5th SJOM – Bejing, 2011

Cone Linear Optimization (CLO)

From LO, SOCO and SDO

Towards Mixed-Integer CLO

Tamás Terlaky

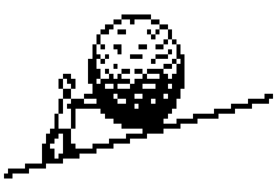
George N. and Soteria Kledaras '87 Endowed Chair Professor.

Chair, Department of Industrial and Systems Engineering

Lehigh University

Outline

- **Motivation:**
  - Data uncertainty in linear inequalities-Robust LO
  - Data uncertainty in linear inequalities
  - Ellipsoidal uncertainty set $\Rightarrow$ norm constraint
  - Eigen- and singular value optimization problems
  - Relaxing integer variables
- **Conic Linear Optimization (CLO):**
  - General Convex Cones
  - Second Order Conic Optimization (SOCO)
  - Semidefinite Optimization (SDO)
- **Interior Point Algorithms for SOCO and SDO:**
- **MISOCO:** Mixed Integer Second Order Conic Optimization



Sisyphus got stuck with a suboptimal solution;

don't let it happen to you! http://www.optimize.com

Robust Linear Optimization
Classic – Polyhedral (scenario) approach

$$(P) \quad \min \quad c^T x \qquad \text{s.t.} \quad a_j^T x - b_j \geq 0 \quad \forall\, j$$

Let (a_j, b_j) be uncertain, it is coming from a polyhedral set (e.g. convex combination of "scenario" data points):

$$\left\{ \begin{pmatrix} a_j \\ -b_j \end{pmatrix} = \sum_{i=1}^{n_j} \begin{pmatrix} a_j^i \\ -b_j^i \end{pmatrix} \lambda_j^i \;\middle|\; \sum_{i=1}^{n_j} \lambda_j^i = 1,\ \lambda_j^i \geq 0 \right\}$$

The inequality $a_j^T x \geq b_j$ must be true for all possible values of $(a_j^T, -b_j)$:

$$\left[\sum_{i=1}^{n_j} \begin{pmatrix} a_j^i \\ -b_j^i \end{pmatrix} \lambda_j^i \right]^T \begin{pmatrix} x \\ 1 \end{pmatrix} \geq 0 \quad \textbf{for all} \quad \sum_{i=1}^{n_j} \lambda_j^i = 1,\ \lambda_j^i \geq 0$$

Infinitely many constraints!

iff $[a_j^i]^T x - b_j^i \geq 0$ for $i = 1, \ldots, n_j$

Finally the problem stays linear as:

$$(RP) \quad \min \quad c^T x \qquad \text{s.t.} \quad [a_j^i]^T x - b_j^i \geq 0 \ \text{for } i = 1, \ldots, n_j \quad \forall\, j$$

Disadvantages: – Huge number of linear inequalities

– Polyhedral uncertainty set not realistic.

Robust Linear Optimization

$$(P) \quad \min \quad c^T x$$
$$\text{s.t.} \quad a_j^T x - b_j \geq 0 \qquad \forall\, j$$

Let (a_j, b_j) be uncertain, it is coming from an ellipsoid (e.g. level set of a distribution):

$$\left\{ \begin{pmatrix} a_j \\ -b_j \end{pmatrix} = \begin{pmatrix} a_j^0 \\ -b_j^0 \end{pmatrix} + Pu \;\middle|\; u \in \mathbb{R}^k,\ u^T u \leq 1 \right\}$$

The inequality $a_j^T x \geq b_j$ must be true for all possible values of $(a_j^T, -b_j)$:

$$\left[\begin{pmatrix} a_j^0 \\ -b_j^0 \end{pmatrix} + Pu \right]^T \begin{pmatrix} x \\ 1 \end{pmatrix} \geq 0 \quad \forall\, u \,:\, u^T u \leq 1 \quad \textbf{iff} \quad [a_j^0]^T x - b_j^0 + \min_{u^T u \leq 1} \left\{ (Pu)^T \begin{pmatrix} x \\ 1 \end{pmatrix} \right\} \geq 0$$

$$[a_j^0]^T x - b_j^0 - \left\| P^T \begin{pmatrix} x \\ 1 \end{pmatrix} \right\|_2 \geq 0$$

This is a nondifferentiable norm constraint: **(See second order cones)**

$$\left\| P^T \begin{pmatrix} x \\ 1 \end{pmatrix} \right\|_2 \leq [a_j^0]^T x - b_j^0.$$

Single nonlinear, norm-constraint!

Eigenvalue Optimization

Given $n \times n$ symmetric matrices $A_1, \ldots, A_m$.

Problem: Find a nonnegative combination of the matrices that has the maximal smallest eigenvalue.

Solution: $$\max \left\{ \lambda \;\middle|\; \begin{array}{r} \sum_{i=1}^{m} A_i y_i - \lambda I \text{ is positive semidefinite} \\ y_i \geq 0 \;\; i = 1, \ldots, m \end{array} \right\}$$

Problem: Find a nonnegative combination of the matrices that has the smallest maximal eigenvalue.

