

Artificial Boundary Method for Calculating the Ship Wave Resistance ^{*†}

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Abstract: We consider the calculation of wave resistance for a ship moving at constant speed near the free surface of the water. This wave resistance can be calculated with a linearized steady potential model. To deal with the unboundedness of the physical domain in the potential flow problem, we introduce one vertical side as the upstream artificial boundary and two vertical sides as the downstream artificial boundaries. On the artificial boundaries, a sequence of high-order global artificial boundary conditions are given. Then the potential flow problem is reduced to a problem defined on a finite computational domain, which is equivalent to a

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variational problem. After solving the variational problem by the finite element method, we obtain the numerical approximation of the potential flow around the ship, which is used to calculate the wave resistance by formula presented in this paper. The numerical examples show the accuracy and efficiency of the proposed numerical scheme given in this paper .

Key words. ship wave, potential flow, global artificial boundary condition, finite element method.

AMS(MOS) subject classifications, 65N05, 76B20

Introduction: Many physical and engineering problems correspond to partial differential equations defined on unbounded physical domain. How to reduce the original problem to a bounded computational domain with high accuracy presents to be a challenge in solving these problems. In the past two decades, there have been many important progress in solving partial differential equations in unbounded domains, see, e.g. [7]-[19]. One of the most popular approach is the artificial boundary method, i.e., by introducing artificial boundaries and setting up artificial boundary conditions on them, we can reduce the original problem to a problem defined on bounded computational domain. In this paper, we would use the artificial boundary method to compute a linearized steady potential flow problem, and use its solution to calculate the ship wave resistance.

A ship moving near the free surface of the water would induce a pattern of trailing gravity waves. This pattern is formed through the work done by the ship

against the wave resistance. Numerical approximation of this wave resistance is a matter of high interest for ship design and marine engineering. One approach to this problem is calculating the wave resistance through a potential model. This model can be simplified by assuming the steady motion of the ship and linearization of the free surface boundary condition. Thus the potential flow around the ship is governed by a linearized steady potential flow problem. After the computation of this linearized steady potential flow problem, the wave resistance can be calculated by the numerical approximation of the potential flow with proper formula. In the recent years, there have been many studies on the ship wave resistance based on this approach [26],[3],[23],[24],[5],[6],[22] which show that this approach is accurate and ensures positive outcome of the wave resistance. The technique for solving the potential problem can be various. In this paper, we are ready to tackle the potential flow problem by the finite element method based on non-symmetric Galerkin variation, and apply the high-order global artificial boundary conditions to handle the unboundedness of the physical domain. Furthermore, We would present the discrete formula for calculating the wave resistance.

We concentrate in applying the high-order global artificial boundary conditions to the potential flow problem and deducing the formula for calculating the wave resistance from the potential flow. Concretely, we consider a 3-D problem for a ship moving at constant speed near the free surface of the water in a three-dimensional channel of constant rectangular cross section .

Let d denote the depth of the water , c denote the width of the channel, U

denote the speed of the body and g denote the acceleration of gravity. We scale the physical quantities by the length d and the velocity \sqrt{gd} . We describe the motion in Cartesian coordinates fixed with respect to the body, where the x -axis points opposite to the forward velocity and z -axis is directed vertically upward, y -axis points to the remaining direction of the right-angle reference frame, $y=0$ corresponds to one side of the channel and $y=c$ to another side of the channel, $z=0$ corresponds to the undisturbed free surface and $z=-1$ to the bottom. $\Omega = \mathbb{R} \times (0, c) \times (-1, 0)$ is the domain occupied by the water. The total velocity potential is split into a free stream potential plus a perturbation potential: $\Phi = \mu x + \phi(x, z)$, where $\mu = U/\sqrt{gd}$ is the Froude number. We would only need the perturbation potential to calculate the wave resistance. Assuming the ship is slender and thin, we can replace the ship by a pressure distribution[24]. By linearizing the boundary condition at the free surface, see whitham[25], we obtain the following problem for the linearized steady perturbation potential on the unbounded domain Ω :

$$\Delta\phi = 0 \quad \text{in } \Omega, \quad (1)$$

together with the boundary conditions

$$(\mu^2\phi_{xx} + \phi_z)|_{z=0} = -\frac{\mu}{\rho\sqrt{g}}P_x \quad -\infty < x < +\infty, 0 < y < c, \quad (2)$$

$$\phi_z|_{z=-1} = 0 \quad -\infty < x < +\infty, 0 < y < c, \quad (3)$$

$$\phi_y|_{y=0} = 0 \quad -\infty < x < +\infty, -1 < z < 0, \quad (4)$$

$$\phi_y|_{y=c} = 0 \quad -\infty < x < +\infty, -1 < z < 0, \quad (5)$$

$$\lim_{x \rightarrow -\infty} \phi = 0, \lim_{x \rightarrow +\infty} \phi \text{ is bounded} \quad -1 < z < 0; \quad (6)$$

where P is the pressure distribution at the free surface. It is reasonable to approximate P as a function having a compact support. This is important for deducting the artificial boundary conditions in this paper.

