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A robust optimal preconditioner for the mixed finite element discretization of elliptic optimal control problems†

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SUMMARY

In this paper we consider the efficient solving of the resulting algebraic system for elliptic optimal control problems with mixed finite element discretization. We propose a block diagonal preconditioner for the symmetric and indefinite algebraic system solved with minimum residual method, which is proved to be robust and optimal with respect to both the mesh size and the regularization parameter. The block diagonal preconditioner is constructed based on an isomorphism between appropriately chosen solution space and its dual for a general control problem with combined state and gradient state observations in the objective functional. Numerical experiments confirm the efficiency of our proposed preconditioner. Copyright © 2010 John Wiley & Sons, Ltd.

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KEY WORDS: Optimal control problem, elliptic equation, mixed finite element method, optimal preconditioner

1. INTRODUCTION

Optimization problem with PDE constraints, including the optimal control problems and inverse problems, play an increasing role in modern science and engineering. The requirement for fast and efficient simulations of such kind of problems also stimulates the development of related fields such as optimization, numerical analysis and numerical linear algebra. Here we refer to the monographs [26], [21] on the theoretical and numerical developments of PDE constrained optimal control problems.

Finite element method is among one of the most popular approaches to solve PDE-constrained optimal control problems, we refer to [21] and [27] for a priori and a posteriori error estimates. For nonstandard finite element method such as mixed method, we refer to [11, 12, 27] for related convergence results for elliptic and Stokes control problems. The first order optimality system of optimal control problems governed by PDEs consists of the state equation, the adjoint equation

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and the control equation which can be viewed as a saddle point problem. The finite element discretization (standard or nonstandard) of optimal control problems usually results in a large scaled indefinite algebraic system with parameter dependency, whose condition number grows when the mesh size and the regularization parameter approach to zero. Solving this system presents significant challenges. In recent years, efficient solver of algebraic system related to optimal control problems attracts a lot of attentions and plenty of efficient algorithms and preconditioners are proposed to solve different type of optimal control problems.

Here we give a short overview on recent developments of efficient solvers for algebraic systems related to the standard finite element discretization of optimal control problems. There are a lot of works on efficient preconditioning for unconstrained optimal control problems. For solving and preconditioning a class of block two-by-two linear systems arising from the Galerkin finite element discretizations of a class of distributed control problems, Bai and his collaborators constructed block-counter-diagonal and block-counter-tridiagonal preconditioning matrices to precondition the Krylov subspace methods such as GMRES in [4], and a preconditioned modified Hermitian and skew-Hermitian splitting iteration scheme in [5]. We also refer to [40] and [41] for various block-triangular preconditioners. In [45] the authors proposed a symmetric indefinite preconditioner for saddle point problems resulted from optimal control problems and showed its robustness with respect to the mesh size and regularization parameter. In [41] the authors proposed a preconditioned conjugate gradient method in nonstandard inner products to solve the saddle-point systems with applications in optimizations. In [30] the authors proposed two block preconditioners for elliptic optimal control problems in either reduced Schur complement system for control variable or saddle point problem and showed their mesh independencies. Zulehner ([52]) proposed a robust block preconditioner for saddle point problems arising from elliptic and Stokes distributed control problems by searching for appropriate nonstandard norms. In [36] Pearson and Wathen proposed a robust block preconditioner with a new approximation of the Schur complement. For efficient multigrid method to solve elliptic optimal control problems we refer to [9, 42, 44, 47].

Additionally, control or state constraints may be incorporated into optimal control problems due to the physical restriction; this will introduce additional difficulties for efficient preconditioning. Stoll and Wathen ([46]) studied the preconditioning for the saddle point problems arising from the primal-dual active set algorithm applied to PDE-constrained optimal control problems. Schiela and Ulbrich ([43]) proposed two strategies for preconditioning linear operator equations that arise in PDE constrained optimal control problem with control or state constraints in the framework of conjugate gradient methods. Herzog and Sachs ([20]) generalized the idea of [45] to solve control problems with pointwise control constraints, mixed control-state constraints and of Moreau-Yosida penalty type by using a preconditioned conjugate gradient method in a nonstandard inner product where the condition numbers for each case were also estimated. In [35] the authors studied the preconditioning technique for state-constrained optimal control problems with Moreau-Yosida penalty. In [15] the authors presented a new multigrid preconditioner for the linear systems arising in the semismooth Newton method solution of certain control-constrained, quadratic distributed optimal control problems. Herzog and Mach proposed in [19] three different preconditioners for elliptic optimal control problems with pointwise state gradient constraints by employing a quadratic penalty approach together with a semismooth Newton iteration, and proved the mesh independency of the spectral properties of the preconditioned linear Newton saddle-point systems.

Due to the close relation between PDE-constrained optimal control problems and parameter idenfitification problems, there are also some attempts for preconditioning inverse problems. In [18] the authors used a variant of symmetric QMR to solve the KKT system of parameter estimation problems in all-at-once approach, an effective preconditioner was obtained by solving the reduced Hessian system approximately. In [32] Nielsen and Mardal studied the efficient preconditioning for optimality system arising from inverse problems and showed that the number of iterations needed to solve the preconditioned problem by the minimal residual method was bounded independently of the mesh parameter, used in the finite element discretization, and increases only moderately as the regularization parameter approaching to zero, see also [33] for the analysis of minimum residual method to solve such kind of saddle point problems.

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The above mentioned results are mainly based on the standard finite element discretizations of the underlying optimal control problems. Mixed finite element method, aiming to recover both the scalar state and the flux simultaneously, also finds many applications in solving optimal control problems, especially for control problems with gradient state observations in the objective functional. In [12] the authors studied the mixed finite element approximations of a linear-quadratic elliptic distributed optimal control problem, while in [11] the superconvergence of mixed finite element method for elliptic optimal control problems was studied. In [17] the authors used the mixed finite element method to approximate the Dirichlet boundary control problems where the mixed variational form can deal with the inhomogeneous Dirichlet boundary condition naturally. Mixed finite element method was also used to solve optimal control problems with gradient state constraints in [13]. Despite the extensive applications of mixed finite element method in optimal control problems, we are not aware of any works on efficient solver for the resulting algebraic systems of such kind of problems. We note that the resulting algebraic system is a large scaled symmetric and indefinite matrix which has a big demand for efficient solvers. In this paper, we intend to fill this gap by proposing robust and optimal a preconditioner for elliptic optimal control problems with mixed finite element discretization.

Generally speaking, the method to solve the algebraic system related to optimal control problems can be classified into two categories. The first approach is to eliminate the state variables and Lagrange multipliers and correspondingly, the state equations and adjoint equations to reduce the system to a Schur complement system involving only the control variables. One can then design preconditioner for this reduced system. On each iteration for solving the control variables one needs to solve the state and adjoint state equations, where the efficient methods for forward PDE such as multigrid method or preconditioner can be incorporated. For related work we refer to [8, 30]. The above elimination procedure is termed a reduced space method, in contrast to a full space method of the second approach, or one-shot approach (also called all-at-once), in which one solves for the state, control, and adjoint state simultaneously. In this case the multigrid algorithms or preconditioner should be designed for this saddle point system which usually yields block type preconditioner. We refer to [41, 42, 45, 52] for more details. We remark that, for the first approach, the efficient preconditioning for the Schur complement system is generally difficult and the subproblems should be solved accurately to guarantee the convergence of the algorithm, and we would expect some kind of convenience via the second approach. Additionally, as mentioned in [31, 38], a block diagonal preconditioner is a natural choice for the saddle point problem in an infinite dimensional Hilbert space. The block diagonal preconditioner for the associated stable discretization problem can be constructed immediately, once the proper inner product is defined on the Hilbert space. Meanwhile, one can construct block triangular preconditioners based on the proper inner product for some special problems. We refer to [28, 29] for the details. Here we adopt the second approach and propose a block diagonal PMinRes ([34]) algorithm for solving elliptic optimal control problems discretized with mixed finite elements under the framework proposed in [31]. The preconditioner covers the case with or without gradient state observations in the objective functional, which is further shown to be robust with respect to both the mesh size and the regularization parameter. Numerical experiments confirm the efficiency of our proposed algorithms.

Finally, we remark that it is a widely used way to (present a problem into its mixed formulation and) solve a mixed system by preconditioning techniques; we refer to, e.g., [37, 49–51] for more discussion. Also, for our present problem, the main computational costs are taken by $H(\mathrm{div})$ solvers, and we refer to, e.g., [2, 3, 16, 22, 24, 25] for many discussions on the solvers and their applications.

The remaining of the paper is organized as follows: In Section 2 we present the linear system generated from the mixed finite element discretization of elliptic optimal control problems. We also give an optimal block diagonal preconditioner for the linear system and the incorporation of Hiptmair-Xu preconditioner ([24]) for solving the $\mathrm{div}^*\mathrm{div}$ subproblems appeared in the block diagonal preconditioner. In Section 3 we give the stability analysis for both the continuous and discrete optimality systems and prove the optimality of the proposed preconditioner. We carry out several numerical experiments in Section 4 to confirm the efficiency of our proposed algorithms. The paper ends with a concluding remark in Section 5.

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In this paper, we will use the following notation. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $\chi_{[0,+\infty)}(x)$ be the characteristic function of the interval $[0, +\infty)$. We denote $H^m(\Omega)$ the usual Hilbert spaces with norms $\|\cdot\|_m$. Let $H_0^m(\Omega)$ be the completion space of functions in $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_m$. Denote $H(\text{div};\Omega) := \{\underset{\sim}{\tau} \in (L^2(\Omega))^2 : \ \text{div}\underset{\sim}{\tau} \in L^2(\Omega)\}$. Denote $L^2_\rho(\Omega)$ the Hilbert space with inner product $(\rho\cdot,\cdot)$ for some positive weight function ρ, and $tL^2(\Omega) \cap wH(\text{div};\Omega)$ the Hilbert space with the inner product $t^2(\cdot,\cdot) + w^2(\text{div}\cdot, \text{div}\cdot)$ for positive constants t and w.

