satisfy

\[
\begin{aligned}
-\Delta \xi \beta_0 &= 0, & \text{in } Z_{bl}, \\
\frac{\partial \beta_0}{\partial n} &= g(\xi_1) - \langle g \rangle, & \text{on } \Sigma, \\
\lim_{\xi_2 \to \infty} \beta_0 &= 0,
\end{aligned}
\]

and

\[
\begin{aligned}
-\Delta \xi \beta_1 &= 0, & \text{in } Z_{bl}, \\
\frac{\partial \beta_1}{\partial n} &= -\frac{\gamma'(\xi_1)}{[1 + (\gamma'(\xi_1))^2]^{1/2}}, & \text{on } \Sigma, \\
\lim_{\xi_2 \to \infty} \beta_1 &= 0,
\end{aligned}
\]

and

\[
\begin{aligned}
-\Delta \xi \beta_2 &= 0, & \text{in } Z_{bl}, \\
\frac{\partial \beta_2}{\partial n} &= \frac{1}{[1 + (\gamma'(\xi_1))^2]^{1/2}} - \frac{1}{r}, & \text{on } \Sigma,
\end{aligned}
\]

and

\[
\lim_{\xi_2 \to \infty} \beta_2 = 0.
\]

Here $\Delta \xi = \partial^2_{\xi_1} + \partial^2_{\xi_2}$. It is well-known that each problem has a unique solution, and the solutions have the following decay properties [18, Theorem 2.2].

**Lemma 3.5.** Let $\beta_0$, $\beta_1$ and $\beta_2$ be the solutions of (3.8), (3.9) and (3.10), respectively. Then, for $i = 0, 1$ and 2, there exist constants $C$ and $\delta$ such that

\[
\|\beta_i\|_{L^\infty(Z_{bl})} + \|\nabla \xi \beta_i\|_{L^\infty(Z_{bl})} \leq Ce^{-\delta \xi_2}. \tag{3.11}
\]

### 3.3. The first order approximation

Denote $\beta_i^\varepsilon(x) = \beta_i(x/\varepsilon)$ for $i = 0, 1, 2$, and define

\[
u^\varepsilon(x) = \beta_0^\varepsilon(x) + \beta_1^\varepsilon \partial_x u^0(x). \tag{3.12}
\]
The first order approximation \( u^1 : = u^0 + \varepsilon u^1 \), which does not satisfy the homogeneous Dirichlet boundary condition as \( u^0 \) on \( \Gamma_D \). It is useful to introduce a corrector \( u^{cr} \) to the first order approximation, which satisfies \( u^{cr} - u^1 \in V(\Omega_0) \) and
\[
a_{\Omega_0}(u^{cr}, v) = 0 \quad \text{for all } v \in V(\Omega_0). \tag{3.13}
\]

The corrector \( u^{cr} \) can be estimated as follows.

**Lemma 3.6.** Let \( u^{cr} \) be the solution of (3.13), then there exists \( C \) that is independent of the size of \( \Omega_0 \) such that
\[
\|
abla u^{cr}\|_{L^2(\Omega_0)} \leq C \left( 1 + \| \nabla u^0 \|_{L^\infty(\Omega_0)} + \| \nabla^2 u^0 \|_{L^2(\Omega_0)} \right). \tag{3.14}
\]

**Proof.** Define a smooth cut-off function \( \rho_\varepsilon \in C_0^\infty(\Omega_0) \) by
\[
\rho_\varepsilon(x) = \begin{cases} 
1, & x \in \Omega_0, \text{dist}(x, \Gamma_D) \geq 2\varepsilon, \\
0, & x \in \Omega_0, \text{dist}(x, \Gamma_D) \leq \varepsilon,
\end{cases}
\]
and \( \| \rho_\varepsilon \|_{L^\infty(\Omega_0)} \leq 1 \) and \( \| \nabla \rho_\varepsilon \|_{L^\infty(\Omega_0)} \leq C/\varepsilon \).

Let \( \eta^\varepsilon(x) = (1 - \rho_\varepsilon(x))u^1(x) \). It is clear to see
\[
\| \nabla u^{cr} \|_{L^2(\Omega_0)} \leq \| \nabla \eta^\varepsilon \|_{L^2(\Omega_0)}. 
\]

A direct calculation gives
\[
\frac{\partial \eta^\varepsilon}{\partial x_i} = -\partial_{x_i} \rho_\varepsilon u^1 + (1 - \rho_\varepsilon)\beta_j^\varepsilon \partial_{x_i}^2 u^0 + (1 - \rho_\varepsilon) \left( \partial_{x_i} \beta_0^\varepsilon + \partial_{x_i} \beta_j^\varepsilon \partial_{x_j} u^0 \right). 
\]