Solution: $$\min \left\{ \lambda \;\middle|\; \begin{array}{r} \lambda I - \sum_{i=1}^{m} A_i y_i \text{ is positive semidefinite} \\ y_i \geq 0 \;\; i = 1, \ldots, m \end{array} \right\}$$

The semidefiteness constraint is not differentiable, not easy to calculate when formulated by explicit functions, e.g., min-eigenvalue, determinant (of minors). **See Semidefinite Optimization.**

Relaxing Binary Variables

Given $z_1, \ldots, z_n$ binary, i.e., $\{0,1\}$ variables with other, e.g., linear constraints.
Problem: Find convex, continuous relaxations of the binary constraints.
Old solution: Let $0 \le z_i \le 1$ for all $i = 1, \ldots, n$ and use branch and bound, branch and cut schemes.
New opportunity to get tighter relaxations: Let $x_i = \frac{2z_1 - 1}{2}$ for all $i = 1, \ldots, n$. Thus gives x_i as a $\{-1, 1\}$ variable. Now

$$n = \sum_{i=1}^{n} x_i^2 = x^T x = \mathbf{Tr}(x^T x) = \mathbf{Tr}(xx^T) = \mathbf{Tr}(X),$$

where $X = xx^T$ is a **rank-1** positive semidefinite matrix with diag$(X) = e$.
Thus, for $x_i \in \{-1, 1\}$ $\forall i$ we may use the relaxation:

$$X_{ii} = 1 \ \forall i \qquad \text{and } X \text{ is positive semidefinite}$$

The semidefiteness constraint is not differentiable, not easy to calculate when formulated by explicit functions, e.g., min-eigenvalue, determinant (of minors). **See Semidefinite Optimization.**

Stability of Optimal Power Flow

$$
\begin{array}{ll}
\text{Min.} & S_b = -(C_d^T P_d - C_s^T P_s) \\
\text{s.t.} & F_{pf}(\delta, V, Q_G, P_s, P_d) = 0 \\
& \sigma_{min}(J_{pf}) \geq \ \sigma_{cpf} \\
& 0 \leq P_s \leq P_{s_{\max}} \\
& 0 \leq P_d \leq P_{d_{\max}} \\
& I_{ij}(\delta, V) \leq I_{ij_{\max}} \\
& I_{ji}(\delta, V) \leq I_{ji_{\max}} \\
& Q_{G_{\min}} \leq Q_G \leq Q_{G_{\max}} \\
& V_{\min} \leq V \leq V_{\max}
\end{array}
$$

The stability of the Power Flow is ensured by a lower bound on the singular value of the Jacobian J_{pf} of the power flow equations:

$$\sigma_{\min}(J_{pf}) \geq \sigma_{cpf}$$

equivalently

$$\lambda_{\min}(J_{pf} J_{pf}^T) \geq \sigma_{cpf}$$

by substitution

$$X - J_{pf} J_{pf}^T = 0$$

and

$X - \sigma_{cpf} I$ **is positive semidefinite**

Again a semidefinite constraint!

One may want maximize σ_{cpf}.

New/Old Convex Optimization Problems
Cone Linear Optimization Problems

Primal-dual pair of CLO problems is given as

$$(P) \quad \min \quad c^T x \qquad\qquad (D) \quad \max \quad b^T y$$

$$\text{s.t.} \quad Ax - b \in \mathcal{C}_1 \qquad\qquad \text{s.t.} \quad c - A^T y \in \mathcal{C}_2^*$$

$$x \in \mathcal{C}_2 \qquad\qquad y \in \mathcal{C}_1^*,$$

where $b, y \in \mathrm{I\!R}^m$, $c, x \in \mathrm{I\!R}^n$, $A : m \times n$ matrix, $\mathcal{C}_1, \mathcal{C}_2$ are convex cones and $\mathcal{C}_i^* = \{s \in \mathrm{I\!R}^n \, : \, x^T s \geq 0, \; \forall x \in \mathcal{C}_i\}$ are the dual cones for $i = 1, 2$.

These are solvable efficiently (in polynomial time) by using Interior Point Methods. LO is based on polyhedral cones.

Be careful! Perfect duality, strict complementarity lost.

Are all convex cones good???

NOT

– New Convex Optimization Problems – Second Order Conic Optimization (SOCO)

The second order cone in $\mathbb{R}^n$ is defined as

$$\mathcal{S}_2^n := \left\{ x \in \mathbb{R}^n \ : \ \|x_{2:n}\| = \sqrt{\sum_{i=2}^{n} x_i^2} \leq x_1 \right\}.$$

The name "ice cream cone" is coming from the 3-dimensional shape of the cone.

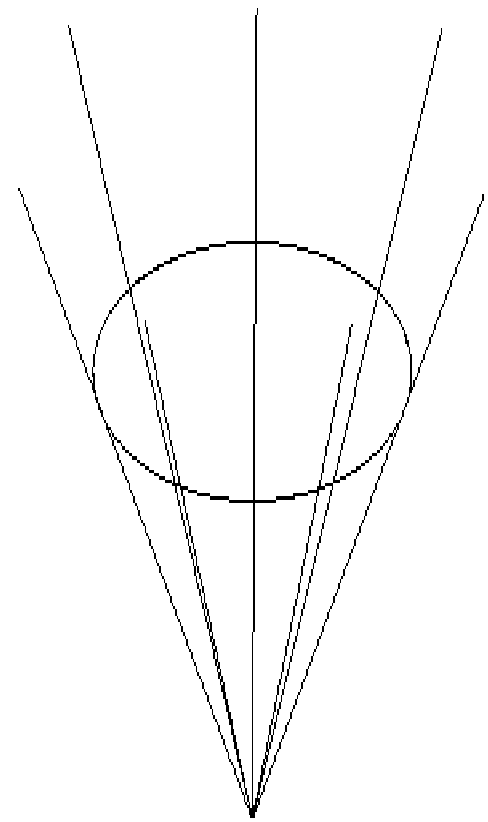
The second order cone is self-dual: $(\mathcal{S}_2^n)^* = \mathcal{S}_2^n$.