The organization of the paper is as follows. In section 2, for the given problem (1)-(6), we introduce the upstream artificial boundary Γ_a , the downstream artificial boundary Γ_b and the auxiliary artificial boundary $\Gamma_{b'}$ and apply the high-order artificial boundary conditions on these artificial boundaries, then the problem (1)-(6) is reduced to a problem defined on a bounded computational domain. In section 3, we get the equivalent variational problem for the reduced problem, which can be solved by the finite element method. In section 4, the formula for calculating the ship wave resistance through the potential flow will be presented. Numerical result is presented and the efficiency and accuracy of the proposed schemes are demonstrated by numerical test in Section 5 and some concluding remarks are given in the final section.

1 The global artificial boundary conditions

Take three constants $a < b' < b$, such that $P_x = 0$ on $x < a$ and $x > b'$. Then we obtain the upstream artificial boundary $\Gamma_a = \{(x, y, z) : x = a, 0 \leq y \leq c, -1 \leq z \leq 0\}$, the downstream artificial boundary $\Gamma_b = \{(x, y, z) : x = b, 0 \leq y \leq c, -1 \leq z \leq 0\}$, and the auxiliary artificial boundary $\Gamma_{b'} = \{(x, y, z) : x = b', 0 \leq y \leq c, -1 \leq$

$z \leq 0$ }. The artificial boundaries Γ_a, Γ_b divide the domain Ω into three parts:

$$\Omega_a = \{(x, y, z) : -\infty < x < a, 0 < y < c, -1 < z < 0\},$$

$$\Omega_T = \{(x, y, z) : a < x < b, 0 < y < c, -1 < z < 0\},$$

$$\Omega_b = \{(x, y, z) : b < x < +\infty, 0 < y < c, -1 < z < 0\};$$

furthermore we denote

$$\Omega_{b'} = \{(x, y, z) : b' < x < +\infty, 0 < y < c, -1 < z < 0\}.$$

1.1 The artificial boundary conditions on the artificial boundaries

We now recall the artificial boundary conditions on artificial boundaries Γ_b and Γ_a [17].

On the downstream artificial boundary Γ_b we have the approximate artificial boundary conditions:

$$\begin{aligned} \frac{\partial \phi}{\partial x} \Big|_{\Gamma_b} &= \sum_{m=0}^M \cos \frac{m\pi y}{c} \left\{ \frac{\mu_0^m \cosh \lambda_0^m (1+z)}{p_c^m [(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m] \sin(\mu_0^m \Delta)} \int_0^c \left\{ \right. \\ &\quad \int_{-1}^0 \left[\frac{\partial \phi}{\partial z} \Big|_{\Gamma_b} \cos(\mu_0^m \Delta) - \frac{\partial \phi}{\partial z} \Big|_{\Gamma_{b'}} \right] \frac{\partial \cosh \lambda_0^m (1+z)}{\partial z} dz \\ &\quad + \left. \left(\frac{m\pi}{c} \right)^2 \int_{-1}^0 \left[\phi \Big|_{\Gamma_b} \cos(\mu_0^m \Delta) - \phi \Big|_{\Gamma_{b'}} \right] \cosh \lambda_0^m (1+z) dz \right\} \cos \frac{m\pi y}{c} dy \\ &\quad - \sum_{k=1}^K \frac{\mu_k^m \cos \lambda_k^m (1+z)}{p_c^m [(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]} \int_0^c \left\{ \int_{-1}^0 \frac{\partial \phi}{\partial z} \Big|_{\Gamma_b} \frac{\partial \cos \lambda_k^m (1+z)}{\partial z} dz \right. \end{aligned}$$

$$+ \left(\frac{m\pi}{c}\right)^2 \int_{-1}^0 \phi \Big|_{\Gamma_b} \cos \lambda_k^m (1+z) dz \Big\} \cos \frac{m\pi y}{c} dy \Big\} \equiv D_b^{MK}(\phi)$$

where M, K are positive integers, the constants $p_c^m, \lambda_k^m, \mu_k^m, c_k^m, d_k^m$ ($m, k \geq 0$) can be found in [17].

