Numer. Math.
DOI 10.1007/s00211-017-0890-x

Numerische Mathematik

Two robust nonconforming H^2-elements for linear strain gradient elasticity

Hongliang Li[1] · **Pingbing Ming**[2,3] · **Zhong-ci Shi**[2,3]

Received: 20 January 2016 / Revised: 6 January 2017

Abstract We propose two finite elements to approximate a boundary value problem arising from strain gradient elasticity, which is a high order perturbation of the linearized elastic system. Our elements are H^2-nonconforming while H^1-conforming. We show both elements converge in the energy norm uniformly with respect to the perturbation parameter.

Mathematics Subject Classification Primary 65N30 · 65N15; Secondary 74K20

1 Introduction

Strain gradient theory, which introduces the high order strain and microscopic parameter into the strain energy density, is one of the most successful approach to characterize the strong size effect of the heterogeneous materials [20]. The origin of this theory can be traced back to Cosserat brothers' celebrated work [13]. Further development

✉ Pingbing Ming
mpb@lsec.cc.ac.cn

Hongliang Li
lihongliang@mtrc.ac.cn

Zhong-ci Shi
shi@lsec.cc.ac.cn

1 Institute of Electronic Engineering, China Academy of Engineering Physics, Mianyang 621900, China

2 The State Key Laboratory of Scientific and Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55, Zhong-Guan-Cun East Road, Beijing 100190, China

3 University of Chinese Academy of Sciences, Beijing 100049, China

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

is related to Mindlin's work on microstructure in linear elasticity [23,24]. However, Mindlin's theory is less attractive in practice because it contains too many parameters. Based on Mindlin's work, Aifantis et al. [3,28] proposed a linear strain gradient elastic model with only one microscopic parameter. This simplified strain gradient theory successfully eliminated the strain singularity of the brittle crack tip field [14].

The strain gradient elastic model of Aifantis' is a perturbed elliptic system of fourth order from the view point of mathematics. To discretize this model by a finite element method, a natural choice is C^1 finite elements such as Argyris triangle [5] and Bell's triangle [7] because this model contains the gradient of strain. We refer to [2,27,32,33] for works in this direction. Alternative approach such as mixed finite element has been employed to solve this model [4]. A drawback of both the conforming finite element method and mixed finite element method is that the number of the degrees of freedom is extremely large and high order polynomial has to be used in the basis function, which is more pronounced for three dimensional problems; See e.g., the finite element for three-dimensional strain gradient model proposed in [27] contains 192 degrees of freedom for the local finite element space.

A common approach to avoid such difficult is to use the nonconforming finite element. For scalar version of such problem, there are a lot of work since the original contribution [26], and we refer to [10,16,30] and the references therein for recent progress. The situation is different for the strain gradient elastic model. The well-posedness of the corresponding boundary value problem hinges on a Korn-like inequality, which will be dubbed as H^2-Korn's inequality. Therefore, a discrete H^2-Korn's inequality has to be satisfied for any reasonable nonconforming finite element approximation. We prove a H^2-Korn's inequality for piecewise vector fields as BRENNER'S seminal H^1 Korn's inequality [9]. Motivated by this discrete Korn's inequality, we propose two nonconforming H^2-finite elements, which are H^1-conforming. Both elements satisfy the discrete H^2-Korn's inequality. We prove that both elements converge in energy norm uniformly with respect to the small perturbation parameter. Numerical results also confirm the theoretic results.

It is worth mentioning that Soh and Chen [29] constructed several noncomforming finite elements for this strain gradient elastic model. Some of them exhibited excellent numerical performance. Their motivation is the so-called C^{0-1} patch test, which is obviously different from ours. It is unclear whether their elements are robust with respect to the small perturbation parameter, which may be an interesting topic for further study. As to various numerical methods based on reformulations of the strain gradient elastic model, we refer to [6,31] and the references therein.

The remaining part of the paper is organized as follows. In the next part, we introduce the linear strain gradient elasticity and prove the well-posedness of its Dirichlet boundary value problem by establishing a H^2-Korn inequality. A discrete H^2-Korn inequality is proved in Sect. 3, and two finite elements are constructed and analyzed in this part. The numerical results can be found in Sect. 4.

Throughout this paper, the generic constant C may differ at different occurrences, while it is independent of the microscopic parameter ι and the mesh size h.

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

2 The Korn's inequality of strain gradient elasticity

The space $L^2(\Omega)$ of the square–integrable functions defined on a bounded polygon Ω is equipped with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|_{L^2(\Omega)}$. Let $H^m(\Omega)$ be the standard Sobolev space [1] and

$$\|v\|_{H^m(\Omega)} = \sum_{k=0}^{m} |v|^2_{H^k(\Omega)} \quad \text{and} \quad |v|^2_{H^k(\Omega)} = \int_\Omega \sum_{|\alpha|=k} |\nabla^\alpha v|^2 \, dx,$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index whose components α_i are nonnegative integers, $|\alpha| = \alpha_1 + \alpha_2$ and $\nabla^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}$. We may drop Ω in the Sobolev norm $\|\cdot\|_{H^m(\Omega)}$ when there is no confusion may occur. The space $H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. In particular,

$$H_0^1(\Omega) := \{\, v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega \,\},$$
$$H_0^2(\Omega) := \{\, v \in H^1(\Omega) \mid v = \partial_n v = 0 \text{ on } \partial\Omega \,\},$$

where $\partial_n v$ is the normal derivative of v. Equally, $\partial_t v$ denotes the tangential derivative of v. The summation convention is used for repeated indices. A comma followed by a subscript, say i, denotes partial differentiation with respect to the spatial variables x_i, i.e., $v_{,i} = \partial v/\partial x_i$.

For any vector-valued function v, its gradient is a matrix-valued function with components $(\nabla v)_{ij} = \partial v_i/\partial x_j$. The symmetric part of a gradient field is also a matrix-valued function defined by

$$\epsilon(v) = \frac{1}{2}(\nabla v + [\nabla v]^T).$$

The anti-symmetric part of a gradient field is defined as $\nabla^{\mathrm{a}} v = \nabla v - \epsilon(v)$. The divergence operator applying to a vector field is defined as the trace of ∇v, i.e., $\nabla \cdot v = \mathrm{tr}\nabla v = \partial v_i/\partial x_i$. The Sobolev spaces $[H^m(\Omega)]^2$, $[H_0^m(\Omega)]^2$ and $[L^2(\Omega)]^2$ of a vector field can be defined in a similar manner as their scalar counterparts, this rule equally applies to their inner products and their norms. For the m-th order tensors A and B, we define the inner product as $A{:}B = \sum_{i_1,\ldots,i_m} A_{i_1} B_{i_1} \ldots A_{i_m} B_{i_m}$.

2.1 Strain gradient elastic model and H$^2-$Korn inequality

The strain gradient elastic model in [3,14,28] is described by the following boundary value problem: For u the displacement vector that solves

$$\begin{cases} (\iota^2 \triangle - I)\,(\mu \triangle u + (\lambda+\mu)\nabla\nabla\cdot u) = f, & \text{in } \Omega, \\ u = \partial_n u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here λ and μ are the Lamé constants, and ι is the microscopic parameter such that $0 < \iota \leq 1$. In particular, we are interested in the regime when ι is close to zero.

