A Class of Fast Algorithms for TV Based Image Reconstruction

Yin Zhang
Department of CAAM
Rice University

Joint work with: Junfeng Yang, Yilun Wang and Wotao Yin
Outline

• Introduction
  – Image formation equation
  – Maximum likelihood estimation
  – Maximum a posteriori estimation
  – Regularization

• A fast alternating algorithm
  – Motivation and algorithm
  – Relation with half-quadratic technique
  – Optimality and convergence results

• Numerical results and extensions
Introduction

- Image formation equation.

\[ f = K \bar{u} + \omega \]

- \( \bar{u} \): original image
- \( K \): convolution operator
- \( \omega \): random noise
- \( f \): observation

Our purpose is to recover \( \bar{u} \) from \( f \) (deconvolve and denoise) as well as possible.
• **Deconvolution is severely ill-conditioned.** Let $\mathbf{F}$ be the 2D Fourier transform matrix. The equation is equivalent to

$$\hat{f} = \hat{K} \hat{u} + \hat{\omega},$$

where $\hat{f} = \mathbf{F} f$ and $\hat{K} = \mathbf{F} K \mathbf{F}^{-1}$ (diagonal). A tempting solution would be

$$u^{\text{direct}} = \mathbf{F}^{-1} (\hat{K}^{-1} \hat{f}) = \mathbf{F}^{-1} (\hat{u} + \hat{K}^{-1} \hat{\omega}).$$

Does this work?
Experiment 1. Blur: ('gaussian',11,5); Noise: \( \mathcal{N}(0, 10^{-8}) \).

Noise is amplified!
Cut off high frequencies (Weiner Filter):

\[
\hat{u}_i = \begin{cases} 
0, & \text{if } |\hat{f}_i / \hat{K}_{ii}| > M; \\
\hat{f}_i / \hat{K}_{ii}, & \text{otherwise.}
\end{cases}
\]

Result of experiment 1 after cutting off some high frequencies:
Statistics Interpretation:

- Maximum likelihood estimation. Given \( \omega \sim \mathcal{N}(0, \sigma^2) \), the MLE of \( \bar{u} \) is

\[
\begin{align*}
    u^{\text{MLE}} &= \arg \max_u \Pr\{f|u\} \\
    &= \arg \min_u (-\log(\Pr\{f|u\})) \\
    &= \arg \min_u \| Ku - f \|^2.
\end{align*}
\]

Thus, MLE, LS and direct inverse are all equivalent. They do not work. When noise is correlated, i.e., \( \omega \sim \mathcal{N}(0, \Sigma) \), MLE becomes weighted LS.
Another Statistics Viewpoint:

- **Maximum a posteriori estimation.** Given $\omega \sim \mathcal{N}(0, \sigma^2)$, the MAP of $\bar{u}$ is

\[
\begin{align*}
\bar{u}^{\text{MAP}} &= \arg \max_u \Pr\{u|f\} \\
&= \arg \max_u \frac{\Pr\{u\}\Pr\{f|u\}}{\Pr\{f\}} \\
&= \arg \min_u \{-\log(\Pr\{u\}) - \log(\Pr\{f|u\})\} \\
&= \arg \min_u \Phi_{\text{prior}}(u) + \| Ku - f \|^2.
\end{align*}
\]

Thus, $\Phi_{\text{prior}}(u)$ enforces some prior constraints on $\bar{u}$, which is called regularization. Question: what kind of prior do we need?
• Regularization.

\[
\min_u \Phi_{\text{reg}}(u) + \mu \| Ku - f \|^2_2
\]

– Tikhonov-like regularization (notice 2-norm squared)

\[
\Phi_{\text{reg}}(u) = \Phi_{\text{Tik}}(u) \triangleq \sum_{j \in J} \| D^{(j)} u \|^2_2,
\]

for some \( J \subset \{0, 1, 2, \ldots\} \), where

* \( D^{(0)} \): identity matrix
* \( D^{(j)}, j = 1, 2 \): the 1st order finite difference matrices
* \( D^{(j)}, j = 3, 4, 5 \): the 2nd order . . . (used by MATLAB “deconvreg”).

The solution satisfies

\[
\left( \sum_{j \in J} (D^{(j)})^\top D^{(j)} + \mu K^\top K \right) u = \mu K^\top f.
\]
Experiment 2. Result of Tikhonov regularization. Blur: ('gaussian',21,11); Noise: $\mathcal{N}(0, 10^{-6})$.

