Submodular Optimization and Approximation Algorithm

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Outline

- Submodular Functions
  - Examples
  - Discrete Convexity
- Submodular Function Minimization
- Approximation Algorithms
- Submodular Function Maximization
- Approximating Submodular Functions

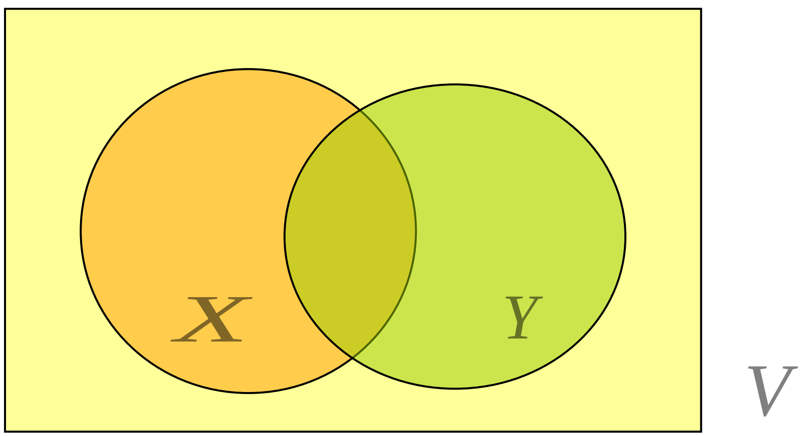
Submodular Functions

V : Finite Set

$f : 2^V \to \mathbf{R}$ $\quad\quad \forall X, Y \subseteq V$

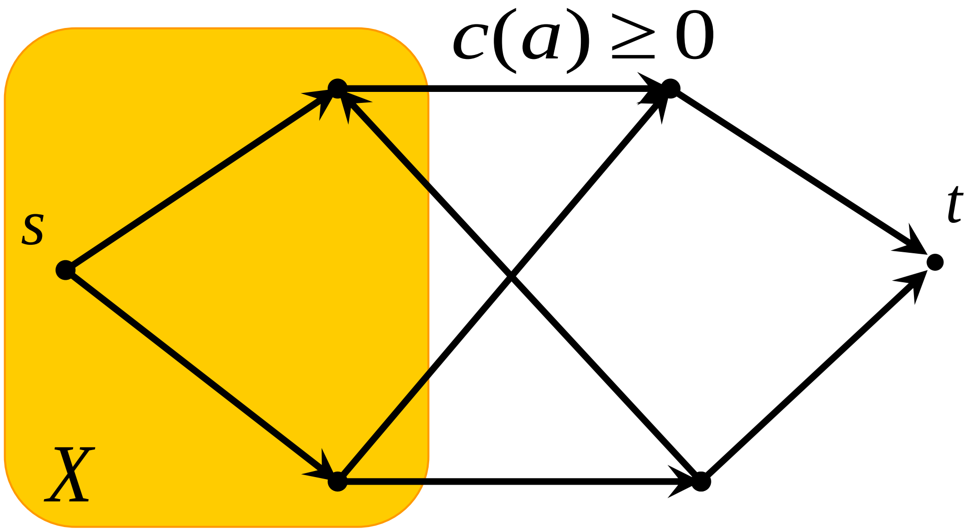
$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

- Cut Capacity Functions
- Matroid Rank Functions
- Entropy Functions



Cut Capacity Function

Cut Capacity $\kappa(X) = \sum \{c(a) \mid a : \text{leaving } X\}$



Max Flow Value = Min Cut Capacity

Matroids

$\mathrm{M} = (V, \Im)$

Whitney (1935)

$\phi \in \Im$

$I \subseteq J \in \Im \Rightarrow I \in \Im$

$\forall I, J \in \Im, |I| < |J| \Rightarrow \exists j \in J - I, I \cup \{j\} \in \Im$

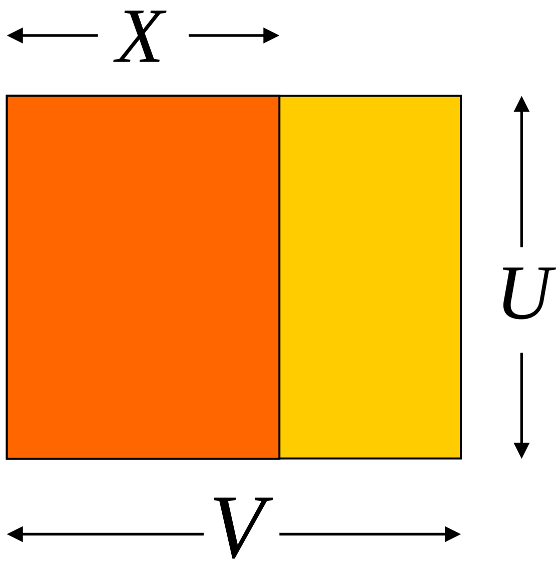
Rank Function

$$\rho(Y) = \max\{|J| : J \subseteq Y, J \in \Im\}$$

Matroid Rank Functions

Matrix Rank Function

$$\rho(X) = \mathrm{rank}\, A[U, X]$$

$A =$ 

$$\rho : 2^V \to \mathbf{Z}_+$$

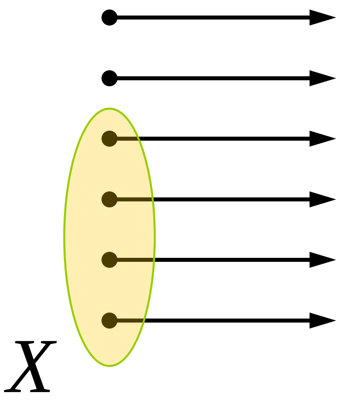
$$\forall X \subseteq V, \rho(X) \le |X|.$$

$$X \subseteq Y \Rightarrow \rho(X) \le \rho(Y).$$

ρ : Submodular.