Optimization problems, where cones $\mathcal{C}_1$ and $\mathcal{C}_2$ are polyhedral or products of second order cones, are *second order cone optimization (SOCO)* problems.

Significance

Norm minimization, robust optimization,

quadratic, and thus portfolio optimization .



The ice-cream cone L^3

SOCO – Optimality

The primal-dual SOCO problem is defined as

$$(SP) \quad \min \; c^T x \quad \text{s.t.} \; Ax = b, \; x \in \times_{j=1}^{k} \mathcal{S}_2^{n_j} \qquad\qquad (SD) \quad \max \; b^T y \quad \text{s.t.} \; A^T y + s = c, \; s \in \times_{j=1}^{k} \mathcal{S}_2^{n_j}.$$

$x^T = ((x^1)^T, ..., (x^j)^T, ..., (x^k)^T) \in \mathbb{R}^n$; and $s^T = ((s^1)^T, ..., (s^j)^T, ..., (s^k)^T) \in \mathbb{R}^n$.

Optimality:

$$Ax = b, \quad x \in \times_{j=1}^{k} \mathcal{S}_2^{n_j}, \qquad A^T y + s = c, \quad s \in \times_{j=1}^{k} \mathcal{S}_2^{n_j}$$

$$(x^j)^T s^j = 0 \Leftrightarrow x^j \circ s^j = 0 \Leftrightarrow \text{Arr}(x^j) s^j = \text{Arr}(s^j) x^j = 0 \;\; \forall j$$

Here we have used the notation:

$$u \circ v = \begin{pmatrix} u^T v \\ u_1 v_{2:n} + v_1 u_{2:n} \end{pmatrix} \quad \text{and} \quad \text{Arr}(u) = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ u_2 & u_1 & & \\ \vdots & & \ddots & \\ u_n & & & u_1 \end{pmatrix}$$

Notes on SOCO

- Convex (conic) optimization problem with "vectors"
- Vector calculus not associative
- Duality gap may exists (next page)
- Strong duality with interior point (Slater) condition
- Second order cones cannot be "combined" into larger second order cones, i.e., $\mathcal{S}_2^{n_1} \times \mathcal{S}_2^{n_2} \neq \mathcal{S}_2^{n_1+n_2}$
- Generalization of LO: $\mathcal{S}_2^1 = \mathbb{R}_+^1$
- Rotated cone $\|x_{2:n}\|^2 \leq x_0 x_1$, $x_0, x_1 \geq 0$
$$\sum_{i=2}^{n} x_i^2 \leq \left(\frac{x_0 + x_1}{2}\right)^2 - \left(\frac{x_0 - x_1}{2}\right)^2$$
- Efficiently solvable by IPMs.

SOCO: Duality gap example

Primal Problem

$$(SP) \quad \min \quad x_2 \quad \text{s.t.} \quad x_1 - x_3 = 0, \quad \sqrt{x_2^2 + x_3^2} \le x_1$$

Dual Problem

$$\begin{aligned} (SD) \quad \max \quad & 0 \cdot y \\ \text{s.t.} \quad & y + s_1 = 0 \\ & 0 \cdot y + s_2 = 1 \\ & -y + s_3 = 0 \\ & \sqrt{s_2^2 + s_3^2} \le s_1 \end{aligned}$$

Primal Optimal solutions:

$x_1 = x_3 \ge 0; \quad x_2 = 0$

Dual problem is infeasible!!

Slater not satisfied $\Longrightarrow$ Zero duality gap may not hold

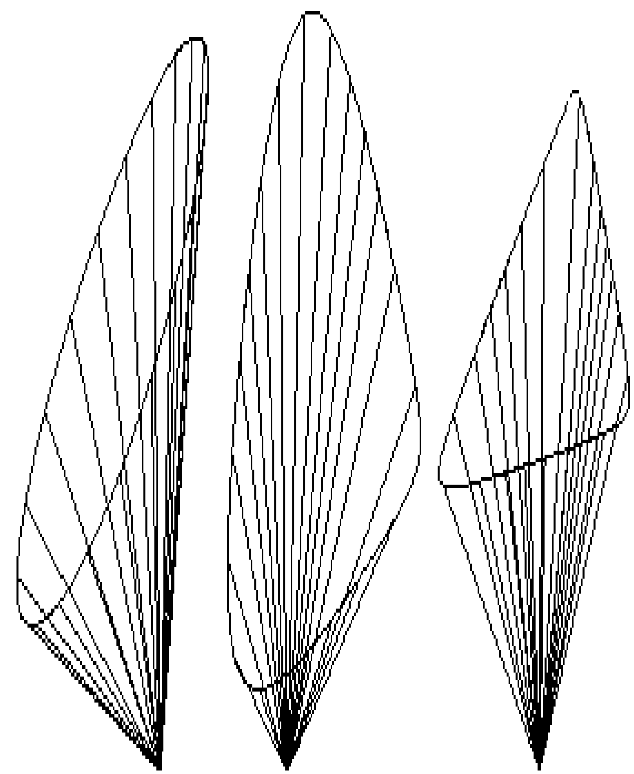
– New Convex Optimization Problems – Semidefinite Optimization – I