Advantages: Not so sensitive to noise, easy to compute.
Disadvantage of Square: Incapable of recovering image discontinuities.

$$\min_{u \in \mathbb{R}^{11}} \phi(u) = \sum_{i} |u_{i+1} - u_i|^2, \text{ s.t. } u_1 = 0, u_{11} = 255.$$
- Total variation regularization (Rudin, Osher and Fatemi, 1992).

\[ \Phi_{\text{reg}}(u) = TV(u) \triangleq \sum_i \| D_i u \|. \]

* \( \| D_i u \| \): the variation of \( u \) at pixel \( i \), where

\[ D_i u = \begin{pmatrix} (D^{(1)}u)_i \\ (D^{(2)}u)_i \end{pmatrix} \in \mathbb{R}^2. \]

* \( \sum_i \) is taken over all pixels.
* The sum represents a 1-norm.
* \( \| \cdot \| \): the 2-norm (isotropic) or the 1-norm (anisotropic).
Advantage of 1-norm: Permits sharp edges in images.

\[
\min_{u \in \mathbb{R}^{11}} TV(u) = \sum_{i} |u_{i+1} - u_i|, \text{ s.t. } u_1 = 0, u_{11} = 255.
\]
Experiment 3. Compare Tikhonov with TV regularization. The same inputs as in experiment 2.

Disadvantages: More expensive in computation, stair-casing effect.
\[ TV/L^2 : \min_u \sum_i \|D_i u\|_2 + \frac{\mu}{2} \|K u - f\|^2. \]

It’s a convex program, large-scale, still ill-conditioned and requires “real-time” processing.

- **Some existing methods.**
  - **Lagged diffusivity method** (Vogel & Oman, 1995). Given \( u^k, u^{k+1} \) is determined by solving
    \[
    \sum_i D_i^\top \frac{D_i u}{\|D_i u^k\|_\alpha} + \mu K^\top (K u - f) = 0,
    \]
    which is a linearization to the optimality condition of
    \[
    \min_u \sum_i \|D_i u\|_\alpha + \frac{\mu}{2} \|K u - f\|^2.
    \]
    Here \( \| \cdot \|_\alpha \triangleq \sqrt{\| \cdot \|^2 + \alpha} \) for some small \( \alpha > 0 \).
    Most earlier methods were based on solving (Euler-Lagrange) PDE.
Iterative Shrinkage/Thresholding based methods (Daubechies, Defrise & De Mol, 2004). Given $u_k$, the original IST method iterates as

$$u_{k+1} = \Psi_\mu \left( u_k - \lambda_k K^\top (K u_k - f) \right),$$

where $\lambda_k > 0$ and

$$\Psi_\mu (\xi) \triangleq \arg \min_u \ TV(u) + \frac{\mu}{2} \| u - \xi \|^2.$$

There exist several variants of IST methods, e.g., TwIST (Bioucas-Dias & Figueiredo, 2007).

- **Second-order cone programming approach** (Goldfarb & Yin, 2005).

- **Iterative Denoising** (Michael Ng et al 2007). Much faster, but ......
A Fast Alternating Algorithm

- **Motivation.** The problem is

\[
\min_u \sum_i \|D_i u\| + \frac{\mu}{2} \|Ku - f\|^2.
\]

By introducing \(w_i \in \mathbb{R}^2\), TV/L^2 is approximated by, for \(\beta \gg 0\),

\[
\min_{w_i, u} \sum_i \left( \|w_i\| + \frac{\beta}{2} \|w_i - D_i u\|^2 \right) + \frac{\mu}{2} \|Ku - f\|^2.
\]

The approximation problem allows very fast alternating minimization.
A simple and important lemma:

**Lemma 1** Given a positive integer $d$. For any $\beta > 0$ and $t \in \mathbb{R}^d$, it holds

$$\max \left\{ \|t\| - \frac{1}{\beta}, 0 \right\} \frac{t}{\|t\|} = \arg \min_{s \in \mathbb{R}^d} \left\{ \|s\| + \frac{\beta}{2} \|s - t\|^2 \right\},$$

where we follow the convention $0 \cdot (0/0) = 0$.

**An important Observation:** Finite differences, $D^{(1)}$ and $D^{(2)}$ can be treated as discrete convolution under suitable boundary conditions.