On the upstream boundary Γ_a we have the approximate artificial boundary conditions:

$$\begin{aligned} \int_{\Gamma_a} \phi dz &= \sum_{k=1}^K \left\{ \frac{\sin \lambda_k^0}{p_c^0 (\lambda_k^0)^3 d_k^0} \int_0^c \left\{ \int_{-1}^0 \frac{\partial \phi}{\partial z} \Big|_{\Gamma_a} \frac{\partial \cos \lambda_k^0 (1+z)}{\partial z} dz \right\} dy \right\} \equiv S_a^K(\phi), \\ \frac{\partial \phi}{\partial n} \Big|_{\Gamma_a} &= - \sum_{m=0}^M \cos \frac{m\pi y}{c} \left\{ \sum_{k=1}^K \frac{\mu_k^m \cos \lambda_k^m (1+z)}{p_c^m [(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]} \int_0^c \left\{ \int_{-1}^0 \frac{\partial \phi}{\partial z} \Big|_{\Gamma_a} \frac{\partial \cos \lambda_k^m (1+z)}{\partial z} dz \right. \right. \\ &+ \left. \left. \left(\frac{m\pi}{c}\right)^2 \int_{-1}^0 \phi \Big|_{\Gamma_a} \cos \lambda_k^m (1+z) dz \right\} \cos \frac{m\pi y}{c} dy \right\} \equiv U_a^{MK}(\phi) \\ R_a^m(\phi) &\equiv \int_0^c \left\{ \int_{-1}^0 \frac{\partial \phi}{\partial z} \Big|_{\Gamma_b} \frac{\partial \cosh \lambda_0^m (1+z)}{\partial z} dz \right. \\ &+ \left. \left(\frac{m\pi}{c}\right)^2 \int_{-1}^0 \phi \Big|_{\Gamma_b} \cosh \lambda_0^m (1+z) dz \right\} \cos \frac{m\pi y}{c} dy = 0, \quad 0 \leq m \leq M' \end{aligned}$$

for $M \geq 0, K \geq 1$ are positive integers.

1.2 The reduced boundary value problem of problem (1)-(6)

Using the artificial boundary conditions given in this section, the problem (1)-(6) is reduced to a boundary value problem on the computational domain Ω_T :

$$\Delta \phi = 0 \quad \text{in } \Omega_T, \quad (7)$$

$$(\mu^2 \phi_{xx} + \phi_z)|_{z=0} = -\frac{\mu}{\rho\sqrt{g}} P_x \quad a < x < b, \quad 0 < y < c, \quad (8)$$

$$\phi_z|_{z=-1} = 0 \quad a < x < b, \quad 0 < y < c, \quad (9)$$

$$\phi_y|_{y=0} = 0 \quad a < x < b, \quad -1 < z < 0, \quad (10)$$

$$\phi_y|_{y=c} = 0 \quad a < x < b, \quad -1 < z < 0, \quad (11)$$

$$R_a^m(\phi) = 0 \quad 0 \leq m \leq M', \quad (12)$$

$$\int_{\Gamma_a} \phi dz = S_a^K(\phi), \quad (13)$$

$$\frac{\partial \phi}{\partial n} \Big|_{\Gamma_a} = U_a^{MK}(\phi), \quad (14)$$

$$\frac{\partial \phi}{\partial n} \Big|_{\Gamma_b} = D_b^{MK}(\phi). \quad (15)$$

Let ϕ be a solution of problem (7)-(15). Then $S_a^K(\phi)$ is constant. Let

$$\tilde{\phi} = \phi - \frac{S_a^K(\phi)}{c}. \quad (16)$$

For $\tilde{\phi}$ the condition (13) is simplified:

$$\int_{\Gamma_a} \tilde{\phi} dz = 0$$

and $\tilde{\phi}$ satisfies the equation (7) and the conditions (8)-(15) except (13). hence we

consider the following simplified problem:

$$\Delta \phi = 0 \quad \text{in } \Omega_T, \quad (17)$$

$$(\mu^2 \phi_{xx} + \phi_z)|_{z=0} = -\frac{\mu}{\rho\sqrt{g}} P_x \quad a < x < b, \quad 0 < y < c, \quad (18)$$

$$\phi_z|_{z=-1} = 0 \quad a < x < b, \quad 0 < y < c, \quad (19)$$

$$\phi_y|_{y=0} = 0 \quad a < x < b, \quad -1 < z < 0, \quad (20)$$

$$\phi_y|_{y=c} = 0 \quad a < x < b, \quad -1 < z < 0, \quad (21)$$

$$R_a^m(\phi) = 0 \quad 0 \leq m \leq M', \quad (22)$$

$$\int_{\Gamma_a} \phi dz = 0, \quad (23)$$

$$\frac{\partial \phi}{\partial n} \Big|_{\Gamma_a} = U_a^{MK}(\phi), \quad (24)$$

$$\frac{\partial \phi}{\partial n} \Big|_{\Gamma_b} = D_b^{MK}(\phi). \quad (25)$$

Suppose $\tilde{\phi}$ is the solution of problem (17)-(25), then $\phi = \tilde{\phi} + \frac{S_a^K(\tilde{\phi})}{c}$ is the solution of problem (7)-(15). In the following section the equivalent variational problem of problem (17)-(25) is given.