Using the decay estimate for \( \beta_j^\varepsilon \) in Lemma 3.5, we may bound \( \| \nabla \eta^\varepsilon \|_{L^2(\Omega_0)} \) as follows. We only estimate the first term, other terms can be bounded similarly.
\[
\| \partial_j \rho_\varepsilon \beta_0^\varepsilon \|_{L^2(\Omega_0)} \leq C\varepsilon^{-2} \left( \int_0^{1-\varepsilon} e^{-25xy/\varepsilon} dx + C\varepsilon^{-2} \int_0^1 \int_{1-2\varepsilon}^{1-\varepsilon} e^{-25xy/\varepsilon} dx \right) \leq C,
\]
where we have used (3.11) and the fact that \( \rho_\varepsilon \) supports in a narrow layer of width \( O(\varepsilon) \). Similarly, we have
\[
\| \partial_{x_i} \rho_\varepsilon \partial_{x_j} u^0 \beta_j^\varepsilon \|_{L^2(\Omega_0)} \leq C \| \nabla u^0 \|_{L^\infty(\Omega_0)},
\]
\[
\| (1 - \rho_\varepsilon) \partial_{x_i}^2 u^0 \beta_j^\varepsilon \|_{L^2(\Omega_0)} \leq C \| \nabla^2 u^0 \|_{L^2(\Omega_0)},
\]
\[
\| (1 - \rho_\varepsilon) \partial_{x_i} \beta_0^\varepsilon \|_{L^2(\Omega_0)} \leq C,
\]
\[
\| (1 - \rho_\varepsilon) \partial_{x_j} u^0 \partial_{x_i} \beta_j^\varepsilon \|_{L^2(\Omega_0)} \leq C \| \nabla u^0 \|_{L^\infty(\Omega_0)}. 
\]

Summing up all the terms, we obtain (3.14) and complete the proof. \( \square \)

The next theorem gives the error estimate for the first order approximation.
Theorem 3.7. Let $u^\varepsilon$ and $u_0$ be the solutions of Problems (2.1) and (3.2), respectively. Let $u^1$ be defined in (3.12). There exists $C$ independent of the size of $\Omega_0$ such that
\[
\|\nabla (u^\varepsilon - u^1_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon (1 + \|\nabla u_0\|_{W^{1,\infty}(\Omega_0)} + \|\nabla^2 u_0\|_{H^1(\Omega_0)}).
\] (3.15)

Proof. For any $v \in V(\Omega_\varepsilon)$, an integration by parts yields
\[
\int_{\Omega_\varepsilon} \nabla u^0 \nabla v \, dx = \int_{\Omega_\varepsilon} f v \, dx + \int_{\Gamma_\varepsilon} \frac{\partial u^0}{\partial n} v \, d\sigma(x),
\]
which together with (2.2) gives
\[
\int_{\Omega_\varepsilon} \nabla (u^\varepsilon - u^0) \nabla v \, dx = \int_{\Gamma_\varepsilon} \left( g - \frac{\partial u^0}{\partial n} \right) v \, d\sigma(x). \tag{3.16}
\]

Next we calculate $\int_{\Omega_\varepsilon} \nabla u^1 \nabla v \, dx$. Under the change of variables $\xi = x/\varepsilon$, $\Omega_\varepsilon$ is mapped onto a domain $D_{\varepsilon,\xi} = \{ \xi \in \mathbb{R}^2 \mid 0 < \xi_1 < 1/\varepsilon, \gamma(\xi_1) < \xi_2 < 1/\varepsilon \}$ with the curved boundary $\Gamma_{\varepsilon,\xi} = \{ \xi \in \mathbb{R}^2 \mid 0 < \xi_1 < 1/\varepsilon, \xi_2 = \gamma(\xi_1) \}$. We denote $D_{0,\xi}$ as the mapped domain of $\Omega_0$ under this map. Notice that for any function $v$, $D_\varepsilon v = \nabla_\varepsilon v + \frac{1}{\varepsilon} \nabla_\xi v$.

Clearly,
\[
\int_{\Omega_\varepsilon} \nabla u^1 \nabla v \, dx = \int_{\Omega_\varepsilon} \nabla_\varepsilon u^1 \nabla v \, dx + \int_{D_{\varepsilon,\xi}} \nabla_\xi u^1 \nabla v \, d\xi.
\]

A direct calculation gives
\[
\int_{D_{\varepsilon,\xi}} \nabla_\xi u^1 \nabla_\xi v \, d\xi = \int_{D_{\varepsilon,\xi}} \left( \nabla_\xi \beta_0 \nabla_\xi v + \nabla_\xi \beta_1 \nabla_\xi \right) \left( \frac{\partial u^0}{\partial x_i} \right) v \, d\xi + \int_{D_{\varepsilon,\xi}} \frac{\partial}{\partial x_i} (\nabla_\xi u^0) \left( \beta_1 \nabla_\xi v - v \nabla_\xi \beta_1 \right) \, d\xi.
\]

Using the definition of $\{\beta_i\}_{i=0}^2$, an integration by parts yields
\[
\int_{D_{\varepsilon,\xi}} \nabla_\xi \beta_0 \nabla_\xi v \, d\xi = \int_{\Gamma_{\varepsilon,\xi}} (g - \langle g \rangle) v \, d\sigma(\xi),
\]
and
\[
\int_{D_{\varepsilon,\xi}} \nabla_\xi \beta_1 \nabla_\xi \left( \frac{\partial u^0}{\partial x_i} \right) \, d\xi = -\int_{\Gamma_{\varepsilon,\xi}} \left( n_1 \frac{\partial u^0}{\partial x_1} + n_2 \frac{\partial u^0}{\partial x_2} \right) v \, d\sigma(\xi) - \frac{1}{r} \int_{\Gamma_{\varepsilon,\xi}} \frac{\partial u^0}{\partial x_2} v \, d\sigma(\xi) = -\frac{1}{\varepsilon} \int_{\Gamma_{\varepsilon,\xi}} \frac{\partial u^0}{\partial \xi_2} v \, d\sigma(\xi) - \frac{1}{r\varepsilon} \int_{\Gamma_{\varepsilon,\xi}} \frac{\partial u^0}{\partial \xi_2} v \, d\sigma(\xi).
\]
Using the fact that $\partial u^0/\partial n = r(g)$ on $\Gamma_0$, we rewrite the last term in the right-hand side of the above identity as