Consequently, $D^{(1)}$ and $D^{(2)}$ and $K$ are circulant matrices under the periodic boundary conditions for $u$, and all can be diagonalized by FFT.
• Our Simple Algorithm:
  
  - $w$-subproblem. Fixing $u$, minimizing w.r.t. $w$ reduces to

  \[
  \min_{w_i} \|w_i\| + \frac{\beta}{2}\|w_i - D_i u\|^2, \quad \forall i.
  \]

  Separate and closed form solutions at all pixels $i$:

  \[
  w_i = \max \left\{ \|D_i u\| - \frac{1}{\beta}, 0 \right\} \frac{D_i u}{\|D_i u\|}, \quad \forall i.
  \]

  Linear time complexity: $O(n^2)$. 

- $u$-subproblem. Fixing $\{w_i\}$, minimizing w.r.t. $u$ reduces to

$$\min_u \frac{\beta}{2} \sum_i \|w_i - D_i u\|^2 + \frac{\mu}{2} \|K u - f\|^2.$$ 

Its normal equations are

$$\left( \sum_i D_i^\top D_i + \frac{\mu}{\beta} K^\top K \right) u = \sum_i D_i^\top w_i + \frac{\mu}{\beta} K^\top f$$

or equivalently

$$\left( \sum_{j=1}^{2} (D^{(j)})^\top D^{(j)} + \frac{\mu}{\beta} K^\top K \right) u = \sum_{j=1}^{2} (D^{(j)})^\top w_j + \frac{\mu}{\beta} K^\top f,$$

where $w_j = \{w_i(j) : i = 1, \ldots, n^2\}$ for $j = 1, 2$.

This system can be solved by 2 FFTs at a cost of $O(n^2 \log n)$. 
- **Continuation/path-following.** Initialize $\beta$ small, and then increase it gradually. The previous solution is used to warm-start the next problem.

Test on continuation: $\beta = 2^0, 2^1, \ldots, 2^{10}$.

Continuation not only accelerates the speed, but also, unexpectedly, enhances solution robustness.
Given $\beta > 0$, we solve the approximation problem by alternately minimizing w.r.t. $w$ and $u$.

- **FTVd** (Fast TV deconvolution). Input $K, f, \mu > 0, \beta_{max} \gg 0$ and $\gamma > 1$; Initialize $\beta = \beta_0 > 0$ and $u = u_0$.

  **While** $\beta \leq \beta_{max}$, **Do**

  1) Solve the approximation to certain accuracy for $u_\beta$.

  2) Update $u \leftarrow u_\beta, \beta \leftarrow \gamma \ast \beta$.

**End Do**
• Relation with half-quadratic technique. Given $\beta > 0$, FTVd solves

$$\min_{u, w} \sum_i \left\{ \|w_i\| + \frac{\beta}{2} \|D_i u - w_i\|^2 \right\} + \frac{\mu}{2} \|K u - f\|^2.$$ 

The above is equivalent to

$$\min_u \sum_i \phi(D_i u) + \frac{\mu}{2} \|K u - f\|^2,$$

where $\phi(t), t \in \mathbb{R}^2$, is defined as

$$\phi(t) = \begin{cases} 
\frac{\beta}{2} \|t\|^2, & \text{if } \|t\| \leq 1/\beta; \\
\|t\| - \frac{1}{2\beta}, & \text{otherwise.}
\end{cases}$$

This is an extension to the half-quadratic transform (German and Yang 1995).
• **Optimality.** A pair \((w, u)\) solves the approximation problem iff

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{w_i}{\|w_i\|} + \beta (w_i - D_i u) = 0 & i \in I_1 \triangleq \{ i : w_i \neq 0 \}, \\
\beta \|D_i u\| & \leq 1 & i \in I_2 \triangleq \{ i : w_i = 0 \}, \\
\end{array} \right. \\
\beta D_i^\top (D u - w) + \mu K^\top (K u - f) = 0.
\end{align*}
\]

Eliminating \(w\), the final equations become

\[
\sum_{i \in I_1} D_i^\top \frac{D_i u}{\|D_i u\|} + \sum_{i \in I_2} D_i^\top h_i + \mu K^\top (K u - f) = 0,
\]

where \(h_i = \beta D_i u\) satisfies \(\|h_i\| \leq 1\), which is an approximation to the optimality condition of TV/L^2.
• **Convergence results.** Let \( D = (D^{(1)}; D^{(2)}) \),

\[
M = D^\top D + (\mu/\beta) \cdot K^\top K \quad \text{and} \quad T = DM^{-1}D^\top.
\]

Assuming \( \mathcal{N}(D) \cap \mathcal{N}(K) = \{0\} \), for fixed \( \beta \) we have

1. The sequence \( \{(w^k, u^k)\} \) generated by FTVd converges to a solution \( (w^*, u^*) \) of the approximation problem.