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

The above boundary value problem may be rewritten into the following variational problem: Find $u \in [H_0^2(\Omega)]^2$ such that

$$a(u, v) = (f, v) \quad \text{for all } v \in [H_0^2(\Omega)]^2, \tag{2}$$

where

$$a(u, v) = (\mathbb{C}\epsilon(u), \epsilon(v)) + (\mathbb{D}\nabla\epsilon(u), \nabla\epsilon(v)),$$

and the fourth-order tensors $\mathbb{C}$ and the sixth-order tensor $\mathbb{D}$ are defined by

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl} \quad \text{and} \quad \mathbb{D}_{ijklmn} = \iota^2\left(\lambda\delta_{il}\delta_{jk}\delta_{mn} + 2\mu\delta_{il}\delta_{jm}\delta_{ln}\right),$$

respectively. Here δ_{ij} is the Kronecker delta function. The third-order tensor $\nabla\epsilon(v)$ is defined as $(\nabla\epsilon(v))_{ijk} = \epsilon_{jk,i}$. We only consider the clamped boundary condition in this paper, the discussion on other boundary conditions can be found in [3,14,28].

The variational problem (2) is well-posed if and only if the bilinear form $a(\cdot, \cdot)$ is coercive over $[H_0^2(\Omega)]^2$.

Theorem 1 *For any $v \in [H_0^2(\Omega)]^2$, there holds*

$$C(\Omega)\left(\|v\|_{H^1}^2 + \iota^2|v|_{H^2}^2\right) \le a(v, v) \le 2(\lambda + \mu)\left(\|v\|_{H^1}^2 + \iota^2|v|_{H^2}^2\right), \tag{3}$$

where $C(\Omega)$ denpends only on μ and the constant C_p in the following Poincaré *inequality,*

$$\|v\|_{L^2} \le C_p\|\nabla v\|_{L^2}.$$

The proof of this theorem essentially depends on the first Korn's inequality [18,19]. For any $v \in [H_0^1(\Omega)]^2$, there holds

$$2\|\epsilon(v)\|_{L^2}^2 \ge \|\nabla v\|_{L^2}^2. \tag{4}$$

The proof of this inequality follows from the following identity

$$|\epsilon(v)|^2 - |\nabla^{\mathrm{a}} v|^2 = |\nabla \cdot v|^2 + \nabla \cdot [(v \cdot \nabla)v - v(\nabla \cdot v)]$$

with the usual notation

$$v \cdot \nabla = \sum_{i=1}^{2} v_i \frac{\partial}{\partial x_i}.$$

Indeed, by the fact $v = 0$ on $\partial\Omega$, the above identity and divergence theorem imply

$$\int_\Omega |\nabla^{\mathrm{a}} v|^2 \,\mathrm{d}x = \int_\Omega |\epsilon(v)|^2 \,\mathrm{d}x - \int_\Omega |\nabla \cdot v|^2 \,\mathrm{d}x \le \int_\Omega |\epsilon(v)|^2 \,\mathrm{d}x,$$

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

which implies the first Korn's inequality (4) by using the algebraic identity

$$|\nabla v|^2 = |\epsilon(v)|^2 + |\nabla^{\mathrm{a}} v|^2.$$

Proof of Theorem 1 By definition, we write

$$a(v, v) = 2\mu\|\epsilon(v)\|_{L^2}^2 + \lambda\|\nabla \cdot v\|_{L^2}^2 + \iota^2\left(2\mu\|\nabla\epsilon(v)\|_{L^2}^2 + \lambda\|\nabla\nabla \cdot v\|_{L^2}^2\right).$$

The upper bound in (3) immediately follows by noting

$$\|\epsilon(v)\|_{L^2}^2 \le \|\nabla v\|_{L^2}^2, \quad \text{and} \quad \|\nabla \cdot v\|_{L^2}^2 \le 2\|\nabla v\|_{L^2}^2.$$

For any $v \in [H_0^2(\Omega)]^2$, we have $\partial_i v \in [H_0^1(\Omega)]^2$ for $i = 1, 2$, we apply the first Korn's inequality (4) to the vector field $\partial_i v$ and obtain

$$2\|\epsilon(\partial_i v)\|_{L^2}^2 \ge \|\nabla\partial_i v\|_{L^2}^2.$$

Using the fact that the strain operator ϵ and the partial gradient operator ∂_i commute, we rewrite the above inequality as

$$\begin{aligned} 2\|\nabla\epsilon(v)\|_{L^2}^2 &= 2\sum_{i=1}^{2}\|\partial_i\epsilon(v)\|_{L^2}^2 = 2\sum_{i=1}^{2}\|\epsilon(\partial_i v)\|_{L^2}^2 \\ &\ge \sum_{i=1}^{2}\|\nabla\partial_i v\|_{L^2}^2 = \|\nabla^2 v\|_{L^2}^2. \end{aligned}$$

Therefore,

$$a(v, v) \ge \mu\left(|v|_{H^1}^2 + \iota^2|v|_{H^2}^2\right),$$

which together with the *Poincaré* inequality leads to the lower bound in (3). □

Proceeding along the same line in [26, §5], we may prove the following regularity results for the solution of Problem (2).

Lemma 1 *There exists C that may depend on Ω but independent of ι such that*

$$|u|_{H^2} + \iota|u|_{H^3} \le C\iota^{-1/2}\|f\|_{L^2}, \tag{5}$$

and

$$\|u - u^0\|_{H^1} \le C\iota^{1/2}\|f\|_{L^2}, \tag{6}$$

where $u^0 \in [H_0^1(\Omega)]^2$ satisfies

$$(\mathbb{C}\epsilon(u^0), \epsilon(v)) = (f, v) \quad \textit{for all} \quad v \in [H_0^1(\Omega)]^2. \tag{7}$$

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

3 The nonconforming finite elements

In this part, we introduce two nonconforming finite elements to approximate the variational problem (2). Let $\mathcal{T}_h$ be a triangulation of Ω with maximum mesh size h. We assume all elements in $\mathcal{T}_h$ is shape-regular in the sense of Ciarlet and Raviart [11]. Denote the set of all the edges in $\mathcal{T}_h$ as $\mathcal{S}(\Omega, \mathcal{T}_h)$. The space of piecewise $[H^m(\Omega, \mathcal{T}_h)]^2$ vector fields is defined by

$$[H^m(\Omega, \mathcal{T}_h)]^2 := \{ v \in [L^2(\Omega)]^2 \mid v|_T \in [H^m(T)]^2, \quad \forall T \in \mathcal{T}_h \},$$

which is equipped with the broken norm

$$\|v\|_{H_h^k} := \|v\|_{L^2} + \sum_{k=1}^{m} \|\nabla_h^k v\|_{L^2},$$

where

$$\|\nabla_h^k v\|_{L^2}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla^k v\|_{L^2(T)}^2$$

with $(\nabla_h^k v)|_T = (\nabla^k v)|_T$. Moreover, $\epsilon_h(v) = (\nabla_h v + [\nabla_h v]^T)/2$.

Brenner [9] established a discrete Korn inequality for any piecewise H^1 vector fields with weak linear continuity across the common surface between two adjacent elements, i.e., for any $v \in [H^1(\Omega, \mathcal{T}_h)]^d$ with $d = 2, 3$ satisfying

$$\int_e [\![v]\!] \cdot p \, \mathrm{d}\tau = 0, \quad p \in [P_1(e)]^d, e \in \mathcal{S}(\Omega, \mathcal{T}_h).$$

There exists a constant C depends on Ω and $\mathcal{T}_h$ but independent of h such that

$$\|v\|_{H_h^1} \leq C \left(\|v\|_{L^2} + \|\epsilon_h(v)\|_{L^2} \right). \tag{8}$$

Here $[P_1(e)]^d$ is the linear vector field over e and $[\![v]\!]$ denotes the jump of v across e with e an edge for $d = 2$ and a face for $d = 3$. This inequality is fundamental to the well-posedness of the discrete problems arising from nonconforming finite element and discontinuous Galerkin method approximation of the linearized elasticity model and Reissner–Mindlin plate model; See [15,25] and [17].