The semidefinite cone in $\mathbb{R}^{n\times n}$ is defined as

$$\mathcal{S}^n := \left\{X \in \mathbb{R}^{n\times n} : X = X^T, z^T X z \geq 0 \,\forall z \in \mathbb{R}^n\right\}$$

i.e. the matrices X are symmetric and positive semidefinite, denoted as $X \succeq 0$. The semidefinite cone is self-dual: $(\mathcal{S}^n)^* = \mathcal{S}^n$.

Optimization problems where the cones $\mathcal{C}_1$ and $\mathcal{C}_2$ are either polyhedral, second order or semidefinite cones are called *semidefinite optimization (SDO) problems*.



3 random 3D cross-sections of S^3_+

– New Convex Optimization Problems – Semidefinite Optimization

Let $A_i,\ i = 1, \cdots, n$ and C, X be $n \times n$ symmetric real matrices, $b, y \in \mathbb{R}^m$ and let Tr$(\cdot)$ denote the trace of a matrix.

The primal-dual SDO problem is defined as

$$
(SP)\quad \min \quad \mathrm{Tr}(CX) \qquad \text{s.t.}\quad \mathrm{Tr}(A_i X) - b_i \ge 0,\ \forall i, \qquad X \succeq 0
$$

$$
(SD)\quad \max \quad b^T y \qquad \text{s.t.}\quad C - \sum_{i=1}^{m} A_i y_i \succeq 0, \qquad y \ge 0.
$$

Optimality: $\mathbf{Tr}(CX) - b^T y = \mathbf{Tr}(XS) = 0 \Leftrightarrow XS = 0$

Significance

Robust optimization, eigenvalue and singular value optimization

Linear matrix inequalities, trust design

Convex relaxation of nonconvex/integer problems

Notes on SDO

- Convex (conic) optimization problem with "matrices"
- Matrix calculus not commutative
- Product of symmetric matrices is not smmetric
- Duality gap may exists
- Strong duality with interior point (Slater) condition
- Semidefinite cones can be "combined" into larger semidefinite cones, i.e., $\mathcal{S}^{n_1} \times \mathcal{S}^{n_2} \subset \mathcal{S}^{n_1+n_2}$
- Generalization of LO: $\mathcal{S}^1 = \mathbb{R}^1_+$ – diagonal matrices
- Not proper generalization of SOCO: $x \in \mathcal{S}^n_2 \Leftrightarrow \mathrm{Arr}(x) \in \mathcal{S}^n$,
  – BUT arrow-head structure cannot be preserved
- Efficiently solvable by IPMs.

Linear Optimization v/s Conic LO

LO	Conic LO
linear objective	linear objective
linear equality constraints	linear equality constraints
linear inequality constraints	conic inequality constraints
perfect duality	perfect duality only with IPC
strictly complementary opt.sol.	maximally complementary opt.sol.
Euclidean linear algebra	matrix and Jordan algebra
$x^T s = 0 \Leftrightarrow xs = 0$	$x^T s = 0 \Leftrightarrow x \circ s = 0$ (SOCO)
	$\mathbf{Tr}(XS) = 0 \Leftrightarrow XS = 0$ (SDO)
	$\mathbf{Tr}(XS) = 0 \Leftrightarrow X^{\frac{1}{2}} S X^{\frac{1}{2}} = S^{\frac{1}{2}} X S^{\frac{1}{2}} = 0$
	$\approx \Leftrightarrow (PXP^T)^{\frac{1}{2}}(P^{-T}SP^{-1})(PXP^T)^{\frac{1}{2}} = \mu I$
	$\approx \Leftrightarrow (PSP^T)^{\frac{1}{2}}(P^{-T}XP^{-1})(PSP^T)^{\frac{1}{2}} = \mu I$

Solvability of CLO problems – Use IPMs

Classic Linear Optimization

Large scale LO problems are solved efficiently.
High performance packages, like (CPLEX, GuRoBi, XPRESS-MP, MOSEK) offer simplex and interior point solvers as well.
Problems solved with 10^7 variables.

SOCO and SDO

Polynomial solvability established.
Traditional software is unable to handle conic constraints.
Specialized software is developed. (SeDuMi, SDPA, SDPT3, CSDP, DSDP, SDP-pack, MOSEK etc.)
SOCO: MOSEK - commercial
SDO: SDPA, CSDP, DSDP
LO-SOCO-SDO: SeDuMi, SDPT3
SOCO: Problems solved with $O(10^6)$ variables.
SDO: solved with $O(10^4)$ dimensional matrices.

http://sedumi.ie.lehigh.edu

The Primal–Dual LO Problems, Central Path

The primal-dual LO problems is given as:

$$\min \; c^T x \qquad\qquad \max \; b^T y$$

$$Ax = b, \quad x \geq 0, \qquad\qquad A^T y + s = c, \quad s \geq 0,$$

where $c, x, s \in \mathcal{R}^n$, $b, y \in \mathcal{R}^m$, $A \in \mathcal{R}^{m \times n}$, $\operatorname{rank}(A) = m$.