2. Finite convergence. \( w^k_L \equiv w^*_L \) in finite number of iterations.

3. \( q \)-linear convergence. For \( k \) sufficiently large, there hold

   (a) \( \|D(u^{k+1} - u^*)\| \leq \sqrt{\|T^2\|_{EE}} \cdot \|D(u^k - u^*)\| \);

   (b) \( \|w^{k+1} - w^*\| \leq \sqrt{\|T^2\|_{EE}} \cdot \|w^k - w^*\| \);

   (c) \( \|u^{k+1} - u^*\|_M \leq \sqrt{\|T_{EE}\|} \cdot \|u^k - u^*\|_M \).

Here \( L = \{i, \|D_iu^*\| < 1/\beta\} \) and \( E = \{1, 2, \ldots, n^2\} \setminus L \).
Numerical results and extensions

- **Restoration of grayscale images.** Kernel: ('gaussian',21,10); Noise: Gaussian white with mean zero and $\text{std}=10^{-3}$. 

Blurry&Noisy. SNR: 6.3dB

Lag.D. SNR: 12.2dB, CPU: 512.1s

FTVd. SNR: 12.6dB, Iter: 11, CPU: 1.9s
Blurry&Noisy. SNR: 7.7dB
Lag.D. SNR: 14.4dB, CPU: 1918.0s
FTVd. SNR: 14.8dB, Iter: 9, CPU: 7.0s

Blurry&Noisy. SNR: 9.1dB
Lag.D. SNR: 15.0dB, CPU: 7306.7s
FTVd. SNR: 15.5dB, Iter: 10, CPU: 31.2s
• **Speed comparison with Lagged Diffusivity method.** Noise: Gaussian, mean zero and std = $10^{-3}$; Blur: (`gaussian',hsize,10).
• **Multichannel image deconvolution.** Let \( u \) be a RGB image. The image formulation equation \( f = Ku + \omega \) becomes

\[
\begin{pmatrix}
    f^r \\
    f^g \\
    f^b
\end{pmatrix} =
\begin{pmatrix}
    K_{rr} & K_{rg} & K_{rb} \\
    K_{gr} & K_{gg} & K_{gb} \\
    K_{br} & K_{bg} & K_{bb}
\end{pmatrix}
\begin{pmatrix}
    u^r \\
    u^g \\
    u^b
\end{pmatrix} +
\begin{pmatrix}
    \omega^r \\
    \omega^g \\
    \omega^b
\end{pmatrix}.
\]

TV is extended to

\[
\text{MTV}(u) \triangleq \sum_i \| (I_3 \otimes D_i) u \|,
\]

where

\[
(I_3 \otimes D_i) u = \begin{bmatrix}
    D^{(1)} u^r, D^{(2)} u^r, D^{(1)} u^g, D^{(2)} u^g, D^{(1)} u^b, D^{(2)} u^b
\end{bmatrix}_i \in \mathbb{R}^6.
\]
Generally, let $u \in \mathbb{R}^{mn^2}$ be a $m$-channel image and $K = [K_{jk}]_{jk=1}^m$ be a cross-channel blurring matrix. The TV/L$^2$ model is extended as

$$
\min_u \sum_i \alpha_i \|G_i u\| + \frac{\mu}{2} \|K u - f\|^2,
$$

where $G_i = I_m \otimes D_i$, and $D_i$ is a 1st and/or higher order local finite difference operator. It is approximated by

$$
\min_{u, w} \sum_i \left( \alpha_i \|w_i\| + \frac{\beta}{2} \|w_i - G_i u\|^2 \right) + \frac{\mu}{2} \|K u - f\|^2.
$$

– Fixing $u$, the minimizer function for $w$ is given explicitly by:

$$
w_i = \max \left\{ \frac{\|G_i u\| - \alpha_i}{\beta}, 0 \right\} \frac{G_i u}{\|G_i u\|}, \quad \forall i.
$$
The \( u \)-subproblem is equivalent to

\[
\left( \sum_{j} (G^{(j)})^{\top} G^{(j)} + \frac{\mu}{\beta} K^{\top} K \right) u = \sum_{j} (G^{(j)})^{\top} w_{j} + \frac{\mu}{\beta} K^{\top} f.
\]