$$
-\frac{1}{r\varepsilon} \int_{\Gamma_{r\varepsilon}} \frac{\partial u^0}{\partial \xi_2} v \, d\sigma(\xi) = -\frac{1}{r\varepsilon} \int_{\Gamma_{r\varepsilon}} \frac{\partial u^0}{\partial \xi_2}(\xi_1,0)v \, d\sigma(\xi)
- \frac{1}{r\varepsilon} \int_{\Gamma_{r\varepsilon}} \left( \frac{\partial u^0}{\partial \xi_2}(\xi_1,\gamma(\xi_1)) - \frac{\partial u^0}{\partial \xi_2}(\xi_1,0) \right)v \, d\sigma(\xi)
= \int_{\Gamma_{r\varepsilon}} (g)v \, d\sigma(\xi) - \frac{1}{r\varepsilon} \int_{D_{0,\varepsilon}\setminus D_{r\varepsilon}} \frac{\partial^2 u^0}{\partial \xi_2^2} \, d\xi.
$$

Combining the above three equations, we obtain

$$
\int_{D_{r\varepsilon}} \nabla_\xi u^1 \nabla_\xi v \, d\xi = \int_{\Gamma_{r\varepsilon}} \frac{1}{\varepsilon} \frac{\partial u^0}{\partial n_\xi} v \, d\sigma(\xi) + \int_{D_{r\varepsilon}} \nabla_\xi u^0 \nabla_\xi v \, d\xi - \varepsilon \int_{\Omega_r} \nabla x u^1 \nabla v \, dx - \varepsilon \int_{\Omega_r} \nabla u^0 \nabla v \, dx.
$$

Denote $e = u^x - u^0 - \varepsilon u^1 - \varepsilon u^tr$, we have the following error expansion:

$$
\int_{\Omega_r} \nabla x \nabla v \, dx = \frac{1}{r} \int_{\Omega_0 \setminus \Omega_r} \frac{\partial^2 u^0}{\partial x_2^2} v \, dx - \varepsilon \int_{\Omega_r} \nabla x u^1 \nabla v \, dx - \varepsilon \int_{\Omega_r} \nabla u^0 \nabla v \, dx - \varepsilon \int_{\Omega_r} \nabla u^0 \nabla v \, dx
$$

By Lemma 3.3, we obtain

$$
\left| \frac{1}{r} \int_{\Omega_0 \setminus \Omega_r} \frac{\partial^2 u^0}{\partial x_2^2} v \, dx \right| \leq C \varepsilon \left\| \nabla^2 u^0 \right\|_{H^1(\Omega_r)} \left\| v \right\|_{H^1(\Omega_r)}.
$$

The second term can be bounded as

$$
\left\| \varepsilon \int_{\Omega_r} \nabla x u^1 \nabla v \, dx \right\| \leq \varepsilon \left\| \nabla x u^1 \right\|_{L^2(\Omega_r)} \left\| \nabla v \right\|_{L^2(\Omega_r)} \leq C \varepsilon \left\| \nabla^2 u^0 \right\|_{L^2(\Omega_r)} \left\| \nabla v \right\|_{L^2(\Omega_r)},
$$

where $C$ depends on $\left\| \beta_i \right\|_{L^\infty}$, which are uniformly bounded.

We transform the last integrand back to $\Omega_r$ as

$$
\int_{D_{r\varepsilon}} \frac{\partial}{\partial \xi_i} (\nabla_\xi u^0) (\beta_i \nabla x v - v \nabla x \beta_i) \, d\xi = \varepsilon \int_{\Omega_r} \frac{\partial}{\partial x_i} (\nabla x u^0) (\beta_i \nabla x v - v \nabla x \beta_i) \, dx.
$$

The first term can be bounded as

$$
\varepsilon \left| \int_{\Omega_r} \frac{\partial}{\partial x_i} (\nabla x u^0) / \beta_i \nabla x v \right| \leq \varepsilon \max_i \left\| \beta_i \right\|_{L^\infty} \left\| \nabla^2 u^0 \right\|_{L^2(\Omega_r)} \left\| \nabla v \right\|_{L^2(\Omega_r)}.
$$