2. LINEAR SYSTEM AND OPTIMAL SOLVER

2.1. Generation of the linear algebraic system

In this paper, we consider the following elliptic distributed optimal control problem

$$\min_{u\in L^2(\Omega)} J(y,u) = \frac{\beta}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|a\nabla y - \underset{\sim}{g}_d\|^2_{(L^2(\Omega))^2} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \tag{1}$$

subject to

$$\begin{cases} -\nabla\cdot a\nabla y = f + u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $y_d \in L^2(\Omega)$ and $\underset{\sim}{g}_d \in (L^2(\Omega))^2$ are the desired states, $\alpha > 0$ is the regularization parameter, β and γ are nonnegative constants such that $\beta + \gamma > 0$, and a is the diffusion coefficient satisfying $C_0 \geqslant a \geqslant c_0 > 0$ for some positive constants C_0 and c_0. We note that $\beta = 0$ corresponds to problem with pure gradient state observation and $\gamma = 0$ corresponds to problem with pure state observation.

It is clear that the above optimization problem is coercive and strictly convex. By standard arguments (see [26]) we can prove that it admits a unique solution and the solution can be characterized by the following first order necessary (also sufficient) optimality conditions:

$$\begin{cases} -\nabla\cdot a\nabla y = f + u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega, \\ -\nabla\cdot a\nabla r = \beta(y - y_d) - \gamma\nabla\cdot a(a\nabla y - \underset{\sim}{g}_d) \quad \text{in } \Omega, \qquad r = 0 \quad \text{on } \partial\Omega, \\ \alpha u + r = 0 \quad \text{in } \Omega, \end{cases} \tag{3}$$

where $r \in H_0^1(\Omega)$ is the so-called adjoint state.

In the following we consider the mixed formulation for the above optimal control problems. Firstly, we consider the mixed variational form of the state equation by introducing the flux $\underset{\sim}{\varphi} = a\nabla y$: find $(y, \underset{\sim}{\varphi}) \in L^2(\Omega) \times H(\text{div};\Omega)$ such that (see [10])

$$\begin{cases} (a^{-1}\underset{\sim}{\varphi}, \underset{\sim}{\tau}) + (y, \text{div}\underset{\sim}{\tau}) = 0 \quad \forall \underset{\sim}{\tau} \in H(\text{div};\Omega), \\ -(\text{div}\underset{\sim}{\varphi}, s) = (f + u, s) \qquad \forall s \in L^2(\Omega), \end{cases} \tag{4}$$

with which we can formulate the following optimal control problems in mixed variational form (see, e.g., [11, 12, 17])

$$\min_{u\in L^2(\Omega)} J(y,u) = \frac{\beta}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|\underset{\sim}{\varphi} - \underset{\sim}{g}_d\|^2_{(L^2(\Omega))^2} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \quad \text{subject to (4)}. \tag{5}$$

Therefore, the first order optimality conditions consist of the following equations

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$$\left\{\begin{array}{llllllll}\alpha(u,z) & & & & +(r,z) & = 0 & \forall\, z\in L^2(\Omega), \\ & \gamma(\underset{\sim}{\varphi},\underset{\sim}{\psi}) & & (a^{-1}\underset{\sim}{\sigma},\underset{\sim}{\psi}) & +(r,\mathrm{div}\underset{\sim}{\psi}) & = (\gamma \underset{\sim}{g}_d,\underset{\sim}{\psi}) & \forall\, \underset{\sim}{\psi}\in H(\mathrm{div};\Omega), \\ & & \beta(y,q) & +(q,\mathrm{div}\underset{\sim}{\sigma}) & & = (\beta y_d, q) & \forall\, q\in L^2(\Omega), \\ & (a^{-1}\underset{\sim}{\varphi},\underset{\sim}{\tau}) & +(y,\mathrm{div}\underset{\sim}{\tau}) & & & = 0 & \forall\, \underset{\sim}{\tau}\in H(\mathrm{div};\Omega), \\ (u,s) & +(\mathrm{div}\underset{\sim}{\varphi},s) & & & & = (-f,s) & \forall\, s\in L^2(\Omega). \end{array}\right. \tag{6}$$

Let $H_h(\mathrm{div}) \subset H(\mathrm{div};\Omega)$ and $L_h^2 \subset L^2(\Omega)$ be some proper conforming mixed finite element spaces. Then we can formulate the discrete optimal control problem in mixed form (see, e.g., [11, 12, 17])

$$\min_{u_h\in L_h^2} J(y_h,u_h) = \frac{\beta}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|\underset{\sim}{\varphi}_h - \underset{\sim}{g}_d\|^2_{(L^2(\Omega))^2} + \frac{\alpha}{2}\|u_h\|^2_{L^2(\Omega)} \tag{7}$$

subject to

$$\left\{\begin{array}{ll} (a^{-1}\underset{\sim}{\varphi}_h,\underset{\sim}{\tau}_h) + (y_h,\mathrm{div}\underset{\sim}{\tau}_h) = 0 & \forall \underset{\sim}{\tau}_h\in H_h(\mathrm{div}), \\ -(\mathrm{div}\underset{\sim}{\varphi}_h, s_h) = (f+u_h, s_h) & \forall s_h\in L_h^2. \end{array}\right. \tag{8}$$

Similar to the continuous case, we can prove the existence of a unique solution for the above discretized optimal control problems. Moreover, a discretized system of first order optimality conditions can be derived analogue to (6):

$$\left\{\begin{array}{llllllll}\alpha(u_h,z_h) & & & & +(r_h,z_h) & = 0 & \forall\, z_h\in L_h^2, \\ & \gamma(\underset{\sim}{\varphi}_h,\underset{\sim}{\psi}_h) & & (a^{-1}\underset{\sim}{\sigma}_h,\underset{\sim}{\psi}_h) & +(r_h,\mathrm{div}\underset{\sim}{\psi}_h) & = \gamma(\underset{\sim}{g}_d,\underset{\sim}{\psi}_h) & \forall\, \underset{\sim}{\psi}_h\in H_h(\mathrm{div}), \\ & & \beta(y_h,q_h) & +(q_h,\mathrm{div}\underset{\sim}{\sigma}_h) & & = \beta(y_d, q_h) & \forall\, q_h\in L_h^2, \\ & (a^{-1}\underset{\sim}{\varphi}_h,\underset{\sim}{\tau}_h) & +(y_h,\mathrm{div}\underset{\sim}{\tau}_h) & & & = 0 & \forall\, \underset{\sim}{\tau}_h\in H_h(\mathrm{div}), \\ (u,s) & +(\mathrm{div}\underset{\sim}{\varphi}_h,s_h) & & & & = (-f,s_h) & \forall\, s_h\in L_h^2. \end{array}\right. \tag{9}$$

In this paper, we choose particularly $H_h(\mathrm{div})$ to be the Raviart-Thomas element space of lowest order, and L_h^2 to be the space of piecewise constants. We refer to [11] and [12] for the convergence of the mixed finite element discretization of above optimal control problem.

Let the set of basis functions of $H_h(\mathrm{div})$ be $\mathfrak{B}_h(\mathrm{div}) := \{\underset{\sim}{\phi}_i : i = 1,....,N\}$ and that of L_h^2 be $\mathfrak{B}_h(L^2) := \{\psi_k : k = 1,....,M\}$. By representing

$$u_h = \sum_{k=1}^{M} U_h(k)\psi_k,\ \underset{\sim}{\varphi}_h = \sum_{i=1}^{N}\Phi_h(i)\underset{\sim}{\phi}_i,\ y_h = \sum_{k=1}^{M} Y_h(k)\psi_k,\ \underset{\sim}{\sigma}_h = \sum_{i=1}^{N}\Sigma_h(i)\underset{\sim}{\phi}_i,\ r_h = \sum_{k=1}^{M} R_h(k)\psi_k, \tag{10}$$

we rewrite (9) in the following equivalent matrix form

$$\begin{pmatrix}\alpha B_h & 0 & 0 & 0 & B_h \\ 0 & \gamma\mathbf{B}_h & 0 & \hat{\mathbf{B}}_h & C_h^T \\ 0 & 0 & \beta B_h & C_h & 0 \\ 0 & \hat{\mathbf{B}}_h & C_h^T & 0 & 0 \\ B_h & C_h & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}U_h \\ \Phi_h \\ Y_h \\ \Sigma_h \\ R_h\end{pmatrix} = \begin{pmatrix}0 \\ G_h^d \\ Y_h^d \\ 0 \\ F_h\end{pmatrix}, \tag{11}$$

where $\mathbf{B}_h = ((\underset{\sim}{\phi}_i,\underset{\sim}{\phi}_j))_{N\times N}$, $\hat{\mathbf{B}}_h = ((a^{-1}\underset{\sim}{\phi}_i,\underset{\sim}{\phi}_j))_{N\times N}$, $B_h = ((\psi_k,\psi_t))_{M\times M}$, and $C_h = ((\psi_k,\mathrm{div}\underset{\sim}{\phi}_j))_{M\times N}$, namely the stiffness and (weighted) mass matrices on $H_h(\mathrm{div})$ and L_h^2, and $G_h^d = [\gamma(\underset{\sim}{g}_d,\underset{\sim}{\phi}_i)]_{N\times 1}$,

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$Y_h^d = [\beta(y_d, \psi_i)]_{M\times 1}$, and $F_h = [-(f, \psi_i)]_{M\times 1}$. In the sequel, we focus ourselves on solving the linear system with respect to

$$\mathcal{A}_{\alpha,\beta,\gamma,h} := \begin{pmatrix} \alpha B_h & 0 & 0 & 0 & B_h \\ 0 & \gamma \mathbf{B}_h & 0 & \hat{\mathbf{B}}_h & C_h^T \\ 0 & 0 & \beta B_h & C_h & 0 \\ 0 & \hat{\mathbf{B}}_h & C_h^T & 0 & 0 \\ B_h & C_h & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