By pre- and post- multiplying \( I_{m} \otimes F \) and its inverse, respectively, the coefficient matrix becomes

\[
\begin{pmatrix}
\Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1m} \\
\Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda_{m1} & \Lambda_{m2} & \cdots & \Lambda_{mm}
\end{pmatrix},
\]

with each \( \Lambda_{ij} \) a diagonal matrix. Thus \( u \)-subproblem is easily solved by FFTs and low complexity Gaussian elimination.
Restoration from cross-channel blur and Gaussian noise:

Original

Blurry&Noisy. SNR: 6.70dB

FTVd: SNR: 18.49dB, t = 4.29s

Original

Blurry&Noisy. SNR: 8.01dB

FTVd: SNR: 19.54dB, t = 16.86s
- **Deconvolution in the presence of impulsive noise.** Cameraman degraded by convolution and 10% salt-and-pepper noise. Right: solution of TV/L$^2$. 
For impulsive noise, the $\ell_1$-norm fidelity is more suitable. We recover $\bar{u}$ as the solution of the TV/L$^1$ model:

$$\min_u \sum_i \|D_i u\| + \mu \|K u - f\|_1.$$ 

The approximation problem is given by

$$\min_{w, z, u} \sum_i \left(\|w_i\| + \frac{\beta}{2} \|w_i - D_i u\|^2\right) + \mu \left(\|z\|_1 + \frac{\gamma}{2} \|z - (K u - f)\|^2\right).$$

Minimization w.r.t. $w$, $z$ and $u$ each is easy!
Restoration from Gaussian blur and salt-and-pepper noise:

Blurry&Noisy: 30% Salt&Pepper

40% Salt&Pepper

50% Salt&Pepper

60% Salt&Pepper

FTVd. $\mu$: 13, t: 15.1s, SNR: 14.16dB

FTVd. $\mu$: 10, t: 13.9s, SNR: 13.21dB

FTVd. $\mu$: 8, t: 13.5s, SNR: 12.35dB

FTVd. $\mu$: 4, t: 16.8s, SNR: 11.08dB
Restoration from cross-channel blur and random-valued noise:

40% RV

50% RV

60% RV

\[ \mu: 8, t: 117s, \text{SNR: 16.04dB} \]

\[ \mu: 4, t: 138s, \text{SNR: 14.06dB} \]

\[ \mu: 2, t: 136s, \text{SNR: 10.60dB} \]
MRI reconstruction. In MR imaging system, MR scanner collects data:

\[ f_p = \mathcal{F}_p \bar{u} + \omega \in \mathcal{C}^M, \quad M \ll N. \]

Without noise, under certain desirable conditions, it holds

\[ \bar{u} = \arg \min_u \{ \text{TV}(u) : \mathcal{F}_p u = f_p \}. \]

In the presence of noise, we recover \( \bar{u} \) via

\[ \min_u \text{TV}(u) + \frac{\mu}{2} \| \mathcal{F}_p u - f_p \|^2. \]

When \( \bar{u} \) has sparse/compressible representation under certain wavelet basis, we recover it via

\[ \min_u \text{TV}(u) + \tau \| \Psi^\top u \|_1 + \frac{\mu}{2} \| \mathcal{F}_p u - f_p \|^2. \]

FTVd can be extended to solve the above TVL\(^1\)-L\(^2\) problem.
Sparse (under TV) image reconstruction. Left to right: Original, Fourier domain samples (9.36%), reconstructed image (RelErr: 4.48%). Gaussian noise with mean zero and std = .01.
Compressible (under wavelet) image reconstruction. Sample ratio: 9.64%; Noise: Gaussian, mean zero, \( \text{std} = .01 \); Left: original brain image; Right: reconstructed (RelErr: 11.58%).
Summary.

- FTVd converges without the assumption of strictly convexity.
- Finite convergence of auxiliary variables is established.
- Linear convergence rate is established and the convergence factor depends on a submatrix.
- FTVd is fast for TV based problem because it fully exploits problem structure and utilizes FFT.
References


[8] J. Yang, Y. Zhang, and W. Yin, *An efficient TVL1 algorithm for deblurring of multichannel images corrupted by impulsive noise*, TR08-12, CAAM, Rice University, Submitted to SISC.

Codes available at:

http://www.caam.rice.edu/~optimization/L1/ftvd

Acknowledgments

- Junfeng Yang has been supported by the Chinese Scholarship Council during his visit to Rice University.

- Wotao Yin has been supported in part by ONR Grant N00014-08-1-1101 and NSF CAREER Award DMS-0748839.

- Yin Zhang has been supported in part by NSF Grant DMS-0811188 and ONR Grant N00014-08-1-1101.
Thank you!