Optimality conditions and the central path are given as:

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= 0, \end{aligned} \qquad\qquad \begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= \mu e, \end{aligned}$$

where $e = (1, \ldots, 1)^T \in \mathcal{R}^n$.

We assume that the Interior Point Condition holds.

Primal-Dual Search-directions for LO

The central path and the Classical Newton direction:

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= \mu e. \end{aligned}$$

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ s\Delta x + x\Delta s &= \mu e - xs, \end{aligned}$$

Scaled Newton direction:

$$\begin{aligned} \bar{A} p_x &= 0, \\ \bar{A}^T \Delta y + p_s &= 0, \\ p_x + p_s &= v^{-1} - v \end{aligned}$$

Proximity Functions:

$$\begin{aligned} \Psi(v) &= \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \log v_i \right) \\ \Psi(v) &= \frac{1}{2} \|v - v^{-1}\|^2. \end{aligned}$$

where $\bar{A} = \frac{1}{\mu} A V^{-1} X$, $V = \mathrm{diag}\,(v)$, $X = \mathrm{diag}\,(x)$ with

$$v := \sqrt{\frac{xs}{\mu}}, \qquad v^{-1} := \sqrt{\frac{\mu}{xs}}, \qquad p_x := \frac{v\Delta x}{x}, \qquad p_s := \frac{v\Delta s}{s}.$$

The Newton System for SDO

$$(SP) \quad \min \quad \mathrm{Tr}(CX) \qquad \text{s.t.} \quad \mathrm{Tr}(A_iX) = b_i,\ \forall i, \quad X \succeq 0$$

$$(SD) \quad \max \quad b^Ty \qquad \text{s.t.} \quad \sum_{i=1}^m A_iy_i + S = C, \quad S \succeq 0.$$

The Newton System for the NT-direction: $P = S^{-\frac{1}{2}}(S^{\frac{1}{2}}XS^{\frac{1}{2}})^{-\frac{1}{2}}S^{-\frac{1}{2}}$

$$\mathrm{Tr}(A_iX) = b_i,\ \forall i \quad X \succeq 0 \qquad\qquad \mathrm{Tr}(A_i\Delta X) = 0,\ \forall i$$

$$\sum_{i=1}^m A_iy_i + S = C \quad S \succeq 0 \qquad\qquad \sum_{i=1}^m A_i\Delta y_i + \Delta S = 0$$

$$\mathbf{Tr}(XS) = 0 \approx\Leftrightarrow XS = \mu I \qquad\qquad X\Delta S + \Delta XS = \mu I - XS$$

$$\Leftrightarrow \Delta X + X\Delta SS^{-1} = \mu S^{-1} - X$$

$H_P(\cdot)$ is a symmetrization: $H_P(X\Delta S + \Delta XS) = \mu I - H_P(XS)$

The Newton System for SOCO

$$
\begin{array}{llll}
(SOP) & \min & c^T x & \\
& \text{s.t.} & Ax & = b, \\
& & x & \in \times_{j=1}^{k} \mathcal{S}_2^{n_j}
\end{array}
\qquad
\begin{array}{llll}
(SOD) & \max & b^T y & \\
& \text{s.t.} & A^T y + s & = c \\
& & s & \in \times_{j=1}^{k} \mathcal{S}_2^{n_j}.
\end{array}
$$

The Newton System for the NT-direction in arrow-head formulation:
$P = S^{-\frac{1}{2}}(S^{\frac{1}{2}} X S^{\frac{1}{2}})^{-\frac{1}{2}} S^{-\frac{1}{2}}$

$$
\begin{array}{rll}
Ax & = b, \ \forall i & x \in \times_{j=1}^{k} \mathcal{S}_2^{n_j} \\
A^T y + s & = c, & s \in \times_{j=1}^{k} \mathcal{S}_2^{n_j}
\end{array}
\qquad
\begin{array}{rcl}
A\Delta x & = & 0, \\
A^T \Delta y + \Delta s & = & 0,
\end{array}
$$

$$
x^j \circ s^j = 0 \approx\Leftrightarrow x^j \circ s^j = \mu e^j \qquad x^j \circ \Delta s^j + \Delta x^j \circ s^j = \mu e^j - x^j \circ s^j
$$

$$
\Leftrightarrow H_P(x^j \circ \Delta s^j + \Delta x^j \circ s^j) = \mu e^j - H_P(x^j \circ s^j)
$$

$H_P(\cdot)$ is a symmetrization operator.

Primal-Dual Interior Point Methods with small and large updates

Input:

A proximity parameter τ; an accuracy parameter $\epsilon > 0$;
an update parameter $0 < \theta < 1$; a variable damping factor α;
(x^0, s^0), $\mu^0 = 1$ s.t. $\Psi(v^0) \leq \tau$.

begin

$x := x^0$; $s := s^0$; $\mu := \mu^0$;

while $n\mu \geq \epsilon$ **do**

begin

$\mu := (1 - \theta)\mu$;

while $\Psi(v) \geq \tau$ **do**

begin

Calculate $\Delta x, \Delta s$;

Do line search for $\Psi(v(\alpha))$;

$x := x + \alpha \Delta x$;

$s := s + \alpha \Delta s$;

end

end

end

Complexity of IPMs for LO

Method	Practice	Large update	Small update
θ	adaptive	$1 - 1/100$	$1/\sqrt{n}$
Iter. bound	max 100	$\mathcal{O}(\vartheta \log \frac{\vartheta}{\epsilon})$	$\mathcal{O}(\sqrt{\vartheta} \log \frac{\vartheta}{\epsilon})$
Performance	Efficient	Efficient	Very poor

"Almost" constant (< 100) number of iterations in practice!