Mardal and Winther [21] improved the above inequality by replacing $[P_1(e)]^d$ by its subspace $[P_{1,-}(e)]^d$ given by

$$[P_{1,-}(e)]^d := \{ v \in [P_1(e)]^d \mid v \cdot t \in \mathrm{RM}(e) \},$$

where t is the tangential vector of edge e, and $\mathrm{RM}(e)$ is the infinitesimal rigid motion on e. In fact, they have proved

$$|v|^2_{H^1_h} \le C \left(\|\epsilon_h(v)\|^2_{L^2} + \|v\|^2_{L^2} + \sum_{e\in\mathcal{S}(\Omega,\mathcal{T}_h)} h_e^{-1} \|[\![\Pi_e v]\!]\|^2_{L^2(e)} \right), \tag{9}$$

where $\Pi_e : [L^2(e)]^d \mapsto [P_{1,-}(e)]^d$ is the L^2 projection.

Our result is an H^2 analog of the discrete Korn's inequality (9).

Theorem 2 *For any $v \in [H^2(\Omega, \mathcal{T}_h)]^2$, there exits C that depends on Ω and $\mathcal{T}_h$ but independent of h such that*

$$\begin{aligned} \|v\|^2_{H^2_h} \le C \Bigg(& \|\nabla_h \epsilon_h(v)\|^2_{L^2} + \|\epsilon_h(v)\|^2_{L^2} + \|v\|^2_{L^2} + \sum_{e\in\mathcal{S}(\Omega,\mathcal{T}_h)} h_e^{-1} \|[\![\Pi_e v]\!]\|^2_{L^2(e)} \\ & + \sum_{i=1}^{2} \sum_{e\in\mathcal{S}(\Omega,\mathcal{T}_h)} h_e^{-1} \|[\![\Pi_e(\partial_i v)]\!]\|^2_{L^2(e)} \Bigg). \end{aligned} \tag{10}$$

The proof follows essentially the same line that leads to Theorem 1.

Proof For any $v \in [H^2(\Omega, \mathcal{T}_h)]^2$, it is clear that $\partial_i v \in [H^1(\Omega, \mathcal{T}_h)]^2$ for $i = 1, 2$. Applying the discrete Korn's inequality (9) to each $\partial_i v$, we obtain

$$\begin{aligned} |v|^2_{H^2_h} &= |\partial_1 v|^2_{H^1_h} + |\partial_2 v|^2_{H^1_h} \\ &\le C \sum_{i=1}^{2} \left(\|\epsilon_h(\partial_i v)\|^2_{L^2} + \|\partial_i v\|^2_{L^2} + \sum_{e\in\mathcal{S}(\Omega,\mathcal{T}_h)} h_e^{-1} \|[\![\Pi_e(\partial_i v)]\!]\|^2_{L^2(e)} \right) \\ &= C \left(\|\nabla_h \epsilon_h(v)\|^2_{L^2} + \|\nabla_h v\|^2_{L^2} + \sum_{i=1}^{2} \sum_{e\in\mathcal{S}(\Omega,\mathcal{T}_h)} h_e^{-1} \|[\![\Pi_e(\partial_i v)]\!]\|^2_{L^2(e)} \right). \end{aligned}$$

Invoking (9) once again, we get (10). □

Motivated by the discrete Korn's inequality (10), we construct two new finite elements that are H^1—conforming but H^2—nonconforming elements. For such elements, the continuity of the tangential derivatives are automatically satisfied, and we only need to deal with the weak continuity of the normal derivative. The finite element space is defined as

$$V_h := \{ v \in [H^1_0(\Omega)]^2 \mid v|_T \in W(T) \text{ for all } T \in \mathcal{T}_h \}.$$

We shall specify two local finite element spaces $W(T)$ in the next two parts.

Given V_h, we find $u_h \in V_h$ such that

$$a_h(u_h, v) = (f, v) \quad \text{for all} \quad v \in V_h, \tag{11}$$

where the bilinear form a_h is defined for any $v, w \in V_h$ as

$$a_h(v, w) := (\mathbb{C}\epsilon(v), \epsilon(w)) + (\mathbb{D}\nabla_h \epsilon(v), \nabla_h \epsilon(w)),$$

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

where the second term is defined in a piecewise manner as

$$(\mathbb{D}\nabla_h\epsilon(v), \nabla_h\epsilon(w)) := \sum_{T\in\mathcal{T}_h}\int_T \mathbb{D}\nabla\epsilon(v)\nabla\epsilon(w)\,\mathrm{d}x.$$

3.1 The first nonconforming element

Define

$$W(T) := [P_2(T)]^2 \oplus bP_2^*(T), \tag{12}$$

where $P_2(T)$ is the quadratic Lagrange element, and $b = \lambda_1\lambda_2\lambda_3$ is the cubic bubble function, and $P_2^*(T) \subset [P_2(T)]^2$ is defined as

$$P_2^*(T) := \{\, v \in [P_2(T)]^2 \;\mid\; v\cdot n|_e \in P_1(e) \quad \text{for all} \quad e \in \partial T\,\}.$$

Next lemma gives the degrees of freedom of this element, which is graphically shown in Fig. 1, and we prove that the degrees of freedom is $W(T)-$unisolvent.

Lemma 2 *The dimension of $W(T)$ is* 21. *Any $w \in W(T)$ is uniquely determined by the following degrees of freedom:*

1. *The values of w at the corners and edge midpoints;*
2. *The moments $\int_e \partial_n(w\cdot t)\,\mathrm{d}\tau$ and $\int_e \partial_n(w\cdot n)\tau^k\,\mathrm{d}\tau$ for $k = 0, 1$ and for all $e \in \partial T$.*

Proof Since $[P_2(T)]^2 \cap bP_2^*(T) = \{0\}$ and $\dim\ P_2^*(T) \geq 9$, we conclude $\dim\ W(T) \geq 21$. It suffices to show that a function $w \in W(T)$ vanishes if all the degrees of freedom are zeros. Note that $w|_e \in [P_2(e)]^2$, with three roots on edge e, then we must have $w|_{\partial T} = 0$. Therefore, we may write $w = bp$ with $p \in P_2^*(T)$. Let e be a fixed edge of T, and denote $b = \lambda_e\lambda_+\lambda_-$ with λ_e the barycentric coordinate functions such that $\lambda_e \equiv 0$ on e, while λ_+ and λ_- the remaining two barycentric

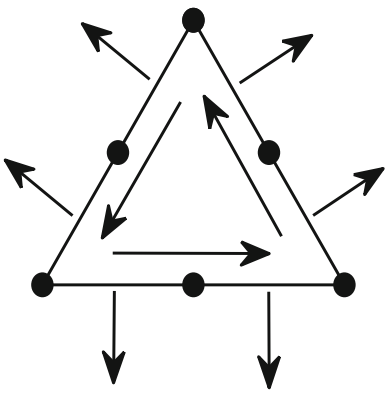

Fig. 1 The degrees of freedom are point evaluations at the vertex and midpoint of each edge, are the moments of the normal derivative of the normal component against P_1 over each edge, are the moments of the normal derivative of the tangential component along each edge

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

coordinate functions. Furthermore, $(\nabla w)|_e = (p\lambda_+\lambda_-)|_e \nabla\lambda_e$. Note that $\lambda_+\lambda_-\partial_n\lambda_e$ is strictly negative in the interior of e. Therefore, the condition

$$0 = \int_e \partial_n(w\cdot n)\tau^k \,\mathrm{d}\tau = \int_e \lambda_+\lambda_- \frac{\partial\lambda_e}{\partial n} p\cdot n\tau^k \,\mathrm{d}\tau$$

implies that for any $e \in \partial T$ and $k = 0, 1$,

$$\int_e p\cdot n\tau^k \,\mathrm{d}\tau = 0. \tag{13}$$

Proceeding along the same line, we obtain, for any $e \in \partial T$,

$$\int_e p\cdot t \,\mathrm{d}\tau = 0. \tag{14}$$

Furthermore, using the fact that $p\cdot n \in P_1(e)$ and (13), we conclude that $p\cdot n \equiv 0$ over ∂T.