Entropy Functions

Information Sources



$$h(\phi) = 0$$

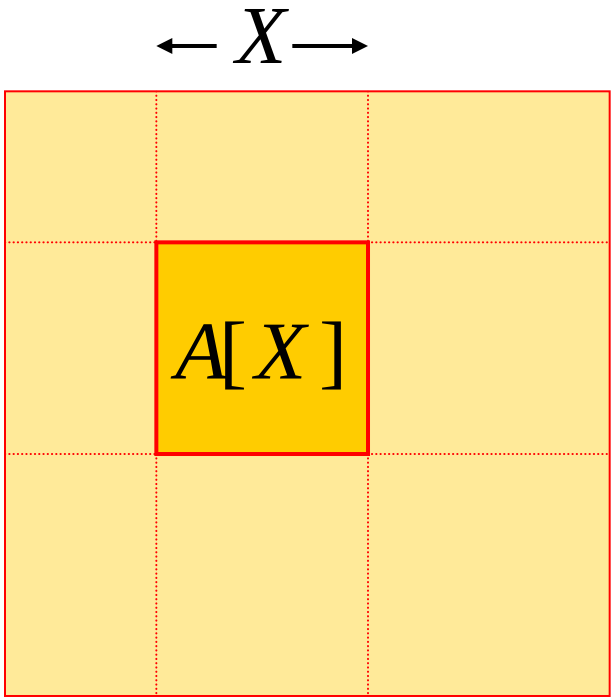
$h(X)$: Entropy of the Joint Distribution