CLO	$\vartheta =$	cost/iteration
LO	# of variables	$O(n^3)$ sparse
SOCO	# of second order cones	$O(n^3)$+update
SDO	dimension of matrix X	$O(n^2m^3)$ dense

Several IPM Solvers for CLO Problems

What made this major advance possible?

Advances in Computers and Software

Computers

- processor speed
- memory
- disk space
- floating point arithmetic
- architecture (cash ...)

Software component/Algorithms

- presolve
- LINEAR ALGEBRA
- sparse factorizations
- symmetric square root
- IPMs, predictor-corrector
- dense and sparse versions

SeDuMi, SDPT3 SDPA-xxx tuned to all three cones

CSDP, DSDP tuned to SDO only

MOSEK, CPLEX are commercial solves for SOCO .

Parallel implementations exist – coming.

MODELING languages – YALMIP (Löfberg); CVx (Boyd/Grant).

Further notes

- Norm and convex quadratic (including portfolio) optimization prob's can be solved with almost the same efficiency as LO.
- Efficient tools to eigenvalue, singular value optimization, LMI's
- CLO based approximation algorithms for nonconvex and combinatorial optimization problems.
- Lots of activity in exploring special structure of conic problems and developing modeling systems that support conic formulation
- First HPC-massively parallel implementations
- Cheap first order methods for very large scale SDO problems.
- Warm start and decomposition/cutting plane algorithms.
- **http://sedumi.ie.lehigh.edu**
  Now: Adding logarithmic objective.

Disjunctive Conic Cuts for Mixed Integer Second Order Cone Optimization (MISOCO)

Tamás Terlaky

Joint work with:
Pietro Belotti, Julio C. Góez, Imre Pólik, Ted Ralphs

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CONTENTS

MIXED INTEGER SECOND ORDER CONE OPTIMIZATION (MISOCO)

$$\begin{aligned} \text{minimize: } & c^T x \\ \text{subject to: } & Ax = b \\ & x \in \mathcal{K} \\ & x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{aligned} \qquad \text{(MISOCO)}$$

where,

- $A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m$
- $\mathbb{L}^n = \{x | x_1 \geq \|x_{2:n}\|\}$
- $\mathcal{K} = \mathbb{L}_1^{n_1} \times \cdots \times \mathbb{L}_k^{n_k}$
- Rows of A are linearly independent

Objectives

- Obtain the convex hull after applying a linear disjunction to a Mixed Integer Second Order Conic Optimization (MISOCO) problem.
- Design Disjunctive Conic Cuts for MISOCO.

Previous work

- Atamtürk and Narayanan (2010), conic cuts for general MISOCO problems.
- Drewes (2009), nonlinear cuts for 0-1 MISOCO problems.
- Krokhmal and Soberanisin(2010), Drewes (2009), Vielma et al. (2008), branch and bound algorithm based on linear outer approximations for Second Order Cones.
- Drewes (2009), Atamtürk and Narayanan (2009), lifting techniques for MISOCO problems.
- Çezik and Iyengar (2005), cuts for mixed 0-1 conic programming.
- Stubbs and Mehrotra (1999), lift-and-project method for 01 mixed convex programming.

APPLICATIONS

- Turbine balancing problems can be modeled as MISOCOs, White (1996).
- The euclidean Steiner tree problem can be formulated as a MISOCO, Fampa and Maculan (2004)
- Computer Vision and Pattern Recognition, Kumar, Torr, and Zisserman (2006).
- Cardinality-constrained portfolio optimization problems, Bertsimas and Shioda (2009).

SINGLE CONE PROBLEM

Let us consider the special case:

$$\begin{aligned} \text{minimize: } & c^T x \\ \text{subject to: } & Ax = b \\ & x \in \mathbb{L}^n \\ & x \in \mathbb{Z}^d \times \mathbb{R}^{n-d}, \end{aligned} \qquad \text{(MISOCO)}$$

- This problem has a single second order cone
- All the variables are in the single second order cone

STEP 1: SOLVE THE RELAXED PROBLEM

Find the optimal solution x^* for the continuous relaxation of the MISOCO problem

$$\begin{array}{lrrrrl} \text{minimize:} & 3x_1 & +2x_2 & +2x_3 & +x_4 & \\ \text{subject to:} & 9x_1 & +x_2 & +x_3 & +x_4 & = 10 \\ & & & (x_1, x_2, x_3, x_4) & \in \mathbb{L}^4 & \\ & & & x_4 & \in \mathbb{Z}. & \end{array}$$

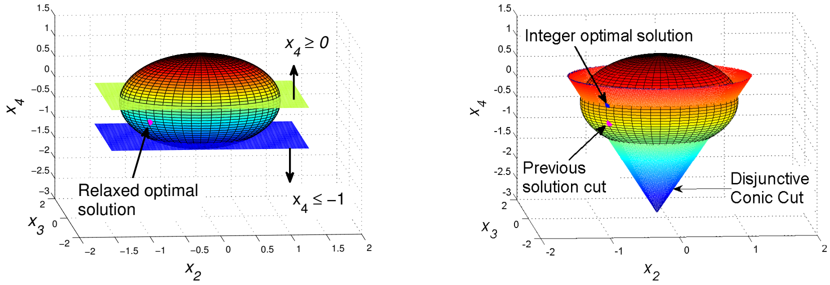
Relaxing the integrality constraint we get the optimal solution:

$$x^* = (1.36, -0.91, -0.91, -0.45),$$

with and optimal objective value: $z^* = 0.00$.