Assume that

$$p = \left(\sum_{|\alpha|=2} a_\alpha \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3},\quad \sum_{|\alpha|=2} b_\alpha \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\right)^T,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ whose components α_i are nonnegative integers and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

Using $(p\cdot n_i)|_{e_i} \equiv 0$, we obtain, for $j = 1, 2, 3$,

$$(a_\alpha, b_\alpha)\cdot n_j = 0 \quad \text{with} \quad \alpha_j = 0. \tag{15}$$

This means that, for each α with one component equals to 2, the vector (a_α, b_α) is orthogonal to the normal directions of two different edges, which immediately implies that such vector (a_α, b_α) must be zero.

Next, using the fact that $\int_{e_i}(p\cdot t)\,\mathrm{d}\tau \equiv 0$, we obtain, for $j = 1, 2, 3$,

$$(a_\alpha, b_\alpha)\cdot t_j = 0, \quad \text{with} \quad \alpha_j = 0, \alpha_k = 1 \quad \text{for} \quad k \neq j.$$

Invoking (15) once again, we conclude that, for each α with only one zero component, (a_α, b_α) is orthogonal to both the normal direction and the tangential direction of the edge indexed with the zero component of α, which must vanish identically. Therefore, all a_α and b_α are zeros, and hence $p \equiv 0$, equivalently, $w \equiv 0$. This completes the proof. □

Using the degrees of freedom given in Lemma 2, we may define a local interpolation operator $\Pi_T : H^2(T) \mapsto W(T)$. The next lemma shows that this operator locally preserves quadratics.

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

Lemma 3

$$\Pi_T v = v, \quad v \in [P_2(T)]^2. \tag{16}$$

Proof Let $\big(T, W(T), \Sigma(T)\big)$ be the finite element triple with $\Sigma(T)$ the degrees of freedom. By construction, $\Sigma(T)$ takes the form as

$$\Sigma(T) = \{d_1^{(l)}, \ldots, d_{12}^{(l)}, d_1^{(m)}, \ldots, d_9^{(m)}\},$$

where $\{d_i^{(l)}\}_{i=1}^{12}$ are the nodal type degrees of freedom, and $\{d_i^{(m)}\}_{i=1}^{9}$ are the moment type degrees of freedom. The basis functions for $[P_2(T)]^2$ and $bP_2^*(T)$ are denoted by $\{\phi_i\}_{i=1}^{12}$ and $\{\psi_i\}_{i=1}^{9}$, respectively.

Define a new set of basis functions

$$\varphi_i = \phi_i - \sum_{j=1}^{9} d_j^{(m)}(\phi_i)\psi_j, \quad i = 1, \ldots, 12.$$

We claim

$$W(T) = \text{span}\{\varphi_1, \cdots, \varphi_{12}, \psi_1, \cdots, \psi_9\}. \tag{17}$$

Note that $d_i^{(l)}(\psi_j) \equiv 0$ because $\psi_j = 0$ on ∂T. We obtain $\{\psi_j\}_{j=1}^{9}$ are the basis functions of $W(T)$ associated with the degrees of freedom $\{d_j^{(m)}\}_{j=1}^{9}$. For any φ_i, there holds

$$d_j^{(l)}(\varphi_i) = d_j^{(l)}(\phi_i) - \sum_{k=1}^{9} d_k^{(m)}(\phi_i) d_j^{(l)}(\psi_k) = d_j^{(l)}(\phi_i) = \delta_{ij},$$

and

$$\begin{aligned} d_j^{(m)}(\varphi_i) &= d_j^{(m)}(\phi_i) - \sum_{k=1}^{9} d_k^{(m)}(\phi_i) d_j^{(m)}(\psi_k) \\ &= d_j^{(m)}(\phi_i) - \sum_{k=1}^{9} d_k^{(m)}(\phi_i)\delta_{jk} = 0. \end{aligned}$$

This verifies the claim (17).

Next, we prove the interpolation operator is locally P_2 invariant. For any $v \in [P_2(T)]^2$, we have the representation

$$v = \sum_{i=1}^{12} d_i^{(l)}(v)\phi_i.$$

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

By definition,

$$\begin{aligned}\Pi_T v &= \sum_{i=1}^{12} d_i^{(l)}(v)\varphi_i + \sum_{j=1}^{9} d_j^{(m)}(v)\psi_j \\ &= \sum_{i=1}^{12} d_i^{(l)}(v)\phi_i - \sum_{j=1}^{9}\sum_{i=1}^{12} d_i^{(l)}(v) d_j^{(m)}(\phi_i)\psi_j + \sum_{j=1}^{9} d_j^{(m)}(v)\psi_j \\ &= \sum_{i=1}^{12} d_i^{(l)}(v)\phi_i - \sum_{j=1}^{9}\left(\sum_{i=1}^{12} d_i^{(l)}(v) d_j^{(m)}(\phi_i) - d_j^{(m)}(v)\right)\psi_j \\ &= \sum_{i=1}^{12} d_i^{(l)}(v)\phi_i = v,\end{aligned}$$

where we have used the identity

$$d_j^{(m)}(v) = d_j^{(m)}\left(\sum_{i=1}^{12} d_i^{(l)}(v)\phi_i\right) = \sum_{i=1}^{12} d_i^{(l)}(v) d_j^{(m)}(\phi_i).$$

This completes the proof. □

The above proof actually provides a constructive way to derive the basis function of this element.

3.2 The second nonconforming element

The second element is almost the same with the first one except that P_2^* is replaced by

$$P_3^*(T) = \{\, w \in [P_3(T)]^2 \mid \nabla \cdot w \in P_0(T),\, w \cdot n|_e \in P_1(e) \text{ for all } e \in \partial T \,\}.$$

Hence,

$$W(T) = [P_2(T)]^2 \oplus b P_3^*(T). \tag{18}$$

Here $P_3^*(T)$ has appeared in [22] to solve Darcy-Stokes flow.

The following lemma gives the degrees of freedom of this element, and we prove it is $W(T)-$unisolvent.

Lemma 4 *The dimension of $W(T)$ is 21. Any $w \in W(T)$ is uniquely determined by the following degrees of freedom:*

1. *The values of w at the corners and edge midpoints;*
2. *The moments $\int_e \partial_n(w\cdot t)\,\mathrm{d}\tau$ and $\int_e \partial_n(w\cdot n)\tau^k\,\mathrm{d}\tau$ for $k = 0, 1$ and for all $e \in \partial T$.*

The proof of this result is slightly different from the direct proof of Lemma 2.