2.2. *An optimal preconditioner of* (12)

A main result of this paper is as follows. Denote $\mathbf{Q}_h = ((\mathrm{div}\underset{\sim}{\phi}_i, \mathrm{div}\underset{\sim}{\phi}_j))_{N\times N}$, $\delta_1 = \max\{\gamma, \chi_{[0,+\infty)}(\beta - \gamma)(\alpha\beta)^{1/2}\}$ and $\delta_2 = \max\{\beta, \gamma\}$. Define $\mathcal{P}_{\alpha,\beta,\gamma,h}$ by

$$\mathcal{P}_{\alpha,\beta,\gamma,h} = \mathrm{diag}\left\{\frac{1}{\alpha}B_h^{-1},\ (\delta_1 \mathbf{B}_h + \alpha \mathbf{Q}_h)^{-1},\ \delta_2^{-1} B_h^{-1}, (\delta_1^{-1}\mathbf{B}_h + \delta_2^{-1}\mathbf{Q}_h)^{-1}, \alpha B_h^{-1}\right\}, \tag{13}$$

i.e.,

$$\mathcal{P}_{\alpha,\beta,\gamma,h} = \begin{cases} \mathrm{diag}\left\{\frac{1}{\alpha}B_h^{-1},\ (\gamma\mathbf{B}_h + \alpha\mathbf{Q}_h)^{-1},\ \frac{1}{\gamma}B_h^{-1}, \gamma(\mathbf{B}_h + \mathbf{Q}_h)^{-1}, \alpha B_h^{-1}\right\}, & \text{if } \gamma > \beta; \\ \mathrm{diag}\left\{\frac{1}{\alpha}B_h^{-1}, ((\alpha\beta)^{1/2}\mathbf{B}_h + \alpha\mathbf{Q}_h)^{-1}, \frac{1}{\beta}B_h^{-1}, ((\alpha\beta)^{-1/2}\mathbf{B}_h + \beta^{-1}\mathbf{Q}_h)^{-1}, \alpha B_h^{-1}\right\}, & \text{if } \beta \geq \gamma \text{ and } \alpha\beta \geq \gamma^2; \\ \mathrm{diag}\left\{\frac{1}{\alpha}B_h^{-1},\ (\gamma\mathbf{B}_h + \alpha\mathbf{Q}_h)^{-1},\ \frac{1}{\beta}B_h^{-1},\ (\gamma^{-1}\mathbf{B}_h + \beta^{-1}\mathbf{Q}_h)^{-1},\ \alpha B_h^{-1}\right\}, & \text{if } \beta \geq \gamma \text{ and } \alpha\beta < \gamma^2. \end{cases}$$

Then $\mathcal{P}_{\alpha,\beta,\gamma,h}$ is a robust preconditioner of $\mathcal{A}_{\alpha,\beta,\gamma,h}$. Indeed, define the condition number κ of $\mathcal{P}_{\alpha,\beta,\gamma,h}\mathcal{A}_{\alpha,\beta,\gamma,h}$ as

$$\kappa(\mathcal{P}_{\alpha,\beta,\gamma,h}\mathcal{A}_{\alpha,\beta,\gamma,h}) := \frac{\max|\lambda(\mathcal{P}_{\alpha,\beta,\gamma,h}\mathcal{A}_{\alpha,\beta,\gamma}h)|}{\min|\lambda(\mathcal{P}_{\alpha,\beta,\gamma,h}\mathcal{A}_{\alpha,\beta,\gamma,h})|},$$

and then we have the theorem below, the proof of which is postponed to next sections.

Theorem 1
The condition number $\kappa(\mathcal{P}_{\alpha,\beta,\gamma,h}\mathcal{A}_{\alpha,\beta,\gamma,h}))$ is bounded uniformly with respect to α, β, γ and h.

Note that the main work of carrying out the preconditioner $\mathcal{P}_{\alpha,\beta,\gamma,h}$ is to invert the matrices like $\gamma\mathbf{B}_h + \alpha\mathbf{Q}_h$ which is the stiffness matrix of the div*div system and etc.. We can employ the Hiptmair-Xu preconditioning technique ([24]) to transforming them to Poisson solvers.

Let H_h^1 be the linear finite element subspace of $H^1(\Omega)$ with basis functions $\mathfrak{B}_h$. Then curl $H_h^1 = \{\underset{\sim}{\tau}_h \in H_h(\mathrm{div}) : \mathrm{div}\underset{\sim}{\tau}_h = 0\}$, where curl $:= (\partial_{x_2}, -\partial_{x_1})$ is perpendicular to ∇. We will write it as $\nabla^\perp$ in the sequel. Denote

- $\mathbf{D}_{\varepsilon,\vartheta,h}$: the diagonal of $\varepsilon\mathbf{B}_h + \vartheta\mathbf{Q}_h$;
- $\mathbf{L}_h$ and $\mathbf{M}_h$: the stiffness matrix of inner product $(\nabla\cdot, \nabla\cdot)$ and the mass matrix of inner product $(\cdot,\cdot)$ corresponding to the basis $\mathfrak{B}_h$ on H_h^1, respectively;
- $\mathbf{P}_{\mathrm{div},h}$: the matrix representation of the nodal interpolation operator Π_h^{div} from $(H_h^1)^2$ to $H_h(\mathrm{div})$ corresponding to $\mathfrak{B}_h$ and $\mathfrak{B}_h(\mathrm{div})$;
- $\mathbf{G}_h$: the matrix representation of operator $\nabla^\perp : H_h^1 \mapsto H_h(\mathrm{div})$ corresponding to the basis $\mathfrak{B}_h$ and $\mathfrak{B}_h(\mathrm{div})$ where $\nabla^\perp := (\partial_{x_2}, -\partial_{x_1})$;
- $\mathbf{C}_h$: the discrete Laplacian (matrix) corresponding to the basis $\mathfrak{B}_h$ on H_h^1.

With this notation, define the Hiptmair-Xu preconditioner as (see, [24])

$$\mathbf{R}_{\varepsilon,\vartheta,h} := (\mathbf{D}_{\varepsilon,\vartheta,h})^{-1} + \mathbf{P}_{\mathrm{div},h}(\vartheta\mathbf{L}_h + \varepsilon\mathbf{M}_h)^{-1}\mathbf{P}_{\mathrm{div},h}^T + \varepsilon^{-1}\mathbf{G}_h(-\mathbf{C}_h)^{-1}\mathbf{G}_h^T. \tag{14}$$

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Lemma 2
([24]) The condition number of $\mathbf{R}_{\varepsilon,\vartheta,h}(\varepsilon\mathbf{B}_h + \vartheta\mathbf{Q}_h)$ is uniformly bounded with respect to ε, ϑ and h.

In real applications, the exact inverse of the Laplacian matrices can be replaced by some norm equivalent solvers, such as multigrid solvers or domain decomposition solvers and as mentioned in [24], the theoretical results in Lemma 2 still hold. We state this exactly in the following corollary. Denote $P_{\vartheta,\varepsilon}$ and P_C the spectral equivalent preconditioners of $\vartheta\mathbf{L}_h + \varepsilon\mathbf{M}_h$ and $-\mathbf{C}_h$ respectively. Let

$$\tilde{\mathbf{R}}_{\varepsilon,\vartheta,h} := (\mathbf{D}_{\varepsilon,\vartheta,h})^{-1} + \mathbf{P}_{\text{div},h}P_{\vartheta,\varepsilon}\mathbf{P}^T_{\text{div},h} + \varepsilon^{-1}\mathbf{G}_hP_C\mathbf{G}_h^T.$$

Corollary 3
The condition number of $\tilde{\mathbf{R}}_{\varepsilon,\vartheta,h}(\varepsilon\mathbf{B}_h + \vartheta\mathbf{Q}_h)$ is uniformly bounded with respect to ε, ϑ and h.

We can inverte $\gamma\mathbf{B}_h + \alpha\mathbf{Q}_h$ and other matrices optimally by the aid of the Hiptmair-Xu preconditioner. Moreover, we can just use Hiptmair-Xu preconditioner in the place of, e.g., $(\gamma\mathbf{B}_h + \alpha\mathbf{Q}_h)^{-1}$. Namely, we have another preconditioner below.

Define $\mathcal{P}'_{\alpha,\beta,\gamma,h}$ by

$$\mathcal{P}'_{\alpha,\beta,\gamma,h} = \text{diag}\left\{\frac{1}{\alpha}B_h^{-1},\ \mathbf{R}_{\delta_1,\alpha,h},\ \delta_2^{-1}B_h^{-1}, \mathbf{R}_{\delta_1^{-1},\delta_2^{-1},h}, \alpha B_h^{-1}\right\}. \tag{15}$$

The theorem below follows from Theorem 1 and Lemma 2.

Theorem 4
The condition number $\kappa(\mathcal{P}'_{\alpha,\beta,\gamma,h}\mathcal{A}_{\alpha,\beta,\gamma,h}))$ is bounded uniformly with respect to α, β, γ and h.

2.3. *A reduced system and its preconditioning*

By eliminating the control variable u, the original problem (6) can be rewritten formally to the following linear system of smaller size

$$\begin{pmatrix} \gamma\underset{\sim}{\text{Id}} & 0 & \hat{\underset{\sim}{\text{Id}}} & \text{div}^* \\ 0 & \beta\text{Id} & \text{div} & 0 \\ \hat{\underset{\sim}{\text{Id}}}^* & \text{div}^* & 0 & 0 \\ \text{div} & 0 & 0 & -\frac{1}{\alpha}\text{Id} \end{pmatrix}\begin{pmatrix} \underset{\sim}{\varphi} \\ y \\ \underset{\sim}{\sigma} \\ r \end{pmatrix} = \begin{pmatrix} \gamma\underset{\sim}{g}_d \\ \beta y_d \\ 0 \\ -f \end{pmatrix}, \tag{16}$$

where we write the variational problem in the formal operator form. Here Id and $\underset{\sim}{\text{Id}}$ denote the identity operators for scalar and vector, $\hat{\underset{\sim}{\text{Id}}}$ denote the operator associated with $(a^{-1}\cdot,\cdot)$ term and div denotes the divergence operator with div* its adjoint.