2 The equivalent variational problem of problem (17)-(25)

Let $H^m(\Omega_T)$ and $H^s(\Gamma_0)$ denote the usual Sobolev spaces on the domain Ω_T and the boundary $\Gamma_0 = \{(x, y, z) | a \leq x \leq b, 0 \leq y \leq c, z = 0\}$ with integer m and real number s [1], we introduce the space:

$$V = \{v | v \in H^1(\Omega_T) \text{ and } v|_{\Gamma_0} \in H^1(\Gamma_0)\}$$

and its subspace

$$U = \{v | v \in V, R_a^m(v) = 0, 0 \leq m \leq M' \text{ and } \int_{-1}^0 v|_{\Gamma_a} dz = 0\}.$$

Then we have the following result:

Theorem 2.1. *The boundary value problem (17)-(25) is equivalent to the following variational problem:*

Find $\phi_{MK} \in U$, such that

$$A_T(\phi_{MK}, \psi) + A_0(\phi_{MK}, \psi) + A_a^{MK}(\phi_{MK}, \psi) + A_b^{MK}(\phi_{MK}, \psi) = F(\psi), \quad \forall \psi \in V \quad (26)$$

where

$$\begin{aligned} A_T(\phi, \psi) &= \int_{\Omega_T} \nabla \phi \cdot \nabla \psi dx dy dz, \\ A_0(\phi, \psi) &= -\mu^2 \int_{\Gamma_0} \frac{\partial \phi(x, y, 0)}{\partial x} \frac{\partial \psi(x, y, 0)}{\partial x} dx dy, \\ A_a^{MK}(\phi, \psi) &= \sum_{m=0}^M \sum_{k=1}^K \left\{ \frac{1}{\mu_k^m p_c^m [(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]} \right. \\ &\quad \left\{ \int_0^c \left\{ \int_{-1}^0 \frac{\partial \phi}{\partial z} \Big|_{\Gamma_a} \frac{\partial \cos \lambda_k^m(1+z)}{\partial z} dz \right. \right. \\ &\quad \left. \left. + \left(\frac{m\pi}{c}\right)^2 \int_{-1}^0 \phi \Big|_{\Gamma_a} \cos \lambda_k^m(1+z) dz \right\} \cos \frac{m\pi y}{c} dy \right\} \\ &\quad \left\{ \int_0^c \left\{ \int_{-1}^0 \frac{\partial \psi}{\partial z} \Big|_{\Gamma_a} \frac{\partial \cos \lambda_k^m(1+z)}{\partial z} dz \right. \right. \\ &\quad \left. \left. + \left(\frac{m\pi}{c}\right)^2 \int_{-1}^0 \psi \Big|_{\Gamma_a} \cos \lambda_k^m(1+z) dz \right\} \cos \frac{m\pi y}{c} dy \right\} \Big\}, \\ A_b^{MK}(\phi, \psi) &= \sum_{m=0}^M \left\{ \left\{ \frac{1}{\mu_0^m p_c^m [(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m]} \sin(\mu_0^m \Delta) \right. \right. \\ &\quad \left\{ \int_0^c \left\{ \int_{-1}^0 \left[\frac{\partial \phi}{\partial z} \Big|_{\Gamma_b} \cos(\mu_0^m \Delta) - \frac{\partial \phi}{\partial z} \Big|_{\Gamma_{b'}} \right] \frac{\partial \cosh \lambda_0^m(1+z)}{\partial z} dz \right. \right. \\ &\quad \left. \left. + \left(\frac{m\pi}{c}\right)^2 \int_{-1}^0 \left[\phi \Big|_{\Gamma_b} \cos(\mu_0^m \Delta) - \phi \Big|_{\Gamma_{b'}} \right] \cosh \lambda_0^m(1+z) dz \right\} \cos \frac{m\pi y}{c} dy \right\} \\ &\quad \left\{ \int_0^c \left\{ \int_{-1}^0 \frac{\partial \psi}{\partial z} \Big|_{\Gamma_b} \frac{\partial \cosh \lambda_0^m(1+z)}{\partial z} dz \right. \right. \\ &\quad \left. \left. + \left(\frac{m\pi}{c}\right)^2 \int_{-1}^0 \psi \Big|_{\Gamma_b} \cosh \lambda_0^m(1+z) dz \right\} \cos \frac{m\pi y}{c} dy \right\} \Big\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^K \left\{ \frac{1}{\mu_k^m p_c^m [(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]} \right. \\
& \quad \left\{ \int_0^c \left\{ \int_{-1}^0 \frac{\partial \phi}{\partial z} \Big|_{\Gamma_b} \frac{\partial \cos \lambda_k^m (1+z)}{\partial z} dz \right. \right. \\
& \quad + \left. \left. \left(\frac{m\pi}{c} \right)^2 \int_{-1}^0 \phi \Big|_{\Gamma_b} \cos \lambda_k^m (1+z) dz \right\} \cos \frac{m\pi y}{c} dy \right\} \\
& \quad \left\{ \int_0^c \left\{ \int_{-1}^0 \frac{\partial \psi}{\partial z} \Big|_{\Gamma_b} \frac{\partial \cos \lambda_k^m (1+z)}{\partial z} dz \right. \right. \\
& \quad + \left. \left. \left(\frac{m\pi}{c} \right)^2 \int_{-1}^0 \psi \Big|_{\Gamma_b} \cos \lambda_k^m (1+z) dz \right\} \cos \frac{m\pi y}{c} dy \right\} \Big\} \Big\}, \\
F(\psi) & = - \int_{\Gamma_0} \frac{\mu}{\rho \sqrt{g}} P_x \psi ds.
\end{aligned}$$