$$h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y)$$

Conditional Mutual Information ≥ 0

Positive Definite Symmetric Matrices

$A =$



$$f(\phi) = 0$$

$$f(X) = \log \det A[X]$$

Ky Fan's Inequality

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

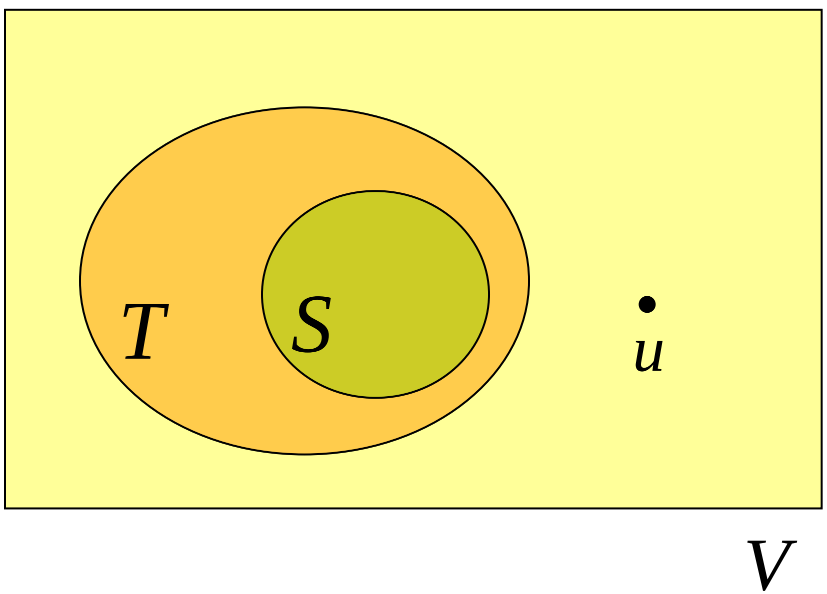
Extension of the Hadamard Inequality

$$\det A \leq \prod_{i \in V} A_{ii}$$

Discrete Concavity

$$S \subseteq T \Rightarrow$$

$$f(S \cup \{u\}) - f(S) \geq f(T \cup \{u\}) - f(T)$$

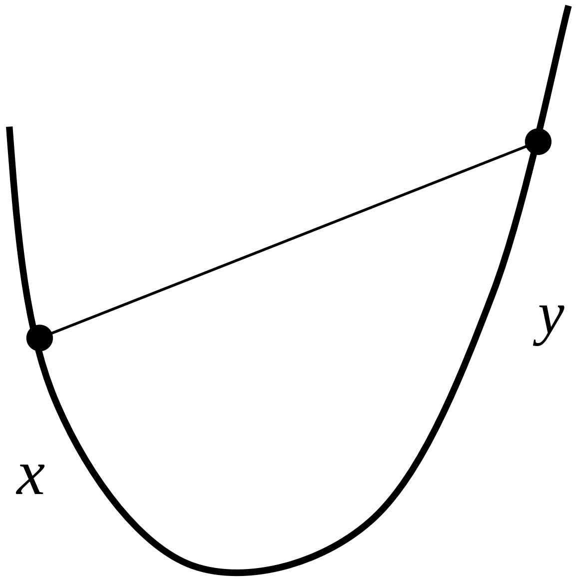


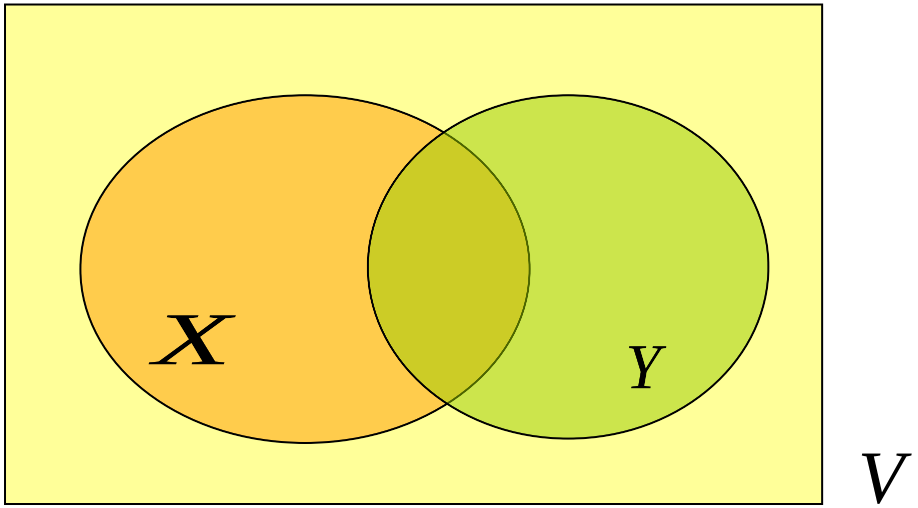
Diminishing Returns

Discrete Convexity

Convex Function

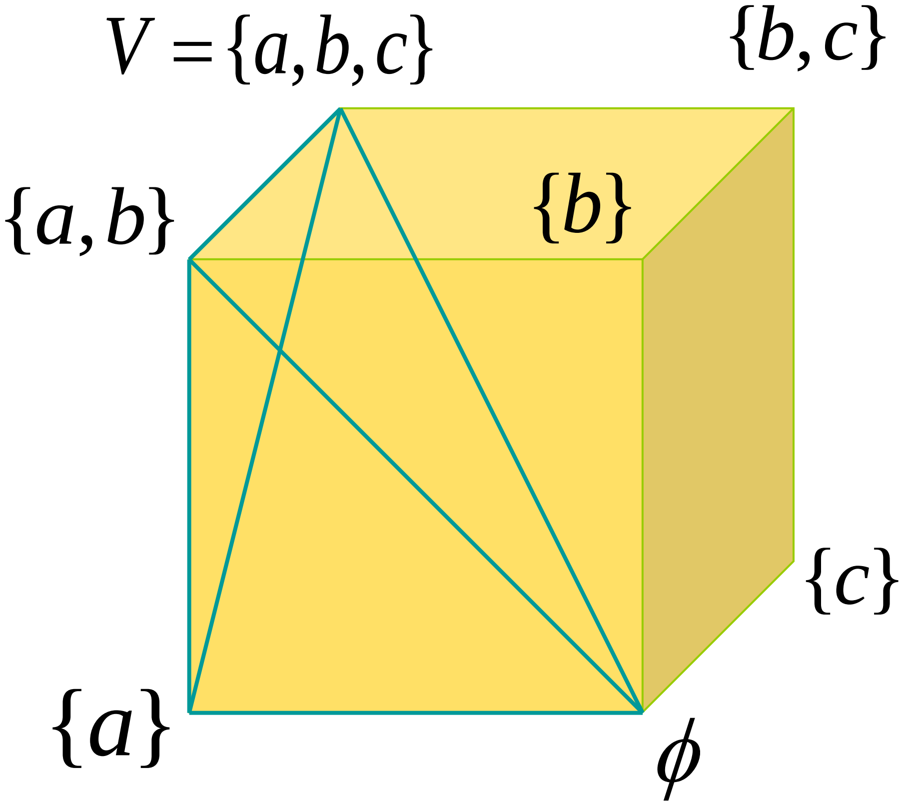
$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$





Discrete Convexity

Lovász (1983)