By introducing finite element spaces with certain basis functions, we generate the linear system corresponding to the discretization of (16) as

$$\begin{pmatrix} \gamma\mathbf{B}_h & 0 & \hat{\mathbf{B}}_h & C_h^T \\ 0 & \beta B_h & C_h & 0 \\ \hat{\mathbf{B}}_h & C_h^T & 0 & 0 \\ C_h & 0 & 0 & -\frac{1}{\alpha}B_h \end{pmatrix}\begin{pmatrix} \Phi_h \\ Y_h \\ \Sigma_h \\ R_h \end{pmatrix} = \begin{pmatrix} \gamma G_h^d \\ \beta Y_h^d \\ 0 \\ F_h \end{pmatrix}. \tag{17}$$

Denote

$$\mathcal{A}^r_{\alpha,\beta,\gamma,h} := \begin{pmatrix} \gamma\mathbf{B}_h & 0 & \hat{\mathbf{B}}_h & C_h^T \\ 0 & \beta B_h & C_h & 0 \\ \hat{\mathbf{B}}_h & C_h^T & 0 & 0 \\ C_h & 0 & 0 & -\frac{1}{\alpha}B_h \end{pmatrix}. \tag{18}$$

Similarly, we can present a preconditioner for $\mathcal{A}^r_{\alpha,\beta,\gamma,h}$. Define $\mathcal{P}^r_{\alpha,\beta,\gamma,h}$ as follows:

$$\mathcal{P}^r_{\alpha,\beta,\gamma,h} := \text{diag}\left\{(\delta_1\mathbf{B}_h + \alpha\mathbf{Q}_h)^{-1},\ \delta_2^{-1}B_h^{-1}, (\delta_1^{-1}\mathbf{B}_h + \delta_2^{-1}\mathbf{Q}_h)^{-1}, \alpha B_h^{-1}\right\}, \tag{19}$$

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i.e.,

$$\mathcal{P}^r_{\alpha,\beta,\gamma,h} = \begin{cases} \operatorname{diag}\left\{(\gamma\mathbf{B}_h+\alpha\mathbf{Q}_h)^{-1},\ \frac{1}{\gamma}B_h^{-1},\ \gamma(\mathbf{B}_h+\mathbf{Q}_h)^{-1},\ \alpha B_h^{-1}\right\}, & \text{if } \gamma>\beta; \\ \operatorname{diag}\left\{((\alpha\beta)^{1/2}\mathbf{B}_h+\alpha\mathbf{Q}_h)^{-1},\ \frac{1}{\beta}B_h^{-1},\ ((\alpha\beta)^{-1/2}\mathbf{B}_h+\beta^{-1}\mathbf{Q}_h)^{-1},\ \alpha B_h^{-1}\right\}, & \text{if } \beta\geq\gamma \text{ and } \alpha\beta\geq\gamma^2; \\ \operatorname{diag}\left\{(\gamma\mathbf{B}_h+\alpha\mathbf{Q}_h)^{-1},\ \frac{1}{\beta}B_h^{-1},(\gamma^{-1}\mathbf{B}_h+\beta^{-1}\mathbf{Q}_h)^{-1},\ \alpha B_h^{-1}\right\}, & \text{if } \beta\geq\gamma \text{ and } \alpha\beta<\gamma^2. \end{cases}$$

Then similar to Theorem 1, we have the theorem below.

Theorem 5
The condition number $\kappa(\mathcal{P}^r_{\alpha,\beta,\gamma,h}\mathcal{A}^r_{\alpha,\beta,\gamma,h})$ is bounded uniformly with respect to α, β, γ and h.

Again, we can use Hiptmair-Xu preconditioner to take the place where inversion of $\mathbf{Q}_h$ needed. Define

$$\mathcal{P}^r_{\alpha,\beta,\gamma,h}{}' := \operatorname{diag}\left\{\mathbf{R}_{\delta_1,\alpha,h},\ \delta_2^{-1}B_h^{-1}, \mathbf{R}_{\delta_1^{-1},\delta_2^{-1},h}, \alpha B_h^{-1}\right\}. \tag{20}$$

The theorem below follows from Lemma 2 and Theorem 5.

Theorem 6
The condition number $\kappa(\mathcal{P}^r_{\alpha,\beta,\gamma,h}{}'\mathcal{A}^r_{\alpha,\beta,\gamma,h})$ is bounded uniformly with respect to α, β, γ and h.

Remark 7
According to Corollay 3, in real application, the block of the form $\mathbf{R}_{\varepsilon,\vartheta,h}$ can be replaced by $\tilde{\mathbf{R}}_{\varepsilon,\vartheta,h}$ which makes the preconditioners more practical and the theoretical results in Theorem 4 and Theorem 6 hold as well.

3. PROOF OF THE ROBUST OPTIMALITY OF THE PRECONDITIONER

In this section, we prove Theorem 1 by presenting stability analysis for the relevant continuous and discretized systems. Theorem 5 is proved the same way. Theorems 4 and 6 follow immediately.

3.1. *Stability analysis of the system* (6)

Firstly, we rewrite (6) in the general form

$$\begin{cases} \alpha(u,z) & & & & +(r,z) & =(l,z) & \forall\, z\in L^2(\Omega), \\ & \gamma(\underset{\sim}{\varphi},\underset{\sim}{\psi}) & & (a^{-1}\underset{\sim}{\sigma},\underset{\sim}{\psi}) & +(r,\operatorname{div}\underset{\sim}{\psi}) & =(\underset{\sim}{f},\underset{\sim}{\psi}) & \forall\, \underset{\sim}{\psi}\in H(\operatorname{div};\Omega), \\ & & \beta(y,q) & +(q,\operatorname{div}\underset{\sim}{\sigma}) & & =(g,q) & \forall\, q\in L^2(\Omega), \\ & (a^{-1}\underset{\sim}{\varphi},\underset{\sim}{\tau}) & +(y,\operatorname{div}\underset{\sim}{\tau}) & & & =(\underset{\sim}{h},\underset{\sim}{\tau}) & \forall\, \underset{\sim}{\tau}\in H(\operatorname{div};\Omega), \\ (u,s) & +(\operatorname{div}\underset{\sim}{\varphi},s) & & & & =(j,s) & \forall\, s\in L^2(\Omega). \end{cases} \tag{21}$$

The stability of the system is constructed as the theorem below.

Theorem 8
Given $(l,\underset{\sim}{f},g,\underset{\sim}{h},j)\in W'_{\alpha,\beta,\gamma}$, there exists a unique $(u,\underset{\sim}{\varphi},y,\underset{\sim}{\sigma},r)\in W_{\alpha,\beta,\gamma}$, such that (21) holds, and

$$\|(u,\underset{\sim}{\varphi},y,\underset{\sim}{\sigma},r)\|_{W_{\alpha,\beta,\gamma}} \eqsim \|(l,\underset{\sim}{f},g,\underset{\sim}{h},j)\|_{W'_{\alpha,\beta,\gamma}} := \sup_{(z,\underset{\sim}{\psi},q,\underset{\sim}{\tau},s)\in W_{\alpha,\beta,\gamma}\setminus\{\mathbf{0}\}} \frac{(l,z)+(\underset{\sim}{f},\underset{\sim}{\psi})+(g,q)+(\underset{\sim}{h},\underset{\sim}{\tau})+(j,s)}{\|(u,\underset{\sim}{\varphi},y,\underset{\sim}{\sigma},r)\|_{W_{\alpha,\beta,\gamma}}}, \tag{22}$$

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where $W_{\alpha,\beta,\gamma}$ is defined by $W_{\alpha,\beta,\gamma} := U_{\alpha,\beta,\gamma} \times V_{\alpha,\beta,\gamma}$, whereas

$$U_{\alpha,\beta,\gamma} = \alpha^{1/2}L^2(\Omega) \times \left[\delta_1^{1/2}L^2(\Omega) \cap \alpha^{1/2}H(\text{div};\Omega)\right] \times \delta_2^{1/2}L^2(\Omega),$$
$$V_{\alpha,\beta,\gamma} = \left[\delta_1^{-1/2}L^2(\Omega) \cap \delta_2^{-1/2}H(\text{div};\Omega)\right] \times \alpha^{-1/2}L^2(\Omega),$$

i.e.,

(1) if $\gamma > \beta$, then

$$U_{\alpha,\beta,\gamma} = \alpha^{1/2}L^2(\Omega) \times \left[\gamma^{1/2}L^2(\Omega) \cap \alpha^{1/2}H(\text{div};\Omega)\right] \times \gamma^{1/2}L^2(\Omega),$$
$$V_{\alpha,\beta,\gamma} = \left[\gamma^{-1/2}L^2(\Omega) \cap \gamma^{-1/2}H(\text{div};\Omega)\right] \times \alpha^{-1/2}L^2(\Omega).$$

(2) if $\beta \geq \gamma$ and $\alpha\beta \geq \gamma^2$, then

$$U_{\alpha,\beta,\gamma} = \alpha^{1/2}L^2(\Omega) \times \left[(\alpha\beta)^{1/4}L^2(\Omega) \cap \alpha^{1/2}H(\text{div};\Omega)\right] \times \beta^{1/2}L^2(\Omega),$$
$$V_{\alpha,\beta,\gamma} = \left[(\alpha\beta)^{-1/4}L^2(\Omega) \cap \beta^{-1/2}H(\text{div};\Omega)\right] \times \alpha^{-1/2}L^2(\Omega).$$

(3) if $\beta \geq \gamma$ and $\alpha\beta < \gamma^2$, then

$$U_{\alpha,\beta,\gamma} = \alpha^{1/2}L^2(\Omega) \times \left[\gamma^{1/2}L^2(\Omega) \cap \alpha^{1/2}H(\text{div};\Omega)\right] \times \beta^{1/2}L^2(\Omega),$$
$$V_{\alpha,\beta,\gamma} = \left[\gamma^{-1/2}L^2(\Omega) \cap \beta^{-1/2}H(\text{div};\Omega)\right] \times \alpha^{-1/2}L^2(\Omega).$$

We postpone the proof of Theorem 8 after some technical lemmas.

