The Basis Partition of the Space of Linear Programs

Gongyun Zhao
Department of Mathematics
National University of Singapore
Republic of Singapore

Thank Josef Stoer for constructive discussions, and thank Wingkeung To, Xingwang Xu, Deqi Zhang and Chengbo Zhu for providing consultations on Geometry and pointing out useful literature.

A relevant paper is available at http://www.math.nus.edu.sg/ matzgy/publist.html.

1 Goals and Motivations

1.1 The space of LP

Linear program (LP):

$$
\begin{array}{llcll}
\min & c^T x & \qquad & \max & b^T y \\
s.t. & Ax = b & & s.t. & A^T y + s = c \\
 & x \geq 0 & & & s \geq 0
\end{array}
$$

where $A \in R^{m \times n}$ is of full row rank.

- (A, b, c) = an LP instance,
- $SLP(n, m)$ = the space of LP, i.e. the collection of (A, b, c).

1.2 The basis partition

For any (A, b, c) there exists a unique index partition (B, N) of $\{1, \ldots, n\}$ such that there is a strictly complementary solution (x, s, y) satisfying

$$Ax = b,\ A^T y + s = c,\ x_B > 0,\ x_N = 0,\ s_B = 0,\ s_N > 0.$$

- (B, N) (or B) is called the basis of (A, b, c). There are a finite number of bases $B_1, \ldots, B_L$.
- $SLP(B_k)$ = the set of all (A, b, c) whose basis is B_k.
- $\{SLP(B_1), \ldots, SLP(B_L)\}$ is called the basis partition of $SLP(n, m)$.

We are particularly interested in exploring structures of the basis partition of $SLP(n, m)$.

Why are we interested in the basis partition of $SLP(n,m)$?

There are already many powerful methods for solving individual linear program. However, we know little about solving problems consisting of infinitely many LPs. Such problems can be solved by virtue of the basis partition.

• If $(A,b,c) \in SLP(B)$ and $|B| = m$, then we have the closed-form optimal solution $x_B = A_B^{-1}b$.

• We can solve a set $\mathcal{P}$ of infinitely many LP instances in the closed-form, if we can determine the partition $\{\mathcal{P}_1, \ldots, \mathcal{P}_L\}$ of $\mathcal{P}$, where

$$\mathcal{P}_k = \mathcal{P} \cap SLP(B_k).$$

(Some $\mathcal{P}_k$ may be empty.)

1.3 Some applications

- Parametric LP:

$$\begin{aligned} f(\omega) = \quad & \min \quad c(\omega)^T x(\omega) \\ & s.t. \quad A(\omega)x(\omega) \leq b(\omega) \end{aligned}$$

where $\omega \in \Omega \subset R^p$ is a parameter. The problem is to find optimal solutions $x^*(\omega)$ and the function f on Ω.

- Bilevel programming:

$$\begin{aligned} & \min_{\omega \in \Omega} & & f(\omega, x(\omega)) \\ x(\omega) = \quad & \arg\min_{x \in R^n} & & c(\omega)^T x \\ & & & A(\omega)x = b(\omega) \\ & & & x \geq 0. \end{aligned}$$

- Differential complementarity problem:

$$\begin{aligned}\dot{\xi} &= f(\xi, x) \\ 0 &\le x \perp A(\xi)x + b(\xi) \ge 0.\end{aligned}$$

  If we can find the closed-form solution $x(\xi)$ of the LCP (LP is a special LCP), then the problem is reduced to

$$\dot{\xi} = f(\xi, x(\xi)).$$

- Operations on the space of LP, e.g.

  Differentiation, such as differential complementarity problems;

  Integration, such as stochastic programming;

  Transformation;

  ... ...

Remarks:

1. Most applications involve only a subset $\mathcal{P} = \{(A(\omega), b(\omega), c(\omega)) : \omega \in \Omega \subset R^p\} \subset SLP(n, m)$.
2. While the dimension of $SLP(n, m)$ can be high, the dimension of parameter set Ω may be low, e.g. $p = 3$.
3. Partition $\{SLP(B_1), \ldots, SLP(B_L)\}$ on $SLP(n, m)$ induces a partition on Ω:
$\Omega(B_i) = \{\omega \in \Omega : (A(\omega), b(\omega), c(\omega)) \in SLP(B_i)\}$, $i = 1, \ldots, L$.
4. Characterization of the basis partition and induced partitions is fundamentally important for solving these problems.