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

Proof Proceeding along the same line that leads to (13) and (14), we obtain that for all $e \in \partial T$,

$$\int_e p \cdot n\tau^k \, \mathrm{d}\tau = 0, k = 0, 1, \quad \text{and} \quad \int_e p \cdot t \, \mathrm{d}\tau = 0. \tag{19}$$

Using the fact that $(p \cdot n)|_{\partial T} \in P_1$ and the first identity of (19), we conclude that $(p \cdot n)|_{\partial T} \equiv 0$, which immediately implies

$$\int_T \nabla \cdot p \, \mathrm{d}x = \int_{\partial T} p \cdot n \, \mathrm{d}\tau = 0.$$

Since $\nabla \cdot p \in P_0(T)$, this implies that p is divergence free. Then there exists a polynomial $\phi \in P_4(T)$ such that $p = \mathrm{curl}\phi$. Furthermore, since

$$\partial_t \phi|_{\partial T} = p \cdot n|_{\partial T} = 0,$$

which implies that ϕ is constant along the edge of T. Without loss of generality, we assume that $\phi|_{\partial T} \equiv 0$. Hence, ϕ is of the form $\phi = b\,\varphi$ with $\varphi \in P_1(T)$. By the second identity of (19), we obtain

$$\int_e \partial_n \phi \, \mathrm{d}\tau = \int_e p \cdot t \, \mathrm{d}\tau = 0.$$

Note that $\partial_n \phi|_e = \lambda_+ \lambda_- \partial_n \lambda_e \, \varphi|_e$ and the fact that $\lambda_+ \lambda_- \partial_n \lambda_e$ is strictly negative in the interior of the edge e, we conclude that φ has a root at each edge, which together with the fact that $\varphi \in P_1(T)$ yields $\varphi \equiv 0$, or equivalently $w = 0$. This completes the proof. □

Proceeding along the same line that leads to (16), we conclude that this nonconforming element also locally preserves quadratics.

Remark 1 Both elements are endowed with the same degrees of freedom. The structure of the local finite element spaces for both element are similar. In fact, the bubble functions can be removed by standard static condensation procedure. Therefore, the resulting method has only 12 degrees of freedom on each element.

For any function $v \in V_h$, we obtain a simplified version of the discrete Korn's inequality (10) without all the jump terms, which may be regarded as a H^2- analog of the discrete Korn's inequality (8).

Lemma 5 *There exists C depends on Ω and $\mathcal{T}_h$, but independent of h such that*

$$\|v\|_{H_h^2} \leq C \left(\|\nabla_h \epsilon(v)\|_{L^2} + \|\epsilon(v)\|_{L^2} \right). \tag{20}$$

Proof For any function $v \in V_h$, we claim that the jump terms in the right-hand side of (10) vanish. Indeed, $[\![\Pi_e v]\!] = 0$ for any $e \in \mathcal{S}(\Omega, \mathcal{T}_h)$ because $v \in [H_0^1(\Omega)]^2$. It remains to verify that for all $e \in \mathcal{S}(\Omega, \mathcal{T}_h)$ and $i = 1, 2$,

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

$$[\![\Pi_e(\partial_i v)]\!] = 0. \tag{21}$$

We write $\partial_i v = \alpha_i \partial_n v + \beta_i \partial_t v$, where α_i and β_i are constants. Hence, it remains to show

$$[\![\Pi_e(\partial_n v)]\!] = 0, \qquad [\![\Pi_e(\partial_t v)]\!] = 0, \quad \forall e \in \mathcal{S}(\Omega, \mathcal{T}_h).$$

Since V_h is H$^1-$ conforming, it is clear that $[\![\Pi_e(\partial_t v)]\!] = 0$.

For any $e \in \mathcal{S}(\Omega, \mathcal{T}_h)$, it is clear that $\mathrm{RM}(e) = P_0(e)$. For any $w \in [P_{1,-}(e)]^2$, there holds

$$w = w_n n + w_t t, \quad w_n \in P_1(e), w_t \in P_0(e).$$

Hence, $[\![\Pi_e(\partial_n v)]\!] = 0$, if and only if

$$\int_e [\![\partial_n (v \cdot n)]\!] \tau^k \, \mathrm{d}\tau = 0, k = 0, 1, \quad \text{and} \quad \int_e [\![\partial_n (v \cdot t)]\!] \, \mathrm{d}\tau = 0.$$

This is true for any $v \in W(T)$ and we prove the claim (21).

For any $v \in V_h$, it follows from (10) that

$$\|v\|_{H_h^2} \leq C \left(\|\nabla_h \epsilon(v)\|_{L^2} + \|\epsilon(v)\|_{L^2} + \|v\|_{L^2} \right).$$

The inequality (20) follows by using the Poincaré's inequality and the first Korn's inequality (4):

$$\|v\|_{L^2}^2 \leq C_p^2 \|\nabla v\|_{L^2}^2 \leq 2C_p^2 \|\epsilon(v)\|_{L^2}^2.$$

□

We are ready to prove the coercivity of the bilinear form a_h over V_h.

Theorem 3 *For any $\iota < 1/\sqrt{2}$, there exists C that depends on the domain Ω and the shape regularity of the triangulation $\mathcal{T}_h$ such that*

$$a_h(v, v) \geq C(\iota^2 \|v\|_{H_h^2}^2 + \|v\|_{H^1}^2), \qquad \forall v \in V_h. \tag{22}$$

Proof For any $v \in V_h$, using (20), we obtain, there exists C that is independent of h such that

$$\begin{aligned} a_h(v, v) &\geq 2\mu \left(\iota^2 \|\nabla_h \epsilon(v)\|_{L^2}^2 + \|\epsilon(v)\|_{L^2}^2 \right) \\ &\geq 2\mu\iota^2 \left(\|\nabla_h \epsilon(v)\|_{L^2}^2 + \|\epsilon(v)\|_{L^2}^2 \right) + \mu \|\epsilon(v)\|_{L^2}^2 \\ &\geq C \left(\iota^2 \|v\|_{H_h^2}^2 + \|v\|_{H^1}^2 \right), \end{aligned}$$

which implies (22), where we have used

$$\|v\|_{H^1}^2 = \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \le (C_p^2+1)\|\nabla v\|_{L^2}^2 \le 2(C_p^2+1)\|\epsilon(v)\|_{L^2}^2,$$

in the last step. □

The following interpolate estimate is a direct consequence of the quadratics invariance of the local finite element spaces $W(T)$; The proof is standard, and we refer to [12] for the details.

Lemma 6 *There exists C independent of h such that for all $v \in [H^k(T)]^2$,*

$$\|v - \Pi_T v\|_{H^j(T)} \le Ch^{k-j}|v|_{H^k(T)}, \quad j = 0, 1, 2, k = 2, 3. \tag{23}$$

A global interpolation operator $I_h : H^k(\Omega) \mapsto V_h$ is defined by $(I_h)|_T = \Pi_T$.

3.3 Convergence analysis

We are ready to prove the main result of this paper.

Theorem 4 *Assume that the weak solution of u of the problem (2) belongs to $[H_0^2(\Omega)]^2 \cap [H^3(\Omega)]^2$. Let u_h be the solution of (11). Then there exists C independent of ι and h such that*

$$|||u - u_h|||_{\iota,h} \le \begin{cases} C(h^2 + \iota h)|u|_{H^3}, \\ Ch(|u|_{H^2} + \iota|u|_{H^3}), \end{cases} \tag{24}$$

where $|||v|||_{\iota,h}^2 := a_h(v, v)$ for any $v \in V_h$.