$\hat{f}$: Linear Interpolation

$\hat{f}$: Convex

$\Updownarrow$

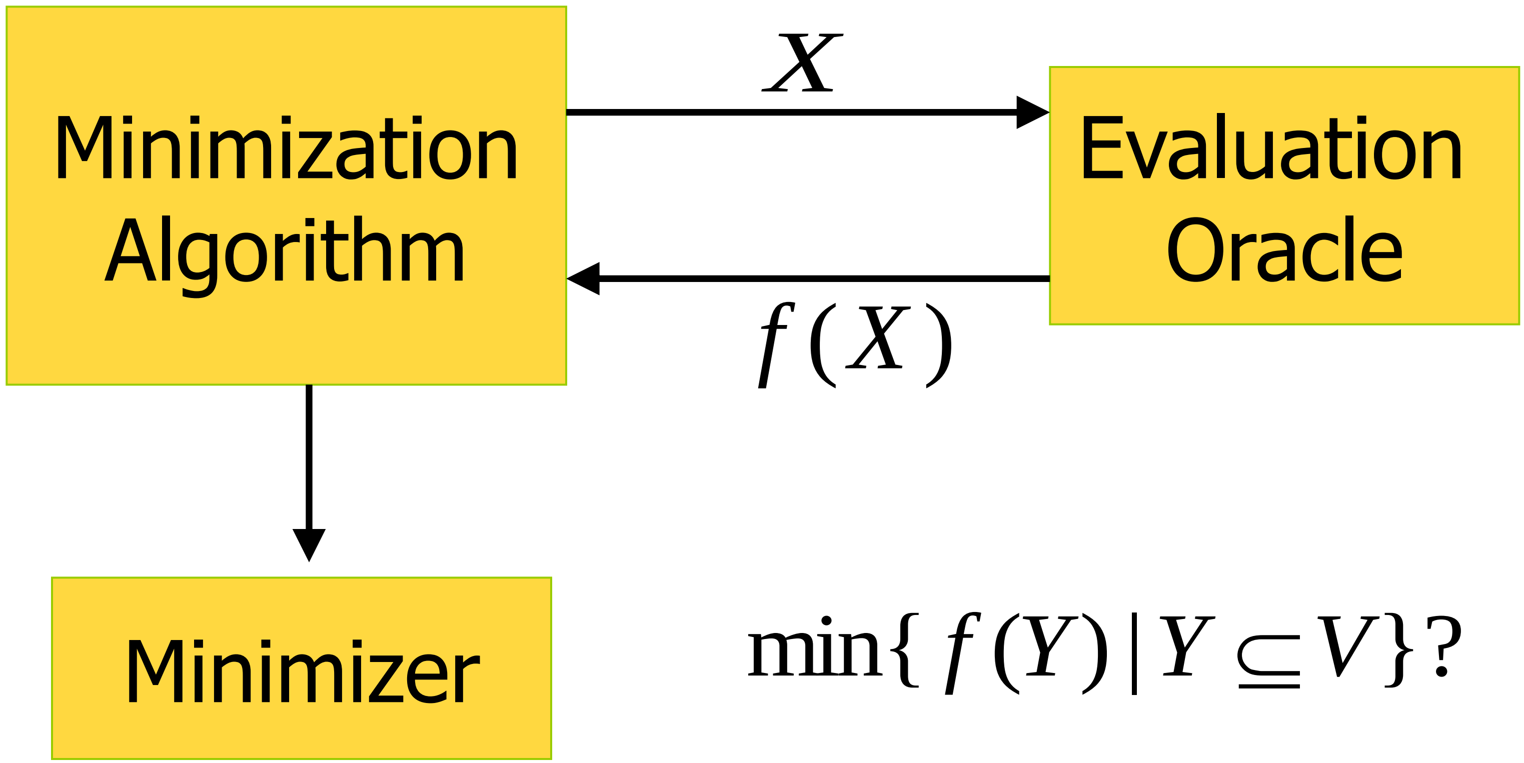
f : Submodular

$\theta \in [0,1]$: Chosen Uniformly at Random

$X := \{v \mid x(v) \geq \theta\}, \qquad \hat{f}(x) = \mathrm{E}[f(X)]$

Submodular Function Minimization

Assumption: $f(\phi) = 0$

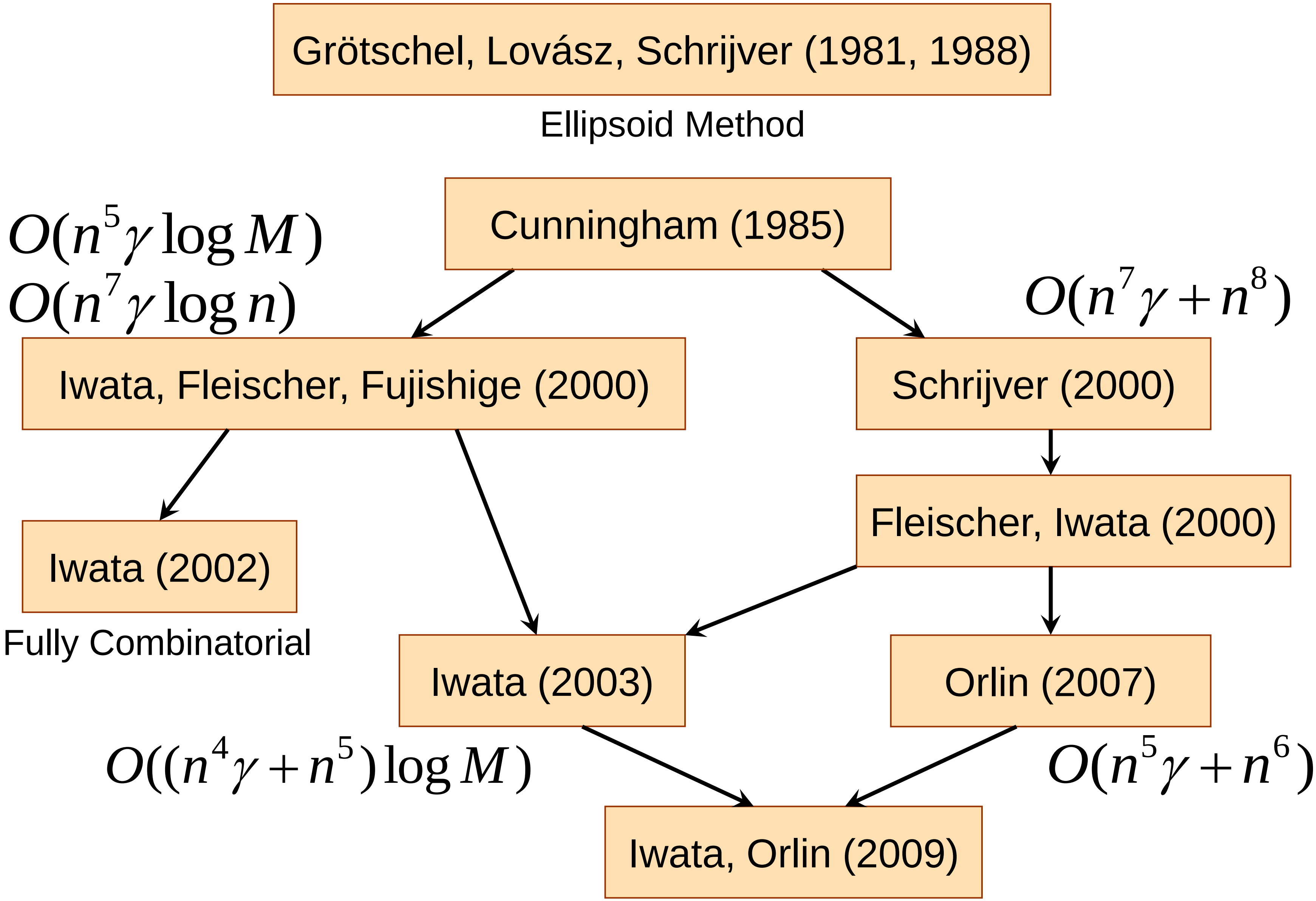


$$\min\{ f(Y) \mid Y \subseteq V \} ?$$

Ellipsoid Method

Grötschel, Lovász, Schrijver (1981)

Submodular Function Minimization



Symmetric Submodular Functions

$$f : 2^V \to \mathbf{R}$$

Symmetric $f(X) = f(V \setminus X), \quad \forall X \subseteq V.$

Symmetric Submodular Function Minimization

$$\min\{ f(X) \mid \phi \neq X \neq V \}?$$

$O(n^3\gamma)$ Queyranne (1998)

Minimum Cut Algorithm by MA-ordering

Nagamochi & Ibaraki (1992)

Minimum Degree Ordering Nagamochi (2007)