By Lemma 3.5, we bound the second term as

$$
\varepsilon \left| \int_{\Omega_r} \frac{\partial}{\partial x_i} (\nabla x u^0) v \nabla x \beta_i \, dx \right| \leq C \left\| \nabla^2 u^0 \right\|_{L^\infty(\Omega_r)} \int_{\Omega_r} e^{-\delta x_2 / \varepsilon} \left| v \right| \, dx
\leq C \left\| \nabla^2 u^0 \right\|_{L^\infty(\Omega_r)} \int_0^\infty e^{-\delta x_2 / \varepsilon} \left( \int_0^1 \left| v \right| \, dx_1 \right) \, dx_2.
$$
By trace inequality, for any $x_2 \in (0, 1)$, we have
\[
\int_0^1 |v(x_1, x_2)| \, dx_1 \leq C \|v\|_{H^1(\Omega_\varepsilon)}.
\]
Combining the above two inequalities, we obtain
\[
\varepsilon \int_{\Omega_\varepsilon} \frac{\partial}{\partial x_i} (\nabla_x u^0) \nabla_x \beta_i \, dx \leq C \varepsilon \left( \|\nabla^2 u^0\|_{L^\infty(\Omega_\varepsilon)} + 1 \right) \|v\|_{H^1(\Omega_\varepsilon)}.
\]
Summing up all the estimates, we obtain that for any $v \in V_0(\Omega)$,
\[
\int_{\Omega_\varepsilon} \nabla e \nabla v \, dx \leq C \varepsilon \left( \|\nabla^2 u^0\|_{H^1(\Omega_0)} + \|\nabla^2 u^0\|_{L^\infty(\Omega_\varepsilon)} + 1 \right) \|v\|_{H^1(\Omega_\varepsilon)}
+ \varepsilon \|\nabla u^c\|_{L^2(\Omega_\varepsilon)} \|\nabla v\|_{L^2(\Omega_\varepsilon)}.
\]
(3.17)
Since $\Gamma_\varepsilon$ is uniformly Lipschitz, we can extend $u^c$ from $\Omega_\varepsilon$ to $\Omega_0$ so that
\[
\|e\|_{H^1(\Omega_0)} \leq C \|e\|_{H^1(\Omega_\varepsilon)},
\]
where $C$ only depends on $\|h^\varepsilon\|_{L^\infty(Y)}$ by Lemma 3.2. Taking $v = e$ in (3.17), we obtain
\[
\|\nabla e\|_{L^2(\Omega_0)}^2 \leq C \varepsilon \left( \|\nabla^2 u^0\|_{H^1(\Omega_0)} + \|\nabla^2 u^0\|_{L^\infty(\Omega_\varepsilon)} + 1 + \|\nabla u^c\|_{L^2(\Omega_\varepsilon)} \right) \|e\|_{H^1(\Omega_\varepsilon)}.
\]
Using Poincaré’s inequality to $e$ because $e \in V_0(\Omega_\varepsilon)$, we obtain
\[
\|\nabla e\|_{L^2(\Omega_\varepsilon)} \leq C \varepsilon \left( \|\nabla^2 u^0\|_{H^1(\Omega_0)} + \|\nabla^2 u^0\|_{L^\infty(\Omega_\varepsilon)} + 1 + \|\nabla u^c\|_{L^2(\Omega_\varepsilon)} \right),
\]
which together with (3.14) yields the desired estimate (3.15). □

4. Error Estimate. We are ready to prove the convergence rate of the proposed MsFEM by the homogenization results in the last section. For any element $\tau_\varepsilon$ with a rough side on $\Gamma_\varepsilon$, we assume that $\tau_\varepsilon$ is contained in its homogenized domain $\tau_0$. Given this assumption, we could apply Theorem 3.7 to each element. In fact, this seemingly restrictive assumption is not essential because Theorem 3.7 remains valid without such assumption. Therefore, the error estimate also holds true without this assumption, which is also confirmed by the numerical examples in the next section.
In addition, to avoid too many technical complexity, the estimate is restricted to the triangular element, while the proof can be generalized to the quadrilateral element with minor modifications.

4.1. Homogenization of multiscale basis functions. We start with some homogenization results for the multiscale basis functions $\Phi_{p, \tau_\varepsilon}^{MS}$. By the homogenization results in last section, we may clarify the zeroth order approximation and the first order approximation of $\Phi_{p, \tau_\varepsilon}^{MS}$, which are denoted by $\Phi_{p, \tau_\varepsilon}^0$ and $\Phi_{p, \tau_\varepsilon}^1$, respectively.

Note that $\Phi_{p, \tau_\varepsilon}^0 - \phi_{p, \tau_\varepsilon} \in V_0(\tau_0)$ and
\[
a_{\tau_0}(\Phi_{p, \tau_\varepsilon}^0, v) = \langle r(\theta)_{p, \tau_\varepsilon}, v \rangle_{\partial \tau_0 \cap \Gamma_0} \quad \text{for all } v \in V_0(\tau_0).
\]
It is clear that $r(\theta)p,\tau_e = \partial_n \phi_{p,\tau_e}$. We conclude that the unique solution of the above problem is $\Phi_{p,\tau_e} = \phi_{p,\tau_e}$.

The first order corrector $\Phi_{p,\tau_e}^1$ of $\Phi_{p,\tau_e}^{MS}$ is given by

$$
\Phi_{p,\tau_e}^1 = \tilde{\phi}_0 \partial_n \phi_{p,\tau_e} + \beta_i \frac{\partial \phi_{p,\tau_e}}{\partial x_i},
$$

where $\tilde{\phi}_0(x) = \tilde{\phi}_0(x/\varepsilon)$ with $\tilde{\phi}_0$ being the solution of

$$
\begin{cases}
-\Delta \tilde{\phi}_0 = 0, & \text{in } Z_{bl}, \\
\partial_n \tilde{\phi}_0 = \frac{1}{r} \left( g(\xi_1)/\langle g \rangle - 1 \right), & \text{on } \Sigma, \\
\lim_{\xi_2 \to \infty} \tilde{\phi}_0 = 0.
\end{cases}
$$

It is clear that $\tilde{\phi}_0 = 0$ if $g = \langle g \rangle$. When $\tilde{\phi}_0 = 0$, the proof is simpler than the case $\tilde{\phi}_0 \neq 0$ but the estimate is same. We only consider the later case in the following.