Proof is standard, which is omitted.

We note that the solution space U is a true subspace of the trial space V . The variational problem (26) is not suitable for obtaining the finite element approximation of the problem (17)-(25) .

Let

$$\psi_1 = 1, \tag{27}$$

$$v_m = \cos \frac{m\pi y}{c} \sin \mu_0^m (x - b') \cosh \lambda_0^m (1+z) \quad 0 \leq m \leq M,$$

$$\psi_{m+2} = \begin{cases} 0, & \{a \leq x \leq b', 0 \leq y \leq c, -1 \leq z \leq 0\} \\ v_m, & \Omega_2 = \{b' \leq x \leq b, 0 \leq y \leq c, -1 \leq z \leq 0\} \end{cases} \tag{28}$$

$$0 \leq m \leq M,$$

$$A(\phi, \psi) = A_T(\phi, \psi) + A_0(\phi, \psi) + A_a^{MK}(\phi, \psi) + A_b^{MK}(\phi, \psi).$$

Then for any $\phi \in U$ we have

$$A_a^{MK}(\phi, \psi_i) = 0 \quad 1 \leq i \leq M+2, \tag{29}$$

$$A_b^{M0}(\phi, \psi_i) = A_b^{MK}(\phi, \psi_i) \quad 1 \leq i \leq M+2. \tag{30}$$

For the variational problem(26), we have the following results, see [17]:

Lemma 2.1. *For any $\phi \in U$ the following equalities hold*

$$A(\phi, \psi_i) = F(\psi_i) \quad \forall \phi \in U \quad 1 \leq i \leq M + 2. \quad (31)$$

Let $V = V^* \oplus \{\psi_1, \psi_2, \dots, \psi_{M+2}\}$. From the lemma 3.1 we know that

Theorem 2.2. *The boundary value problem (17)-(25) is equivalent to the following variational problem*

Find $\phi_{MK} \in U$, such that

$$A(\phi_{MK}, \psi) = F(\psi), \quad \forall \psi \in V^*. \quad (32)$$

Suppose U_h and V_h^* are the finite element subspaces of U and V^* , then we obtain the finite element approximation of the problem (32) :

Find $\phi_h^{MK} \in U_h$, such that

$$A(\phi_h^{MK}, \psi_h) = F(\psi_h), \quad \forall \psi_h \in V_h^*. \quad (33)$$

After solving the problem (33) we obtain the approximate solution ϕ_h^{MK} of the original problem (1)-(6) on the computational domain Ω_T

3 Formula for calculating the ship wave resistance

The wave resistance can be calculated by integrating the normal pressure forces over the ship, which can be transformed into the integration of the potential flow on a

plane section downstream by some proper manipulation . Papers [23],[24] provide the expression for calculating the wave resistance in the following integrating form:

$$F_x = -\frac{1}{2}\rho\left\{\int_{-1}^0\int_0^c [(\phi_x)^2 - (\phi_y)^2 - (\phi_z)^2]dydz - \mu^2\int_0^c (\phi_x)^2|_{z=0}dy\right\}\Big|_{x=r}, \quad (34)$$

where ϕ is the potential flow around the ship, r can be any number greater than b , i.e. the value of F_x have no connection with the choice of r provided $r > b$.

We would transfer this expression to the discrete formula suitable for the numerical result of the potential flow.

We again use the seperated variable expression for the downstream potential flow:

$$\begin{aligned} \phi(x, y, z) = & \widetilde{\alpha}_0 + \sum_{m=0}^{\infty} \cos \frac{m\pi y}{c} \left\{ [\alpha_0^m \cos \mu_0^m(x-b) + \beta_0^m \sin \mu_0^m(x-b)] \cosh \lambda_0^m(1+z) \right. \\ & \left. + \sum_{k=1}^{\infty} \beta_k^m e^{-\mu_k^m(x-b)} \cos \lambda_k^m(1+z) \right\} \quad (x, y, z) \in \overline{\Omega}_{b'}. \end{aligned} \quad (35)$$