Submodular Partition

Minimize $\sum_{i=1}^{k} f(V_i)$

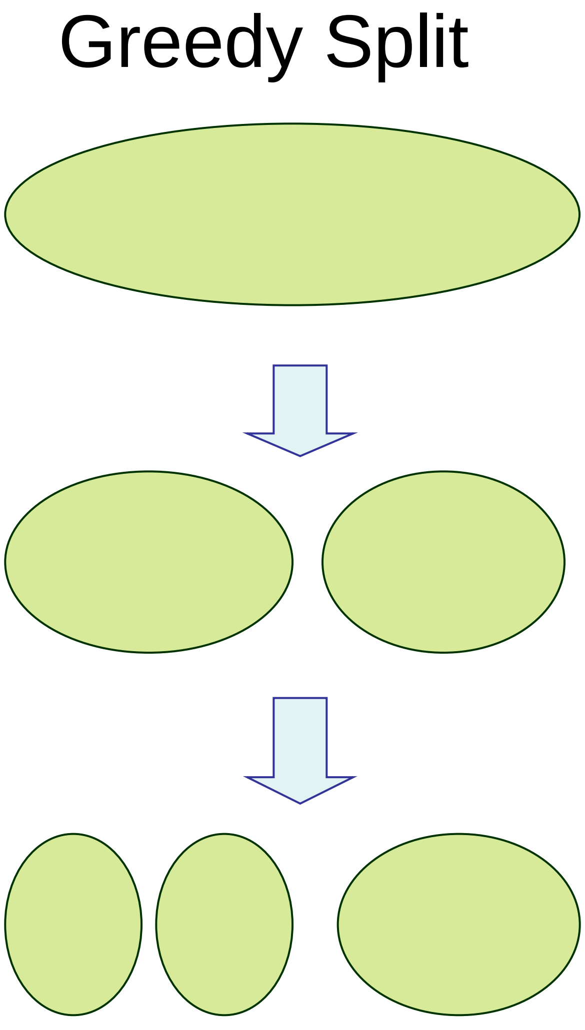
subject to $V = V_1 \cup \cdots \cup V_k$

$V_i \cap V_j = \phi \ (i \neq j)$

f : Monotone or Symmetric

$2(1-1/k)$-Approximation

Zhao, Nagamochi, Ibaraki (2005)



Questions

What Kind of Approximation Algorithms
Can Be Extended to Optimization Problems
with Submodular Cost or Constraints ?

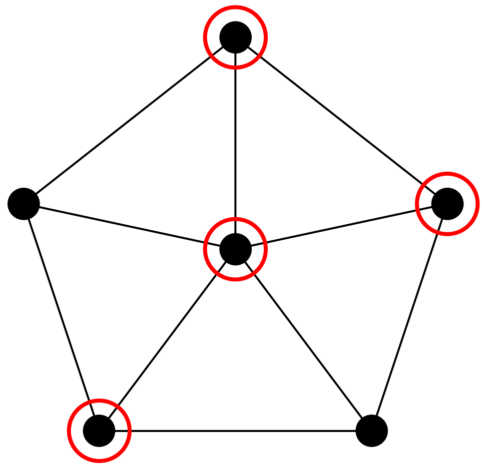
Cf. Submodular Flow (Edmonds & Giles, 1977)

How Can We Exploit Discrete Convexity
in Design of Approximation Algorithms?

Cf. Ellipsoid Method
(Grötschel, Lovász & Schrijver, 1981)

Submodular Vertex Cover

Graph $G = (V, E)$



Submodular Function

$$f : 2^V \to \mathbf{R}_+$$

Find a Vertex Cover $S \subseteq V$ Minimizing $f(S)$

2-Approximation Algorithm

Goel, Karande, Tripathi, Wang (FOCS 2009)

Iwata & Nagano (FOCS 2009)

Relaxation Problem

Convex Programming Relaxation (CPR)

Minimize $\hat{f}(x)$

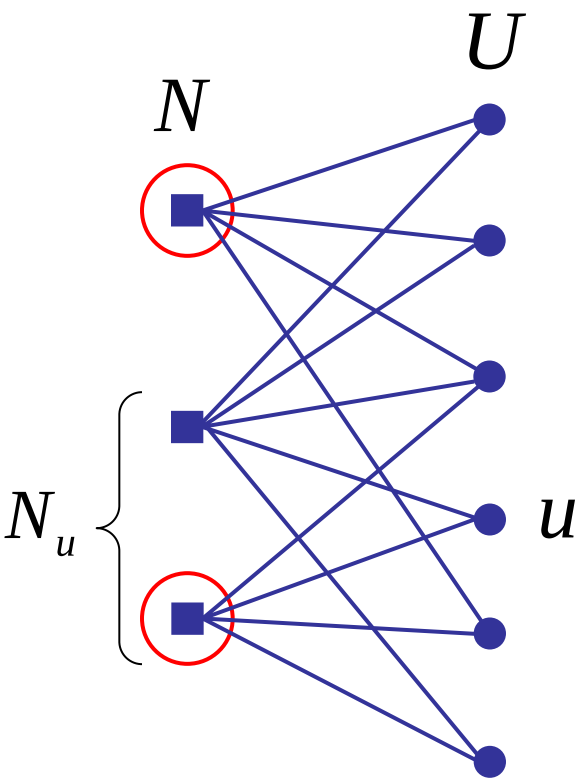
subject to $x(u)+x(v)\geq 1 \quad (\forall e=(u,v)\in E)$

$$x(v)\geq 0 \quad (\forall v\in V)$$

CPR has a half-integral optimal solution.