STEP 2: FIND A DISJUNCTION $a^T x \leq \beta \bigvee a^T x \geq \beta$ VIOLATED BY $x^* = (1.36, -0.91, -0.91, -0.45)$

The disjunction $x_4 \leq -1 \bigvee x_4 \geq 0$ is violated by x^*



Step 3: Apply the disjunction and Convexify

$$
\begin{array}{rrrrrrrrl}
\text{min:} & 3x_1 & +2x_2 & +2x_3 & +x_4 & & & & \\
\text{s.t:} & 9x_1 & +x_2 & +x_3 & +x_4 & & & & = 10 \\
 & & -0.04x_2 & -0.04x_3 & -3.56x_4 & +x_5 & & & = 10.14 \\
 & & -6.28x_2 & -6.28x_3 & +0.14x_4 & & +x_6 & & = 1.65 \\
 & & 6.36x_2 & -6.36x_3 & & & & +x_7 & = 0 \\
\end{array}
$$

$$
(x_1, x_2, x_3, x_4) \in \mathbb{L}^4
$$

$$
(x_5, x_6, x_7) \in \mathbb{L}^3
$$

$$
x_4 \in \mathbb{Z}.
$$

The constraints in red represent the disjunctive conic cut.
An integer optimal solution is obtained after adding one cut:

$$
x^* = (1.32,\ -0.93,\ -0.93,\ 0.00,\ 10.06,\ -10.06,\ 0.00),
$$

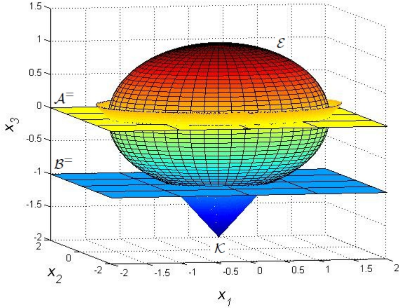
with and optimal objective value: $z^* = 0.24$.

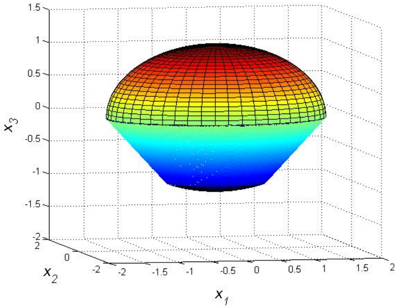
Convex hull of the intersection of a disjunction and a convex set

Consider a closed convex set $\mathcal{E}$ and two halfspaces

$$\mathcal{H}_1 = \{x \in \mathbb{R}^n : a^\top x \leq \alpha\} \text{ and } \mathcal{H}_2 = \{x \in \mathbb{R}^n : b^\top x \leq \beta\},$$

such that they do not intersect inside $\mathcal{E}$, i.e., $\mathcal{E} \cap \mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$.
Denote $\mathcal{H}_1^= = \{x \in \mathbb{R}^n : a^\top x = \alpha\}$, and $\mathcal{H}_2^= = \{x \in \mathbb{R}^n : b^\top x = \beta\}$.
If $\exists$ a convex cone $\mathcal{K}$ s.t. $\mathcal{H}_1^= \cap \mathcal{E} = \mathcal{K} \cap \mathcal{H}_1^=$ and $\mathcal{H}_2^= \cap \mathcal{E} = \mathcal{K} \cap \mathcal{H}_2^=$ are bounded, then $\operatorname{conv}(\mathcal{E} \cap (\mathcal{H}_1 \cup \mathcal{H}_2)) = \mathcal{E} \cap \mathcal{K}$.





INTERSECTION OF AN AFFINE SPACE AND A SECOND ORDER CONE

Consider an affine subspace $\mathcal{H} = \{x|Ax = b\}$ and $x_0 \in \mathcal{H}$.
Let $H \perp A$ be s.t rank$([H, A^T]) = n$, & columns of H are orthonormal.
We can write $\mathcal{H} = \{x|x = x_0 + Hz, \ \forall z \in \mathbb{R}\}$.
Then, there exist a matrix $Q \in \mathbb{R}^{n-m \times n-m}$, $q \in \mathbb{R}^{n-m}$, $\rho \in \mathbb{R}$, s.t.

$$\mathcal{H} \cap \mathbb{L}^n = \{y|x = x_0 + Hz \text{ with } z^\top Qz + 2q^\top z + \rho \leq 0\}.$$

Further, Q has at most one negative eigenvalue.
Define a quadric as the set $\mathcal{Q} = \{z|z^\top Qz + 2q^\top z + \rho \leq 0\}$,
which we also denote as $\mathcal{Q} = (Q, q, \rho)$.