We choose the wave-containing part only, the potential flow can be approximated as:

$$\phi(x, y, z) \approx \widetilde{\alpha}_0 + \sum_{m=0}^{\infty} \cos \frac{m\pi y}{c} [\alpha_0^m \cos \mu_0^m(x-b) + \beta_0^m \sin \mu_0^m(x-b)] \cosh \lambda_0^m(1+z). \quad (36)$$

We have

$$\begin{aligned} \phi_x &= \sum_{m=0}^{\infty} \mu_0^m \cos \frac{m\pi y}{c} [-\alpha_0^m \sin \mu_0^m(x-b) + \beta_0^m \cos \mu_0^m(x-b)] \cosh \lambda_0^m(1+z), \\ \phi_y &= \sum_{m=0}^{\infty} -\left(\frac{m\pi}{c}\right) \sin \frac{m\pi y}{c} [\alpha_0^m \cos \mu_0^m(x-b) + \beta_0^m \sin \mu_0^m(x-b)] \cosh \lambda_0^m(1+z), \end{aligned}$$

$$\phi_z = \sum_{m=0}^{\infty} \lambda_0^m \cos \frac{m\pi y}{c} [\alpha_0^m \cos \mu_0^m (x-b) + \beta_0^m \sin \mu_0^m (x-b)] \sinh \lambda_0^m (1+z);$$

so

$$\int_{-1}^0 \int_0^c (\phi_x)^2 dy dz = \sum_{m=0}^{\infty} (\mu_0^m)^2 H_m \frac{e^{2\lambda_0^m} - e^{-2\lambda_0^m} + 4\lambda_0^m}{8\lambda_0^m} [-\alpha_0^m \sin \mu_0^m (x-b) + \beta_0^m \cos \mu_0^m (x-b)]^2, \quad (37)$$

$$\int_{-1}^0 \int_0^c (\phi_y)^2 dy dz = \sum_{m=0}^{\infty} \left(\frac{m\pi}{c}\right)^2 H_m \frac{e^{2\lambda_0^m} - e^{-2\lambda_0^m} + 4\lambda_0^m}{8\lambda_0^m} [\alpha_0^m \cos \mu_0^m (x-b) + \beta_0^m \sin \mu_0^m (x-b)]^2, \quad (38)$$

$$\int_{-1}^0 \int_0^c (\phi_z)^2 dy dz = \sum_{m=0}^{\infty} (\lambda_0^m)^2 H_m \frac{e^{2\lambda_0^m} - e^{-2\lambda_0^m} - 4\lambda_0^m}{8\lambda_0^m} [\alpha_0^m \cos \mu_0^m (x-b) + \beta_0^m \sin \mu_0^m (x-b)]^2, \quad (39)$$

$$\mu^2 \int_0^c (\phi_x)^2|_{z=0} dy = \sum_{m=0}^{\infty} \mu^2 (\mu_0^m)^2 H_m \cosh^2 \lambda_0^m [-\alpha_0^m \sin \mu_0^m (x-b) + \beta_0^m \cos \mu_0^m (x-b)]^2; \quad (40)$$

where

$$H_m = \begin{cases} 1 & m = 0 \\ \frac{1}{2} & m > 0 \end{cases}.$$

Denote:

$$K_{1,m} = (\mu_0^m)^2 H_m \frac{e^{2\lambda_0^m} - e^{-2\lambda_0^m} + 4\lambda_0^m}{8\lambda_0^m} \quad m \geq 0,$$

$$K_{2,m} = \left(\frac{m\pi}{c}\right)^2 H_m \frac{e^{2\lambda_0^m} - e^{-2\lambda_0^m} + 4\lambda_0^m}{8\lambda_0^m} \quad m \geq 0,$$

$$K_{3,m} = (\lambda_0^m)^2 H_m \frac{e^{2\lambda_0^m} - e^{-2\lambda_0^m} - 4\lambda_0^m}{8\lambda_0^m} \quad m \geq 0,$$

$$K_{4,m} = \mu^2 (\mu_0^m)^2 H_m \cosh^2 \lambda_0^m \quad m \geq 0.$$

We can find

$$K_{4,m} = K_{1,m} + K_{2,m} + K_{3,m} \quad m \geq 0, \quad (41)$$

which can also be written as

$$K_{4,m} - K_{1,m} = K_{2,m} + K_{3,m} \quad m \geq 0. \quad (42)$$

Put (37)-(40) into (34) and notice (42), we have

$$\begin{aligned} F_x &= -\frac{1}{2}\rho \sum_{m=0}^{\infty} \{(\alpha_0^m)^2 K_{1,m} - (\beta_0^m)^2 K_{2,m} - (\beta_0^m)^2 K_{3,m} - (\alpha_0^m)^2 K_{4,m}\} \\ &= -\frac{1}{2}\rho \sum_{m=0}^{\infty} \{(\beta_0^m)^2 K_{1,m} - (\alpha_0^m)^2 K_{2,m} - (\alpha_0^m)^2 K_{3,m} - (\beta_0^m)^2 K_{4,m}\} \\ &= \frac{1}{2}\rho \sum_{m=0}^{\infty} \{(\beta_0^m)^2 [K_{4,m} - K_{1,m}] + (\alpha_0^m)^2 [K_{2,m} + K_{3,m}]\} \\ &= \frac{1}{2}\rho \sum_{m=0}^{\infty} \{[(\beta_0^m)^2 + (\alpha_0^m)^2] [K_{2,m} + K_{3,m}]\}. \end{aligned} \quad (43)$$