→ 2-Approximation Algorithm

Submodular Cost Set Cover



Find $X \subseteq N$ Covering U with Minimum $f(X)$.

$$\eta := \max_{u \in U} |N_u|$$

η-Approximation

Rounding Algorithm

Primal-Dual Algorithm

Koufogiannakis & Young (ICALP 2009)

Greedy η -Approximation Algorigm for Monotone Submodular Functions

Submodular Multiway Partition

Chekuri & Ene (FOCS 2011)

Minimize $\sum_{i=1}^{k} f(V_i)$

subject to $V = V_1 \cup \cdots \cup V_k$

$V_i \cap V_j = \phi \ (i \neq j)$

$s_i \in V_i \quad (i = 1, \ldots, k)$

f : Nonnegative Submodular

Relaxation Problem

Convex Programming Relaxation

Minimize $\displaystyle\sum_{i=1}^{k} \hat{f}(x_i)$

subject to $\displaystyle\sum_{i=1}^{k} x_i(v) = 1 \quad (\forall v \in V)$

$$x_i(s_i) = 1 \quad (i = 1, \ldots, k)$$

$$x_i(v) \geq 0 \quad (i = 1, \ldots, k, \forall v \in V)$$

Ellipsoid Method ⟶ x_i^* : Optimal Solution

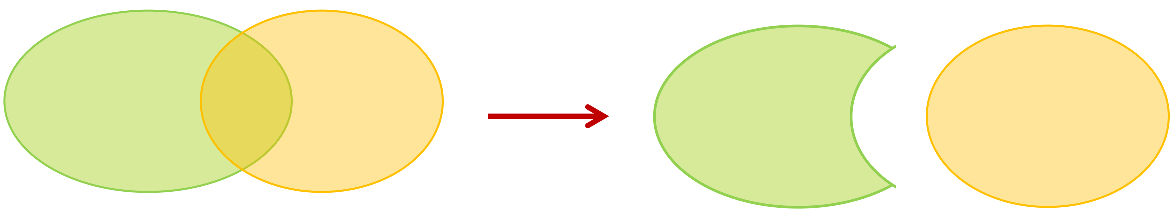
Rounding Scheme

f : Symmetric Submodular

$\theta \in [0,1]$: Chosen Uniformly at Random

$$A_i := \{v \mid x_i(v) \geq \theta\} \quad (i = 1,\ldots,k), \qquad U := V - \bigcup_{i=1}^{k} A_i$$

Uncross $(A_1,\ldots,A_k)$



Return $(A_1,\ldots,A_{k-1},A_k \cup U)$

→ 3/2-Approximate Solution

Rounding Scheme

f : Nonnegative Submodular

$\delta \in (\frac{1}{2}, 1]$: Chosen Uniformly at Random

$A_i := \{v \mid x_i(v) \geq \theta\} \quad (i = 1, \ldots, k), \qquad U := V - \bigcup_{i=1}^{k} A_i$

Return $(A_1, \ldots, A_{k-1}, A_k \cup U)$

$\longrightarrow$ 2-Approximate Solution

Improvement over the $(k-1)$-Approximation by Zhao, Nagamochi, & Ibaraki (2005)

Submodurlar Function Maximization

Approximation Algorithms

Nemhauser, Wolsey, Fisher (1978)

Monotone Submodular Function

Cardinality Constraint

(1−1/e)-Approximation

Calinescu, Chekuri, Pál, Vondrák (IPCO 2007)

Vondrák (STOC 2008)

Monotone Submodular Function

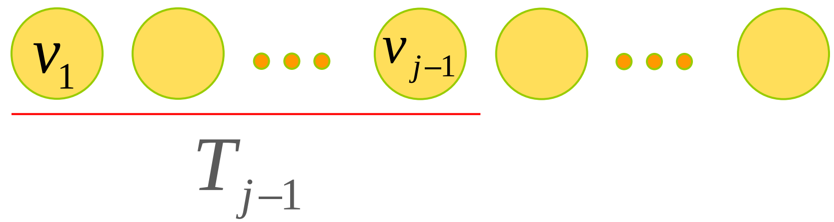
Matroid Constraint

(1−1/e)-Approximation

Cardinality Constraint

Maximize $f(S)$ subject to $|S| \leq k$.

Greedy Algorithm



$$T_0 := \phi$$

$$\left.\begin{aligned} v_j &:= \arg\max\{ f(T_{j-1} \cup \{v\}) \mid v \in V \setminus T_{j-1} \} \\ T_j &:= T_{j-1} \cup \{v_j\} \end{aligned}\right\} j = 1, \ldots, k$$

$$f(T_k) \geq \left(1 - 1/\mathrm{e}\right) f(S), \quad \forall S : |S| \leq k$$

Matroid Constraint

Matroid $\mathrm{M} = (V, \mathfrak{I})$

Monotone Submodular Function f

Maximize $f(S)$

subject to $S \in \mathfrak{I}$

$(1-1/e)$ -Approximation

Vondrák (STOC 2008)

Multilinear Extension

$p \in [0,1]^V$

R_p : Random Subset $\qquad \Pr\{v \in R_p\} = p(v)$

$$\bar{f}(p) := \mathrm{E}[f(R_p)] = \sum_{X \subseteq V} f(X) \prod_{v \in X} p(v) \prod_{u \in V \setminus X} p(u)$$