Lemma 9
[48, Sec. 10] There exists orthogonal decomposition of $H(\text{div};\Omega)$, which reads

$$H(\text{div};\Omega) = \nabla^{\perp}H^1(\Omega) \oplus (\nabla^{\perp}H^1(\Omega))^{\perp},$$

where $(\nabla^{\perp}H^1(\Omega))^{\perp}$ is orthogonal to $\nabla^{\perp}H^1(\Omega)$ in both the $L^2_\rho(\Omega)$ inner product for any positive weight function ρ and the $H(\text{div};\Omega)$ product.

Furthermore, for $\underset{\sim}{\varphi} \in H(\text{div};\Omega)$, if $\underset{\sim}{\varphi} = \underset{\sim}{\varphi}_1 + \underset{\sim}{\varphi}_2$ with $\underset{\sim}{\varphi}_1 \in \nabla^{\perp}H^1(\Omega)$, and $\underset{\sim}{\varphi}_2 \in (\nabla^{\perp}H^1(\Omega))^{\perp}$, then $\text{div}\underset{\sim}{\tau}_1 = 0$ and $c_\rho(\|\underset{\sim}{\tau}_1\|_\rho^2 + \|\text{div}\underset{\sim}{\tau}_2\|_0^2) \leq \|\underset{\sim}{\tau}\|^2_{H(\text{div};\Omega)} \leq C_\rho(\|\underset{\sim}{\tau}_1\|_\rho^2 + \|\text{div}\underset{\sim}{\tau}_2\|_0^2)$, i.e., $c_\rho(\|\underset{\sim}{\tau}_1\|_\rho^2 + \|\text{div}\underset{\sim}{\tau}\|_0^2) \leq \|\underset{\sim}{\tau}\|^2_{H(\text{div};\Omega)} \leq C_\rho(\|\underset{\sim}{\tau}_1\|_\rho^2 + \|\text{div}\underset{\sim}{\tau}\|_0^2)$ with c_ρ and C_ρ uniformly in $H(\text{div};\Omega)$.

In the sequel, for $\underset{\sim}{\varphi}, \underset{\sim}{\tau} \in H(\text{div};\Omega)$, we always denote their decompositions by $\underset{\sim}{\varphi} = \underset{\sim}{\varphi}_1 + \underset{\sim}{\varphi}_2$ and $\underset{\sim}{\tau} = \underset{\sim}{\tau}_1 + \underset{\sim}{\tau}_2$, with $\underset{\sim}{\varphi}_1, \underset{\sim}{\tau}_1 \in \nabla^{\perp}H^1(\Omega)$, and $\underset{\sim}{\varphi}_2, \underset{\sim}{\tau}_2 \in (\nabla^{\perp}H^1(\Omega))^{\perp}$.

Lemma 10
For each $y \in L^2(\Omega)$, there exist $\underset{\sim}{\tau}_y \in H(\text{div};\Omega)$ such that,

$$\text{div}\underset{\sim}{\tau}_y = y \quad \text{and} \quad \|\underset{\sim}{\tau}_y\|_0 \leq C_\Omega\|y\|_0,$$

where C_Ω is a positive constant uniform for $L^2(\Omega)$ and $H(\text{div},\Omega)$.

Proof
We prove the lemma by a constructive approach. Let $(\underset{\sim}{p}, \hat{u}) \in H(\text{div};\Omega) \times L^2(\Omega)$ be such that

$$\begin{cases} (\underset{\sim}{p}, \underset{\sim}{q}) + (\hat{u}, \text{div}\underset{\sim}{q}) &= 0 \qquad \forall\, \underset{\sim}{q} \in H(\text{div};\Omega), \\ \qquad (\text{div}\underset{\sim}{p}, \hat{v}) &= (y, \hat{v}) \quad \forall\, \hat{v} \in L^2(\Omega). \end{cases}$$

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Then we have

$$\|\underset{\sim}{p}\|_{H(\mathrm{div};\Omega)} + \|\hat{u}\|_0 \le C_\Omega \|y\|_0$$

for some constant C_Ω. Setting $\underset{\sim}{\tau}_y = \underset{\sim}{p}$ we obtain the desired result. □

Proof of Theorem 8 Define bilinear forms

$$\mathcal{A}((u,\underset{\sim}{\varphi},y),(z,\underset{\sim}{\psi},q)) := \alpha(u,z) + \gamma(\underset{\sim}{\varphi},\underset{\sim}{\psi}) + \beta(y,q)$$

for any $(u,\underset{\sim}{\varphi},y),(z,\underset{\sim}{\psi},q) \in L^2(\Omega)\times H(\mathrm{div};\Omega)\times L^2(\Omega)$ and

$$\mathcal{B}((z,\underset{\sim}{\psi},q),(\underset{\sim}{\tau},s)) := (a^{-1}\underset{\sim}{\psi},\underset{\sim}{\tau}) + (q,\mathrm{div}\underset{\sim}{\tau}) + (z,s) + (\mathrm{div}\underset{\sim}{\psi},s)$$

for any $(z,\underset{\sim}{\psi},q) \in L^2(\Omega)\times H(\mathrm{div};\Omega)\times L^2(\Omega)$ and $(\underset{\sim}{\tau},s) \in H(\mathrm{div};\Omega)\times L^2(\Omega)$. Then problem (6) can be reformulated as the following saddle point problem: Find $((u,\underset{\sim}{\varphi},y),(\underset{\sim}{\sigma},r)) \in (L^2(\Omega)\times H(\mathrm{div};\Omega)\times L^2(\Omega))\times(H(\mathrm{div};\Omega)\times L^2(\Omega))$ such that

$$\begin{cases} \mathcal{A}((u,\underset{\sim}{\varphi},y),(z,\underset{\sim}{\psi},q)) + \mathcal{B}((z,\underset{\sim}{\psi},q),(\underset{\sim}{\sigma},r)) & = (l,z) + (\underset{\sim}{f},\underset{\sim}{\psi}) + (g,q), \\ \mathcal{B}((u,\underset{\sim}{\varphi},y),(\underset{\sim}{\tau},s)) & = (\underset{\sim}{h},\underset{\sim}{\tau}) + (j,s) \end{cases} \tag{23}$$

holds for any $(z,\underset{\sim}{\psi},q) \in L^2(\Omega)\times H(\mathrm{div};\Omega)\times L^2(\Omega)$ and any $(\underset{\sim}{\tau},s) \in H(\mathrm{div};\Omega)\times L^2(\Omega)$. We follow the standard approach (see [10]) to prove the theorem case by case. Actually it is quite direct to verify the continuity of $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ for the cases. Then, define $\ker(\mathcal{B}) := \Big\{(z,\underset{\sim}{\psi},q) \in U : \mathcal{B}((z,\underset{\sim}{\psi},q),(\underset{\sim}{\tau},s)) = 0,\ \forall\,(\underset{\sim}{\tau},s) \in V\Big\}$. For any $(u,\underset{\sim}{\varphi},y) \in \ker(\mathcal{B})$ it holds that

$$\begin{cases} (a^{-1}\underset{\sim}{\varphi},\underset{\sim}{\tau}) + (y,\mathrm{div}\underset{\sim}{\tau}) = 0,\ \ \forall\underset{\sim}{\tau} \in H(\mathrm{div};\Omega), \\ (u,s) + (\mathrm{div}\underset{\sim}{\varphi},s) = 0,\ \ \forall s \in L^2(\Omega), \end{cases} \tag{24}$$

which implies $\mathrm{div}\underset{\sim}{\varphi} = -u$. We are going to check the coercivity of $\mathcal{A}(\cdot,\cdot)$ on $ker(\mathcal{B})$ and the inf-sup condition case by case. For simplicity, we drop the subscript α,β,γ below without ambiguity.

Case I: $\gamma > \beta$ Given $(u,\underset{\sim}{\varphi},y) \in \ker(\mathcal{B})$, set $\underset{\sim}{\tau} = \underset{\sim}{\tau}_y$ as in (24), then by Lemma 10 we have

$$(y,y) = (y,\mathrm{div}\underset{\sim}{\tau}_y) = -(a^{-1}\underset{\sim}{\varphi},\underset{\sim}{\tau}_y) \leqslant c_0^{-1}\|\underset{\sim}{\varphi}\|_0\|\underset{\sim}{\tau}_y\|_0 \leqslant c_0^{-1}C_\Omega\|\underset{\sim}{\varphi}\|_0\|y\|_0.$$

Hence, $\|y\|_0 \leqslant c_0^{-1}C_\Omega\|\underset{\sim}{\varphi}\|_0$. Therefore,

$$\begin{aligned} \alpha\|u\|_0^2 + \gamma\|\underset{\sim}{\varphi}\|_0^2 + \alpha\|\mathrm{div}\underset{\sim}{\varphi}\|_0^2 + \gamma\|y\|_0^2 &\le \alpha\|u\|_0^2 + \gamma\|\underset{\sim}{\varphi}\|_0^2 + \alpha\|u\|_0^2 + \gamma c_0^{-2}C_\Omega^2\|\underset{\sim}{\varphi}\|_0^2 \\ &\le \max\{1 + c_0^{-2}C_\Omega^2, 2\}(\alpha\|u\|_0^2 + \gamma\|\underset{\sim}{\varphi}\|_0^2) \\ &\le \max\{1 + c_0^{-2}C_\Omega^2, 2\}\mathcal{A}((u,\underset{\sim}{\varphi},y),(u,\underset{\sim}{\varphi},y)), \end{aligned}$$

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which implies the coercivity of $\mathcal{A}$ on ker($\mathcal{B}$).