Proof By the the theorem of Berger, Scott, and Strang [8], we have

$$|||u - u_h|||_{\iota,h} \le \inf_{v\in V_h} |||u - v|||_{\iota,h} + \sup_{w\in V_h} \frac{E_h(u, w)}{|||w|||_{\iota,h}}, \tag{25}$$

where $E_h(u, w) = a_h(u, w) - (f, w)$.

By the interpolate estimate (23), we obtain

$$\inf_{v\in V_h} |||u - v|||_{\iota,h} \le |||u - I_h u|||_{\iota,h} \le \begin{cases} C(h^2 + \iota h)|u|_{H^3}, \\ Ch(|u|_{H^2} + \iota|u|_{H^3}). \end{cases} \tag{26}$$

Next, we focus on the estimate of the consistency error. We write $\kappa_{ijk} = (\nabla\epsilon(u))_{ijk} = \partial_{x_i}\epsilon_{jk}(u)$. The stress and couple stress are defined by $\sigma = \mathbb{C}\epsilon(u)$ and $\tau = \mathbb{D}\nabla\epsilon(u)$, respectively. Or

$$\sigma_{ij} = \mathbb{C}_{ijkl}\epsilon_{kl}(u) \quad \text{and} \quad \tau_{ijk} = \mathbb{D}_{ijklmn}\kappa_{lmn}(u).$$

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

By the symmetry of the tensors $\mathbb{C}$ and $\mathbb{D}$, there holds

$$\sigma_{ij} = \sigma_{ji} \quad \text{and} \quad \tau_{ijk} = \tau_{ikj}.$$

By the chain rule and the symmetry of $\mathbb{C}$ and $\mathbb{D}$, we obtain, on each element T and for any $w \in V_h$,

$$\begin{aligned}\mathbb{C}\epsilon(u) : \epsilon(w) + \mathbb{D}\nabla\epsilon(u) : \nabla\epsilon(w) &= \sigma_{jk}\epsilon_{jk}(w) + \tau_{ijk}\kappa_{ijk}(w)\\ &= \sigma_{jk}w_{k,j} + \tau_{ijk}w_{k,ij}\\ &= \big((\sigma_{jk} - \tau_{ijk,i})w_k\big)_{,j} - (\sigma_{jk,j} - \tau_{ijk,ij})w_k\\ &\quad + (\tau_{ijk}w_{k,j})_{,i}.\end{aligned}$$

Using the above representation and integration by parts, we obtain

$$\begin{aligned}&\int_T \mathbb{C}\epsilon(u) : \epsilon(w) + \mathbb{D}\nabla\epsilon(u) : \nabla\epsilon(w)\,\mathrm{d}x\\ &= \int_T (\tau_{ijk,ij} - \sigma_{jk,j})w_k\,\mathrm{d}x\\ &\quad + \int_{\partial T} n_j(\sigma_{jk} - \tau_{ijk,i})w_k\,\mathrm{d}\tau + \int_{\partial T} n_i\tau_{ijk}w_{k,j}\,\mathrm{d}\tau.\end{aligned} \tag{27}$$

Using the fact $w_{k,j} = n_j\partial_n w_k + t_j\partial_t w_k$ and

$$n_i t_j \tau_{ijk}\partial_t w_k = \partial_t(n_i t_j \tau_{ijk}w_k) - \partial_t(n_i t_j \tau_{ijk})w_k,$$

we obtain

$$\int_{\partial T} n_i\tau_{ijk}w_{k,j}\,\mathrm{d}\tau = \int_{\partial T} n_i n_j \tau_{ijk}\partial_n w_k\,\mathrm{d}\tau - \int_{\partial T}(n_i t_j \tau_{ijk})w_k\,\mathrm{d}\tau, \tag{28}$$

where we have used the fact that the contour integration of tangential derivative along the element boundary is zero. By (27), (28) and the continuity of w, we obtain

$$\begin{aligned}E_h(u, w) &= \sum_{T\in\mathcal{T}_h}\int_{\partial T} n_i n_j \tau_{ijk}\partial_n w_k\,\mathrm{d}\tau\\ &= \sum_{e\in\mathcal{S}(\Omega,\mathcal{T}_h)}\int_e n_i n_j \tau_{ijk}[\![\partial_n w_k]\!]\,\mathrm{d}\tau,\end{aligned}$$

where $\tau_{ijk} = \iota^2\sigma_{jk,i}$.

By

$$\int_e [\![\partial_n(w\cdot n)]\!]\,\mathrm{d}\tau = 0 \quad \text{and} \quad \int_e [\![\partial_n(w\cdot t)]\!]\,\mathrm{d}\tau = 0,$$

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

we obtain, for $k = 1, 2$,

$$\int_e [\![\partial_n w_k]\!] \, \mathrm{d}\tau = 0.$$

Employing the standard trace inequality and scaling argument, we obtain

$$|E_h(u, w)| \leq Ch|\tau|_{H^1} |\partial_n w|_{H_h^1} \leq C\iota^2 h |u|_{H^3} |w|_{H_h^2}.$$

Substituting the above estimate and (26) into (25), we obtain (24). □

Combining the error estimate (24) and the regularity results in Lemma 1, proceeding along the same line of [26, Theorem 5.2], we could obtain the following $\iota-$independent error estimate

$$|\!|\!| u - u_h |\!|\!|_{\iota,h} \leq Ch^{1/2} \|f\|_{L^2}, \tag{29}$$

where C is independent of ι and h. We leave the details to the interested readers.

4 Numerical example

In this section we provide two numerical examples that show the accuracy of the proposed elements, and the robustness of the elements with respect to the microscaopic parameter ι. The first example is performed on the uniform mesh, while the second one is performed on the nonuniform mesh. As a first step toward understanding the

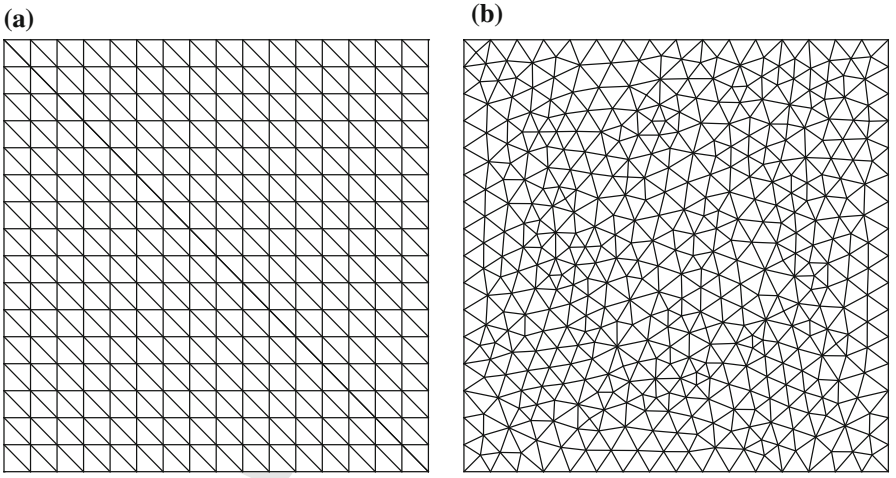

Fig. 2 Plots of the mesh. **a** Is the uniform triangulations with $h = 1/16$. **b** Is the nonuniform mesh with maximum mesh size $h = 1/16$

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

Table 1 The convergence rate of the first element over uniform mesh with $\lambda = \mu = 1$