For any $\tau \in \mathcal{T}_h$, we define the MsFEM interpolant of $u^\varepsilon$ as

$$
\Pi_h u^\varepsilon := \sum_{p \in \mathcal{S}(\tau)} u^0(x_p) \Phi_{p,\tau_e}^{MS}.
$$

It is clear to see $\Pi_h u^\varepsilon$ reduces to the standard linear Lagrange interpolant of $u^0$, which is denoted by $\pi_h u^0$ when $\tau$ has no side on $\Gamma_e$. For element $\tau_e$ with a rough side, we define the first-order approximation of the MsFEM interpolant by

$$
(\Pi_h u^\varepsilon)_1 := \sum_{p \in \mathcal{S}(\tau_e)} u^0(x_p) \left( \phi_{p,\tau_e} + \varepsilon \Phi_{p,\tau_e}^1 \right).
$$

The interpolate estimate is based on the following decomposition

$$
u^\varepsilon - \Pi_h u^\varepsilon = (u^\varepsilon - u^\varepsilon_1) + (u^\varepsilon_1 - (\Pi_h u^\varepsilon)_1) + ((\Pi_h u^\varepsilon)_1 - \Pi_h u^\varepsilon) .
$$

The following lemma is a direct consequence of Theorem 3.7.

**Lemma 4.1.** For any rough-sided element $\tau_e$, we have

$$
\| \nabla \Pi_h u^\varepsilon - \nabla (\Pi_h u^\varepsilon)_1 \|_{L^2(\tau_e)} \leq C \varepsilon (1 + \| \nabla u^0 \|_{L^\infty(\tau_e)}).
$$

By definition, we rewrite $(\Pi_h u^\varepsilon)_1$ as

$$
(\Pi_h u^\varepsilon)_1 = \pi_h u^0 + \varepsilon \tilde{\phi}_0 \frac{\partial \pi_h u^0}{\partial n} + \varepsilon \beta_i \frac{\partial \pi_h u^0}{\partial x_i},
$$

and

$$
u^\varepsilon_1 = u^0 + \varepsilon \tilde{\phi}_0 \frac{\partial u^0}{\partial n} + \varepsilon \beta_i \frac{\partial u^0}{\partial x_i}.
$$

A direct consequence of the representations (4.3) and (4.4) is
For any rough-sided element $\tau$, we have
\[
\| \nabla u^e_1 - \nabla (\Pi \hat{u}^e) \|_{L^2(\tau)} \leq C (h_{\tau} + \varepsilon) \| D^2 u^0 \|_{L^2(\tau)}.
\]

Proof. A direct calculation gives that for $i = 1, 2$,
\[
\frac{\partial}{\partial x_i} (u^e_1 - \nabla (\Pi \hat{u}^e)_1) = \frac{\partial}{\partial x_i} (u^0 - \pi_h u^0) + \varepsilon \frac{\partial \tilde{\beta}_0}{\partial x_i} \frac{\partial}{\partial n} (u^0 - \pi_h u^0)
\]
\[
+ \frac{\varepsilon}{h} \frac{\partial^2 u^0}{\partial n \partial x_i} + \frac{\varepsilon}{\partial x_i} \frac{\partial}{\partial x_j} (u^0 - \pi_h u^0) + \varepsilon \frac{\partial \beta^e}{\partial x_i} \frac{\partial^2 u^0}{\partial x_i \partial x_j}.
\]

Note that $\tilde{\beta}_0$ satisfies the same decay estimate (3.11) as $\beta_0$, and proceeding along the same line that leads to Lemma 3.6, we obtain
\[
\int_{\tau} \left| \nabla \tilde{\beta}_0 \partial_n (u^0 - \pi_h u^0) \right|^2 \, dx \leq C \varepsilon^{-2} \int_{\tau \cap \Gamma} \left| \nabla (u^0 - \pi_h u^0) \right|^2 \, dx \int_0^h e^{-2x/\varepsilon} \, dx
\]
\[
\leq C \varepsilon^{-1} \int_{\tau \cap \Gamma} \left| \nabla (u^0 - \pi_h u^0)(x_1, x_2) \right|^2 \, dx_1.
\]

By the trace inequality, we get
\[
\left\| \nabla \tilde{\beta}_0 \partial_n (u^0 - \pi_h u^0) \right\|_{L^2(\tau)} \leq C \varepsilon^{-1/2} \left\| \nabla (u^0 - \pi_h u^0) \right\|_{L^2(\tau)}^{1/2} \left\| \nabla^2 (u^0 - \pi_h u^0) \right\|_{L^2(\tau)}^{1/2}
\]
\[
\leq C (h_{\tau}/\varepsilon)^{1/2} \left\| \nabla^2 u^0 \right\|_{L^2(\tau)}.
\]

Proceeding along the same line that leads to the above inequality, we obtain
\[
\left\| \nabla \beta_i \partial_n (u^0 - \pi_h u^0) \right\|_{L^2(\tau)} \leq C (h_{\tau}/\varepsilon)^{1/2} \left\| \nabla^2 u^0 \right\|_{L^2(\tau)}.
\]