UNI-PARAMETRIC FAMILY OF QUADRICS $\mathcal{Q}(\tau)$

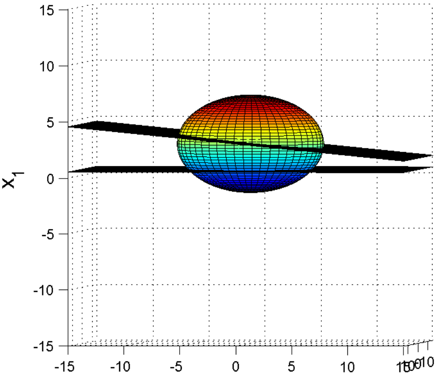
Given two hyperplanes $\mathcal{H}_1 = \{z | a_1^\top z = \alpha_1\}$ and $\mathcal{H}_2 = \{z | a_2^\top z = \alpha_2\}$.
Let $\mathcal{Q} = (Q, q, \rho)$ be a quadric where Q is positive definite.
The family of quadrics having the same intersection with $\mathcal{H}_1$ and $\mathcal{H}_2$ as the quadric $\mathcal{Q}$ is parametrized by $\tau \in \mathbb{R}$ as $\mathcal{Q}(\tau)$, where

$$\begin{aligned} Q(\tau) &= Q + \tau \frac{a_1 a_2^T + a_2 a_1^T}{\omega} \\ q(\tau) &= q - \tau \frac{\alpha_2 a_1 + \alpha_1 a_2}{\omega} \\ \rho(\tau) &= \rho + 2\tau \frac{\alpha_1 \alpha_2}{\omega}, \end{aligned}$$

where

$$\omega = \begin{cases} 2 a_1^T a_2 & \text{if } a_1^T a_2 \neq 0 \\ 1 & \text{if } a_1^T a_2 = 0. \end{cases}$$

Uni-parametric family of quadrics $\mathcal{Q}(\tau)$



Sequence of quadrics $x^\top Q(\tau)x + 2q(\tau)^\top x + \rho(\tau) \leq 0$, for $-106.863 \leq \tau \leq 1617$

UNI-PARAMETRIC FAMILY OF QUADRICS $\mathcal{Q}(\tau)$

Range	Description
$\tau = -8.9946, \tau = 1617$	*Paraboloid*
$\tau = -106.863, \tau = -9.581$	*Cones*
$-8.9946 < \tau < 1617$	*Ellipsoids*
$\tau > 1617$	*Two sheets hyperboloids*
$-106.863 < \tau < -8.9946$	*One sheet hyperboloids*
$\tau < -106.863,$	*Two sheets hyperboloids*

Behavior of the quadrics for different ranges of τ

THE DISJUNCTIVE CONIC CUT FOR PARALLEL DISJUNCTIONS

Theorem

Let $\mathcal{A}_1 = \{x | a_1^\top x = \alpha_1\}$ *and* $\mathcal{A}_2 = \{x | a_2^\top x = \alpha_2\}$*, be two parallel hyperplanes where* $a_1 = \gamma a_2$*.*
The disjunctive conic cut is the quadric generated by

$$\mathcal{Q}(\hat{\tau}) = (Q(\hat{\tau}), q(\hat{\tau}), \rho(\hat{\tau})),$$

where $\hat{\tau}$ *is the larger root of equation*

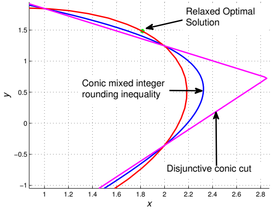
$$q(\tau)^\top Q(\tau) q(\tau) - \rho(\tau) = 0.$$

OUR DISJUNCTIVE CONIC CUT IS NEW

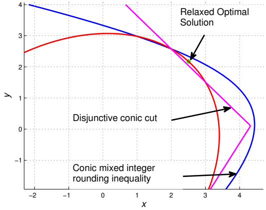
Atamtürk and Narayanan designed a conic mixed integer rounding inequality.
Our disjunctive conic cut is different, sometimes stronger.
Consider the problem:

$$
\begin{array}{llllll}
\text{minimize:} & -x & -y & & & \\
\text{subject to:} & x & +y & +2t & = & \alpha \\
& \multicolumn{3}{l}{\sqrt{(x-\frac{4}{3})^2+(y-1)^2}} & \leq & t \\
& \multicolumn{5}{c}{x \in \mathbb{Z}, y \in \mathbb{R}}
\end{array}
$$

In this particular example our disjunctive conic cut is stronger.



Case with $\alpha = 4.66$



Case with $\alpha = 8$

CONCLUSIONS

- We developed a new disjunctive conic cut for MISOCO.
- It is algebraically simple to find the disjunctive conic cut for MISOCO problems.

Next steps

- Develop disjunctive conic cuts for the case when Q is not positive definite.
- Develop a prototype branch-and-cut framework for solving MISOCO problems using disjunctive conic cuts.
- Develop strategies which, and how many disjunctive conic cuts to generate when several cones are in the problem.
- Develop a comprehensive branch-and-cut framework for solving MISOCO problems using disjunctive conic cuts.

Far future work

- Develop disjunctive conic cuts for Semidefinite Optimization.