$\iota \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
1e0	2.37e−1	1.38e−1	7.31e−2	3.73e−2	1.87e−2	9.38e−3
Rate		0.79	0.91	0.97	0.99	1.00
1e−1	1.81e−1	1.04e−1	5.47e−2	2.78e−2	1.40e−2	6.99e−3
Rate		0.81	0.92	0.98	0.99	1.00
1e−2	3.44e−2	1.66e−2	8.28e−3	4.15e−3	2.07e−3	1.04e−3
Rate		1.06	1.00	1.00	1.00	1.00
1e−3	1.87e−2	5.25e−3	1.54e−3	5.35e−4	2.26e−4	1.07e−4
Rate		1.83	1.77	1.53	1.25	1.08
1e−4	1.85e−2	4.96e−3	1.27e−3	3.22e−4	8.30e−5	2.29e−5
Rate		1.90	1.96	1.98	1.96	1.86
1e−5	1.85e−2	4.95e−3	1.27e−3	3.19e−4	7.98e−5	2.00e−5
Rate		1.90	1.97	1.99	2.00	2.00

Table 2 The convergence rate of the second element over uniform mesh with $\lambda = \mu = 1$

$\iota \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
1e0	2.73e−1	1.67e−1	9.22e−2	4.76e−2	2.40e−2	1.20e−2
Rate		0.70	0.86	0.95	0.99	1.00
1e−1	2.10e−1	1.27e−1	6.91e−2	3.55e−2	1.79e−2	8.97e−3
Rate		0.73	0.88	0.96	0.99	1.00
1e−2	4.01e−2	2.04e−2	1.05e−2	5.30e−3	2.66e−3	1.33e−3
Rate		0.97	0.96	0.98	1.00	1.00
1e−3	2.13e−2	6.10e−3	1.83e−3	6.54e−4	2.84e−4	1.36e−4
Rate		1.80	1.74	1.48	1.20	1.06
1e−4	2.10e−2	5.73e−3	1.48e−3	3.75e−4	9.70e−5	2.70e−5
Rate		1.87	1.95	1.98	1.95	1.85
1e−5	2.10e−2	5.73e−3	1.47e−3	3.71e−4	9.29e−5	2.33e−5
Rate		1.87	1.96	1.99	2.00	2.00

size effect of the heterogeneous materials, we test the proposed elements for a benchmark problem with smooth solution. Tests for realistic problems will appear in the forthcoming work.

Let $\Omega = (0, 1)^2$ and

$$u_1 = (\exp(\cos 2\pi x) - e)(\exp(\cos 2\pi y) - e),$$
$$u_2 = (\cos 2\pi x - 1)(\cos 4\pi y - 1).$$

The force f is obtained by (1).

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

Table 3 The convergence rate of the first element over nonuniform mesh with $\lambda = \mu = 1$

$\iota \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
1e0	1.26e−1	6.80e−2	3.61e−2	1.87e−2	9.57e−3	4.84e−3
Rate		0.89	0.91	0.94	0.97	0.98
1e−1	9.43e−2	5.08e−2	2.69e−2	1.40e−2	7.14e−3	3.61e−03
Rate		0.89	0.92	0.95	0.97	0.98
1e−2	1.66e−2	7.92e−3	4.05e−3	2.08e−3	1.06e−3	5.35e−4
Rate		1.07	0.97	0.96	0.97	0.99
1e−3	8.24e−3	2.24e−3	6.79e−4	2.52e−4	1.12e−4	5.46e−5
Rate		1.88	1.73	1.43	1.16	1.04
1e−4	8.09e−3	2.09e−3	5.33e−4	1.36e−4	3.56e−5	1.01e−5
Rate		1.95	1.97	1.97	1.94	1.81
1e−5	8.09e−3	2.08e−3	5.31e−4	1.34e−4	3.38e−5	8.48e−6
Rate		1.96	1.97	1.98	1.99	1.99

Table 4 The convergence rate of the first element over nonuniform mesh with $\lambda = 10$ and $\mu = 1$

$\iota \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
1e0	1.09e−1	5.87e−2	3.12e−2	1.63e−2	8.36e−3	4.24e−3
Rate		0.89	0.91	0.94	0.96	0.98
1e−1	8.22e−2	4.43e−2	2.35e−2	1.23e−2	6.30e−3	3.19e−3
Rate		0.89	0.91	0.94	0.96	0.98
1e−2	1.43e−2	6.93e−3	3.58e−3	1.85e−3	9.47e−4	4.79e−4
Rate		1.05	0.96	0.95	0.97	0.98
1e−3	6.67e−3	1.82e−3	5.66e−4	2.18e−4	9.96e−5	4.88e−5
Rate		1.87	1.69	1.38	1.13	1.03
1e−4	6.52e−3	1.67e−3	4.25e−4	1.09e−4	2.87e−5	8.40e−6
Rate		1.97	1.97	1.97	1.92	1.77
1e−5	6.52e−3	1.67e−3	4.23e−4	1.07e−4	2.69e−5	6.75e−6
Rate		1.97	1.98	1.99	1.99	1.99

First, the triangulation of the unit square for the uniform mesh is illustrated in Fig. 2_a. In Tables 1 and 2, we report the convergence rates for both elements in the energy norm $|||u - u_h|||_{\iota,h}/|||u|||_{\iota,h}$ for different values of ι and h with $\lambda = \mu = 1$. We observe that the convergence rate appears to be linear when ι is large, while it turns out to be quadratic when ι is close to zero, which is consistent with the theoretical prediction in the estimate (24).

Next, we test both elements over a nonuniform mesh. The initial mesh is generated by the function "initmesh" of the partial differential equation toolbox of MATLAB. The initial mesh consists of 872 triangles and 469 vertices, and the maximum mesh size is $h = 1/16$; See Fig. 2_b. In Tables 3, 4, 5 and 6, we report the convergence rate of both elements in the energy norm when $\lambda = \mu = 1$ and $\lambda = 10$, $\mu = 1$. It seems

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

Table 5 The convergence rate of the second element over nonuniform mesh with $\lambda = \mu = 1$

$\iota \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
1e0	2.50e−1	1.44e−1	8.06e−2	4.29e−2	2.20e−2	1.11e−02
Rate		0.80	0.84	0.91	0.96	0.98
1e−1	1.88e−1	1.08e−1	6.02e−2	3.20e−2	1.64e−2	8.30e−3
Rate		0.80	0.84	0.91	0.96	0.98
1e−2	3.14e−2	1.65e−2	9.04e−3	4.77e−3	2.44e−3	1.23e−3
Rate		0.92	0.87	0.92	0.97	0.99
1e−3	1.34e−2	3.79e−3	1.24e−3	5.11e−4	2.48e−4	1.24e−4
Rate		1.82	1.62	1.27	1.04	1.00
1e−4	1.31e−2	3.43e−3	8.85e−4	2.28e−4	6.06e−5	1.82e−5
Rate		1.93	1.96	1.96	1.91	1.74
1e−5	1.31e−2	3.43e−3	8.80e−4	2.23e−4	5.61e−5	1.41e−5
Rate		1.93	1.96	1.98	1.99	1.99

Table 6 The convergence rate of the second element over nonuniform mesh with $\lambda = 10$ and $\mu = 1$

$\iota \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
1e0	2.47e−1	1.35e−1	7.41e−2	3.99e−2	2.09e−2	1.07e−2
Rate		0.87	0.86	0.89	0.93	0.97
1e−1	1.87e−1	1.02e−1	5.59e−2	3.00e−2	1.58e−2	8.07e−3
Rate		0.88	0.86	0.89	0.93	0.97
1e−2	3.19e−2	1.59e−2	8.53e−3	4.55e−3	2.37e−3	1.21e−3
Rate		1.01	0.90	0.90	0.94	0.97
1e−3	1.43e−2	3.92e−3	1.25e−3	5.04e−4	2.43e−4	1.23e−4
Rate		1.87	1.65	1.31	1.05	0.99
1e−4	1.41e−2	3.63e−3	9.32e−4	2.39e−4	6.34e−5	1.88e−5
Rate		1.96	1.96	1.96	1.92	1.75
1e−5	1.41e−2	3.63e−3	9.29e−4	2.35e−4	5.92e−5	1.49e−5
Rate		1.96	1.97	1.98	1.99	1.99

the convergence rate is the same with that over the uniform mesh. The first element is slightly more accurate than the second one, in particular over the nonuniform mesh.