Up to date, there is no good tool for characterizing the basis partition.

2 A new tool for characterizing the basis partition — A dynamical system on the space of projection matrices

2.1 Mapping (A, b, c) to M

- The central path of (A, b, c), $(x(t), s(t), y(t))$, is defined by

$$
\begin{aligned}
& x \circ s = e^{-t}\mathbb{1} \\
& Ax = b, \quad A^T y + s = c \\
& x \geq 0, s \geq 0.
\end{aligned}
$$

$\mathbb{1}$ = the vector of all ones (regardless of dimension).

- The projection matrix associated with (A, b, c) is defined by

$$\Gamma(A, b, c) = \lfloor x(0) \rceil A^T (A \lfloor x(0) \rceil^2 A^T)^{-1} A \lfloor x(0) \rceil, \tag{2.1}$$

where $\lfloor x \rceil = Diag(x)$.

Γ maps the space of linear programs $SLP(n, m)$ onto the space of projection matrices $G(n, m)$.

$$G(n, m) = \{M \in S^n \mid MM = M, \; rank(M) = m\}$$

is the Grassmann manifold.

S^n is the space of symmetric $n \times n$-matrices.

2.2 The dynamical system on $G(n, m)$

The following differential equation is a key for studying the basis partition of SLP.

$$\frac{dM}{dt} = M\lfloor M\mathbb{1}\rceil + \lfloor M\mathbb{1}\rceil M - 2M\lfloor M\mathbb{1}\rceil M =: h(M). \tag{2.2}$$

Denote by $M(t, M_0)$ the solution of $M' = h(M)$ with $M(0) = M_0$.

Let $x(t)$ be the central path of (A, b, c). Then

$$M(t) = \lfloor x(t)\rceil A^T(A\lfloor x(t)\rceil^2 A^T)^{-1}A\lfloor x(t)\rceil, \quad t \in R,$$

is the solution of $M' = h(M)$ with $M(0) = \Gamma(A, b, c)$.

2.3 Defining the basis of (A, b, c) by $M' = h(M)$

- $M \in G(n, m)$ is an equilibrium point if $h(M) = 0$.

$M \in G(n, m)$ is an equilibrium if and only if
$M = \begin{pmatrix} M_B & 0 \\ 0 & M_N \end{pmatrix}$ with $M_B \mathbb{1}_B = \mathbb{1}_B$ and $M_N \mathbb{1}_N = 0$.

Given (A, b, c), let $M^0 = \Gamma(A, b, c)$.
If $\lim_{t\to+\infty} M(t, M^0) = \begin{pmatrix} \bar{M}_B & 0 \\ 0 & \bar{M}_N \end{pmatrix}$ with $\bar{M}_B \mathbb{1}_B = \mathbb{1}_B$ and $\bar{M}_N \mathbb{1}_N = 0$, then $\{B, N\}$ is the basis of (A, b, c).

$$
\begin{array}{ccccc}
SLP(n,m) \ni (A,b,c) & \Longleftrightarrow & M(0) = \Gamma(A,b,c) \in G(n,m) \\
x(t) & & M(t) \\
\downarrow & & \downarrow \\
\bar{x} & \Longleftrightarrow & \bar{M} \\
& \{B,N\} &
\end{array}
$$

Basis partition of $SLP(n,m) \Longleftrightarrow$ Basis partition of $G(n,m)$.