By truncating the infinite series in (43), we get

$$F_x \approx \frac{1}{2}\rho \sum_{m=0}^{M'} \{[(\beta_0^m)^2 + (\alpha_0^m)^2] [K_{2,m} + K_{3,m}]\}, \quad (44)$$

where M' is the number used in the upstream artificial boundary condition (22), $\{\alpha_0^m\}, \{\beta_0^m\}$ are the coefficients in the expression (35) or (36).

(44) is the formula for calculating the wave resistance with the numerical result of the potential flow. Formula (44) retains two properties of expression (34). First, the value of the wave resistance always hold positive in (34). It can be easily checked $K_{2,m}$ and $K_{3,m}$ always hold positive for $m \geq 0$, so formula (44) ensures positive outcome of the calculation of the wave resistance. Secondly, the value of the wave resistance is disrelated with the choice of r in (34). This is obvious in (44) since the value of $\{\alpha_0^m\}, \{\beta_0^m\}$ have no relation with the choice of the x-coordinate value.

4 Numerical Results

In this section, we present the numerical result of the ship wave resistance for the Wigley model 1805 A and give some numerical test to prove the effectiveness and accuracy of our global artificial boundary conditions.

4.1 Solution of a 3-D problem

In this subsection, we consider the calculation of the wave resistance for the Wigley model 1805 A [23] moving at constant speed near the free surface of the water in a three-dimensional channel of constant rectangular cross section . The hull shape for this model is defined by

$$y - \frac{c}{2} = \pm \frac{B}{2} \left(1 - \left(\frac{z}{D}\right)^2\right) \left(1 - \left(\frac{2x}{L}\right)^2\right) \left(1 - \xi \left(\frac{2x}{L}\right)^2\right) \quad (45)$$

for $|x| < \frac{L}{2}$, $-D < z < 0$ with $\xi = 0.6$.

L, B, D are the length, beam and draft of the ship respectively, in our computation, we choose $L = 0.4, B = 0.05, D = 0.05$.

We would use in this calculation a relatively simple stationary pressure model for the potential problem, that is, the pressure distribution on the free surface is taken as the water pressure due to the existence of a unmoving ship. But we should mention that our global artificial boundary conditions are independent with the choice of the concrete mathematical model for the potential problem. As long as the free surface boundary condition in the far field can be approximated as a homogeneous one, then our artificial boundary conditions can work well.

The linearized free boundary condition is defined on the undisturbed free surface of water which is assumed in the mid-height of the ship[3].