The remaining terms may be bounded as follows,
\[
\left\| \nabla (u^0 - \pi_h u^0) \right\|_{L^2(\tau)} \leq C h_{\tau} \left\| \nabla^2 u^0 \right\|_{L^2(\tau)},
\]
and
\[
\left\| \tilde{\beta}_0 \partial_n (\nabla u^0) \right\|_{L^2(\tau)} \leq C \left\| \nabla^2 u^0 \right\|_{L^2(\tau)}, \quad \left\| \beta_i \partial_n (\nabla u^0) \right\|_{L^2(\tau)} \leq C \left\| \nabla^2 u^0 \right\|_{L^2(\tau)}.
\]

Combining the above estimates, we obtain the desired estimate. $\blacksquare$

For any element $\tau$ without rough edge, $\Pi_h u^e = \pi_h u^0$. The decomposition (4.1) is replaced by $u^e - \Pi_h u^e = u^e - u^e_1 + u^0 - \pi_h u^0 + \varepsilon u^1$. Therefore, we need the a priori estimate for $u^1$ over elements without rough edge. We divide the elements into three groups: the elements with one rough edge belong to $T^1_h$, the elements with one vertex on the rough boundary belong to $T^2_h$, and the remaining elements belong to $T^3_h$.

Lemma 4.3. When $\tau \in T^2_h$, then
\[
\sum_{\tau \in T^2_h} \left\| \nabla u^1 \right\|_{L^2(\tau)}^2 \leq C h^{-1} \left( 1 + \left\| \nabla u^0 \right\|_{L^2(\Omega_1)}^2 \right) + C \sum_{\tau \in T^2_h} \left\| \nabla^2 u^0 \right\|_{L^2(\tau)}^2.
\]
When $\tau \in T_h^3$, then
\[
\sum_{\tau \in T_h^3} \left\| \nabla u^1 \right\|_{L^2(\tau)}^2 \leq C \frac{h}{\varepsilon} \left( 1 + \left\| \nabla u^0 \right\|_{W^{1,\infty}(\Omega_h)}^2 \right) + C \sum_{\tau \in T_h^3} \left\| \nabla^2 u^0 \right\|_{L^2(\tau)}^2. \tag{4.6}
\]

**Proof.** For an element $\tau \in T_h^3$, we assume that the two sides intersect with $\Gamma_\varepsilon$ are given explicitly by $x_1 = \alpha_1 x_2$ and $x_1 = \alpha_2 x_2$, with $|\alpha_i| \leq c_1$ and the bound $c_1$ depends only on the minimal angle of $\tau$. Using (3.11), a direct calculation gives
\[
\left\| \nabla \beta_0^\varepsilon \right\|_{L^2(\tau)} \leq C \varepsilon^{-1} \left( \int_\tau e^{-25x_2/\varepsilon} dx \right)^{1/2} \leq C \varepsilon^{-1} \left( \int_0^\infty (\alpha_2 - \alpha_1) x_2 e^{-25x_2/\varepsilon} dx_2 \right)^{1/2} \leq C.
\]
A direct calculation gives that for $i = 1, 2$, there holds
\[
\left\| \nabla (\partial_x, u^0 \beta_i^\varepsilon) \right\|_{L^2(\tau)} \leq C \left( \left\| \nabla u^0 \right\|_{L^\infty(\tau)} + \left\| \nabla^2 u^0 \right\|_{L^2(\tau)} \right).
\]
Combining the above estimates and using the fact that the cardinality of $T_h^3$ is $O(h^{-1})$, we obtain (4.5).

Using the facts that the triangulation is regular, for the element in $k$-th layer, there exists a constant $c_0$ such that $c_0 kh \leq \text{dist}(\tau, \Gamma_\varepsilon) \leq c_0 (k + 1) h$. By (3.11), a direct calculation gives
\[
\left\| \nabla \beta_0^\varepsilon \right\|_{L^2(\tau)} \leq C \varepsilon^{-1} \left( \int_\tau e^{-25x_2/\varepsilon} dx \right)^{1/2} \leq Ch\varepsilon^{-1} \left( \int_{c_0 kh}^{c_0 (k+1) h} e^{-25x_2/\varepsilon} dx_2 \right)^{1/2} \leq C (h/\varepsilon)^{1/2} \exp(-c_0 \delta kh/\varepsilon).
\]
Proceeding along the same line that leads to the above estimate, we have for $i = 1, 2$,
\[
\left\| \nabla (\partial_x, u^0 \beta_i^\varepsilon) \right\|_{L^2(\tau)} \leq C (h/\varepsilon)^{1/2} \exp(-c_0 \delta kh/\varepsilon) \left\| \nabla u^0 \right\|_{L^\infty(\tau)} + C \left\| \nabla^2 u^0 \right\|_{L^2(\tau)}
\]
A combination of the above estimates leads to
\[
\left\| \nabla u^1 \right\|_{L^2(\tau)} \leq C \exp(-c_0 \delta kh/\varepsilon) (h/\varepsilon)^{1/2} \left( 1 + \left\| \nabla u^0 \right\|_{L^\infty(\tau)} \right) + C \left\| \nabla^2 u^0 \right\|_{L^2(\tau)}.
\]
Summing up all the elements in $T_h^3$ leads to (4.6). \[\Box\]

### 4.2. Interpolation error estimate

The next theorem gives the interpolate error estimate.