2.4 The partition on $G(n,m)$

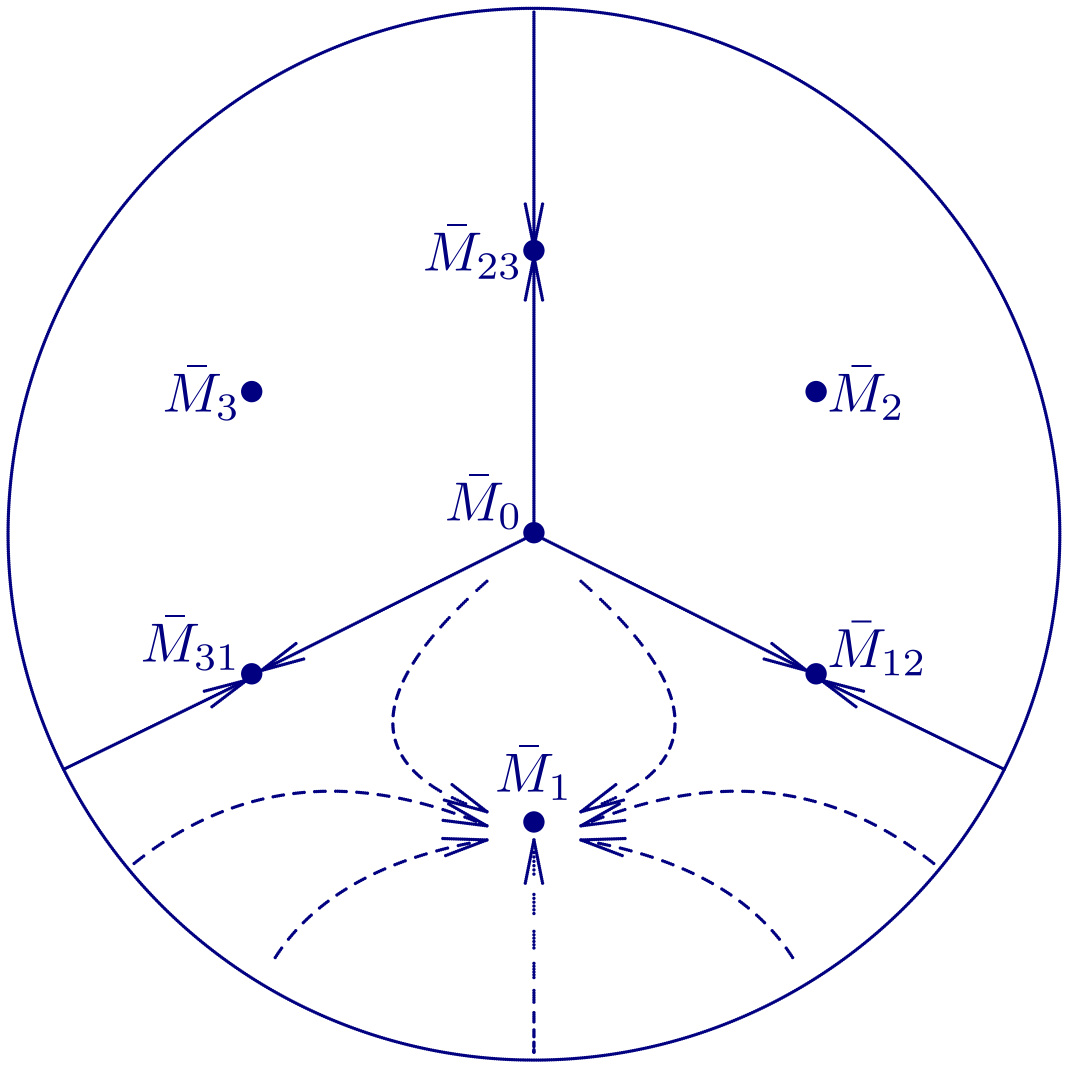
- M_∞ is an (asymptotically) stable point of $M' = h(M)$ on $G(n,m)$ if there exists a neighborhood $\mathcal{N}(M_\infty) \subset G(n,m)$ such that for any $M_0 \in \mathcal{N}(M_\infty)$, $M(t, M_0) \to M_\infty$ as $t \to +\infty$.
- The largest neighborhood $\mathcal{N}(M_\infty)$ possessing the above property is called the attraction region of M_∞.

There are $\binom{n}{m}$ stable points in the form of

$$\bar{M} = \begin{pmatrix} I_B & 0 \\ 0 & 0_N \end{pmatrix}, \qquad |B| = m.$$

$G(n,m)$ consists of $\binom{n}{m}$ disjoint attraction regions. Each region is associated with a stable point.

Equilibria, attraction regions and boundaries in $G(3,1)$:



where

$$\bar{M}_0 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\bar{M}_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \ \bar{M}_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \ \bar{M}_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$\bar{M}_{12} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \bar{M}_{23} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \ \bar{M}_{31} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We will study the basis partition of $SLP(n, m)$ via the partition of $G(n, m)$ that is characterized by $M' = h(M)$.