Concave along Nonnegative Direction

Convex along the Direction of $\chi_u - \chi_v$

Continuous Relaxation

Maximize	$\bar{f}(p)$
subject to	$p \in \mathrm{MP}(\rho)$

Matroid Polytope

$$\mathrm{MP}(\rho) = \mathrm{conv}\{\chi_J \mid J \in \mathfrak{I}\}$$

$$= \{y \mid y \in \mathrm{R}_+^V, \forall X \subseteq V, y(X) \le \rho(X)\}$$

Continuous Greedy Process

$$y \in \mathrm{MP}(\rho)$$

$$z(y) \in \arg\max\{ z \cdot \nabla f \mid z \in \mathrm{MP}(\rho)\}$$

Differential Equation

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, t} = z(y)$$

Initial Value

$$y(0) = 0$$

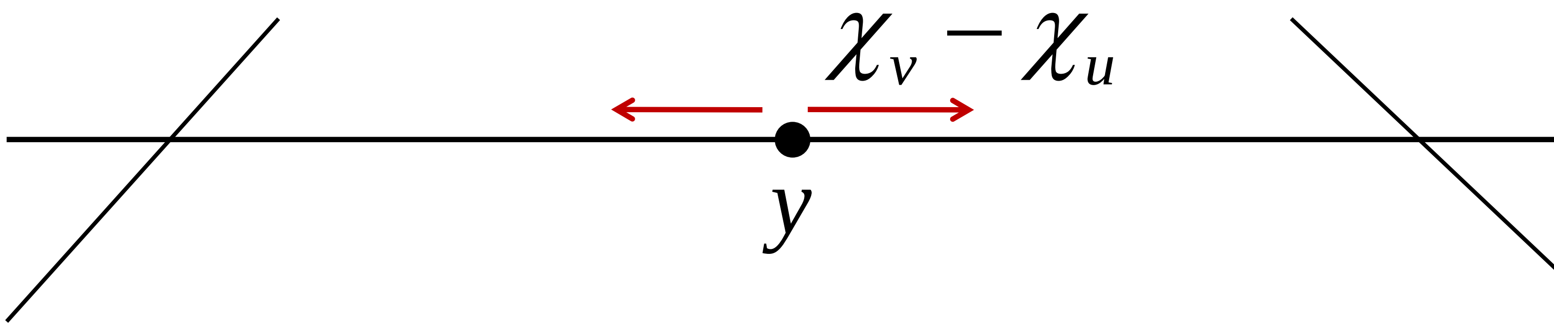
$$y(1) \in \mathrm{MP}(\rho), \qquad \bar{f}(y(1)) \geq (1 - 1/\mathrm{e})\, \bar{f}(y^*)$$

y^* : Optimal Solution

Pipage Rounding

Given $y \in \mathrm{MP}(\rho)$,

Find $S \in \mathfrak{I}$ such that $\mathrm{E}[\bar{f}(S)] \geq \bar{f}(y)$



$\mathrm{E}[y'] = y$

$\mathrm{E}[\bar{f}(y')] \geq \bar{f}(y)$

Jensen's Inequality

Submodular Welfare Problem

Utility Functions $f_1, \ldots, f_k$ (Monotone, Submodular)

Maximize $\sum_{i=1}^{k} f_i(V_i)$

subject to $V = V_1 \cup \cdots \cup V_k$

$V_i \cap V_j = \phi \ (i \neq j)$

$(1 - 1/e)$ -Approximation Vondrák (2008)

Submodurlar Function Maximization

Approximation Algorithms

Feige, Mirrokni, Vondrák (FOCS 2007)

Nonnegative Submodular Function

2/5-Approximation

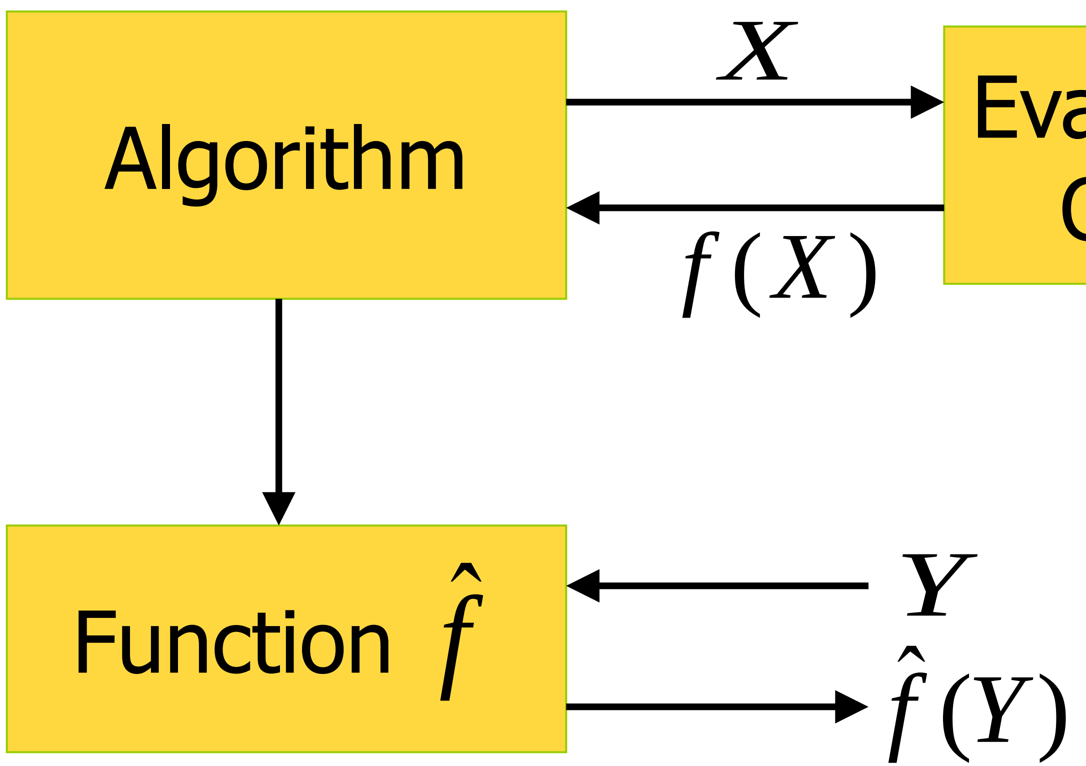
Lee, Mirrokni, Nagarajan, Sviridenko (STOC 2009)

Nonnegative Submodular Function

1/4-Approximation (Matroid Constraint)

1/5-Approximation (Knapsack Constraints)

Approximating Submodular Functions



Approximating Submodular Functions

Goemans, Harvey, Iwata & Mirrokni (2009)

$$f(\phi) = 0, \quad f(X) \geq 0, \forall X \subseteq V.$$

Construct a set function $\hat{f}$ such that

$$\hat{f}(X) \leq f(X) \leq \alpha(n)\hat{f}(X), \quad \forall X \subseteq V.$$

For what function α is this possible?