Now, given $(\underset{\sim}{\tau}, s) \in V$, set $y = \gamma^{-1}\mathrm{div}\underset{\sim}{\tau}$, $\underset{\sim}{\varphi} = \gamma^{-1}\underset{\sim}{\tau}_1$, and $u = \alpha^{-1}s$, then by Lemma 9 we have

$$\begin{aligned}
\mathcal{B}((u,\underset{\sim}{\varphi},y),(\underset{\sim}{\tau},s)) &= (a^{-1}\underset{\sim}{\varphi},\underset{\sim}{\tau}) + (y,\mathrm{div}\underset{\sim}{\tau}) + (u,s) + (\mathrm{div}\underset{\sim}{\varphi},s)\\
&= \gamma^{-1}(a^{-1}\underset{\sim}{\tau}_1,\underset{\sim}{\tau}) + (\gamma^{-1}\mathrm{div}\underset{\sim}{\tau},\mathrm{div}\underset{\sim}{\tau}) + \alpha^{-1}(s,s)\\
&= (\gamma^{-1}(\|\underset{\sim}{\tau}_1\|_\rho^2 + \|\mathrm{div}\underset{\sim}{\tau}\|_0^2) + \alpha^{-1}\|s\|_0^2)\\
&\geq \min\{C_\rho^{-1},1\}(\gamma^{-1}(\|\underset{\sim}{\tau}\|_0^2 + \|\mathrm{div}\underset{\sim}{\tau}\|_0^2) + \alpha^{-1}\|s\|_0^2)\\
&= \min\{C_\rho^{-1},1\}\|(\underset{\sim}{\tau},s)\|_V^2
\end{aligned}$$

and

$$\alpha\|u\|_0^2 + \gamma\|\underset{\sim}{\varphi}\|_0^2 + \alpha\|\mathrm{div}\underset{\sim}{\varphi}\|_0^2 + \gamma\|y\|_0^2 = \alpha^{-1}\|s\|_0^2 + \gamma^{-1}\|\underset{\sim}{\tau}_1\|_0^2 + \gamma^{-1}\|\mathrm{div}\underset{\sim}{\tau}\|_0^2 \leq \max\{1, C_0c_\rho^{-1}, c_\rho^{-1}\}\|(\underset{\sim}{\tau},s)\|_V^2.$$

Therefore, we have

$$\sup_{(u,\underset{\sim}{\varphi},y)\in U}\frac{\mathcal{B}((u,\underset{\sim}{\varphi},y),(\underset{\sim}{\tau},s))}{\|(u,\underset{\sim}{\varphi},y)\|_U} \geqslant \frac{\min\{C_\rho^{-1},1\}}{\sqrt{\max\{1,C_0c_\rho^{-1},c_\rho^{-1}\}}}\|(\underset{\sim}{\tau},s)\|_V.$$

The inf-sup condition is proved.

Case II. $\gamma \leq \beta$**, and** $\alpha\beta \geq \gamma^2$ Again, for $\forall(u,\underset{\sim}{\varphi},y) \in \ker(\mathcal{B})$, set $\underset{\sim}{\tau} = \underset{\sim}{\varphi}$ in (24), then we have

$$(\alpha\beta)^{1/2}(a^{-1}\underset{\sim}{\varphi},\underset{\sim}{\varphi}) = -(\alpha\beta)^{1/2}(y,\mathrm{div}\underset{\sim}{\varphi}) = (\alpha\beta)^{1/2}(y,u) \leqslant 2(\alpha\|u\|_0^2 + \beta\|y\|_0^2),$$

which in turn implies

$$(\alpha\beta)^{1/2}(\underset{\sim}{\varphi},\underset{\sim}{\varphi}) \leqslant 2C_0(\alpha\|u\|_0^2 + \beta\|y\|_0^2).$$

Therefore,

$$\begin{aligned}
\alpha\|u\|_0^2 + (\alpha\beta)^{1/2}\|\underset{\sim}{\varphi}\|_0^2 + \alpha\|\mathrm{div}\underset{\sim}{\varphi}\|_0^2 + \beta\|y\|_0^2 &\leq \alpha\|u\|_0^2 + 2C_0(\alpha\|u\|_0^2 + \beta\|y\|_0^2) + \alpha\|u\|_0^2 + \beta\|y\|_0^2\\
&\leq \max\{2+2C_0, 1+2C_0\}(\alpha\|u\|_0^2 + \beta\|y\|_0^2)\\
&\leq \max\{2+2C_0, 1+2C_0\}\mathcal{A}((u,\underset{\sim}{\varphi},y),(u,\underset{\sim}{\varphi},y)).
\end{aligned}$$

This gives the coercivity of $\mathcal{A}$ on ker($\mathcal{B}$).

Now, given $(\underset{\sim}{\tau}, s) \in V$, set $y = \beta^{-1}\mathrm{div}\underset{\sim}{\tau} - (\alpha\beta)^{-1/2}s$, $\underset{\sim}{\varphi} = (\alpha\beta)^{-1/2}\underset{\sim}{\tau}$ and $u = \alpha^{-1}s$, and we are led to

$$\begin{aligned}
&\mathcal{B}((u,\underset{\sim}{\varphi},y),(\underset{\sim}{\tau},s))\\
= \;& (a^{-1}\underset{\sim}{\varphi},\underset{\sim}{\tau}) + (y,\mathrm{div}\underset{\sim}{\tau}) + (u,s) + (\mathrm{div}\underset{\sim}{\varphi},s)\\
= \;& (\alpha\beta)^{-1/2}(a^{-1}\underset{\sim}{\tau},\underset{\sim}{\tau}) + (\beta^{-1}\mathrm{div}\underset{\sim}{\tau} - (\alpha\beta)^{-1/2}s,\mathrm{div}\underset{\sim}{\tau}) + \alpha^{-1}(s,s) + (\alpha\beta)^{-1/2}(s,\mathrm{div}\underset{\sim}{\tau})\\
\geq \;& \min\{C_0^{-1},1\}((\alpha\beta)^{-1/2}\|\underset{\sim}{\tau}\|_0^2 + \beta^{-1}\|\mathrm{div}\underset{\sim}{\tau}\|_0^2 + \alpha^{-1}\|s\|_0^2) = \min\{C_0^{-1},1\}\|(\underset{\sim}{\tau},s)\|_V^2
\end{aligned}$$

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and

$$
\begin{aligned}
& \alpha\|u\|_0^2 + (\alpha\beta)^{1/2}\|\underset{\sim}{\varphi}\|_0^2 + \alpha\|\mathrm{div}\underset{\sim}{\varphi}\|_0^2 + \beta\|y\|_0^2 \\
= \; & \alpha^{-1}\|s\|_0^2 + (\alpha\beta)^{-1/2}\|\underset{\sim}{\tau}\|_0^2 + \beta^{-1}\|\mathrm{div}\underset{\sim}{\tau}\|_0^2 + \beta\|\beta^{-1}\mathrm{div}\underset{\sim}{\tau} - (\alpha\beta)^{-1/2}s\|_0^2 \\
\leq \; & \alpha^{-1}\|s\|_0^2 + (\alpha\beta)^{-1/2}\|\underset{\sim}{\tau}\|_0^2 + \beta^{-1}\|\mathrm{div}\underset{\sim}{\tau}\|_0^2 + 2\beta(\beta^{-2}\|\mathrm{div}\underset{\sim}{\tau}\|_0^2 + (\alpha\beta)^{-1}\|s\|_0^2) \leq 3\|(\underset{\sim}{\tau}, s)\|_V^2.
\end{aligned}
$$

Therefore, it holds

$$
\sup_{(u,\underset{\sim}{\varphi},y)\in U} \frac{\mathcal{B}((u,\underset{\sim}{\varphi},y),(\underset{\sim}{\tau},s))}{\|(u,\underset{\sim}{\varphi},y)\|_U} \geqslant \frac{\min\{C_0^{-1},1\}}{\sqrt{3}}\|(\underset{\sim}{\tau},s)\|_V.
$$

This proves the inf-sup condition. Thus we finish the proof of the second case.

Case III. $\gamma \leq \beta$**, and** $\alpha\beta < \gamma^2$**.** For $\forall(u,\underset{\sim}{\varphi},y) \in \ker(\mathcal{B})$, by (24) we have

$$
\begin{aligned}
\alpha\|u\|_0^2 + \gamma\|\underset{\sim}{\varphi}\|_0^2 + \alpha\|\mathrm{div}\underset{\sim}{\varphi}\|_0^2 + \beta\|y\|_0^2 &= \alpha\|u\|_0^2 + \gamma\|\underset{\sim}{\varphi}\|_0^2 + \alpha\|u\|_0^2 + \beta\|y\|_0^2 \\
&\leq 2\mathcal{A}((u,\underset{\sim}{\varphi},y),(u,\underset{\sim}{\varphi},y)).
\end{aligned}
$$

This gives the coercivity desired.