3 Basic properties

For any equilibrium point M, Jacobian $Dh(M) : S^n \to S^n$ has the form:

$$\begin{aligned} Dh(M)U &= M\lfloor U\mathbb{1}\rceil + \lfloor U\mathbb{1}\rceil M - 2M\lfloor U\mathbb{1}\rceil M \\ &\quad +U\lfloor M\mathbb{1}\rceil(I-2M) + (I-2M)\lfloor M\mathbb{1}\rceil U. \end{aligned}$$

Denote

$$h_M(d) = M\lfloor d\rceil + \lfloor d\rceil M - 2M\lfloor d\rceil M.$$

Equilibria are clustered into a number of connected sets which are submanifolds. For each B and $m_B = rank(M_B)$, there is a unique submanifold $G^c(B, m_B)$, called an equilibrium cluster.

Theorem 1. The eigenvalues λ and eigenvectors U of the Jacobian $Dh(M)$ at $M = \begin{pmatrix} M_B & 0 \\ 0 & M_N \end{pmatrix} \in G^c(B, m_B)$:

$\lambda = 1$:

$$U = \begin{pmatrix} h_{M_B}(d_B) & U_0 \\ U_0^T & h_{M_N}(d_N) \end{pmatrix}$$

with $M_B d_B = 0$, $M_N d_N = d_N$, $M_B U_0 = 0$ and $U_0 M_N = U_0$.

$\lambda = -1$:

$$U = \begin{pmatrix} -h_{M_B}(U_0 \mathbb{1}_N) & U_0 \\ U_0^T & -h_{M_N}(U_0^T \mathbb{1}_B) \end{pmatrix}$$

with $M_B U_0 = U_0$ and $U_0 M_N = 0$.

$\lambda = 0$: U are tangents to $G^c(B, m_B)$ at M.

Denote

$$n_B = \text{ number of rows in } M_B, \quad m_B = rank(M_B),$$
$$n_N = \text{ number of rows in } M_N, \quad m_N = rank(M_N).$$

Eigenvalue	Number of linearly independent eigenvectors
$\lambda = 1$	$n_B - m_B + m_N + (n_B - m_B)m_N$
$\lambda = -1$	$m_B(n_N - m_N)$
$\lambda = 0$	$(m_B - 1)(n_B - m_B) + (n_N - m_N - 1)m_N$

Total: $m(n - m)$

Theorem 2. One-to-one correspondences between

(i) $M \in G(n,m)$;

(ii) Path $M(\cdot)$;

(iii) Equilibrium-eigenvector pair $(\bar{M}, U^+)$ for $\lambda = 1$;

(iv) Equilibrium-eigenvector pair $(\hat{M}, U^-)$ for $\lambda = -1$.

Corollary. Partition of $G(n,m) \Leftrightarrow$ Partition of $\{(\bar{M}, \bar{U})\}$.

Remark: The latter is easier to describe because it is decomposed into $\bar{M}$ and $\bar{U}$.

Question: How to determine if $(\bar{M}, \bar{U}) \in G(B)$?

Theorem 3. For any $(\bar{M}, \bar{U})$ we can construct a (A, b, c) in terms of $(\bar{M}, \bar{U})$.

Corollary.

$$(\bar{M}, \bar{U}) \longrightarrow (A, b, c) \longrightarrow x^* \longrightarrow B,$$