Algorithm with $\alpha(n) = O(\sqrt{n}\log n)$ for monotone submodular functions

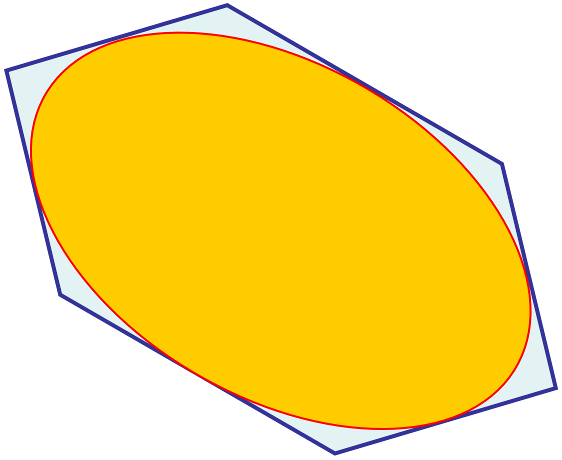
Ellipsoidal Approximation

K : Centrally Symmetric Convex Body

$$(x \in K \Rightarrow -x \in K)$$

$E(A)$: Maximum Volume Inscribed Ellipsoid

(The John Ellipsoid)



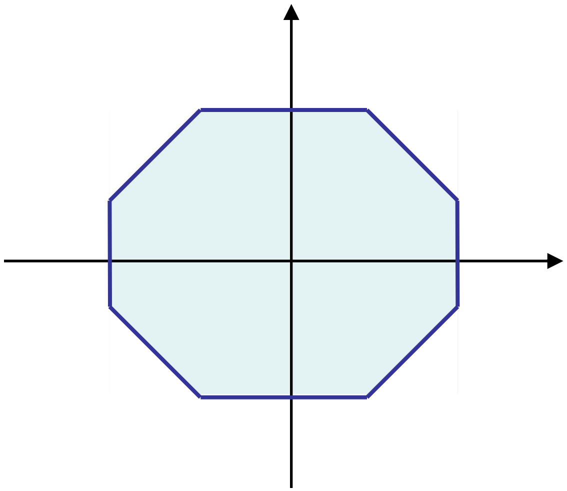
$$E(A) \subseteq K \subseteq \sqrt{n}E(A)$$

Symmetrized Polymatroids

f : Monotone Submodular Function $f(\phi) = 0$

$$P(f) = \{x \mid x \in \mathbf{R}_+{}^V, \sum_{v \in S} x(v) \le f(S), \forall S \subseteq V\}$$

$$Q(f) = \{x \mid x \in \mathbf{R}^V, \sum_{v \in S} | x(v) | \le f(S), \forall S \subseteq V\}$$



John Ellipsoid of $Q(f)$

Axis-Aligned $E(D)$

D : Diagonal

Symmetrized Polymatroids

$E(D)$: John Ellipsoid of $Q(f)$ $\qquad p(u) = \dfrac{1}{D_{uu}} \quad (u \in V)$

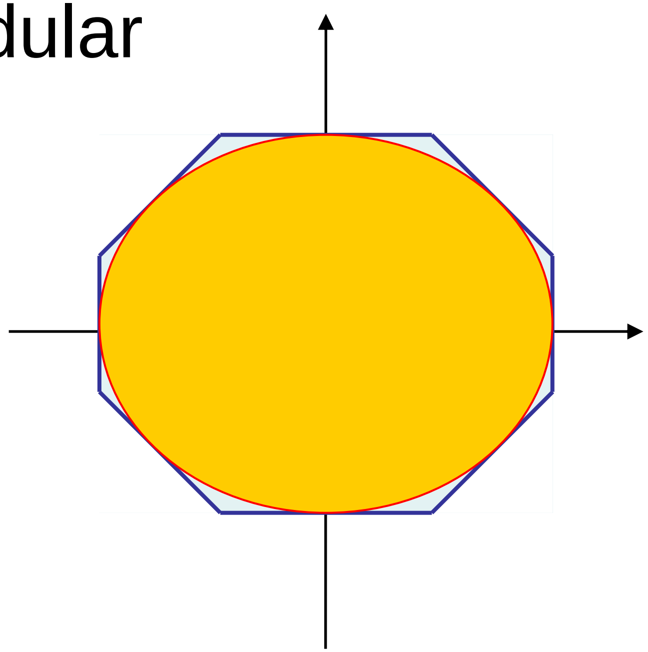
$$\hat{f}(S) := \sqrt{\sum_{u \in S} p(u)} = \max\{\chi_S y \mid y \in E(D)\}$$

$\hat{f}$: Monotone Submodular

$$f(S) = \max\{\chi_S y \mid y \in Q(f)\}$$

$$E(D) \subseteq Q(f) \subseteq \sqrt{n}E(D)$$

$$\hat{f}(S) \leq f(S) \leq \sqrt{n}\,\hat{f}(S)$$



Submodular Load Balancing

Svitkina & Fleischer (2008)

$f_1, \ldots, f_m$: Monotone Submodular Functions

$$\min_{\{V_1, \ldots, V_m\}} \max_j f_j(V_j) ?$$

$$f_j(X) := \sum_{v \in X} p_j(v)$$ → Scheduling

2-Approximation Algorithm

Lenstra, Shmoys, Tardos (1990)

$O(\sqrt{n} \log n)$-Approximation Algorithm

Summary

- Examples and Discrete Convexity of Submodular Functions.
- Combinatorial Strongly Polynomial Algorithms for Submodular Function Minimization.
- Design of Approximation Algorithms Using Discrete Convexity.
- Constant Factor Approximation Algorithms for Submodular Function Maximization.
- Approximating Submodular Functions Everywhere.