**THEOREM 4.4.** Let $u^\varepsilon$ be the solution of the problem (2.1), we have
\[
\left\| \nabla (u^\varepsilon - \Pi h u^\varepsilon) \right\|_{L^2(\Omega_h)} \leq C \varepsilon \left( \left\| \nabla^2 u^0 \right\|_{H^1(\Omega_h)} + \left\| \nabla u^0 \right\|_{W^{1,\infty}(\Omega_h)} \right) + Ch \left\| \nabla^2 u^0 \right\|_{L^2(\Omega_h)} + C \varepsilon h^{-1/2} \left( \left\| \nabla u^0 \right\|_{L^\infty(\Omega_h)} + 1 \right). \tag{4.7}
\]

**Proof.** We start from the following decomposition (4.1). Using Lemma 4.2,
\[
\sum_{\tau \in T_h^3} \left\| \nabla u^1_1 - \nabla (\Pi h u^\varepsilon)_1 \right\|_{L^2(\tau)}^2 \leq C (\varepsilon + h)^2 \sum_{\tau \in T_h^3} \left\| \nabla^2 u^0 \right\|_{L^2(\tau)}^2.
\]

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For \( \tau \in T^2_h \), we have \( u_1^\tau - (\Pi_h u^\tau)_1 = u^0 - \pi_h u^0 + \varepsilon u^1 \). Therefore,

\[
\sum_{\tau \in T^2_h} \| \nabla u_1^\tau - \nabla [(\Pi_h u^\tau)_1] \|^2_{L^2(\tau)} \leq 2 \sum_{\tau \in T^2_h} \| \nabla (u^0 - \pi_h u^0) \|^2_{L^2(\tau)} + 2\varepsilon^2 \sum_{\tau \in T^2_h} \| \nabla u^1 \|^2_{L^2(\tau)}.
\]

Using Lemma 4.3, we obtain

\[
\sum_{\tau \in T^2_h} \| \nabla u_1^\tau - \nabla [(\Pi_h u^\tau)_1] \|^2_{L^2(\tau)} \leq C(\varepsilon + h)^2 \sum_{\tau \in T^2_h} \| \nabla^2 u^0 \|^2_{L^2(\tau)} + \frac{C\varepsilon^2}{h} (1 + \| \nabla u^0 \|^2_{L^\infty(\tau)}).
\]

Proceeding along the same line that leads to the above estimate, we obtain

\[
\sum_{\tau \in T^3_h} \| \nabla u_1^\tau - \nabla [(\Pi_h u^\tau)_1] \|^2_{L^2(\tau)} \leq C(\varepsilon + h)^2 \sum_{\tau \in T^3_h} \| \nabla^2 u^0 \|^2_{L^2(\tau)} + \frac{C\varepsilon^2}{h} (1 + \| \nabla u^0 \|^2_{L^\infty(\tau)}).
\]

Summing up all the above estimates, we obtain

\[
\| \nabla u_1^\tau - \nabla [(\Pi_h u^\tau)_1] \|_{L^2(\Omega)} \leq C(\varepsilon + h) \| \nabla^2 u^0 \|_{L^2(\Omega)} + C\varepsilon h^{-1/2} (1 + \| \nabla u^0 \|_{L^\infty(\Omega)}).
\]

By Lemma 4.1,

\[
\| \nabla \Pi_h u^\tau - \nabla [(\Pi_h u^\tau)_1] \|_{L^2(\Omega)}^2 = \sum_{\tau \in T^3_h} \| \nabla \Pi_h u^\tau - \nabla [(\Pi_h u^\tau)_1] \|_{L^2(\tau)}^2 \leq C\varepsilon^2 \sum_{\tau \in T^3_h} \| \nabla u^0 \|_{L^\infty(\tau_0)} \leq \frac{C\varepsilon^2}{h} \| \nabla u^0 \|_{L^\infty(\Omega)}.
\]

Finally, the term \( \| \nabla (u^\tau - u^\tau)_1 \|_{L^2(\Omega)} \) can be bounded by Theorem 3.7. Summing up all the estimates we obtain the desired estimate (4.7). \( \square \)

Using the above interpolation estimate, we obtain the main result of this paper.

**Theorem 4.5.** Let \( u^\tau \) and \( u_h \) be the solutions of Problem (2.1) and Problem (2.6), respectively. Then

\[
\| \nabla (u^\tau - u_h) \|_{L^2(\Omega)} \leq C\varepsilon \left( \| \nabla^2 u^0 \|_{H^1(\Omega_0)} + \| \nabla u^0 \|_{W^{1,\infty}(\Omega_0)} \right) + C h \| \nabla^2 u^0 \|_{L^2(\Omega_0)} + C\varepsilon h^{-1/2} \| \nabla u^0 \|_{L^\infty(\Omega_0)} + 1.
\]
Remark 4.6. We have not estimated the $L^2$ error of the method, because the $H^1$ error estimate is not optimal with respect to the regularity of the data. Standard dual argument only yields a suboptimal convergence rate as the original MsFEM [24]. This would be a topic for further study.

5. Numerical Examples. In this section, we perform three numerical experiments to verify the convergence rate and efficiency of the proposed method. We solve Problem (2.1) for different rough domains, different source terms and different boundary fluxes. The implementation of the method is similar to the standard MsFEM [23, 24]. The cell problem (2.3) is numerically solved to only for elements with a rough side. In our simulations below, we use $P_1$ element to solve (2.3) with quasi-uniform mesh with the subgrid mesh size around $\varepsilon/20$.