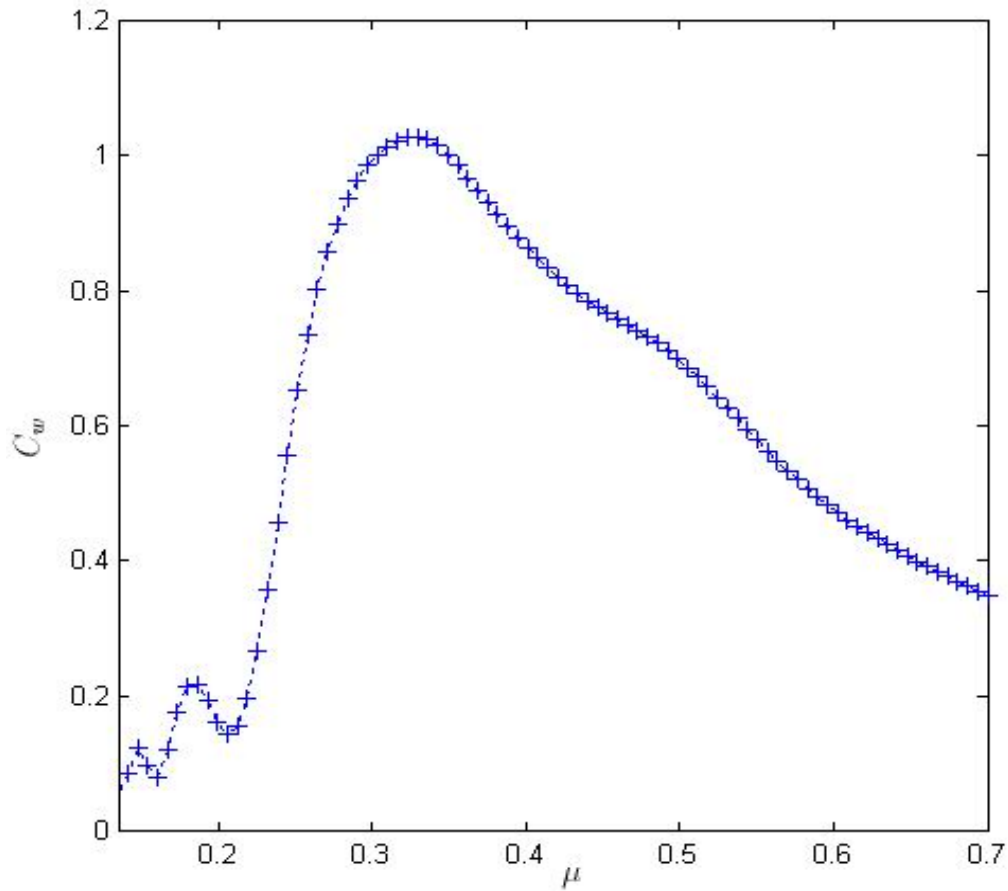


Figure 1: Numerical result of the drag curve for the Wigley model 1805 A .

The mesh used in the computation is nonuniform in z -direction. Let h represents the mesh spacings in x -direction. We shall always take $b' = b - h$ in the following. The mesh spacings in x - and y -directions are 0.025 , 0.02 respectively.

After we numerically solve the potential problem, the wave resistance F_x is calculated by the numerical approximation of the potential flow.

The circular Froude coefficient is defined as

$$C_w = \frac{250}{\pi} \frac{F_x}{\Omega_{\text{ship}}^{2/3} \rho g \mu^2},$$

where $\Omega_{\text{ship}} = \frac{44}{225} LBD$ is the volume of the wetted part of the ship.

Take $a = -0.25, b = 0.25, c = 1$, Figure 4 shows the numerical result of the drag curve for the Wigley model 1805 A , which is in good agreement with the results found in [23],[24],[22],[6] .

4.2 Test of the artificial boundary condition on downstream artificial boundaries

We shall test the influence of the location of the artificial boundary Γ_b . We take $a = -0.3$ and $b = 0.3, 0.5, 0.6, 0.8, 0.9$ and 1.0 , respectively. We choose $\mu = 0.4, M = 60, K = 20, M' = 60$. Let ϕ denote the “exact solution” which is the finite element solution of (33) when $b = 1.0$. Table 1 shows the relative errors of $\phi - \phi_h$ in L_∞ -norm, L_2 -norm , H^1 -norm and the wave resistances calculated from them for different location of the artificial boundary Γ_b , where Ω_0 is the bounded computational domain Ω_T when $b = 0.3$. As shown in table 1, the influence caused by different location of artificial boundary Γ_b is very small. Therefore for a given accuracy, it is possible to use a small bounded computational domain. So the using of global artificial boundary conditions can save computational cost greatly.

Table 1: The effect of the location of the artificial boundary Γ_b

Errors	$b = 0.3$	$b = 0.5$	$b = 0.6$	$b = 0.8$	$b = 0.9$
$\max \phi - \phi_h / \max \phi $	2.1710E-2	1.3366E-2	1.5141E-2	1.3441E-2	9.6702E-3
$\ \phi - \phi_h\ _{0,\Omega_0} / \ \phi\ _{0,\Omega_0}$	2.9874E-2	2.0295E-2	1.8855E-2	9.6861E-3	5.0122E-3
$\ \phi - \phi_h\ _{1,\Omega_0} / \ \phi\ _{1,\Omega_0}$	2.8937E-2	2.2215E-2	2.0913E-2	1.2276E-2	6.4914E-3
$ F_x - F_x^\infty / F_x $	7.4479E-4	1.9118E-3	4.6823E-3	1.9122E-4	2.1485E-4

5 Conclusions

A sequence of high-order global artificial boundary conditions at the downstream and upstream artificial boundaries are applied for the linearized steady potential flow around a body moving at constant speed near the free surface of a liquid . Then the potential flow problem is reduced to a problem defined on a finite computational domain, which is equivalent to a variational problem. The variational problem can be solved by the finite element method. Then the numerical approximation for

the potential flow problem is obtained, which can be used to calculate the wave resistance with formula presented in this paper. Numerical examples show that our global artificial boundary conditions are very effective and the numerical scheme for calculating the ship wave resistance is accurate. Summarizing this paper, We can make some remarks:

- Our global artificial boundary conditions are very effective in that the influence caused by different location of artificial boundary is very small. Therefore we can choose a small bounded computational domain to get high accuracy.
- Numerical experiments demonstrate the accuracy of our numerical scheme and the effectiveness of the nonuniform mesh.
- The wave resistance calculation formula in this paper can ensure a positive result.

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