Given $(\underset{\sim}{\tau}, s) \in V$, if we set $y = \beta^{-1}\mathrm{div}\underset{\sim}{\tau} - \gamma^{-1}s$, $\underset{\sim}{\varphi} = \gamma^{-1}\underset{\sim}{\tau}$ and $u = \alpha^{-1}s$, we can deduce

$$
\begin{aligned}
\mathcal{B}((u,\underset{\sim}{\varphi},y),(\underset{\sim}{\tau},s)) &= (a^{-1}\underset{\sim}{\varphi},\underset{\sim}{\tau}) + (y,\mathrm{div}\underset{\sim}{\tau}) + (u,s) + (\mathrm{div}\underset{\sim}{\varphi},s) \\
&= \gamma^{-1}(a^{-1}\underset{\sim}{\tau},\underset{\sim}{\tau}) + (\beta^{-1}\mathrm{div}\underset{\sim}{\tau} - \gamma^{-1}s,\mathrm{div}\underset{\sim}{\tau}) + \alpha^{-1}(s,s) + \gamma^{-1}(s,\mathrm{div}\underset{\sim}{\tau}) \\
&\geq \min\{C_0^{-1},1\}(\gamma^{-1}\|\underset{\sim}{\tau}\|_0^2 + \beta^{-1}\|\mathrm{div}\underset{\sim}{\tau}\|_0^2 + \alpha^{-1}\|s\|_0^2) = \min\{C_0^{-1},1\}\|(\underset{\sim}{\tau},s)\|_V^2
\end{aligned}
$$

and

$$
\begin{aligned}
& \alpha\|u\|_0^2 + \gamma\|\underset{\sim}{\varphi}\|_0^2 + \alpha\|\mathrm{div}\underset{\sim}{\varphi}\|_0^2 + \beta\|y\|_0^2 \\
= \; & \alpha^{-1}\|s\|_0^2 + \gamma^{-1}\|\underset{\sim}{\tau}\|_0^2 + \alpha\gamma^{-2}\|\mathrm{div}\underset{\sim}{\tau}\|_0^2 + \beta\|\beta^{-1}\mathrm{div}\underset{\sim}{\tau} - \gamma^{-1}s\|_0^2 \\
\leq \; & \alpha^{-1}\|s\|_0^2 + \gamma^{-1}\|\underset{\sim}{\tau}\|_0^2 + \beta^{-1}\|\mathrm{div}\underset{\sim}{\tau}\|_0^2 + 2\beta(\beta^{-2}\|\mathrm{div}\underset{\sim}{\tau}\|_0^2 + \gamma^{-2}\|s\|_0^2) \leq 3\|(\underset{\sim}{\tau},s)\|_V^2.
\end{aligned}
$$

Therefore,

$$
\sup_{(u,\underset{\sim}{\varphi},y)\in U} \frac{\mathcal{B}((u,\underset{\sim}{\varphi},y),(\underset{\sim}{\tau},s))}{\|(u,\underset{\sim}{\varphi},y)\|_U} \geqslant \frac{\min\{C_0^{-1},1\}}{\sqrt{3}}\|(\underset{\sim}{\tau},s)\|_V.
$$

This proves the inf-sup condition. We thus finish the proof of the third case.

Combining the above three cases we complete the proof of the theorem. □

3.2. Stability analysis of (9)

Define $U_{\alpha,\beta,\gamma,h}$ by the product of $L_h^2 \times H_h(\mathrm{div}) \times L_h^2$ equipped with the same topology as $U_{\alpha,\beta,\gamma}$ and $V_{\alpha,\beta,\gamma,h}$ by the product of $H_h(\mathrm{div}) \times L_h^2$ equipped with the same topology as $V_{\alpha,\beta,\gamma}$.

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Theorem 11
Assume the two items below are true:

1. there exists a $C > 0$, such that given $y_h \in L_h^2$, there exists a $\underset{\sim}{\tau}_{y,h} \in H_h(\mathrm{div})$, such that $\mathrm{div}\underset{\sim}{\tau}_{y,h} = y_h$ and $\|\underset{\sim}{\tau}_{y,h}\|_{0,\Omega} \leqslant C\|y_h\|_{0,\Omega}$;

2. there exists a $C > 0$, such that $\|\underset{\sim}{\tau}_h\|_{\mathrm{div},\Omega} \leqslant C\|\mathrm{div}\underset{\sim}{\tau}_h\|_{0,\Omega}$ for $\underset{\sim}{\tau}_h \in (\nabla^\perp H_h^1)^\perp$.

Then (9) induces an isomorphism between $W_{\alpha,\beta,\gamma,h} := U_{\alpha,\beta,\gamma,h} \times V_{\alpha,\beta,\gamma,h}$ to its dual.

The proof is the same as that of Theorem 8, and we omit it here.

Remark 12
The two assumptions of Theorem 8 hold for the Raviart-Thomas element space of lowest order ($H_h(\mathrm{div})$) and piecewise constants(L_h^2). (c.f. [1, 23].)

Proof of Theorem 1 Based on Theorem 11, for any $\mathbf{Z} \in \mathbb{R}^{3M+2N} \setminus \{\mathbf{0}\}$, we have

$$|\mathbf{Z}^t \mathcal{A}_{\alpha,\beta,\gamma,h}\mathbf{Z}| \leqslant C_1 \mathbf{Z}^t \mathcal{P}_{\alpha,\beta,\gamma,h}^{-1}\mathbf{Z}$$

and

$$\sup_{\mathbf{T}\in\mathbb{R}^{3M+2N}\setminus\{\mathbf{0}\}} \frac{\mathbf{T}^t \mathcal{A}_{\alpha,\beta,\gamma,h}\mathbf{Z}}{\sqrt{\mathbf{Z}^t \mathcal{P}_{\alpha,\beta,\gamma,h}^{-1}\mathbf{Z}}\sqrt{\mathbf{T}^t \mathcal{P}_{\alpha,\beta,\gamma,h}^{-1}\mathbf{T}}} \geqslant C_2.$$

This implies that $|\lambda(\mathcal{P}_{\alpha,\beta,\gamma,h}\mathcal{A}_{\alpha,\beta,\gamma,h})|$ is bounded from above and from below away from 0. This finishes the proof. □

4. NUMERICAL EXPERIMENTS

In this section we carry out several numerical experiments to confirm the efficiency of our proposed block diagonal preconditioners. We use minimum residual method (MINRES) to solve the symmetric and indefinite linear system. For different mesh size and regularization parameter, we list the condition numbers of preconditioned matrix $\mathcal{P}^r_{\alpha,\beta,\gamma,h}\mathcal{A}^r_{\alpha,\beta,\gamma,h}$, the iteration numbers for MINRES without preconditioner and with block diagonal preconditioner $\mathcal{P}^r_{\alpha,\beta,\gamma,h}$, and the iteration numbers for MINRES with block diagonal preconditioner $\mathcal{P}^r_{\alpha,\beta,\gamma,h}{}'$ (where the $\mathrm{div}^*\mathrm{div}$ subproblems are replaced by the Hiptmair-Xu preconditioners). For all the experiments we set the tolerance for the residual of MINRES algorithm as $1.0e-8$ and consider only $a = 1$ in (2).

In the following, we consider three numerical examples:

- the first one with only state observation, i.e. $\beta = 1$ and $\gamma = 0$;
- the second one with only gradient state observations, i,e. $\beta = 0$ and $\gamma = 1$;
- the last one with both state and gradient state observations, i.e. $\beta = 1$ and $\gamma = 1$.

The numerical results presented in the following part are the results associated with the reduced system (17). For the system (11), we do the same numerical tests with the corresponding preconditioners and they give the similar numerical performances as the reduced case. Hence, we omit them here for conciseness. Meanwhile, we also carry out the numerical experiments on non-convex domains like “L-”shape domains, and the numerical performances are almost the same. We omit them for conciseness as well.

Example 13
Let $\Omega = (0, 1)^2$, we set $\beta = 1$ and $\gamma = 0$, $y_d = \sin(\pi x_1)\sin(\pi x_2)$ and $f = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$.

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Table I. Condition number of $\mathcal{P}^r_{\alpha,\beta,\gamma,h}\mathcal{A}^r_{\alpha,\beta,\gamma,h}$ versus Dofs and α for Example 13.

α \ Nodes	41	145	545	2113
$1.0e-1$	2.8940	2.8903	2.8894	2.8891
$1.0e-2$	3.2155	3.2161	3.2162	3.2163
$1.0e-3$	3.2378	3.2532	3.2571	3.2581
$1.0e-4$	3.1938	3.2479	3.2627	3.2665
$1.0e-5$	3.0839	3.2111	3.2521	3.2636
$1.0e-6$	2.8782	3.1146	3.2223	3.2561
$1.0e-7$	2.2008	2.9312	3.1411	3.2314

Table II. Iteration number versus Dofs and α for Example 13 with direct MINRES.

α \ Nodes	41	145	545	2113	8321
$1.0e-1$	46	190	450	898	1788
$1.0e-2$	46	147	290	582	1164
$1.0e-3$	56	168	318	593	1146
$1.0e-4$	134	491	818	1334	2426
$1.0e-5$	173	923	3052	5452	6601
$1.0e-6$	126	568	2191	6296	12803
$1.0e-7$	92	285	928	2959	8716

Table III. Iteration number of PMinRes versus Dofs and α for Example 13 with preconditioner $\mathcal{P}^r_{\alpha,\beta,\gamma,h}$.

α \ Nodes	41	145	545	2113	8321	33025
$1.0e-1$	18	18	18	18	18	14
$1.0e-2$	22	22	21	20	20	18
$1.0e-3$	25	26	23	22	22	18
$1.0e-4$	30	30	29	26	26	26
$1.0e-5$	30	35	32	30	28	24
$1.0e-6$	34	38	38	36	34	32
$1.0e-7$	30	38	38	36	34	30

In the first example we do not consider gradient observation. It is clear that y approaches to y_d as $\alpha \to 0$ and thus u approaches to zero. We test this example with fixed α but increased number of nodes or with fixed number of nodes but decreased α, respectively. In Table I we list the condition numbers of the preconditioned matrix $\mathcal{P}^r_{\alpha,\beta,\gamma,h}\mathcal{A}^r_{\alpha,\beta,\gamma,h}$, with respect to the mesh size and the regularization parameter. We can observe the independency of the condition numbers with respect to the two parameters which show the robustness of the proposed preconditioner.