$$\Rightarrow (\bar{M}, \bar{U}) \in G(B).$$

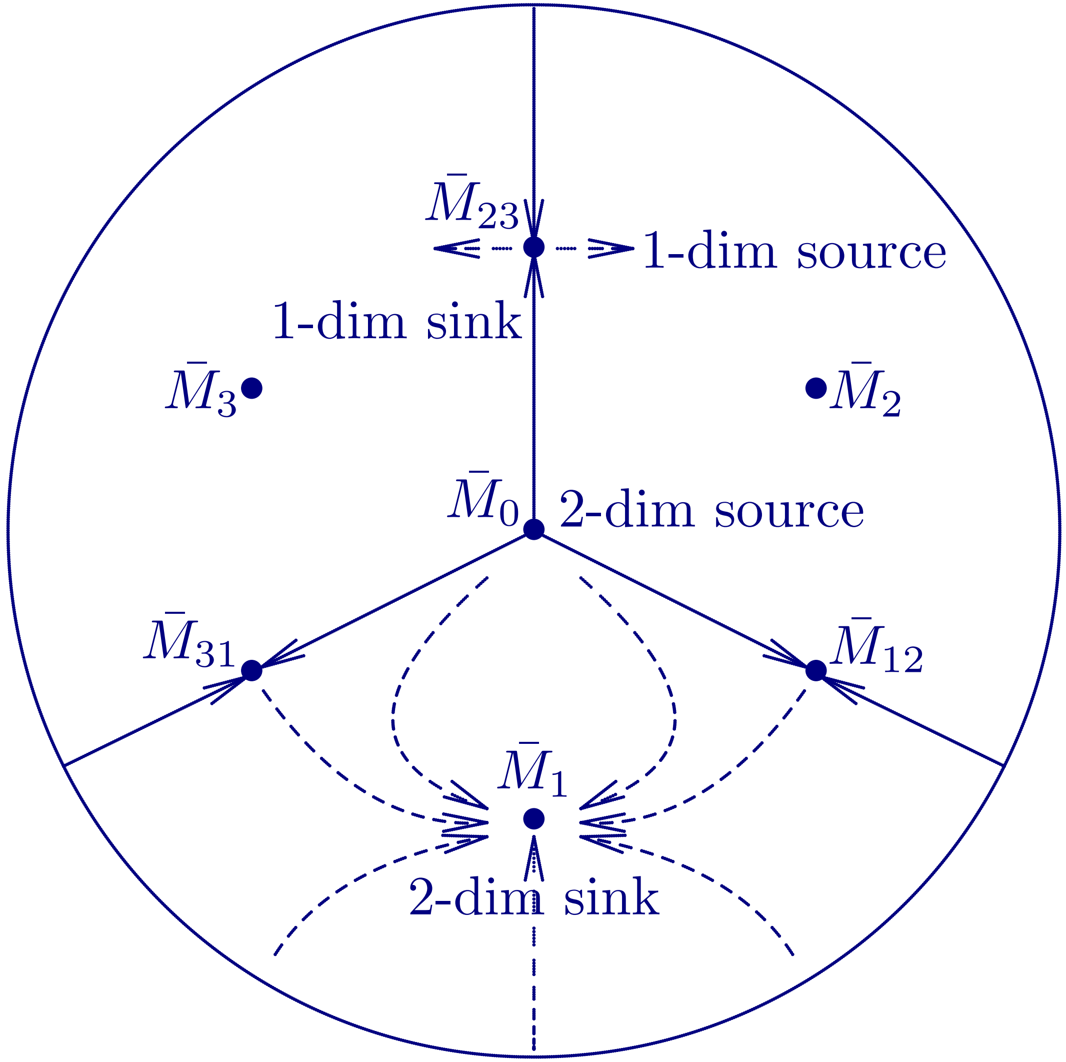
4 Sources, Sinks, and Their Dimensions

$E^{\pm}(M) :=$ eigenspace of $Dh(M)$ for $\lambda = \pm 1$.

$\text{Source}(B, m_B) := \{(M, U^+) : M \in G^c(B, m_B), U^+ \in E^+(M)\}$

$\text{Sink}(B, m_B) := \{(M, U^-) : M \in G^c(B, m_B), U^- \in E^-(M)\}$.

$$\begin{aligned} \dim(Source) &= \dim G^c + \dim E^+ \\ \dim(Sink) &= \dim G^c + \dim E^- \end{aligned}$$



The circle is a 2-dim source

4.1 Important sources and sinks

$$\bar{n} := \dim G(n, m) = m(n - m).$$

- $\bar{n}$-dim sources:

Source-0:

$$\begin{aligned} G^c &= \{M \ : \ M\mathbb{1} = 0\} \\ E^+ &= \{h_M(d) \ : \ Md = d\}. \end{aligned}$$

Source-1:

$$\begin{aligned} G^c &= \{M \ : \ M\mathbb{1} = \mathbb{1}\} \\ E^+ &= \{h_M(d) \ : \ Md = 0\}. \end{aligned}$$

Remark: (Source-0) $\cup$ (Source-1) $\simeq G(n, m)$.

Partitions of source-0 and source-1 $\Leftrightarrow$ Partition of $G(n, m)$.