First Example In this example, the rough domain is given by
\[ \Omega_\varepsilon = \{ x \in \mathbb{R}^2 \mid 0 < x_1 < 1, \varepsilon \gamma(x_1/\varepsilon) < x_2 < 1 \}, \]
where $\gamma(y_1) = (\cos(2\pi y_1) - 1)/10$. The rough boundary is given by
\[ \Gamma_\varepsilon = \{ x \in \mathbb{R}^2 \mid 0 < x_1 < 1, x_2 = \varepsilon \gamma(x_1/\varepsilon) \}, \]
and $\Gamma_D = \partial \Omega_\varepsilon \setminus \Gamma_\varepsilon$. We choose $f = 1$ and $g = 0$ in (2.1), and set homogeneous Dirichlet boundary condition on $\Gamma_D$ and $\varepsilon = 1/128$.

The grid in this example is the uniform triangular grid as the right subfigure in Fig. 2.1 with the mesh size $h$ varying from $1/5$ to $1/160$. The errors are measured by
\[ \text{Err}_{L^2} = \| u_h - \tilde{u} \|_{L^2(\Omega_\varepsilon)}, \quad \text{Err}_{H^1} = \| \nabla u_h - \nabla \tilde{u} \|_{L^2(\Omega_\varepsilon)}. \]
Here $\tilde{u}$ is a solution computed on an adaptive refined mesh with mesh size $h \approx 10^{-4}$ by linear finite element.

Fig. 5.1 shows two different scenarios of the convergence behaviour of the method. When the mesh size $h$ is larger than the roughness parameter $\varepsilon$, the method is first order in the $H^1$ semi-norm and second order in the $L^2$ norm. When $h$ is commensurate with $\varepsilon$, the method degenerates due to the resonance error. This is consistent with our theoretical prediction, at least for the $H^1$ error.

Second example In this example, the domain $\Omega_\varepsilon$ is the same with that in the first example. Unlike the first example, we choose $f = 0$ and an inhomogeneous boundary flux $g_\varepsilon = (1 - \cos(2\pi x_1/\varepsilon))/2$. In addition, we impose an inhomogeneous Dirichlet boundary value $u(x) = (1 - x_2)/2$ on $\Gamma_D$.

Uniform triangular grid with the mesh size $h$ varying from $1/5$ to $1/80$ is employed in this example. Fig. 5.2 shows the convergence behavior of the method. It is clear that the method has nearly optimal convergence rate for both the $H^1$ semi-norm and the $L^2$ norm when $h = 1/40 > 2\varepsilon$, while the resonance error gets to dominate and the convergence rate degenerates when $h$ is approximately $1/80$. 

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In this example, we use linear finite element to solve the homogenized problem

\[
\begin{aligned}
-\Delta u^0 &= 0 & \text{in } \Omega_0, \\
u^0 = \frac{1-x_2}{2} & & \text{on } \Gamma_D, \\
\frac{\partial u^0}{\partial n} = \frac{r}{2} & & \text{on } \Gamma_0.
\end{aligned}
\]

where $\Omega_0 = \{x \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < 1\}$, $\Gamma_0 = \{(x_1,0) : 0 < x_1 < 1\}$ and $r = \int_0^1 [1 + (\gamma'(y_1))^2]^{1/2} \, dy_1 \approx 1.01$. It seems that the linear finite element method is less accurate as MsFEM. This is due to the fact that the homogenization errors dominate as the mesh is refined. This degeneracy of the convergence rate is more significant for the $L^2$ error.

**Third Example** In this example, we test the problem with a non-periodic rough boundary, which is not covered by our theoretical results, while the method works as well. The domain is

\[
\Omega_\varepsilon = \left\{ x \in \mathbb{R}^2 \mid 0 < x_1 < 1, \varepsilon \frac{\gamma(x_1) - 1}{10} < x_2 < 1 \right\}
\]

with $\gamma$ an oscillating function defined as follows. We firstly divide the interval $(0,1)$ uniformly as $0 = s_0 < s_1 < \cdots < s_M = 1$ with $M = 1/\varepsilon = 128$. The function $\gamma$ is set to be a piecewise continuous linear function over such partition, with $\gamma(s_i)$, $i = 0, \cdots, M$, a series of pseudo-random numbers between 0 and 1 generated by a standard C++ library function. The rough boundary

\[
\Gamma_\varepsilon = \left\{ x \in \mathbb{R}^2 \mid 0 < x_1 < 1, x_2 = \varepsilon(\gamma(x_1) - 1)/10 \right\}.
\]
We choose $f = 1, g = 0$, and impose a homogeneous Dirichlet boundary condition on $\Gamma_D$. We use a uniform triangular grid with the mesh size $h$ varies from $1/5$ to $1/160$. The results for MsFEM is reported in Fig. 5.3. Similar to the previous examples, we get optimal convergence rate when $h > 2\varepsilon$. The resonance errors becomes dominate for the $L^2$ error when $h = 1/80$, but still small for the $H^1$ error even when $h = 1/160$. This might indicate the resonance error for the $L^2$ error is more pronounced.

REFERENCES

Fig. 5.3. Convergence behavior of the method for the third example.


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