In Table II we give the iteration numbers of MINRES without any preconditioners. We can see that the iteration numbers grow as the mesh size decreases and the regularization parameter goes to zero. We also show in Table III the results with preconditioner $\mathcal{P}^r_{\alpha,\beta,\gamma,h}$, where the iteration numbers keep stable as the mesh size decreases. We observe a slightly dependency of the iteration numbers on the regularization parameter α that is due to the fact that our preconditioner is α-dependent and the stopping criteria of MINRES could be chosen as α-dependent, which is not realized in our numerical experiments. When evaluating the preconditioner one needs to solve $\mathrm{div}^*\mathrm{div}$ subproblems which may be costly when the number of DOFs is large, this can be alleviated by the Hiptmair-Xu preconditioner. In Table IV we give the results with the preconditioner $\mathcal{P}^r_{\alpha,\beta,\gamma,h}{}'$. Compared to Table III we have larger iteration numbers in this case but is paid back with cheaper computational cost.

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Table IV. Iteration number of PMinRes versus Dofs and α for Example 13 with preconditioner $\mathcal{P}^r_{\alpha,\beta,\gamma,h}{}'$.

α \ Nodes	41	145	545	2113	8321	33025
$1.0e-1$	42	70	78	82	82	84
$1.0e-2$	44	72	80	84	86	88
$1.0e-3$	44	74	84	88	90	92
$1.0e-4$	43	73	85	87	91	93
$1.0e-5$	45	61	75	81	85	87
$1.0e-6$	53	62	65	75	79	81
$1.0e-7$	47	91	65	65	73	77

Table V. Condition number of $\mathcal{P}^r_{\alpha,\beta,\gamma,h}\mathcal{A}^r_{\alpha,\beta,\gamma,h}$ versus Dofs and α for Example 14.

α \ Nodes	41	145	545	2113
$1.0e-1$	2.6180	2.6180	2.6180	2.6180
$1.0e-2$	2.6180	2.6180	2.6180	2.6180
$1.0e-3$	2.6180	2.6180	2.6180	2.6180
$1.0e-4$	2.6180	2.6180	2.6180	2.6180
$1.0e-5$	2.6180	2.6180	2.6180	2.6180
$1.0e-6$	2.6180	2.6180	2.6180	2.6180
$1.0e-7$	2.6180	2.6180	2.6180	2.6180

Table VI. Iteration number versus Dofs and α for Example 14 with direct MinRes.

α \ Nodes	41	145	545	2113	8321
$1.0e-1$	47	294	702	1446	2906
$1.0e-2$	63	420	878	1583	3080
$1.0e-3$	105	565	1434	2974	5784
$1.0e-4$	117	591	1282	3048	6874
$1.0e-5$	133	636	1485	2917	5830
$1.0e-6$	145	688	1575	3201	6133
$1.0e-7$	153	715	1611	3319	6556

Example 14
Let $\Omega = (0,1)^2$, we set $\beta = 0$ and $\gamma = 1$, $\underset{\sim}{g}_d = (\pi\cos(\pi x_1)\sin(\pi x_2), \pi\sin(\pi x_1)\cos(\pi x_2))$ and $f = 2\pi^2\sin(\pi x_1)\sin(\pi x_2)$.

In the second example we consider only gradient observations of the state. It is clear that ∇y approaches to $\underset{\sim}{g}_d$ as $\alpha \to 0$ and thus u approaches to zero. As in Example 13 we also test this example with fixed α but increased number of nodes or with fixed number of nodes but decreased α, respectively. We list the condition numbers of the preconditioned matrix $\mathcal{P}^r_{\alpha,\beta,\gamma,h}\mathcal{A}^r_{\alpha,\beta,\gamma,h}$ in Table V with respect to the mesh size and the regularization parameter. Although the condition numbers are almost the same for different cases as shown in Table V, they may differ if we consider more significant digits. We also list in Table VI, VII, VIII the iteration numbers of MINRES without preconditioners, with preconditioners $\mathcal{P}^r_{\alpha,\beta,\gamma,h}$ and $\mathcal{P}^r_{\alpha,\beta,\gamma,h}{}'$, resepctively. We can observe the similar phenomenon as in Example 13.

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Table VII. Iteration number of PMinRes versus Dofs and α for Example 14 with block diagonal preconditioner $\mathcal{P}^r_{\alpha,\beta,\gamma,h}$.

α \ Nodes	41	145	545	2113	8321	33025
$1.0e-1$	15	13	12	11	11	9
$1.0e-2$	17	17	15	13	13	12
$1.0e-3$	16	18	19	17	15	13
$1.0e-4$	12	15	18	17	16	14
$1.0e-5$	11	12	12	15	16	16
$1.0e-6$	11	11	11	10	11	13
$1.0e-7$	12	9	9	11	10	10

Table VIII. Iteration number of PMinRes versus Dofs and α for Example 14 with preconditioner ${\mathcal{P}^r_{\alpha,\beta,\gamma,h}}'$.

α \ Nodes	41	145	545	2113	8321	33025
$1.0e-1$	39	54	58	62	64	105
$1.0e-2$	33	49	56	58	62	111
$1.0e-3$	36	44	49	55	59	61
$1.0e-4$	37	55	52	47	51	113
$1.0e-5$	37	58	62	55	49	187
$1.0e-6$	36	58	63	60	57	109
$1.0e-7$	36	58	63	61	60	57

Table IX. Condition number of $\mathcal{P}^r_{\alpha,\beta,\gamma,h}\mathcal{A}^r_{\alpha,\beta,\gamma,h}$ versus Dofs and α for Example 15.

α \ Nodes	41	145	545	2113
$1.0e-1$	2.6949	2.6936	2.6932	2.6932
$1.0e-2$	2.6442	2.6441	2.6440	2.6440
$1.0e-3$	2.6215	2.6215	2.6215	2.6215
$1.0e-4$	2.6184	2.6184	2.6184	2.6184
$1.0e-5$	2.6181	2.6181	2.6181	2.6181
$1.0e-6$	2.6180	2.6180	2.6180	2.6180
$1.0e-7$	2.6180	2.6180	2.6180	2.6180

Example 15
Let $\Omega = (0,1)^2$, we set $\beta = \gamma = 1$, $y_d = \sin(\pi x_1)\sin(\pi x_2)$, $\underset{\sim}{g}_d = (\pi\cos(\pi x_1)\sin(\pi x_2),$ $\pi\sin(\pi x_1)\cos(\pi x_2))$ and $f = 2\pi^2\sin(\pi x_1)\sin(\pi x_2)$.

In the last example we consider problem with both state and gradient state observations. As in two previous examples we test this example with fixed α but increased number of nodes or with fixed number of nodes but decreased α, respectively. We list the condition numbers of the preconditioned matrix $\mathcal{P}^r_{\alpha,\beta,\gamma,h}\mathcal{A}^r_{\alpha,\beta,\gamma,h}$ in Table IX with respect to the mesh size and the regularization parameter. We also list in Table X, XI, XII the iteration numbers of MINRES without preconditioners, with preconditioners $\mathcal{P}^r_{\alpha,\beta,\gamma,h}$ and ${\mathcal{P}^r_{\alpha,\beta,\gamma,h}}'$, resepctively. We can observe the similar phenomenon as in previous two examples.

Table X. Iteration number versus Dofs and α for Example 15 with direct MinRes.

α \ Nodes	41	145	545	2113	8321
$1.0e-1$	46	180	393	790	1566
$1.0e-2$	85	238	387	676	1265
$1.0e-3$	92	317	828	1414	2169
$1.0e-4$	100	385	1004	2030	3702
$1.0e-5$	115	496	1197	2297	4412
$1.0e-6$	136	593	1395	2729	5138
$1.0e-7$	147	650	1539	3006	5750

Table XI. Iteration number of PMinRes versus Dofs and α for Example 15 with preconditioner $\mathcal{P}^r_{\alpha,\beta,\gamma,h}$.

α \ Nodes	41	145	545	2113	8321	33025
$1.0e-1$	18	16	16	16	14	12
$1.0e-2$	18	19	19	17	17	16
$1.0e-3$	16	20	21	18	16	14
$1.0e-4$	10	13	15	16	14	12
$1.0e-5$	7	7	10	11	12	10
$1.0e-6$	6	7	7	7	8	10
$1.0e-7$	5	5	6	6	6	6

Table XII. Iteration number of PMinRes versus Dofs and α for Example 15 with preconditioner $\mathcal{P}^r_{\alpha,\beta,\gamma,h}{}'$.

α \ Nodes	41	145	545	2113	8321	33025
$1.0e-1$	44	64	70	73	77	79
$1.0e-2$	38	51	60	65	67	70
$1.0e-3$	30	46	49	55	58	60
$1.0e-4$	29	49	48	49	51	54
$1.0e-5$	30	53	55	51	47	49
$1.0e-6$	28	53	58	56	52	47
$1.0e-7$	28	53	58	57	56	53

5. CONCLUDING REMARKS

In this paper, we study preconditioning the linear systems in saddle point formulation generated from the mixed finite element discretization of optimal control problems. We present optimal preconditioners for the systems, which are robust and uniformly optimal with respect to the parameters including mesh size. Both theoretical analysis and numerical verification are given. By the aid of the Hiptmair-Xu preconditioner, the work of carrying out the preconditioners can be transformed to Poisson solvers, and the cost can be shown optimal.

In current paper we only consider the two dimensional case, the extension to three diemensional case is also possible with slight modifications. In real application, the Poisson solver in the implementation of the Hiptmair-Xu preconditioner can be replaced by geometric or algebraic multigrid algorithms or other efficient preconditioners. This can reduce the computational time of our proposed preconditioner significantly and the possible increased MINRES iterations will be compensated by substantially saved computational time.

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