- $(\bar{n}-1)$-dim sinks:

p-sink:

$$\begin{aligned} G^c &= \{M = \begin{pmatrix} I_{m-1} & 0 \\ 0 & uu^T \end{pmatrix} \;:\; u^T u = 1, u^T \mathbb{1} = 0\} \\ E^+ &= \{\begin{pmatrix} 0 & U_0 \\ U_0^T & -h_{uu^T}(U_0^T \mathbb{1}) \end{pmatrix} \;:\; U_0 u = 0\}. \end{aligned}$$

d-sink:

$$\begin{aligned} G^c &= \{M = \begin{pmatrix} I_{m+1} - vv^T & 0 \\ 0 & 0 \end{pmatrix} \;:\; v^T v = 1, v^T \mathbb{1} = 0\} \\ E^+ &= \{\begin{pmatrix} -h_{vv^T}(U_0 \mathbb{1}) & U_0 \\ U_0^T & 0 \end{pmatrix} \;:\; U_0 u = 0\}. \end{aligned}$$

Remark: Paths converging into p-sink or d-sink comprise $(\bar{n}-1)$-dim boundary.

5 Geometrical properties of attraction regions and their boundaries

5.1 Outlines

- There are $\binom{n}{m-1}$ pieces of p-boundary represented by $\begin{pmatrix} I_B & 0 \\ 0 & uu^T \end{pmatrix}$ with $|B| = m-1$ and $\binom{n}{m+1}$ pieces of d-boundary represented by $\begin{pmatrix} I_B - vv^T & 0 \\ 0 & 0 \end{pmatrix}$ with $|B| = m+1$.
- Each attraction region $G(B)$ has $(n-m)$ disjoint pieces of p-boundary and m disjoint pieces d-boundary,
- Each piece of p(d)-boundary intersets all pieces of d(p)-boundary.

5.2 Representation of p(d)-boundaries on sources

Consider

$$\text{Source-0} = \{(M, U) \ : \ M \in G^0,\ U \in E^+(M)\}.$$

(Source-1 can be considered analogously.)

Construct $\binom{n}{m}$ special points $\pi^0(B) \in G^0$ for all $|B| = m$.

For $|\bar{B}| = m$, let $B_{ij} = \bar{B} - i + j$, $i \in \bar{B}$ and $j \in \bar{N}$, be adjacent bases.

Denote

$$(I_{\bar{B}}, \zeta_{ij}) := \pi^0(B_{ij})_{\bar{B}}^{-1} \pi^0(B_{ij}).$$

- Given $k \in \bar{B}$, for any $A_{\bar{N}} = \sum_{j\in\bar{N}} \sum_{i\in\bar{B}} t_{ij}\zeta_{ij}$ with $\sum_{j\in\bar{N}} \sum_{i\in\bar{B}} t_{ij} = 1$,

$$A = (I_{\bar{B}}, A_{\bar{N}}), \quad b \geq 0 \text{ with } b_k = 0, \quad c = \mathbb{1}$$

is on the p-boundary which converges to $\begin{pmatrix} I_{\bar{B}-k} & 0 \\ 0 & uu^T \end{pmatrix}$.

- Given $k \in \bar{N}$, if $\sum_{i\in\bar{B}} t_{il} > 0 \; \forall\, l \in \bar{N} - k$, $\sum_{l\in\bar{N}-k} \sum_{i\in\bar{B}} t_{il} = 1$, and $\beta \in R^{m+1}_{++}$, then

$$A = (I_{\bar{B}}, \sum_{l\in\bar{N}-k} \sum_{i\in\bar{B}} t_{il}\zeta_{il}), \quad b = A_{\bar{B}+k}\beta, \quad c = \mathbb{1}$$

is on the d-boundary which converges to $\begin{pmatrix} I_{\bar{B}+k} - vv^T & 0 \\ 0 & 0 \end{pmatrix}$.

6 Conclusions

- We wish to characterize the basis partition of the SLP. This is of fundamental importance for solving problems which require closed-form solutions of a set of infinitely many LP instances.
- Our new tool, a dynamical system $M' = h(M)$, defines a partition on $G(n, m)$ which corresponds to the basis partition of SLP.
- We have found some properties about the dynamical system and the partition, e.g. eigenvalues/vectors of $Dh(M)$, one-to-one correspondence between paths and equilibrium-eigenvector pairs, sources and sinks, and some characterization of attraction regions and their boundaries.