5 Conclusion

We prove a Korn-like inequality and its discrete analog for the strain gradient elastic problem, which is crucial for the well-posedness of the underlying variational problems as the Korn's inequality for the linearized elasticity. Guided by the discrete Korn's inequality, we constructed two nonconforming elements that converge uniformly in the microscopic parameter with optimal convergence rate. Numerical experiments

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

validate the theoretical results. The extension of the elements to three dimensional problem and to high order would be very interesting and challenging. Applications of these elements to realistic problem in strain gradient plasticity is another topic deserves further pursuit. We leave all these issues in a forthcoming work.

Acknowledgements The work of Li was supported by Science Challenge Project No. TZ 2016003. The work of Ming was partially supported by the National Natural Science Foundation of China for Distinguished Young Scholars 11425106, and National Natural Science Foundation of China Grants 91630313, and by the support of CAS NCMIS. The work of Shi was partially supported by the National Natural Science Foundation of China Grant 11371359. We are grateful to the anonymous referees for their valuable suggestions.

References

1. Adams, R., Fournier, J.: Sobolev Spaces, 2nd edn. Academic, New York (2003)
2. Akarapu, S., Zbib, H.: Numerical analysis of plane cracks in strain-gradient elastic materials. Int. J. Fract. **141**, 403–430 (2006)
3. Altan, S., Aifantis, E.: On the structure of the mode III crack-tip in gradient elasticity. Scr. Metal. Mater. **26**, 319–324 (1992)
4. Amanatidou, E., Aravas, N.: Mixed finite element formulations of strain-gradient elasticity problems. Comput. Methods Appl. Mech. Eng. **191**, 1723–1751 (2006)
5. Argyris, J., Fried, I., Scharpf, D.: The Tuba family of plate elements for the matrix displacement method. Aeronaut. J. R. Aeronaut. Soc. **72**, 701–709 (1968)
6. Askes, H., Aifantis, E.: Numerical modeling of size effects with gradient elasticity-formulation, meshless discretization and examples. Int. J. Frac. **117**, 347–358 (2002)
7. Bell, K.: A refined triangular plate bending element. Int. J. Numer. Methods Eng. **1**, 101–122 (1969)
8. Berger, A., Scott, R., Strang, G.: Approximate boundary conditions in the finite element method. Symp. Math. **X**, 295–313 (1972)
9. Brenner, S.: Korn's inequalities for piecewise H^1 vector fields. Math. Comp. **73**, 1067–1087 (2004)
10. Brenner, S., Neilan, M.: A C^0 interior penalty method for a fourth order elliptic singular perturbation problem. SIAM J. Numer. Anal. **49**, 869–892 (2011)
11. Ciarlet, P.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
12. Ciarlet, P., Raviart, P.A.: General Lagrange and Hermite interpolation in R^n with applications to finite element methods. Arch. Rational Mech. Anal. **46**, 177–199 (1972)
13. Cosserar, E., Cosserat, F.: Theorie des corps deformables. Herman et fils, Paris (1909)
14. Exadaktylos, G., Aifantis, E.: Two and three dimensional crack problems in gradient elasticity. J. Mech. Behav. Mater. **7**, 93–118 (1996)
15. Falk, R.: Nonconforming finite element methods for the equation of linear elasticity. Math. Comput. **57**, 529–550 (1991)
16. Guzmán, J., Leykekhman, D., Neilan, M.: A family of non-conforming elements and the analysis of Nitsche's method for a singularly perturbed fourth order problem. Calcolo **49**, 95–125 (2012)
17. Hansbo, P., Larson, M.: Discontinuous Galerkin methods for incompressible and nearly incompressibly elasticity by Nitsche's method. Comput. Methods Appl. Mech. Eng. **191**, 1895–1908 (2002)
18. Korn, A.: Solution générale du problème d'équilibre dans la théorie de l'élasticité dans le cas où les efforts sont donnés à la surface. Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (2) **10**, 165–269 (1908)
19. Korn, A.: Über einige ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen. Bull. Intern. Cracov. Akad. Umiejetnosci. (Classe Sci Math Nat) pp 706–724 (1909)
20. Ma, Q., Clarke, D.: Size dependent hardness of silver single crystals. J. Mater. Res. **10**, 853–863 (1995)
21. Mardal, K., Winther, R.: An observation on Korn's inequality for nonconforming finite element methods. Math. Comput. **75**, 1–6 (2006)
22. Mardal, K., Tai, X., Winther, R.: A robust finite element method for Darcy–Stokes flow. SIAM J. Numer. Anal. **40**, 1605–1631 (2002)
23. Mindlin, R.: Microstructure in linear elasticity. Arch. Rational Mech. Anal. **10**, 51–78 (1964)

Journal: 211 Article No.: 0890 TYPESET DISK LE CP Disp.:2017/4/29 Pages: 21 Layout: Small-X

24. Mindlin, R., Eshel, N.: On the first strain-gradient theories of linear elasticity. Int. J. Solid Struct. **4**, 109–124 (1968)
25. Ming, P.B., Shi, Z.C.: Nonconforming rotated Q_1 element for Reissner–Mindlin plate. Math. Model Methods Appl. Sci. **11**, 1311–1342 (2001)
26. Nilssen, T., Tai, X., Winther, R.: A robust nonconforming H^2-element. Math. Comput. **70**, 489–505 (2001)
27. Papanastasiou, S.A., Zervos, A., Vardoulakis, I.: A three-dimensional C^1 finite element for gradient elasticity. Int. J. Numer. Eng. **135**, 1396–1415 (2009)
28. Ru, C., Aifantis, E.: A simple approach to solve boundary-value problems in gradient elasticity. Acta Mech. **101**, 59–68 (1993)
29. Soh, A.K., Chen, W.: Finite element formulations of strain gradient theory for microstructures and the C^{0-1} patch test. Int. J. Numer. Methods Eng. **61**, 433–454 (2004)
30. Wang, L., Wu, Y., Xie, X.: Uniformly stable rectangular elements for fourth order elliptic singular perturbation problems. Numer. Methods Partial Differ. Equ. **29**, 721–737 (2013)
31. Wei, Y.: A new finite element method for strain gradient theories and applications to fracture analyses. Eur. J. Mech. A Solids **25**, 897–913 (2006)
32. Zervos, A., Papanastasiou, P., Vardoulakis, I.: Modelling of localisation and scale effect in thick-walled cylinders with gradient elastoplasticity. Int. J. Soilds Struct. **38**, 5081–5095 (2001)
33. Zervos, A., Papanastasiou, S.A., Vardoulakis, I.: Two finite element discretizations for gradient elasticity. J. Eng. Mech. **135**, 203–213 (2009)

Journal: **211** Article No.: **0890** TYPESET DISK LE CP Disp.:**2017/4/29** Pages: **21** Layout: **Small-X**